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A NEW METHOD FOR COMPUTING A COLUMN REDUCED POLYNOMIAL MATRIX

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ABSTRACT

A new algorithm is presented for computing a column reduced form of a given full column rank polynomial matrix. The method is based on reformulating the problem as a problem of constructing a minimal basis for the rigth nullspace of a polynomial matrix closely related to the original one. The latter problem can easily be solved in a numerically reliable way. Two examples illustrating the method are included.

KEYWORDS : Polynomial matrix, column reduced, minimal polynomial basis, numerical method.

1. INTRODUCTION

The problem of constructing a column reduced form of a given polynomial matrix P(s) is well understood from an algebraic point of view (see e.g. Wolovich [6]). The approach indicated in [6] consists of applying elementary column operations to P(s) until a polynomial matrix R(s) is formed with leading column coefficient matrix having full column rank. However, it is well known that this method is not recommanded from a numerical point of view (see Van Dooren [5]). In this paper we propose a new method for making a given polynomial matrix column reduced which is expected to have much better numerical properties than the method in [6]. Our approach exists of reformulating the original problem as a problem of constructing a minimal polynomial basis for the right nullspace of a polynomial matrix closely related to the original one. The latter problem can be solved in a numerically reliable way using the algorithm presented by Beelen and Veltkamp [1],[2]. We show that a column reduced form of the original matrix can easily be obtained from the constructed minimal polynomial basis. We conclude with an example to illustrate our method.

2. NOTATIONS

In this paper we consider matrices over the ring of complex polynomials $\mathbb{C}[s]$. If P(s) is a polynomial matrix of size mxn with entries $p_{ij}(s)$ then the degree $\partial(P)$ of P(s) is given by $\partial(P) = \max_{i,j} \text{degr}(p_{ij}(s))$.

The j-th column degree of P(s) is defined by $\partial_{cj}(P) = \max_{i} \text{degr}(p_{ij}(s))$.

The leading column coefficient matrix of a polynomial matrix P(s) of full column rank is the unique constant matrix $\Gamma_c(P)$ such that each column of $\Gamma_c(P)$ contains the coefficients of the highest power of s occuring in the corresponding column of P(s). A full column rank polynomial matrix is called column reduced if its leading column coefficient matrix has full column rank.

3. PROBLEM FORMULATION AND SOLVING METHOD

Let P(s) be an mxn polynomial matrix. It is assumed that P(s) has full column rank for almost all s in $\mathbb{C}$. Our goal is to construct a unimodular matrix U(s) such that R(s)=P(s)*U(s) is column reduced, i.e., the leading column coefficient matrix $\Gamma_c(R)$ of R(s) has full column rank. In other words, we have to solve

$$\begin{pmatrix} P(s) & , & -I_m \end{pmatrix} * \begin{pmatrix} U(s) \\ R(s) \end{pmatrix} = 0 \tag{1}$$

such that U(s) is unimodular and R(s) is column reduced.
Here I_m denotes the mxm identity matrix.
Clearly, Eq. (1) suggests that the problem might be solved by constructing

a minimal polynomial basis (MPB) (see Forney [3]) for the right nullspace $\mathrm{Ker}\begin{pmatrix} P(s) & , & -I_m \end{pmatrix}$ of $\begin{pmatrix} P(s) & , & -I_m \end{pmatrix}$. Unfortunately, this does not always yield a solution as can be seen from the next example.

<u>**Example 3.1**</u>

Let $P(s) = \begin{pmatrix} 1 & s \\ 1 & s+1 \end{pmatrix}$. Clearly, P(s) is not column reduced.

Now $\begin{pmatrix} U(s) \\ R(s) \end{pmatrix} = \begin{pmatrix} I_n \\ P(s) \end{pmatrix}$ is an MPB for $\mathrm{Ker}\begin{pmatrix} P(s) & , & -I_m \end{pmatrix}$ but obviously R(s)

is not column reduced.
So in order to find a solution to our problem we need an MPB having special properties. Therefore, we first mention some lemmas concerning MPB's.

<u>**Lemma 3.1**</u>

Let $\begin{pmatrix} U(s) \\ R(s) \end{pmatrix}$ be an MPB for $\mathrm{Ker}\begin{pmatrix} P(s) & , & -I_m \end{pmatrix}$. Then U(s) is unimodular.

<u>**Proof.**</u> Since $\begin{pmatrix} U(s) \\ R(s) \end{pmatrix}$ is an MPB, $\begin{pmatrix} U(s) \\ R(s) \end{pmatrix}$ has full column rank for all s

in $\mathbb{C}$, i.e., U(s) and R(s) are right coprime (see Kailath [4]). Clearly, U(s)=I*U(s) and R(s)=P(s)*U(s). Thus, U(s) is a common right divisor of U(s) and R(s), i.e., U(s) is unimodular.

Q.E.D.

The next two lemmas will be useful for solving our problem.

Lemma 3.2

Let $A(s)$ and $B(s)$ be polynomial matrices of dimensions $k\times n$ and $m\times n$, respectively. If $\begin{pmatrix} A(s) \\ B(s) \end{pmatrix}$ is column reduced and $\partial_{cj}(A) < \partial_{cj}(B)$, $1\leq j\leq n$, then $B(s)$ is column reduced.

Proof. We have $\Gamma_c\left(\begin{pmatrix} A \\ B \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \Gamma_c(B) \end{pmatrix}$ since $\partial_{cj}(A) < \partial_{cj}(B)$, $1\leq j\leq n$. Since $\begin{pmatrix} A(s) \\ B(s) \end{pmatrix}$ is column reduced, $\Gamma_c\left(\begin{pmatrix} A \\ B \end{pmatrix}\right)$ has full column rank. Thus $\Gamma_c(B)$ has full column rank, i.e., $B(s)$ is column reduced.

Q.E.D.

Lemma 3.3

Let $P(s)$ be an $m\times n$ full column rank polynomial matrix of degree d. Let $\alpha \geq 0$ be an integer and $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ be an MPB for $\mathrm{Ker}\left(s^\alpha P(s)\ ,\ -I_m \right)$. Then

1. $s^{-\alpha} Z_\alpha(s)$ is a full column rank polynomial matrix ;

2. $\displaystyle\sum_{j=1}^{n} \partial_{cj}(s^{-\alpha} Z_\alpha) \leq \sum_{j=1}^{n} \partial_{cj}(P)$; (2)

3. $\partial(Y_\alpha) \leq (3n-1)d$. (3)

Proof. See Appendix A. Q.E.D.

Now we can formulate the main result of this paper.

Theorem 3.1

Let $P(s)$ be an $m\times n$ full column rank polynomial matrix of degree d. Let

$\alpha > (3n-1)d$ be an integer and $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ be an MPB for $\text{Ker}\begin{pmatrix} s^\alpha P(s) \ , -I_m \end{pmatrix}$. Then

1. $Y_\alpha(s)$ is unimodular ; (4)

2. $s^{-\alpha}Z_\alpha(s) = P(s)Y_\alpha(s)$; (5)

3. $s^{-\alpha}Z_\alpha(s)$ is a column reduced polynomial matrix. (6)

Proof. Ad 1: Apply Lemma 3.1. Ad 2: Trivial.

Ad 3: $s^{-\alpha}Z_\alpha(s)$ is a full column rank polynomial matrix due to 1 and 2. By Lemma 3.3 and hypothesis we have for each j, $1 \leq j \leq n$

$$\partial_{cj}(Y_\alpha) \leq \partial(Y_\alpha) \leq (3n-1)d < \alpha. \tag{7}$$

Furthermore, $Z_\alpha(s) = s^\alpha P(s)Y_\alpha(s)$. Thus

$$\partial_{cj}(Z_\alpha) \geq \alpha \ , \ 1 \leq j \leq n. \tag{8}$$

Combination of (7) and (8) yields

$$\partial_{cj}(Y_\alpha) < \partial_{cj}(Z_\alpha) \ , \ 1 \leq j \leq n. \tag{9}$$

Since $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ is a minimal polynomial basis, we have that $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ is column reduced. Using Lemma 3.2 we find that $Z_\alpha(s)$ (and hence also $s^{-\alpha}Z_\alpha(s)$) is column reduced. Q.E.D.

4. ALGORITHMS

From Theorem 3.1. we can now derive the following algorithm for constructing a column reduced form of a full column rank polynomial matrix P(s).

Algorithm I

1. Choose $\alpha > (3n-1)d$.

2. Compute an MPB $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ for $\mathrm{Ker}\begin{pmatrix} s^\alpha P(s) & , & -I_m \end{pmatrix}$ using the algorithm in [2].

Result : $P(s)Y_\alpha(s) = s^{-\alpha}Z_\alpha(s)$ with $Y_\alpha(s)$ unimodular and $s^{-\alpha}Z_\alpha(s)$ column reduced.

End of Algorithm I

We note that in step 2 the polynomial matrix $Q(s) = \begin{pmatrix} s^\alpha P(s) & , & -I_m \end{pmatrix}$ of degree $d+\alpha$ has to be expressed as $\sum_{k=0}^{d+\alpha} s^k Q_k$ (see [2]). The coefficient matrices Q_k are then used to construct a pencil sE-A which is transformed to a generalized Schur form. Hereafter an MPB can easily be computed by exploiting the special structure of the Schur form. As indicated in [2] the amount of computational effort is completely determined by the dimensions of sE-A and thus by the number of coefficient matrices Q_k. Consequently, in step 1 we have to choose $\alpha > (3n-1)d$ as small as possible. Moreover, the bound $(3n-1)d$ can be replaced by a smaller one as can be seen from the proof of Lemma 3.3. However, we shall not discuss this aspect in detail since we found that in many cases step 2 already yields a solution when $\alpha \ll (3n-1)d$. Therefore, we propose the following alternative for algorithm I.

Algorithm II

1. Initialization $\alpha := 0$;

   Compute an MPB $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ for $\mathrm{Ker}\begin{pmatrix} s^\alpha P(s) & , & -I_m \end{pmatrix}$;

   Determine the leading column coefficient matrix $\Gamma_{c,\alpha}$ of $Z_\alpha(s)$;

2. while not ($\Gamma_{c,\alpha}$ has full column rank) do

begin $\alpha := \alpha + 1$;

Compute an MPB $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ for $\mathrm{Ker}\left(s^\alpha P(s) \ , \ -I_m \right)$;

Determine the leading column coefficient matrix $\Gamma_{c,\alpha}$ of $Z_\alpha(s)$

end

Result : $P(s)Y_\alpha(s) = s^{-\alpha}Z_\alpha(s)$ with $Y_\alpha(s)$ unimodular and $s^{-\alpha}Z_\alpha(s)$ column reduced.

End of Algorithm II

5. NUMERICAL EXAMPLES

We present two numerical examples of computing a column reduced form of a given full column rank polynomial matrix using Algorithm II. The computations have been carried out on a VAX-750 computer with relative machine precision EPS $\approx 2^{-56} \approx 0.14*10^{-16}$. The computed coefficients below are correct up to 16 digits.

Example 5.1

Consider the polynomial matrix P(s) as given in Kailath [4] p. 386, i.e.,

$$P(s) = \begin{pmatrix} (s+1)^2(s+2)^2 & -(s+1)^2(s+2) \\ 0 & (s+2) \end{pmatrix} . \qquad (10)$$

We found as an MPB for Ker (P(s), -I) the matrix $\begin{pmatrix} Y_0(s) \\ Z_0(s) \end{pmatrix}$ where

$$\begin{pmatrix} Y_0(s) \\ \hline Z_0(s) \end{pmatrix} = \left(\begin{array}{c|c} \alpha & \delta-\beta s \\ \alpha(s+2) & (\gamma+2\delta)+(\delta-2\beta)s-\beta s^2 \\ \hline 0 & -\gamma(s^3+4s^2+5s-2) \\ \alpha(s^2+4s+4) & (2\gamma+4\delta)+(4\delta+\gamma-4\beta)s+(\delta-4\beta)s^2-\beta s^3 \end{array} \right) . \qquad (11)$$

Here α, β, γ and δ are some nonzero constants.

Clearly, $\Gamma_c(Z_0) = \begin{pmatrix} 0 & -\gamma \\ \alpha & -\beta \end{pmatrix}$ has full column rank. So, Algorithm II already ends after the first step. By Theorem 3.1, $Z_0(s)$ is a column reduced form of P(s).

By introducing the unimodular matrix $U(s) = \begin{pmatrix} 1/\alpha & (\beta s - \delta)/\alpha\gamma \\ 0 & 1/\gamma \end{pmatrix}$, we see that $Y_0(s)U(s) = \begin{pmatrix} 1 & 0 \\ s+2 & 1 \end{pmatrix}$ and $Z_0(s)U(s) = \begin{pmatrix} 0 & s^3+4s^2+5s+2 \\ s^2+4s+4 & s+2 \end{pmatrix}$.

We note that these results are given in [4].

Example 5.2

Consider the unimodular matrix P(s) given by

$$P(s) = \begin{pmatrix} 0 & s^2 & 1 \\ 0 & 1 & 0 \\ 1 & s+7 & s^2+7s+3 \end{pmatrix}. \tag{12}$$

Using Algorithm II it turned out that the steps α=0, 1 and 2 did not yield a full column rank matrix $\Gamma_{c,\alpha}$. In case α=3 we found

$$\begin{pmatrix} Y_3(s) \\ \hline Z_3(s) \end{pmatrix} = \left(\begin{array}{c|c|c} \lambda & \mu + \nu s+\mu s^2 & -s+3s^2+7s^3+s^4 \\ 0 & 0 & 1 \\ 0 & -\mu & -s^2 \\ \hline 0 & -\mu s^3 & 0 \\ 0 & 0 & s^3 \\ \lambda s^3 & -2\mu s^3 & 7s^3 \end{array}\right). \tag{13}$$

Here λ, μ and ν are some nonzero constants.

Now $\Gamma_c(Z_3) = \begin{pmatrix} 0 & -\mu & 0 \\ 0 & 0 & 1 \\ \lambda & -2\mu & 7 \end{pmatrix}$ is invertible. By Theorem 3.1, $s^{-3}Z_3(s)$ is a column reduced form of P(s) and the corresponding column transformation matrix is $Y_3(s)$.

Remark 5.1

Note that if P(s) is a unimodular matrix of size mxm, then a column reduced form is the identity I_m. So, in this case we do not need any algorithm for computing a column reduced form. Notice also that in Example 5.2 we can easily compute the inverse of P(s) being $Y_3(s)*\Gamma_3^{-1}(Z_3)$ since $s^{-3}Z_3(s)=\Gamma_3(Z_3)$.

Remark 5.2

Note that in Example 5.2 we have (3n-1)d=16. When applying Algorithm I to this example we have to compute an MPB for the kernel of a polynomial matrix of degree at least 17. This requires much more computational effort than when using Algorithm II which ends after 4 steps.

6. CONCLUDING REMARKS

In this paper we have presented a new method for computing a column reduced form of a given full column polynomial matrix. Two algorithms are proposed. The numerical aspects of both are identical. However, the latter one is preferred to the first. This preference is based on the experimentally found number of operations needed. However, there is still a lack of theoretical foundation for these observations.
The numerical qualities of the method are completely determined by those of the algorithm in [2] for constructing an MPB for the kernel of a polynomial matrix. Although no complete backward stability of this algorithm can be proven, upper bounds for the roundoff errors can be derived. However, several numerical experiments indicate that these bounds are too generous and the computed results agree with the exact ones within the order of machine precision.

Appendix A : Proof of Lemma 3.3

Ad 1: Y_α is unimodular by Lemma 3.1 and P(s) has full column rank. Hence, $s^{-\alpha}Z(s) = P(s)Y_\alpha(s)$ is a full column rank polynomial matrix.

Ad 2: Since $\begin{pmatrix} I_n \\ s^\alpha P(s) \end{pmatrix}$ is a polynomial basis for $\mathrm{Ker}\begin{pmatrix} s^\alpha P(s) & , & -I_m \end{pmatrix}$ and $\begin{pmatrix} Y_\alpha(s) \\ Z_\alpha(s) \end{pmatrix}$ is a minimal one, we have

$$\sum_{j=1}^{n} \partial_{cj}\left(\begin{pmatrix} Y_\alpha \\ Z_\alpha \end{pmatrix}\right) \le \sum_{j=1}^{n} \partial_{cj}\left(\begin{pmatrix} I_n \\ s^\alpha P \end{pmatrix}\right). \tag{A1}$$

Since P(s) has full column rank, we find for each j

$$\partial_{cj}\left(\begin{pmatrix} I_n \\ s^\alpha P \end{pmatrix}\right) = \partial_{cj}(s^\alpha P) = \partial_{cj}(P) + \alpha . \tag{A2}$$

Furthermore, since $s^{-\alpha}Z_\alpha(s)$ has full column rank we have

$$\partial_{cj}(s^{-\alpha}Z_\alpha) = \partial_{cj}(Z_\alpha) - \alpha \le \partial_{cj}\left(\begin{pmatrix} Y_\alpha \\ Z_\alpha \end{pmatrix}\right) - \alpha. \qquad \text{(A3)}$$

Combination of equations (A1),(A2) and (A3) yields

$$\sum_{j=1}^{n} \partial_{cj}(s^{-\alpha}Z_\alpha) \le \sum_{j=1}^{n} \partial_{cj}\left(\begin{pmatrix} Y_\alpha \\ Z_\alpha \end{pmatrix}\right) - \alpha n \le \sum_{j=1}^{n} \partial_{cj}\left(\begin{pmatrix} I_n \\ s^{\alpha}P \end{pmatrix}\right) - \alpha n = \sum_{j=1}^{n} \partial_{cj}(P). \qquad \text{(A4)}$$

Ad 3: Since P(s) has full column rank, $P^H(s)P(s)$ is invertible. Then $W(s)=\det((P^H(s)P(s)) * \{ P^H(s)P(s) \}^{-1}$ is a polynomial matrix.
By Cramer's rule we have

$$W_{ij}(s) = (-1)^{i+j}\det(M_{ji}(s)) , \quad 1\le i\le n,\ 1\le j\le n, \qquad \text{(A5)}$$

where $M_{ji}(s)$ is the ji-th minor of $P^H(s)P(s)$ having size (n-1)x(n-1).
Hence, we easily find

$$\partial(W) \le (n-1)2d. \qquad \text{(A6)}$$

Since $P(s)Y_\alpha(s) = s^{-\alpha}Z_\alpha(s)$ we have

$$\det(P^H(s)P(s))*Y_\alpha(s) = W(s)*P^H(s)*s^{-\alpha}Z_\alpha(s). \qquad \text{(A7)}$$

So

$$\partial(\det(P^HP)) + \partial(Y_\alpha) \le \partial(W) + \partial(P^H) + \partial(s^{-\alpha}Z_\alpha). \qquad \text{(A8)}$$

Combination of (A4), (A6) and (A8) finally yields :

$$\partial(Y_\alpha) \le \partial(W) + \partial(P^H) + \partial(s^{-\alpha}Z_\alpha) \le (n-1)2d + d + nd \qquad \text{(A9)}$$

Q.E.D.

7. REFERENCES

[1] Th. Beelen, "New algorithms for computing the Kronecker structure of

a pencil with applications to systems and control theory", Ph. D. thesis, Eindhoven University of Technology, the Netherlands, 1987.

[2] Th. Beelen, G. Veltkamp, "Numerical computation of a coprime factorization of a transfer function matrix", to appear in Systems & Control Letters, 1987.

[3] G.D. Forney, "Minimal bases of rational vector spaces with applications to multivariable linear systems", SIAM J. Control, Vol. 13, pp. 493-520, 1975.

[4] T. Kailath, Linear Systems, Prentice Hall, Englewood Cliffs, N.J., 1980.

[5] P. Van Dooren, "The generalized eigenstructure problem in linear system theory", IEEE Trans. Aut. Contr., Vol AC-26, p. 111-129, 1981.

[6] W. Wolovich, Linear Multivariable Systems, Springer Verlag, New York, 1974.