Roots, iterations and logarithms of formal automorphisms

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ROOTS, ITERATIONS AND LOGARITHMS

of formal automorphisms

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Abstract. In this paper it is proved that having a logarithm is equivalent to having roots of arbitrary order in the group of automorphisms of a formal power series ring, and in algebraic subgroups, too.

INTRODUCTION

The question whether an automorphism of a complex formal power series ring has an iteration, or may be embedded in a one parameter group has received much attention lately. The original setting concerned the possibility of embedding an automorphism in a complex analytic one dimensional Lie group (LEWIS [4, section 4] , STERNBERG [12, section 2]). In the recent terminology this is expressed as having a complex analytic iteration. At this moment the equivalence of the following statements has been proven:

THEOREM 1 *Let* F *be an automorphism of the complex formal power series ring* $\mathbb{C}[[x_1,\ldots,x_m]]$.*Then the following statements are equivalent:*

i F *has a complex analytic iteration.*

ii F *is conjugate to an automorphism in smooth normal form.*

iii F *has a real continuous iteration.*

iv F *has a rational continuous iteration.*

v F *is the exponential of a derivation of* $\mathbb{C}[[x_1,\ldots,x_m]]$

vi F *has pairwise commuting roots of all orders.*

vii F *is conjugate to an automorphism in normal form which has roots of all orders in normal form.*

The notions mentioned in this theorem are explained in the sequel. The proofs of the various equivalences may be found in: REICH - SCHWAIGER [11, satz 4] for *i* ⇔ *ii*, BUCHER [1] for *i* ⇔ *iii*, PRAAGMAN [7, Theorem 5] for *i* ⇔ *ii* ⇔ *iii* ⇔ *iv* ⇔ *v*, PRAAGMAN [8, Theorem 3] for *iv* ⇔ *v*, MEHRING [5, satz 1.10] for *i* ⇔ *vi*, and PRAAGMAN [10, theorem 6] for *v* ⇔ *vii*.

In the first section I shall prove that the condition pairwise commuting in *vi* may be deleted. In fact it will turn out that having roots of all orders is equivalent to having a rational iteration, which "lies between" *iv* and *vi*. In the remainder of the paper these results are generalized to algebraic subgroups of the automorphism group of a complete local ring.

§1 ROOTS AND ITERATIONS

DEFINITIONS. Denote the ring of formal power series over $\mathbb{C}$ in m indeterminates, $\mathbb{C}[[x_1,\ldots,x_m]]$ by $\mathcal{O}$, let $\mathfrak{m}$ be its maximal ideal, and equip $\mathcal{O}$ with the topology of coefficientwise convergence: $\sum v_\alpha(n)x^\alpha \to \sum v_\alpha x^\alpha$ for $n \to \infty$ if and only if $v_\alpha(n) \to v_\alpha$ for $\alpha \in \mathbb{N}_0^m$. Here the sum is taken over all m-tuples $\alpha \in \mathbb{N}_0^m$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$. Let L be the ring of all filtration preserving $\mathbb{C}$-linear mappings: if $L \in L$ then $L\mathfrak{m}^h \subset \mathfrak{m}^h$ for all $h \in \mathbb{N}$. Equip L with the topology of pointwise convergence, then L is complete and $A = \mathrm{Aut}_{\mathbb{C}}\mathcal{O}$ and $\mathcal{D} = \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathfrak{m})$ are closed subspaces. The map exp: $L \to L$, $\exp L = \sum_{n=0}^{\infty} L^n/n!$ is well-defined and maps $\mathcal{D}$ into A (PRAAGMAN [7, section 2]).

LIE STRUCTURE. In fact A has the structure of an infinite dimensional complex analytic Lie group, and $\mathcal{D}$ of its Lie algebra. Let G be an arbitrary group containing $\mathbb{Z}$, if necessary with some topological or differentiable structure.

Definition. Let $F \in A$. Then

i F *has a logarithm* if $F \in \exp(\mathcal{D})$.

ii F *has a* G*-iteration* if there exists a $\lambda \in \mathrm{Hom}(G,A)$ such that $\lambda(1) = F$.

iii F is called *divisible* if F has a $\frac{1}{h}\mathbb{Z}$-iteration for all $h \in \mathbb{N}$ (i.e. if F has roots of all orders).

Note that a complex analytic iteration is an analytic $\mathbb{C}$-iteration in the above terminology and so on. A weaker form of THEOREM 1 in this terminology would be "F *has a logarithm if and only if* F *has a* $\mathbb{Q}$*-iteration*."

DIVISIBLE AUTOMORPHISMS. In the proof of LEMMA 1 below I need some results from algebraic geometry. To be able to apply these I have to define group homomorphisms $p_k\colon A \to Gl_{l(k)}\mathbb{C}$, where $l(k) = \dim_{\mathbb{C}} \mathcal{O}/m^k$. Define p_k by $(p_k F)x^\alpha \bmod m^k = Fx^\alpha \bmod m^k$. Then $p_k(A)$ is an algebraic subgroup of $Gl_{l(k)}\mathbb{C}$.

LEMMA 1. *Let* $F \in A$ *be divisible, and let* $h \in \mathbb{N}$. *There exists a divisible* h-th *root of* F , *i.e. a* $G \in A$, *divisible, satisfying* $G^h = F$.

Proof. Let $k \in \mathbb{N}$. Assume there exists a divisible $G_k \in A$ such that $p_k G_k^h = p_k F$ (note that this holds trivially for $k = 1$, take $G_1 = I$). Define:

$$Z_n = \{T \in A \mid p_{k+1}T^{n!h} = p_{k+1}F \text{ and } p_k T^{n!} = p_k G_k\}, \text{ and}$$

$$\varphi_n\colon A \to A \text{ by } \varphi_n T = T^n.$$

$p_{k+1}\varphi_n = \varphi_n p_{k+1}$ obviously, and since $p_{k+1}Z_n$ is an algebraic subset of $p_{k+1}A$ it follows that $p_{k+1}\varphi_n Z_n$ is a constructible subset of $p_{k+1}A$ (HUMPREYS [3, theorem 4.4]). One easily verifies that $\varphi_{n+1}Z_{n+1} \subset \varphi_n Z_n$ so the $p_{k+1}\varphi_n Z_n$ form a nested sequence of constructible sets in $p_{k+1}A$. Hence $\bigcap_{n\in\mathbb{N}} p_{k+1}\varphi_n Z_n$ is nonempty (OORT [6, lemma 2]), and there exists a G_{k+1} in $\bigcap_{n\in\mathbb{N}} \varphi_n Z_n$. Clearly G_{k+1} is divisible and satisfies $p_{k+1}G_{k+1} = p_{k+1}F$, and $p_k G_{k+1} = G_k$.

Define $G = \lim_{k\to\infty} G_k$, then $p_k G^h = p_k G_k^h = p_k F$, so $G^h = F$. To see that G is divisible, fix $n \in \mathbb{N}$ and define

$$W_{n,k} = \{T \in A \mid T^n = G_k\} .$$

$W_{n,k}$ is nonempty for all k and $W_{n,k+1} \subset W_{n,k}$. $p_1(W_{n,k})$ is a constructible subset of $p_1 A$, and hence as above $\bigcap_k W_{n,k}$ is nonempty. Let $T \in \bigcap_k W_{n,k}$ then $T^n = G$ and G is divisible.

COROLLARY. *Let* $F \in A$ .F *is divisible if and only if* F *has a* $\mathbb{Q}$*-iteration.*

Proof. $\lambda(1) = F$, and by induction $\lambda(1/n!)$ is a divisible n-th root of $\lambda(1|(n-1)!)$.

If $p/q \in \mathbb{Q}$ define $\lambda(p/q) = \lambda(p(q-1)!/q!)$, then $\lambda \in \mathrm{Hom}(\mathbb{Q},A)$.

So together with THEOREM 1 one has:

THEOREM 2. F *has a logarithm if and only if* F *is divisible.*

§2 ALGEBRAIC SUBGROUPS OF FORMAL AUTOMORPHISMS

ALGEBRAIC SUBGROUPS. In this section I have gathered a number of well-known results that will be needed in §3. First a defenition: a closed subgroup H of A is called *algebraic* if $p_k H$ is an algebraic group for all $k \in \mathbb{N}$.

Examples.

1. Let I be an ideal in $\mathcal{O}$, then $A_I = \{T \in A \mid TI = I\}$ is algebraic: for all k $p_k A_I$ is the intersection of $p_k A$ with the algebraic group leaving $I/m^k \cap I$ invariant.
2. $A(I) = \{T \in A \mid Tv = v \bmod I$ for all $v \in \mathcal{O}\}$ is an algebraic subgroup of $A(I)$.
3. $T = \{T \in A \mid Tx^\alpha = t_\alpha x^\alpha$ for all $\alpha \in \mathbb{N}_0^m\}$ is algebraic. In fact $T \cong p_k T \cong (\mathbb{C}^*)^m$ for all $k \geq 2$ by sending T, defined by $Tx_i = t_i x_i$, to $\varphi T = (t_1,\dots,t_m)$.
4. Let G be a linear algebraic group contained in $Gl_m\mathbb{C}$. Sending $e_i,\dots,e_m$ to $x_1,\dots,x_m$ identifies G with a subgroup of A.

NORMAL FORMS AND JORDAN DECOMPOSITION. Let $F \in A$. Then there exists a $T \in A$ such that $T^{-1}FT$ is in normal form: there exist $\lambda_1,\dots,\lambda_m \in \mathbb{C}^*$ such that:

$$T^{-1}FTx^\alpha = \sum_{\beta\in\mathbb{N}_0^m} f_{\alpha\beta}x^\beta ,$$

where $f_{\alpha\alpha} = \lambda^\alpha$ for all α, and $f_{\alpha\beta} = 0$ if either $\beta < \alpha$ lexicographically or $\lambda^\alpha \neq \lambda^\beta$.(PRAAGMAN [7, theorem 3]). Define ${}^sF$, the topologically semisimple part of F, by ${}^sFTx^\alpha = \lambda^\alpha Tx^\alpha$ and ${}^uF$, the topologically nilpotent part of F, by ${}^uFTx^\alpha = \lambda^{-\alpha}\sum_\beta f_{\alpha\beta}Tx^\beta$. Then $[{}^sF,{}^uF] = {}^sF{}^uF - {}^uF{}^sF = 0$. (If $T \in T$ is given by $Tx_i = t_ix_i$, then $L \in L$ defined by $Lx^{\alpha} = \sum_{\beta} l_{\alpha\beta} x^{\beta}$ commutes with T if and only if $t^{\alpha} \neq t^{\beta}$ implies $l_{\alpha\beta} = 0$). Since $p_k{}^sF \cdot p_k{}^uF$ is the Jordan decomposition of p_kF, it follows that if $F \in H$, an algebraic subgroup of A, then $p_k{}^sF$ and $p_k{}^uF \in p_kH$ (HUMPREYS [3, theorem 15.3]) and hence ${}^sF$ and ${}^uF \in H$ (compare PRAAGMAN [7, theorem 2]).

ALGEBRAIC LIE ALGEBRA'S. Let H be an algebraic subgroup of A, and identify its Lie algebra $\mathfrak{h}$ with a subalgebra of $\mathcal{D}$. (Then $\mathfrak{h}$ is an algebraic sub Lie algebra of $\mathcal{D}$).

Examples.

1. Let I be an ideal in $\mathcal{O}$. Then $\mathcal{D}_I = \{D \in \mathcal{D} \mid DI \subset I\}$ is the Lie algebra associated to A_I.
2. $\mathcal{D}(I) = \{D \in \mathcal{D} \mid D\mathcal{O} \subset I\}$ is the Lie algebra of $A(I)$.
3. $\mathfrak{t} = \{D \in \mathcal{D} \mid Dx^{\alpha} = d_{\alpha} x^{\alpha}$ for all $\alpha \in \mathbb{N}_0^m\}$ is the Lie algebra of T.

Since exp: $p_k\mathfrak{h} \to p_kH$ for all k (VARADARAJAN [13, section 2.10]), clearly exp $\mathfrak{h} \subset H$.

§3 ITERATIONS AND LOGARITHMS IN ALGEBRAIC SUBGROUPS

PROBLEM FORMULATION. Let H be an algebraic subgroup of A, and $\mathfrak{h}$ its Lie algebra. $F \in H$ is called *divisible in* H if it has roots of all orders within H: $\forall n \in \mathbb{N} : \exists G \in H$ such that $G^n = F$. The question considered in this section is whether $\exp \mathfrak{h}$ consists of all elements divisible in H, as was proved for A in §1. The problem is tackled in two steps. First I prove that having a logarithm is equivalent to having a rational iteration in H, and then a slight modification of LEMMA 1 yields that being divisible in H is equivalent to having a rational iteration.

Concentrating on the first problem, note that for $F \in H$, $\log {}^uF = -\sum_{j=1}^{\infty} (I-F)^j/j$ is welldefined, $\log {}^uF \in \mathcal{D}$ and $\exp \log {}^uF = {}^uF$ (PRAAGMAN [7, theorem 4]). And since $p_k \log {}^uF = \log p_k {}^uF$, and $\log p_k {}^uF \in p_k\mathfrak{h}$ (HUMPREYS [3, section 15.1]) it follows that $\log {}^uF \in \mathfrak{h}$. So if one finds a $D \in \mathfrak{h}$ such that $\exp D = {}^sF$, and $[D, \log {}^uF] = 0$, then $D + \log {}^uF$ is a logarithm of F in $\mathfrak{h}$.

Further note that all properties that are considered here are invariant under internal automorphisms of A: $\exp (T^{-1}DT) = T^{-1}(\exp D)T$; if $\lambda \in \mathrm{Hom}(\mathbb{Q},H)$ is an iteration of F, then $T^{-1}\lambda T \in \mathrm{Hom}(\mathbb{Q},T^{-1}HT)$ is an iteration of $T^{-1}FT \in T^{-1}HT$, an algebraic subgroup conjugated to the algebraic subgroup H. Since $[\lambda(t),\lambda(s)] = 0$ for all $s,t \in \mathbb{Q}$ there exists a T such that $T^{-1}\lambda(t)T$ is in normal form for all $t \in \mathbb{Q}$. (PRAAGMAN [7, lemma 2]).

Containing these two simplifications of the problem, it becomes clear that special attention should be given to algebraic subgroups of T.

DIAGONIZABLE GROUPS. Let H be a connected algebraic subgroup of T, then $H \cong (\mathbb{C}^*)^h$ for some $h \leq m$, and $\mathfrak{h} \cong \mathbb{C}^h$ (HUMPREYS [3, theorem 16.2]), and there exists a $\tau \in \mathrm{Hom}((\mathbb{C}^*)^h, (\mathbb{C}^*)^m)$ such that for all $F \in H$ there exists a $\rho \in (\mathbb{C}^*)^h$ such that $Fx_i = \tau_i(\rho)x_i$, and similarly $\sigma \in \mathrm{Hom}(\mathbb{C}^h, \mathbb{C}^m)$ such that for all $D \in \mathfrak{h}$ there exists a $\nu \in \mathbb{C}^h$ such that $Dx_i = \sigma(\nu)x_i$. Moreover $\exp \sigma(\nu) = \tau(\exp \nu)$.
If $F \in H$, then $Fx_i = \tau_i(\rho)x_i$ for some $\rho \in (\mathbb{C}^*)^h$. Take a $\nu \in \mathbb{C}^h$ such that $\exp \nu = \rho$, and define $D \in \mathfrak{h}$ by $Dx_i = \sigma_i(\nu)x_i$, then $(\exp D)x_i = \exp \sigma_i(\nu)x_i = \tau_i(\rho)x_i = Fx_i$. So $\exp \mathfrak{h} = H$ and any choice of $\log \rho$ will do.

RATIONAL CHARACTERS. To determine which choice of the logarithm will be the right one, I will use the following lemma, or more precisely its corollary:

LEMMA 2. $\mathrm{Hom}(\mathbb{Q};\mathbb{C}^*)$ *is torsionfree.*

Proof. Let $\lambda \in \mathrm{Hom}(\mathbb{Q},\mathbb{C}^*)$ satisfy $\lambda^k(t) = 1$ for all $t \in \mathbb{Q}$. Then $\lambda(\mathbb{Q})$ is a finite subgroup of $\mathbb{C}^*$. But $\mathbb{Q}$ being divisible, so is $\lambda(\mathbb{Q})$, hence $\lambda(\mathbb{Q}) = 1$.

Corollary. Every finitely generated subgroup of $\mathrm{Hom}(\mathbb{Q},\mathbb{C}^*)$ is free.

CHOOSING THE LOGARITHM. The following proposition yields the right choice of the logarithm: a derivation which commutes with maps commuting with the iteration.

PROPOSITION 1. *Let* $F \in T$, *and let* $\lambda \in \mathrm{Hom}(\mathbb{Q},T)$ *be an iteration for* F. *There exists a* $D \in \mathfrak{t}$ *such that* $\exp D = F$, *and* $[D,L] = 0$ *for all* $L \in L$

satisfying $[L, \lambda(t)] = 0$ *for all* $t \in \mathbb{Q}$.

Proof. Let H be the algebraic closure of the subgroup of T generated by the set $\{\lambda(t) \mid t \in \mathbb{Q}\}$. H is connected since $\lambda(t)$ has roots of all orders in H for all t. Let $\lambda(t)$ be given by $\lambda(t)x_i = \lambda_i(t)x_i$, then $\lambda_i \in \mathrm{Hom}(\mathbb{Q},\mathbb{C}^*)$. Let Λ be the free subgroup of $\mathrm{Hom}(\mathbb{Q},\mathbb{C}^*)$ generated by $\lambda_1,\ldots,\lambda_m$, and let $\rho_1,\ldots,\rho_h$ be a free set of generators of Λ. Choose $\nu_1,\ldots,\nu_h$ such that $\exp \nu_i = \rho_i(1)$, and let $\gamma(1),\ldots,\gamma(m) \in \mathbb{Z}^d$ be defined by $\lambda_i = \rho^{\gamma(i)}$. Define $D \in \mathfrak{h}$ by

$$Dx_i = \langle \nu, \gamma(i)\rangle \; x_i \qquad \text{with } \langle\alpha,\beta\rangle = \sum \alpha_i\beta_i.$$

Then clearly $\exp D = F$. Now if $[L,\lambda(t)] = 0$ for all t, and $Lx^\alpha = \sum_\beta l_{\alpha\beta}x^\beta$, then $\lambda^\alpha \neq \lambda^\beta$ implies $l_{\alpha\beta} = 0$. Since $\langle\alpha,\mu\rangle \neq \langle\beta,\mu\rangle$, where $\mu_i = \langle\nu,\gamma(i)\rangle$ implies that $\lambda^\alpha \neq \lambda^\beta$, it follows immediately that $[L, \lambda(t)] = 0$ for all t implies $[L,D] = 0$.

Remark. Note that $D \in \mathfrak{h}$.

LOGARITHMS, ITERATIONS... The theorem which follows is:

THEOREM 3. *Let* $F \in H$, *an algebraic subgroup of* A. *Then* $F \in \exp \mathfrak{h}$ *if and only if there exists a rational iteration* $\lambda \in \mathrm{Hom}(\mathbb{Q},H)$ *of* F *in* H.

Proof. If $F = \exp D$, then clearly $\lambda \in \mathrm{Hom}(\mathbb{Q},C^*)$ defined by $\lambda(t) = \exp tD$ is an iteration of F in H, so assume $\lambda \in \mathrm{Hom}(\mathbb{Q},H)$ with $\lambda(1) = F$ is given. Let ${}^s\lambda \in \mathrm{Hom}(\mathbb{Q},H)$ be defined by $({}^s\lambda)(t) = {}^s(\lambda(t))$, then ${}^s\lambda$ is a rational iteration of ${}^sF$. Since $[{}^s\lambda(s),{}^s\lambda(t)] = 0$ for all $s,t \in \mathbb{Q}$ there exists a $T \in A$ such that $T^s\lambda T^{-1} \in \mathrm{Hom}(\mathbb{Q},T)$.
Let $D \in \mathfrak{h}$ such that $\exp T^{-1}DT = T^{-1}\ {}^sFT$, and $[T^{-1}DT,L] = 0$ for all L with $[T^{-1}\ {}^s\lambda(t)T,L] = 0$ for all $t \in \mathbb{Q}$. So from $[{}^s\lambda(t),{}^uF] = 0$

follows that $[D, {}^{u}F] = 0$ and hence $[D, \log {}^{u}F] = 0$. Therefore $\exp(D + \log {}^{u}F) = \exp D \cdot \exp \log {}^{u}F = {}^{s}F \cdot {}^{u}F = F$.

...AND ROOTS. Note that in LEMMA 2 only the algebraicity of $p_k A$ played a rôle. The argument works equally well for subgroups H with $p_k H$ algebraic. So from this modification of LEMMA 2 and THEOREM 3 immediately follows:

THEOREM 4. *Let* $F \in H$ *, an algebraic subgroup of* A*. Then* $F \in \exp \mathfrak{h}$ *if and only if* F *is divisible in* H*.*

Remark. Note that THEOREM 4 in particular holds for linear algebraic groups.

§4 MISCELLANEOUS REMARKS

BEHAVIOR UNDER MORPHISMS. Let G be a (possibly infinite dimensional) complex Lie group, H an algebraic subgroup of A, and φ a morphism of Lie groups from H onto G. Then $d\varphi$ maps $\mathfrak{h}$ onto $\mathfrak{g}$, and $\varphi \circ \exp = \exp \circ d\varphi$ (VARADARAJAN 13,[theorem 2.10.3]). Clearly φ maps divisible automorphisms onto divisible elements of G, but under circumstances it is also the other way around:

LEMMA 3. *Let* $\ker\varphi$ *be an algebraic subgroup of* H, *and let* g *be divisible in* G. *Then there exists an* F, *divisible in* H, *such that* $\varphi(F) = g$.

Proof. (sketch). For all $n \in \mathbb{N}$ let $V_n = \{T \in \varphi^{-1}(g) \mid \exists T' \in H$ such that $(T')^{n!} = T\}$. Then $p_k V_n$ is constructible for all k and all n, $V_{n+1} \subset V_n$, so using the same argument as in LEMMA 1, the intersection of the V_n is nonempty, which yields the desired F.

As an immediate consequence it follows that divisibility in G is equivalent to having an automorphism. Consider the special case where $G = \text{Aut } R$, $R = \mathcal{O}/I$. Then the exact sequence

$$0 \to A(I) \to A_I \to \text{Aut } R \to 0$$

yields with LEMMA 3 the results of PRAAGMAN [9, chapter VId] : divisibility in Aut R is equivalent with having a logarithm, and even more: the same holds for algebraic subgroups of Aut R, the images of algebraic subgroups of A_I.

THEOREM 5. *Let* I *be an ideal of* $\mathcal{O}$, *and* $R \cong \mathcal{O}/I$. *Then divisibility in any algebraic subgroup of* Aut R *is equivalent to having a logarithm in its Lie algebra.*

PROAFFINE ALGEBRAIC GROUPS. Let K be the $\mathbb{C}$-algebra of polynomials in the variables $X_{i\alpha}$, $i \in \{1,\dots,m\}$, $\alpha \in \mathbb{N}_0^m$, $K = \bigcup_{l \in \mathbb{N}} \mathbb{C}[\bigcup_{|\alpha| \leq l} X_{i\alpha}]$. Then (A,K) defines the structure of a proaffine algebraic group over $\mathbb{C}$ (HOCHSCHILD-MOSTOW [2, section 2]), and a similar construction is possible for Aut R, and the algebraic subgroups of A, coincide with those defined by this proaffine algebraic structure.

The question considered here is whether the results of §3 extend to proaffine algebraic groups.

From HOCHSCHILD-MOSTOW [2, theorem 2.1] it follows that any proaffine algebraic group is isomorphic to a projective limit of linear algebraic groups and morphisms, and vice versa:

PROPOSITION 2. Let G be group. Then G may be equipped with the structure of a proaffine algebraic group if and only if there exists a partially ordered set Γ, linear algebraic groups G_γ, for all $\gamma \in \Gamma$, and for all $\gamma,\delta \in \Gamma$ with $\delta < \gamma$ morphisms of algebraic groups $\pi_{\delta\gamma} : G_\delta \to G_\gamma$ satisfying for $\eta < \delta < \gamma$: $\pi_{\eta\delta}\pi_{\delta\gamma} = \pi_{\eta\gamma}$, such that $G = \varprojlim_{\gamma \in \Gamma} G_\gamma$.

One verifies without any difficulty that $\mathfrak{g}$, the Lie algebra of G is isomorphic to the inverse limit of the Lie algebras of the group G_γ: $\mathfrak{g} = \varprojlim \mathfrak{g}_\gamma$. If $x \in G$ or $\mathfrak{g}$ x_γ is its image in $G_\gamma(\mathfrak{g}_\gamma)$ under the canonical projection. Further $\exp: \mathfrak{g} \to G$ satisfies $\exp(\varprojlim x_\gamma) = \varprojlim (\exp x_\gamma)$.

As before $g \in G$ is said to have a logarithm if $g \in \exp \mathfrak{g}$, to have a rational iteration if there exists a $\lambda \in \mathrm{Hom}(\mathbb{Q},G)$ such that $\lambda(1) = g$, and to be divisible if for all $n \in \mathbb{N}$ there exists an $f_n \in G$ such that $f_n^n = g$. As a new concept call g *normally divisible* if for all $n,m \in \mathbb{N}$ $[{}^s f_n, {}^s f_m] = 0$ where ${}^s x$ is defined as $\varprojlim {}^s x_\gamma$.

Finally, define ${}^u x$ as $\varprojlim {}^u x_\gamma$, and let $\sigma(g) = \bigcup_\gamma \sigma(g_\gamma)$ be the spectrum of g: $\sigma(g_\gamma)$ is the set of eigenvalues of g_γ considered as a linear map.

THEOREM 6. *Let G be a complex proaffine algebraic group, $G \cong \lim_{\gamma \in \Gamma} G_\gamma$. Suppose there exists a $\gamma \in \Gamma$ such that for all $g \in G$ the subgroup $\sum(g)$ of $\mathbb{C}^*$ generated by $\sigma(g)$ equals $\sum(g_\gamma)$, the subgroup generated by $\sigma(g_\gamma)$. Then $g \in G$ has a logarithm if and only if it is normally divisible.*

Proof. Let $f_n^n = g$ and $[{}^s f_n, {}^s f_m] = 0$. Then the algebraic subgroup H of G generated by the ${}^s f_n$ is isomorphic to H_γ. Since g is divisible ${}^s g$ lies in the component of identity H^0 of H. $H^0 \cong (\mathbb{C}^*)^h$ for some $h \in \mathbb{N}$, and ${}^s g \to (\lambda_1,\ldots,\lambda_h) \in (\mathbb{C}^*)^h$. Now any choice of $\log \lambda_i$ yields a $d \in \mathfrak{h}$ with $\exp d = {}^s g$, and $[d, {}^u g] = 0$. Hence $g = \exp (d + \log {}^u g)$.

THEOREM 7. *Let $\Gamma \subset \mathbb{N}$ as an ordered set. Then $g \in G$ is divisible if and only if g has a rational iteration.*

Proof. Completely analogous to the proof of the COROLLARY of LEMMA 1.

Remarks.

1. Combination of THEOREMS 6 and 7 does not yield an extension of THEOREM 5. Any $\varprojlim G_n$, with $\sigma(g) = \sigma(g_n)$ may be embedded in an automorphism group of a complete local ring.
2. In fact one could take any totally ordered set Γ in THEOREM 7, since Γ always contains a countable subset Γ' such that $\varprojlim_{\gamma \in \Gamma} = \varprojlim_{\gamma \in \Gamma'}$. To prove this, however, something about the ordinality

of the set of all linear algebraic groups over $\mathbb{C}$ should be said, and to my feeling this falls beyond the scope of this paper.

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