

# On the dimension of bivariate periodic spline spaces type-1 triangulation

**Citation for published version (APA):**

Morsche, ter, H. G. (1988). *On the dimension of bivariate periodic spline spaces type-1 triangulation*. (RANA : reports on applied and numerical analysis; Vol. 8805). Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/1988

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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**Eindhoven University of Technology**  
**Department of Mathematics and Computing Science**

RANA 88-05

April 1988

**ON THE DIMENSION OF BIVARIATE  
PERIODIC SPLINE SPACES  
TYPE-1 TRIANGULATION**

by

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# On the dimension of bivariate periodic spline spaces type-1 triangulation

by

H.G. ter Morsche

## 1. Introduction

The extensive literature on spline functions shows that the problem of periodic spline interpolation is thoroughly studied in the univariate case, especially when the knots are equally spaced on the real line. In this case the number of knots in one period equals the dimension of the space of periodic spline functions, and the interpolating periodic spline function can be computed without matrix inversions. Moreover, the Fast Fourier transform is a powerful tool in these circumstances.

The main reason that the theory of univariate spline functions with equidistant knots is so elegant is based on the fact that the corresponding spline-space is spanned by translates of a fixed compactly supported spline function, known as a B-spline function. The situation differs completely when the problem is lifted to more dimensions. Even the most straight forward generalization may cause difficulties. For instance, let us consider double-periodic bivariate cubic spline functions with global smoothness  $C^1$  on a rectangular mesh in  $\mathbb{R}^2$ .

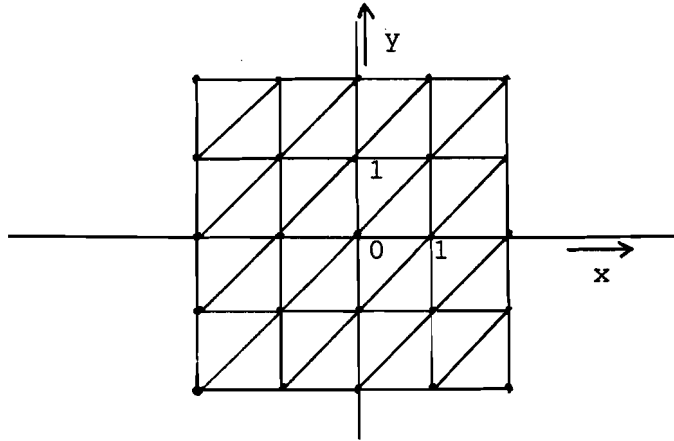
By a rectangular mesh, we mean a subdivision of the  $x$ - $y$  plane in squares bounded by the mesh-lines  $x = \mu_1$ ,  $y = \mu_2$ , which pass through the lattice points  $\underline{\mu} = (\mu_1, \mu_2) \in \mathbb{Z}^2$ .

On each square of the rectangular mesh a cubic spline function coincides with a bivariate polynomial of degree at most three, and it is differentiable on  $\mathbb{R}^2$  with continuous first order partial derivatives.

It turns out that in this case the dimension of the space of  $(m, n)$ -periodic cubic splines, i.e., cubic spline functions  $s$  for which  $s(x+m, y) = s(x, y) = s(x, y+n)$  for all  $(x, y) \in \mathbb{R}^2$  with integer  $m$  and  $n$ , is equal to  $2m + 2n + 1$ . So, if we like to interpolate an  $(m, n)$ -periodic function at the lattice points  $\underline{\mu} \in \mathbb{Z}^2$  then the number of interpolation conditions  $m \cdot n$  differs generally from the dimension of the space of candidates for the interpolating function; the answer to the

question of unisolvence of the interpolation problem must unfortunately be negative.

The aim of this paper is to compute the dimension of various spaces of  $(m, n)$ -periodic spline functions with respect to a three direction mesh, also called a type-1 triangulation, as given below



The mesh lines defines a uniform triangulation of  $\mathbb{R}^2$ .

Evidently, the dimension we are looking for depends on the degree and the smoothness of the bivariate spline functions. Therefore, we introduce the spline space  $S_k^\rho$  as follows.

**Definition 1.1.** Let  $k$  and  $\rho$  be nonnegative integers. Then a spline function  $s$  belongs to  $S_k^\rho$  if and only if

- i) on each triangle of the type-1 triangulation,  $s$  coincides with a bivariate polynomial of total degree at most  $k$ .
- ii)  $s \in C^\rho(\mathbb{R}^2)$ , i.e., all the partial derivatives of  $S$  up to the order  $\rho$  exist and are continuous on  $\mathbb{R}^2$ .

Our interest is focussed on the  $(m, n)$ -periodic functions in  $S_k^\rho$ . The set of these functions will be denoted by  $S_{k,\underline{n}}^\rho$ , where  $\underline{n} = (n, m)$ .

It appears that as in the one-dimensional situation (cf Schoenberg [3]) the so-called exponential or eigen splines play an important rôle to understand the space  $S_k^\rho$ . The exponential splines may be interpreted as eigen functions of some shift operators; they can be defined as follows. If  $(z_1, z_2)$  is a pair of complex numbers and  $s \in S_k^\rho$  a nontrivial spline function satisfying

$$(1.1) \quad \begin{aligned} s(x+1, y) &= z_1 s(x, y), \\ s(x, y+1) &= z_2 s(x, y), \end{aligned} \quad (x, y) \in \mathbb{R}^2$$

then  $s$  is called an *exponential spline*.

The importance of the exponential spline will be shown in the next section. There we shall prove that  $S_{k,\underline{n}}^\rho$  is spanned by a basis consisting only of  $\underline{n}$ -periodic exponential splines. For the computation of  $\dim S_{k,\underline{n}}^\rho$ , the dimension of the space  $S_{k,\underline{n}}^\rho$ , we are thus interested in the quantity

$$(1.2) \quad \dim S_{z_1, z_2},$$

denoting the dimension of the subspace  $S_{z_1, z_2} \subset S_k^p$  of exponential spline functions with  $(z_1, z_2)$  fixed. This quantity will be computed in Section 3. Finally, a complete answer to our problem of the computation of  $\dim S_{k, n}^p$  will be given in the final section.

## 2. Exponential splines

The discrete Fourier transform, a basic tool in translation invariant periodic function spaces, can be applied in our problem with success. The bivariate discrete Fourier transform of an  $\underline{n}$ -periodic sequence ( $\underline{n} = (n, m)$ ) of (complex) numbers  $(a_{\underline{\mu}})$  ( $\underline{\mu} \in \mathbb{Z}^2$ ) is an  $\underline{n}$ -periodic sequence  $(Fa)_{\underline{\mu}}$  given by

$$(2.1) \quad (Fa)_{\underline{\mu}} = \frac{1}{N} \sum_{\underline{v} \in R_{\underline{n}}} a_{\underline{v}} e^{-2\pi i (\underline{v}, \underline{\mu}/\underline{n})},$$

where the summation runs over the square.

$$(2.2) \quad R_{\underline{n}} = \{ \underline{v} = (v_1, v_2) \in \mathbb{Z}^2 \mid 0 \leq v_1 \leq m-1, 0 \leq v_2 \leq n-1 \}.$$

Moreover,  $(\underline{v}, \underline{\mu}/\underline{n}) := \frac{v_1 \mu_1}{m} + \frac{v_2 \mu_2}{n}$  and  $N := nm$ .

It is well known that the inverse discrete Fourier transform can be written as follows

$$a_{\underline{\mu}} = \sum_{\underline{v} \in R_{\underline{n}}} (Fa)_{\underline{v}} e^{2\pi i (\underline{v}, \underline{\mu}/\underline{n})}.$$

Now let  $s \in S_{k, n}^p$ , and let the function  $(Fs)_{\underline{\mu}}$  be defined by

$$(Fs)_{\underline{\mu}}(x, y) = \frac{1}{N} \sum_{\underline{v} \in R_{\underline{n}}} s(x-v_1, y-v_2) e^{-2\pi i (\underline{v}, \underline{\mu}/\underline{n})}$$

It can easily be verified that  $(Fs)_{\underline{\mu}} \in S_{k, n}^p$ . But the more important observation is based on the following relations:

$$(2.3) \quad \begin{cases} (Fs)_{\underline{\mu}}(x+1, y) = e^{2\pi i \mu_1/m} (Fs)_{\underline{\mu}}(x, y), \\ (Fs)_{\underline{\mu}}(x, y+1) = e^{2\pi i \mu_2/n} (Fs)_{\underline{\mu}}(x, y). \end{cases}$$

In other words, if  $(Fs)_{\underline{\mu}}$  is not the null function, then  $(Fs)_{\underline{\mu}}$  is an  $\underline{n}$ -periodic exponential spline function with  $z_1 = e^{2\pi i \mu_1/m}$  and  $z_2 = e^{2\pi i \mu_2/n}$ .

The function  $s$  can be recovered from the functions  $(Fs)_{\underline{\mu}}$  ( $\underline{\mu} \in R_{\underline{n}}$ ) by inverse discrete Fourier transform.

$$s(x_1, x_2) = \sum_{v \in R_n} (F s)_v(x_1, x_2).$$

In fact, we have shown that the set of  $n$ -periodic exponential splines in  $S_k^p$  spans the space  $S_{k,n}^p$  in the sense that every  $s \in S_{k,n}^p$  is a finite linear combination of  $n$ -periodic exponential splines. The next question is to compute the number of elements in an independent set of  $n$ -periodic exponential splines that spans the space  $S_{k,n}^p$ . For this, we will investigate the subspace  $S_{z_1, z_2}$  in  $S_k^p$  in more detail.

### 3. The space $S_{z_1, z_2}$

The contents of this section is rather technical; we do not explain all the details. The basic idea is to represent an arbitrary polynomial  $p \in \Pi_k$ , i.e., the set of bivariate polynomials of total degree at most  $k$ , by means of univariate (generalized) Euler-Frobenius polynomials or Bernoulli polynomials. Let  $z \in \mathbb{C}$  be given with  $z \neq 1$ , and let  $i$  be a positive integer. We introduce here the Euler-Frobenius polynomial  $x \mapsto E_i(z, x)$  of degree  $i$  as the unique  $i$ -th degree polynomial satisfying

$$(3.1) \quad E_i(z, x+1) - z E_i(z, x) = \frac{x^i}{i!} (1-z).$$

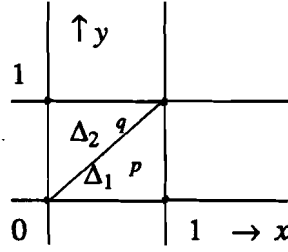
The Bernoulli polynomial  $x \mapsto B_i(x)$  is defined here as the unique polynomial of degree  $i$  satisfying

$$(3.2) \quad \begin{cases} B_0(x) \equiv 1, \\ B_i(x+1) - B_i(x) = \frac{x^{i-1}}{(i-1)!} \quad (i \geq 1), \\ \int_0^1 B_i(x) dx = 0 \quad (i \geq 1). \end{cases}$$

For notational purposes, it is sometimes convenient to write  $E_i(1, x) = B_i(x)$ . If  $(z_1, z_2)$  is a pair of complex numbers and  $p \in \Pi_k$ , then  $p$  can be written as

$$(3.3) \quad p(x, y) = \sum_{\alpha_i + j \leq k} \alpha_{i,j} E_i(z_1, x) E_j(z_2, y).$$

Now let  $s$  be an exponential spline function in  $S_k^p$  and  $p$  and  $q$  its polynomial pieces on the triangles  $\Delta_1$  and  $\Delta_2$  in the three direction mesh as given in the figure below.



Since  $s$  is an exponential spline function in  $C^p(\mathbb{R}^2)$ , it is necessary that the following conditions are satisfied.

$$(3.4) \quad \begin{cases} (i) & (\partial_x^l p)(1, y) = z_1 (\partial_x^l q)(0, y), \\ (ii) & (\partial_y^l q)(x, 1) = z_2 (\partial_y^l p)(x, 0), (l=0, 1, \dots, p), \\ (iii) & q(x, y) = p(x, y) + r(x, y) \\ & \text{with } r(x, y) = (x-y)^{p+1} r_1(x, y) \text{ and } r_1 \in \Pi_{k-p-1} \end{cases}$$

The partial derivatives to  $x$  and  $y$  are denoted by  $\partial_x$  and  $\partial_y$ , respectively.

On the other hand, if  $p$  and  $q$  satisfy (3.4) then due to (1.1) there is a unique exponential spline  $s$  which coincides with the polynomial  $p$  on  $\Delta_1$  and with  $q$  on  $\Delta_2$ .

It follows from (3.4) and the representation (3.3) that

$$(3.5) \quad \begin{cases} \sum_{0 \leq i+j \leq k} \alpha_{i,j} (E_i^{(l)}(z_1, 1) - z_1 E_i^{(l)}(z_1, 0)) E_j(z_2, y) = z_1 (\partial_x^l r)(0, y), \\ \sum_{0 \leq i+j \leq k} \alpha_{i,j} (E_j^{(l)}(z_2, 1) - z_2 E_j^{(l)}(z_2, 0)) E_i(z_1, x) = -(\partial_y^l r)(x, 1), \\ (l=0, 1, \dots, p). \end{cases}$$

These relations may be simplified by using the properties of the polynomials  $E_i$ . To do so we must distinguish the two cases:

$A : z_1 \neq 1$  and  $z_2 \neq 1$ ,  $B : z_1 = 1$  or  $z_2 = 1$ .

$A : z_1 \neq 1$  and  $z_2 \neq 1$ .

The polynomials  $E_i(z_1, x)$  and  $E_i(z_2, y)$  are both Euler-Frobenius polynomials. So, by using (3.1), we may write

$$(3.6) \quad E_i^{(l)}(z, 1) - z E_i^{(l)}(z, 0) = \begin{cases} 0 & (l \neq i), \\ 1-z & (l=i). \end{cases}$$

Relations (3.5) can now be replaced by

$$\left\{ \begin{array}{l} (1-z_1) \sum_{j=0}^{k-l} \alpha_{l,j} E_j(z_2, y) = z_1 (\partial_x^l r)(0, y), \\ (1-z_2) \sum_{i=0}^{k-l} \alpha_{i,l} E_i(z_1, x) = -(\partial_y^l r)(x, 1) \quad (l=0, 1, \dots, \rho), \\ r(x, y) = (x-y)^{\rho+1} r_1(x, y), \quad r_1 \in \Pi_{k-\rho-1} \end{array} \right.$$

Even a further simplification can be made by using (3.6) again. We obtain

$$(3.7) \quad \left\{ \begin{array}{l} (1-z_1)(1-z_2) \alpha_{l,j} = z_1 \{ \partial_y^j \partial_x^l r(0, 1) - z_2 \partial_y^j \partial_x^l r(0, 0) \}, \\ (1-z_1)(1-z_2) \alpha_{i,l} = z_1 \partial_y^l \partial_x^i r(0, 1) - \partial_y^l \partial_x^i r(1, 1), \\ l=0, 1, \dots, \rho; j=0, 1, \dots, k-l; i=0, 1, \dots, k-l, \\ r(x, y) = (x-y)^{\rho+1} r_1(x, y), \quad r_1 \in \Pi_{k-\rho-1}. \end{array} \right.$$

Let us formulate what we have reached up to now. If  $p$  and  $q$  are the polynomial pieces of an exponential spline in  $S_k^\rho$  based on the pair  $(z_1, z_2)$ , then numbers  $\alpha_{i,j}$  and a polynomial  $r_1 \in \Pi_{k-\rho-1}$  exist such that (3.7) is satisfied. On the other hand, if we can find numbers  $\alpha_{i,j}$  and a polynomial  $r_1 \in \Pi_{k-\rho-1}$  satisfying (3.7), then a polynomial  $p$  can be constructed by means of (3.3), and subsequently a polynomial  $q$  by means of (3.4) iii. These polynomials are used as a restriction of a spline function  $s$  to the two triangles  $\Delta_1$  and  $\Delta_2$ . The function  $s$  can then be extended to the whole  $x-y$  plane on base of (1.1) giving an exponential spline in  $S_k^\rho$ .

If  $r_1$  is known then it follows immediately from (3.7) that the numbers  $\alpha_{i,j}$  with  $i \leq \rho$  or  $j \leq \rho$  are determined. The remaining numbers  $\alpha_{i,j}$  ( $i+j \leq k, i \geq \rho+1, j \geq \rho+1$ ) can be chosen freely. If  $a_+ := \max(a, 0)$  then the total number of free  $\alpha_{i,j}$  is equal to

$$\frac{1}{2} (k-2\rho-1)_+ (k-2\rho).$$

In fact, we have computed the dimension of the space of exponential splines corresponding to the pair  $(z_1, z_2)$  for which the polynomial  $r$  vanishes. But, this is precisely the space of exponential splines corresponding to the rectangular mesh, where the diagonal mesh lines are omitted. Apparently, a basis of this space is given by

$$\{E_i(x, z_1)E_j(y, z_2) \mid i+j \leq k, i \geq \rho+1, j \geq \rho+1\}.$$

Note that in any case

$$\dim S_{z_1, z_2} \leq \dim \Pi_{k-\rho-1} + \frac{1}{2} (k-2\rho-1)_+ (k-2\rho),$$

which is equal to zero if  $\rho \geq k$ . This is not surprising; if  $\rho \geq k$  then  $S_k^\rho = \Pi_k$  and the space of polynomials contains no exponentials except the constants.

To each number  $\alpha_{i,j}$  there corresponds one or two equations in (3.7) involving  $\alpha_{i,j}$ . In order to avoid contradictions, we have to impose some conditions on the polynomial  $r_1$ . The numbers  $\alpha_{i,j}$  with  $i+j \leq k, i \leq \rho$  and  $j \leq \rho$  occur twice in (3.7). We conclude that a necessary and sufficient



condition for the existence of the numbers  $\alpha_{i,j}$  may be written in the following form

$$(3.8) \quad \begin{cases} L_{i,j} r = 0, \\ i = 0, 1, 2, \dots, \rho; j = 0, 1, 2, \dots, \rho; i + j \leq k, \\ r(x, y) = (x-y)^{\rho+1} r_1(x, y), r_1 \in \Pi_{k-\rho-1}. \end{cases}$$

where the linear functional  $L_{i,j}$  is defined by

$$(3.9) \quad L_{i,j} r = (\partial_x^i \partial_y^j r)(1, 1) - z_1 z_2 (\partial_x^i \partial_y^j r)(0, 0).$$

We wish to express  $L_{i,j} r$  in terms of the polynomial  $r_1$ . A straight forward computation will show that

$$(3.10) \quad L_{i,j} r = (\rho+1)! (-1)^j \sum_{i_1+j_1=i+j-\rho-1} \binom{i}{i_1} \binom{j}{j_1} (-1)^{j_1} L_{i_1, j_1} r_1$$

For some values of  $(i, j)$ , (3.10) is an empty sum; the corresponding condition  $L_{i,j} r = 0$  may then be omitted. The values of  $(i, j)$  giving a nonempty condition belong to the set

$$(3.11) \quad \Omega_k^\rho = \{(i, j) \in \mathbb{Z}^2 \mid \rho+1 \leq i+j \leq k, i \leq \rho, j \leq \rho\}$$

By setting  $\tilde{L}_{i,j} r_1 = L_{i,j} r$ , we define linear functionals  $\tilde{L}_{i,j}$  on the polynomial space  $\Pi_{k-\rho-1}$ . Due to (3.10) the linear functionals  $\tilde{L}_{i,j}$  can be expressed in terms of  $L_{i_1, j_1}$  as follows.

$$(3.12) \quad \tilde{L}_{i,j} = (-1)^j (\rho+1)! \sum_{i_1+j_1=i+j-\rho-1} \binom{i}{i_1} \binom{j}{j_1} (-1)^{j_1} L_{i_1, j_1}.$$

It is clear from the foregoing discussion that

$$(3.13) \quad \dim S_{z_1, z_2} = \dim \Pi_{k-\rho-1} - \dim \langle \tilde{L}_{i,j} \mid (i, j) \in \Omega_k^\rho \rangle + \frac{1}{2} (k-2\rho-1)_+ (k-2\rho)$$

Here  $\langle \tilde{L}_{i,j} \mid (i, j) \in \Omega_k^\rho \rangle$  is the space of linear functionals on  $\Pi_{k-\rho-1}$  spanned by the set  $\{\tilde{L}_{i,j} \mid (i, j) \in \Omega_k^\rho\}$ .

Since, every  $\tilde{L}_{i,j}$  belongs to  $\langle L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k - \rho - 1 \rangle$ , we shall investigate the dependence of the functionals  $L_{i,j}$  first. We have to distinguish two cases now to wit :  $z_1 z_2 \neq 1$  and  $z_1 z_2 = 1$ .

**Lemma 3.1.**

If  $z_1 z_2 \neq 1$  then  $\{L_{i,j} \mid 0 \leq i_1 + j_1 \leq k - \rho - 1\}$  is an independent set of functionals on  $\Pi_{k-\rho-1}$ .

**Proof.** Note that  $\dim \Pi_{k-\rho-1} = \#\{L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k - \rho - 1\}$ , so it suffices to prove that if  $q \in \Pi_{k-\rho-1}$  and  $L_{i_1, j_1} q = 0$  for all  $(i_1, j_1)$ , then  $q \equiv 0$ . So, let  $g \in \Pi_{k-\rho-1}$  be such that  $L_{i_1, j_1} g = 0$  for all  $(i_1, j_1)$ . Then all the partial derivatives of the polynomial  $g(x+1, y+1) - z_1 z_2 g(x, y)$  vanish at  $(0, 0)$ . Hence,  $g(x+1, y+1) = z_1 z_2 g(x, y)$  for all  $(x, y)$ . Since  $z_1 z_2 \neq 1$ , the polynomial  $g$  must be identically zero.  $\square$

In the previous proof, the assumption  $z_1 z_2 \neq 1$  is used only for the implication:  $g(x+1, y+1) = z_1 z_2 g(x, y)$  for all  $(x, y)$  then  $g \equiv 0$ . If  $z_1 z_2 = 1$ , then it can easily be shown that the property  $g(x+1, y+1) = g(x, y)$  for all  $(x, y)$  implies  $g(x, y) = h(x-y)$  where  $h$  is a univariate polynomial of degree at most  $k-\rho-1$ . We denote by  $V$  the space of all these polynomials  $g$  in  $\Pi_{k-\rho-1}$ . Evidently,  $\dim V = k-\rho$ . If  $i_1 + j_1 = k-\rho-1$  then  $\partial_x^{i_1} \partial_y^{j_1} r_1$  is a constant. In this case the functional  $L_{i_1, j_1}$  is the null functional on  $\Pi_{k-\rho-1}$ . The set  $\{L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k-\rho-1\}$  is dependent. However if we delete the null functionals, then according to the following lemma the remaining set will be independent.

Lemma 3.2.

If  $z_1 z_2 = 1$  then  $\{L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k-\rho-2\}$  is an independent set of functionals on  $\Pi_{k-\rho-1}$ .

Proof Let  $g \in \Pi_{k-\rho-1}$  be such that  $L_{i_1, j_1} g = 0$  for all  $0 \leq i_1 + j_1 \leq k-\rho-2$ . Then all the partial derivatives of the polynomial  $g(x+1, y+1) - g(x, y)$  vanish at  $(0, 0)$ . Hence  $g \in V$ . Therefore,  $\dim \langle L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k-\rho-2 \rangle = \dim \Pi_{k-\rho-1} - \dim V = \binom{k-\rho}{2}$ , i.e., the number of elements in  $\{L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k-\rho-2\}$ . This proves the Lemma.  $\square$

Now we return to the functionals  $\tilde{L}_{i, j}$ . They are described as linear combinations of  $L_{i_1, j_1}$  with  $i_1 + j_1 = i + j - \rho - 1$  (cf.3.12). So  $\tilde{L}_{i, j}$  is the null functional for  $i + j = k, z_1 z_2 = 1$ . This means that the conditions for  $r_1 \in \Pi_{k-\rho-1}$  may be summed up by

$$(3.14) \quad \begin{cases} \tilde{L}_{i, j} r_1 = 0 & ((i, j) \in \Omega_k^{\rho}; z_1 z_2 \neq 1), \\ \tilde{L}_{i, j} r_1 = 0 & ((i, j) \in \Omega_{k-1}^{\rho}; z_1 z_2 \neq 1). \end{cases}$$

We subdivide the collection of conditions into disjoint sets  $\Lambda_c$  ( $c = \rho + 1, \dots, k$ ) given as follows.

$$\Lambda_c = \{\tilde{L}_{i, j} \mid i + j = c, i \leq \rho, j \leq \rho\}$$

As a consequence of Lemma 3.1 and 3.2 linear functionals stemming from different nonempty sets  $\Lambda_c$  are independent. Therefore, we have to consider linear functionals  $\tilde{L}_{i, j}$  within one definite nonempty set  $\Lambda_c$ . A functional  $\tilde{L}_{i, j}$  in  $\Lambda_c$  can be written as

$$(3.15) \quad \tilde{L}_{i, j} = (-1)^j (\rho + 1)! \sum_{i_1 + j_1 = c - \rho - 1} P_{j_1}(i) L_{i_1, j_1}$$

where  $P_{j_1}$  is a polynomial of degree  $c - \rho - 1$ ;

$$P_{j_1}(t) = \frac{(-1)^{j_1} t(t-1) \cdots (t-i_1+1)(c-t) \cdots (c-t-j_1+1)}{i_1! j_1!} \quad (t \in \mathbb{R}),$$

$$(i_1 + j_1 = c - \rho - 1).$$

The polynomials  $P_{j_1}$  are independent, which can be shown as follows. Let  $\sum_{j_1=0}^{c-\rho-1} \alpha_{j_1} P_{j_1}(t) = 0$  for

all  $t \in \mathbb{R}$ . Then by a successive substitution of  $t=c, t=c-1, \dots, t=\rho+2$  we find  $\alpha_0 = \alpha_1 = \dots = \alpha_{c-\rho-2} = 0$ , and thus  $\alpha_{c-\rho-1} = 0$ . This proves the independence of the polynomials  $P_{j_1}$ .

Since every functional in  $\Lambda_c$  is a linear combination of the independent functionals  $\{L_{i_1, j_1} \mid i_1 + j_1 = c - \rho - 1\}$ , we have that  $\dim \langle \tilde{L}_{i,j} \mid \tilde{L}_{i,j} \in \Lambda_c \rangle \leq c - \rho$ . On the other hand there holds that  $\dim \langle \tilde{L}_{i,j} \mid \tilde{L}_{i,j} \in \Lambda_c \rangle \leq |\Lambda_c|$ , where  $|\Lambda_c|$  denotes the number of elements in  $\Lambda_c$ . We shall show now that indeed

$$(3.16) \quad \dim \langle \tilde{L}_{i,j} \mid \tilde{L}_{i,j} \in \Lambda_c \rangle = \min (|\Lambda_c|, c - \rho).$$

For this, it is sufficient to prove that any collection of no more than  $c - \rho$  functionals in  $\Lambda_c$  is independent. So let  $\{\tilde{L}_{i,j} \in \Lambda_c \mid i \in I\}$  be such that  $|I| \leq c - \rho$ , and assume  $\sum_{i \in I} \alpha_i \tilde{L}_{i,j} r_1 = 0$  for all  $r_1 \in \Pi_{k-\rho-1}$ . Then

$$\sum_{i_1+j_1=c-\rho-1} (L_{i_1, j_1} r_1) \sum_{i \in I} \alpha_i P_{j_1}(i) = 0 \quad (r_1 \in \Pi_{k-\rho-1}).$$

Because of the independence of the functionals  $L_{i_1, j_1}$  one has

$$\sum_{i \in I} \alpha_i P_{j_1}(i) = 0 \quad (j_1 = 0, 1, \dots, c - \rho - 1).$$

However, the polynomials  $P_{j_1}$  are independent and  $|I| \leq c - \rho$ , which implies  $\alpha_i = 0$  ( $i \in I$ ). So we have proved relation (3.16).

Relation (3.16) combined with (3.13) gives the expression:

$$(3.17) \quad \dim S_{z_1, z_2} = \frac{1}{2} (k - 2\rho - 1)_+ (k - 2\rho) + \dim \Pi_{k-\rho-1} - \begin{cases} \sum_{c=\rho+1}^k \min (|\Lambda_c|, c - \rho) (z_1 z_2 \neq 1), \\ \sum_{c=\rho+1}^{k-1} \min (|\Lambda_c|, c - \rho) (z_1 z_2 = 1). \end{cases}$$

The last part of our treatment of the case  $z_1 \neq 1$  and  $z_2 \neq 1$  consists of an evaluation of the expressions in (3.17). It turns out that

$$(3.18) \quad \sum_{c=\rho+1}^k \min (|\Lambda_c|, c - \rho) = \frac{1}{2} (k - \rho)_+ (k - \rho + 1) - (k - k_\rho)_+ (k_\rho + k - 3\rho) + \frac{1}{2} (k - 2\rho)_+ (k - 2\rho - 1),$$

where  $k_\rho = [\frac{3\rho+1}{2}]$ , i.e., the integer part of  $\frac{1}{2} (3\rho + 1)$ .

**Lemma 3.3.** For all  $k = 1, 2, \dots, ; \rho = 0, 1, 2, \dots$ , and  $(z_1, z_2) \in \mathbb{C}^2$  with  $z_1 \neq 1$  and  $z_2 \neq 1$ , there holds

$$\dim S_{z_1, z_2} = \begin{cases} (k-k_\rho)_+ (k_\rho+k-3\rho) & (z_1 z_2 \neq 1) \\ (k-1-k_\rho)_+ (k_\rho+k-1-3\rho) + (k-\rho)_+ + (k-2\rho-1)_+ & (z_1 z_2 = 1). \end{cases}$$

The previous theorem is the ultimate result in the case  $z_1 \neq 1, z_2 \neq 1$ . Now we will continue with the situation  $z_1 \neq 1$  or  $z_2 = 1$ .

$B : z_1 = 1$  or  $z_2 = 1$

First, we consider the case  $z_1 z_2 \neq 1$ . Hence  $z_1 \neq 1$  or  $z_2 \neq 1$ . Without any restriction we may assume  $z_1 \neq 1, z_2 = 1$ . The transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

maps the type-1 triangulation onto itself, moreover, if  $s(x, y)$  is an exponential spline in  $S_k^\rho$  based on the pair  $(z_1, z_2)$ , then  $s(T(x, y))$  is again an exponential spline in  $S_k^\rho$  based on  $(z_1, (z_1 z_2)^{-1}) = (z_1, z_1^{-1})$ . Since  $z_1 \neq 1, z_1 z_1^{-1} = 1$ , it is clear that Lemma 3.3 may be applied. The result is

$$\dim S_{z_1, z_2} = (k-1-k_\rho)_+ + (k_\rho+k-1-3\rho) + (k-\rho)_+ + (k-2\rho-1)_+ .$$

The situation  $z_1 = z_2 = 1$  differs from the previous cases; it can not be obtained by a simple transformation. Instead of generalized Euler-Frobenius polynomials, we have to use the Bernoulli polynomials  $B_i$  (cf. (3.2)). From (3.4) it follows that a sufficient and necessary condition for the polynomial pieces  $p$  and  $q$ ,

$$p(x, y) = \sum_{0 \leq i+j \leq k} \alpha_{i,j} B_i(x) B_j(y),$$

$$q = p + r$$

to be a restriction of an exponential spline  $s$  with  $z_1 = z_2 = 1$  is given by

$$\left\{ \begin{array}{l} \sum_{0 \leq i+j \leq k} \alpha_{i,j} (B_i^{(l-1)}(1) - B_i^{(l-1)}(0)) B_j(y) = (\partial_x^{l-1} r)(0, y), \\ \sum_{0 \leq i+j \leq k} \alpha_{i,j} (B_j^{(l-1)}(1) - B_j^{(l-1)}(0)) B_i(x) = -(\partial_y^{l-1} r)(x, 1), \\ l = 1, \dots, \rho + 1, \\ r(x, y) = (x-y)^{\rho+1} r_1(x, y). \end{array} \right.$$

By using the property that  $B_i^{(l-1)}(1) - B_i^{(l-1)}(0) = 0$  ( $i \neq l$ ),  $B_0^{(l-1)}(1) = B_0^{(l-1)}(0)$  for all  $l$ , we may replace the system above by the equivalent system

$$\left\{ \begin{array}{l} \sum_{j=0}^{k-l} \alpha_{l,j} B_j(y) = (\partial_x^{l-1} r)(0, y), \\ \sum_{i=0}^{k-l} \alpha_{i,l} B_i(x) = -(\partial_y^{l-1} r)(x, 1), \\ l = 1, \dots, \rho + 1, \\ r(x, y) = (x-y)^{\rho+1} r_1(x, y). \end{array} \right.$$

A further simplification can be made, by using (3.3) again. We have

$$(3.19) \quad \left\{ \begin{array}{l} \alpha_{l,j} = (\partial_y^{j-1} \partial_x^{l-1} r)(0, 1) - (\partial_y^{j-1} \partial_x^{l-1} r)(0, 0), \\ \alpha_{i,l} = (\partial_x^{i-1} \partial_y^{l-1} r)(0, 1) - (\partial_x^{i-1} \partial_y^{l-1} r)(1, 1), \\ i, j = 1, \dots, k-l, \\ \alpha_{l,0} = \int_0^1 (\partial_x^{l-1} r)(0, y) dy, \\ \alpha_{0,l} = -\int_0^1 (\partial_y^{l-1} r)(x, 1) dx, \\ (\partial_x^{l-1} r)(0, y) \in \Pi_{k-l}, (\partial_y^{l-1} r)(x, 1) \in \Pi_{k-l} \\ l = 1, \dots, \rho + 1, \\ r(x, y) = (x-y)^{\rho+1} r_1(x, y). \end{array} \right.$$

Let us first consider the solutions for the rectangular mesh, i.e., solutions corresponding to  $r = 0$ .

Then

$$\alpha_{l,0} = \alpha_{0,l} = 0 \quad (l = 1, \dots, \rho + 1), \text{ and,}$$

$$\alpha_{l,j} = \alpha_{i,l} = 0 \quad (l = 1, \dots, \rho + 1; i = 1, \dots, k-l; j = 1, \dots, k-l)$$

The remaining  $\alpha_{i,j}$  can be chosen freely, the total number of which is given by

$$(3.20) \quad 1 + 2(k - \rho - 1)_+ + \frac{1}{2}(k - 2\rho - 3)_+(k - 2\rho - 2),$$

which is equal to the dimension of the space of the exponential splines ( $z_1 = z_2 = 1$ ) with respect to the rectangular mesh.

Note that the numbers  $\alpha_{i,j}$  for  $i = 1, \dots, \rho + 1, j = 1, \dots, \rho + 1$  and  $i + j \leq k$  occur twice in (3.19). Therefore, a necessary and sufficient condition for the polynomial  $r$  to be a solution of (3.19) is given by

$$(3.21) \quad \left\{ \begin{array}{l} r(x, y) = (x-y)^{\rho+1} r_1(x, y), \\ (\partial_x^i r)(0, y) \in \Pi_{k-i-1}, (\partial_y^j r)(x, 1) \in \Pi_{k-i-1}, j, j = 0, 1, \dots, \rho. \\ \partial_x^i \partial_y^j r(1, 1) = \partial_x^i \partial_y^j r(0, 0), \\ i = 0, 1, \dots, \rho; j = 0, 1, \dots, \rho \end{array} \right.$$

The next step is to convert the conditions (3.21) for the polynomial  $r$  into conditions for the polynomial  $r_1$ . The computation is straight forward. We only present here the result. It turns out that

$r_1$  must satisfy the conditions

$$(3.22) \quad \begin{cases} \partial_x^i \partial_y^j r_1(0, 0) = 0, \\ i \leq \rho \text{ or } j \leq \rho, i + j = k - \rho - 1, \\ \tilde{L}_{i,j} r_1 = 0 \\ i, j = 0, 1, \dots, \rho; i + j \leq k - 2. \end{cases}$$

Here the functional  $\tilde{L}_{i,j}$  is defined in (3.12), where

$L_{i_1, j_1} r_1 = \partial_x^{i_1} \partial_y^{j_1} r_1(1, 1) - \partial_x^{i_1} \partial_y^{j_1} r_1(0, 0)$ . Let the functional  $L'_{i,j}$  be defined by

$$L'_{i,j} r_1 = \partial_x^i \partial_y^j r_1(0, 0).$$

We prove now that the collection of functionals

$$\{L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k - \rho - 3\} \cup \{L'_{i,j} \mid i \leq \rho \text{ or } j \leq \rho, i + j = k - \rho - 1\}$$

is independent on  $\Pi_{k-\rho-1}$ . As a consequence of Lemma 3.2 one has that the collection  $\{L_{i_1, j_1} \mid 0 \leq i_1 + j_1 \leq k - \rho - 3\}$  is independent on  $\Pi_{k-\rho-2}$ . Note that  $L'_{i,j} g = 0$  for all  $g \in \Pi_{k-\rho-2}$ .

So, we only have to prove that the collection

$\{L_{i,j} \mid i \leq \rho \text{ or } j \leq \rho, i + j = k - \rho - 1\}$  is independent on  $\Pi_{k-\rho-1}$ . However, this follows easily from the observation that  $\{L'_{i,j} \mid i + j = k - \rho - 1\}$  is independent on  $\Pi_{k-\rho-1}$ .

The total number of functionals in  $\{L'_{i,j} \mid i \leq \rho \text{ or } j \leq \rho, i + j = k - \rho - 1\}$  is equal to  $(k - \rho)_+ - (k - 3\rho - 2)_+$ . Hence

$$\dim \langle L'_{i,j} \mid i \leq \rho \text{ or } j \leq \rho, i + j = k - \rho - 1 \rangle = (k - \rho)_+ - (k - 3\rho - 2)_+.$$

Since

$$\dim \langle \tilde{L}_{i,j} \mid 0 \leq j, j \leq \rho, i + j \leq k - 2 \rangle = \sum_{c=p+1}^{k-2} \min(|\Lambda_c|, c - \rho),$$

one has that for  $z_1 = z_2 = 1$

$$\begin{aligned} \dim S_{z_1, z_2} &= 1 + 2(k - \rho - 1)_+ + \frac{1}{2} (k - 2\rho - 3)_+ (k - 2\rho - 2) + \dim \Pi_{k-\rho-1} + \\ &\quad - (k - \rho)_+ + (k - 3\rho - 2)_+ - \sum_{c=p}^{k-2} \min(|\Lambda_c|, c - \rho). \end{aligned}$$

By using (3.18) this expression can be simplified. We have

$$\dim S_{z_1, z_2} = 1 + 3(k - \rho - 1)_+ + (k - 3\rho - 2)_+ + (k - k_\rho - 2)(k_\rho + k - 3\rho - 2).$$

This completes case B.

We summarize our results in the next theorem.

**Theorem 3.4.** For all  $k = 1, 2, \dots$ ;  $\rho = 0, 1, 2, \dots$ , and  $(z_1, z_2) \in \mathcal{C}$  there holds

$$\dim S_{z_1, z_2} = \begin{cases} (k - k_\rho)_+ (k_\rho + k - 3\rho) & (1 \notin \{z_1, z_2, z_3\}), \\ (k - \rho)_+ + (k - 1 - k_\rho)_+ (k_\rho + k - 1 - 3\rho) & (1 \text{ occurs once in } \{z_1, z_2, z_3\}), \\ 1 + 3(k - \rho - 1)_+ + (k - 3\rho - 2)_+ + (k - k_\rho - 2)_+ (k_\rho + k - 3\rho - 2) & (\{1\} = \{z_1, z_2, z_3\}). \end{cases}$$

The comparison between smoothness and degree of a spline function is of importance with respect to the existence of spline functions in  $S_k^\rho$  having a compact support.

If  $s$  is a compactly supported spline function then

$$\sum_{(v_1, v_2) \in \mathbf{Z}^2} s(x - v_1, y - v_2) z_1^{v_1} z_2^{v_2}$$

is a nontrivial spline function showing that  $\dim S_{z_1, z_2} > 0$  for all  $(z_1, z_2) \in \mathcal{C}^2$ . As a consequence of Theorem 3.4,  $k$  must then satisfy  $k > k_\rho$ . This observation can also be derived from results published in a paper of de Boor and Höllig [1], where the number  $k_\rho$  is fundamental for the study of the approximation order of  $S_k^\rho$ .

#### 4. The dimension of periodic spline spaces

It is now easy to compute the dimension of the space  $S_{k, \underline{n}}^\rho$  consisting of the  $\underline{n} = (n, m)$  periodic function in  $S_k^\rho$ . The space  $S_{k, \underline{n}}^\rho$  has a basis of exponential splines (cf. Section 2). Therefore, in order to compute  $\dim S_{k, \underline{n}}^\rho$ , we only have to add the dimensions of the spaces of exponential splines  $S_{z_1, z_2}$  where

$$(z_1, z_2) = (e^{2\pi i u/m}, e^{2\pi i v/n}) \quad (u = 0, 1, \dots, m-1; v = 0, 1, \dots, n-1).$$

The total number of pairs  $(u, v)$  corresponding to the various cases can be listed as follows

$$\begin{array}{ll} (m-1)(n-1) - (gcd(m, n) - 1), & (z_1 \neq 1, z_2 \neq 1, z_1 z_2 \neq 1), \\ n-1 & (z_1 = 1, z_2 \neq 1), \\ m-1 & (z_2 = 1, z_1 \neq 1), \\ gcd(m, n) - 1 & (z_1 z_2 = 1, z_1 \neq 1), \\ 1 & (z_1 = z_2 = 1). \end{array}$$

Here  $gcd(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ . Our final result is the contents of the next theorem.

**Theorem 4.1.** Let  $k = 0, 1, \dots$  and  $\rho = 0, 1, \dots$ . Then the dimension of the space of  $\underline{n}$ -periodic splines ( $\underline{n} = (n, m)$ ) in  $S_k^\rho$  is given by

$$\begin{aligned} \dim S_{k,n}^{\rho} = & (mn - n - m + 2 - g c d(m, n)) (k - k_{\rho})_+ (k_{\rho} + k - 3\rho) + \\ & (m + n + g c d(m, n) - 3) ((k - \rho)_+ + (k - 1 - k_{\rho})_+ (k_{\rho} + k - 1 - 3\rho)) + \\ & 1 + 3(k - \rho - 1)_+ + (k - 3\rho - 2)_+ + (k - k_{\rho} - 2)_+ (k_{\rho} + k - 3\rho - 2) . \end{aligned}$$

The complicated formula for the dimension of  $S_{k,n}^{\rho}$  can be simplified for the situation, when  $k = k_{\rho} + 1$ . This is the smallest degree  $k$  corresponding to a given smoothness  $\rho$  such that  $S_{k,n}^{\rho}$  contains finitely supported splines. Examples of finitely supported splines in  $S_{k,n}^{\rho}$  are the so-called Box-splines (cf. de Boor, Höllig [1]).

If  $k = k_{\rho} + 1$  then

$$\dim S_{k_{\rho}+1,n}^{\rho} = \begin{cases} mn + l(m + n + g c d(m, n)) & (\rho = 2l) , \\ 2mn + 2 + l(m + n + g c d(m, n)) & (\rho = 2l + 1) . \end{cases}$$

For periodic cubic splines in  $S_3^1$  ( $\rho = 1, k_{\rho} = 2$ ), this formula reduces to

$$\dim S_{3,n}^1 = 2mn + 2 ,$$

which agrees with a result given in Ter Morsche [2], where the problem of periodic bivariate spline interpolation is studied.

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