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Abstract

We consider properties of second-order operators $H = -\sum_{i,j=1}^d \partial_i \, c_{ij} \, \partial_j$ on $\mathbf{R}^d$ with bounded real symmetric measurable coefficients. We assume that $C = (c_{ij}) \geq 0$ almost everywhere, but allow for the possibility that C is singular. We associate with H a canonical self-adjoint viscosity operator H_0 and examine properties of the viscosity semigroup $S^{(0)}$ generated by H_0. The semigroup extends to a positive contraction semigroup on the L_p-spaces with $p \in [1, \infty]$. We establish that it conserves probability, satisfies L_2 off-diagonal bounds and that the wave equation associated with H_0 has finite speed of propagation. Nevertheless $S^{(0)}$ is not always strictly positive because separation of the system can occur even for subelliptic operators. This demonstrates that subelliptic semigroups are not ergodic in general and their kernels are neither strictly positive nor Hölder continuous. In particular one can construct examples for which both upper and lower Gaussian bounds fail even with coefficients in $C^{2-\varepsilon}(\mathbf{R}^d)$ with $\varepsilon > 0$.

August 2004

AMS Subject Classification: 35Hxx, 35J70, 47A52.

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1 Introduction

Our intention is to investigate global properties of second-order operators H with real measurable coefficients on $\mathbf{R}^d$. We consider operators in divergence form formally given by

$$H = -\sum_{i,j=1}^{d} \partial_i \, c_{ij} \, \partial_j \tag{1}$$

where $\partial_i = \partial/\partial x_i$. The coefficients c_{ij} are assumed to be real L_∞-functions and the corresponding matrix $C = (c_{ij})$ is assumed to be symmetric and positive-definite almost-everywhere. Since the classical work of Nash [Nas] and De Giorgi [DeG] the theory of such operators is well developed under the additional hypothesis of strong ellipticity, i.e., the assumption

$$C \geq \mu \, I > 0 \tag{2}$$

almost-everywhere. The principal result of this theory is the local Hölder continuity of weak solutions of the associated elliptic and parabolic equations. In Nash's approach the Hölder continuity of the elliptic solution is derived as a corollary of continuity of the parabolic solution and the latter is established by an iterative argument from good upper and lower bounds on the fundamental solution. Aronson [Aro] subsequently improved Nash's bounds and proved that the fundamental solution of the parabolic equation, the heat kernel, satisfies Gaussian upper and lower bounds. Specifically the kernel K of the semigroup S generated by H is a symmetric function over $\mathbf{R}^d \times \mathbf{R}^d$ satisfying bounds

$$a' \, G_{b';t}(x-y) \leq K_t(x\,;y) \leq a \, G_{b;t}(x-y) \tag{3}$$

uniformly for $x, y \in \mathbf{R}^d$ and $t > 0$ where $G_{b;t}$ is the usual Gaussian function, $G_{b;t}(x) = t^{-d/2} e^{-b|x|^2 t^{-1}}$, and $a, a', b, b' > 0$. (Background material on the Nash–De Giorgi theory can be found in the books and reviews [Dav2] [Gia1] [Gia2] [Stro2] [Stro3]. The derivation of Hölder continuity from the Aronson bounds is well described in [FaS] and a clear statement of the equivalence of estimates for elliptic and parabolic solutions is given in [Aus1].)

In contrast to the Nash–De Giorgi theory we examine operators for which the strong ellipticity assumption (2) is not satisfied. Part of our work requires nothing other than the ellipticity property $C \geq 0$ but we also analyze operators which satisfy a condition of subellipticity. The strong ellipticity bound (2) on the coefficients is equivalent to the operator bound

$$H \geq \mu \, \Delta \tag{4}$$

on $L_2(\mathbf{R}^d)$ where $\Delta = -\sum_{i=1}^{d} \partial_i^2$ is the usual self-adjoint Laplacian. The subelliptic condition which we consider is

$$H \geq \mu \, \Delta^\gamma - \nu \, I \tag{5}$$

where $\mu > 0$, $\nu \geq 0$ and $\gamma \in \langle 0, 1]$. This subellipticity condition first arose in Hörmander's work [Hör1] on the characterization of hypoelliptic operators as sums of squares of C^∞-vector fields satisfying a fixed rank condition. This work was extended by Rothschild and Stein [RoS] and the relation between the rank r of the vector fields and the order γ was clarified. In fact $\gamma = 1/r \in \langle 0, 1/2] \cup \{1\}$. Subsequently Fefferman and Phong [FeP] (see also [FeS] [San] and [OlR]) analyzed operators with smooth coefficients satisfying the inequality without assuming they could be expressed as sums of squares of vector fields.

They established that the subellipticity condition could be characterized by properties of the intrinsic geometry. All this analysis was of a local nature. Later Kusuoka and Stroock [KuS] examined global properties of operators of the form (1) under various assumptions on the local geometry and positivity of the corresponding semigroup kernels (see, for example, Theorems (2.6), (3.1) and (3.9) of [KuS]). These results could then be applied to sums of squares of vector fields satisfying a uniform version of Hörmander's rank condition (see [KuS] Theorems (3.20) and (3.24)). It is notable that many of the estimates of Fefferman–Phong and Kusuoka–Stroock only depend on the C^2-norm of the coefficients. One can, however, establish a broad range of examples for which the Fefferman–Phong characterization of subellipticity and the lower bounds on the kernel fail if the coefficients are not in $C^2(\mathbf{R}^d)$. In particular the global behaviour of the kernel is quite different to the smooth situation.

The results of Fefferman–Phong indicate that the local behaviour is governed by the intrinsic geometry associated with the subelliptic operator. The Kusuoka–Stroock philosophy, explained in the introduction to [KuS], is based on the idea that the detailed geometry is blurred with passing time and that the semigroup kernel should resemble the standard Gaussian $G_{b,t}$ for large time. Our results establish that this is not the case for a large class of subelliptic operators whose coefficients are less than twice differentiable. Local properties often persist and dictate the global behaviour.

The phenomenon which distinguishes between general subelliptic operators with measurable coefficients and those of the Hörmander type is the possibility of separation. For example, in one-dimension the operator $H = -d\,c\,d$ satisfies the subelliptic condition (5) if c has an isolated zero $c(x) \asymp x^{2(1-\gamma)}$ as $x \to 0$ with $\gamma \in \langle 0, 1/2 \rangle$. Nevertheless H separates into a direct sum of two operators acting on the left and right half-lines, respectively. Then the corresponding kernel cannot be strictly positive nor uniformly continuous. More complicated separation phenomena occur if c has several zeros or for operators in higher dimensions.

The theory of elliptic operators in divergence form, and in non-divergence form, has a long and complex history. A partial perspective on modern aspects of the fundamental theory can be obtained from the books [Fri] [Gia2] [GiT] [Hör2] [Hör3] [Hör4] [Hör5] [Tay] [Tre] and references therein. Probabilistic methods and stochastic analysis have been applied to the analysis of elliptic operators and relevant information can be found in [Stro1] [StV]. More recently the theory has been extended to the setting of Dirichlet spaces (see, for example, [BiM] [Stu1] [Stu2] [Stu3]).

Although the theory of strongly elliptic operators in divergence form is now well understood and systematically developed the same cannot be said of the theory of degenerate elliptic operators. Despite much interest in degenerate operators (see, for example, [BiM] [Fra] [FKS] [FrL] [FLW] [LaM] [MuV] [Mus] [Tru] [Var] and references therein) there is no commonly accepted definition of degeneracy. Various conditions of positivity, integrability and regularity of the lowest eigenvalue of the coefficient matrix C have been proposed and studied as measures of degeneracy. The main aim of many of the investigations have been to prove Hölder continuity of solutions or to derive Poincaré or Sobolev style inequalities, properties analogous to those of strongly elliptic operators. Our results are of a different nature. We examine situations in which the corresponding heat kernels are not even continuous (see Section 6). Therefore many of the regularity conditions analyzed in the previous works are not satisfied. We stress, however, that one can nevertheless obtain many positive results for operators with irregular and degenerate coefficients, e.g., L_2 off-diagonal bounds, finite speed of propagation of the corresponding wave equation and lower bounds for the kernel of high powers of the resolvent.

2 Elliptic operators

The first problem in the analysis of the elliptic operators (1) is the rigorous definition of H as a positive self-adjoint operator on $L_2(\mathbf{R}^d)$. This is a delicate problem for degenerate operators although the delicacies are often overlooked. The usual approach in operator theory is by quadratic forms. First one introduces the elliptic form

$$h(\varphi) = \sum_{i,j=1}^{d} (\partial_i\varphi, c_{ij}\partial_j\varphi) \tag{6}$$

where $D(h) = \bigcap_{i=1}^{d} D(\partial_i) = L_{2;1}(\mathbf{R}^d) = D(\Delta^{1/2})$. Then h is positive, symmetric and densely-defined. Therefore if h is closed there is a uniquely defined positive self-adjoint operator H such that $D(H) \subseteq D(h)$, $D(H^{1/2}) = D(h)$ and $h(\varphi) = \|H^{1/2}\varphi\|_2^2$ (see, for example, [Kat] Section VI.2). Alternatively if h is closable then one can define H in a similar manner through the closure of h. Therefore the first onus of any careful investigation is to establish closure properties of the quadratic form h.

If the coefficients of the operator satisfy the strong ellipticity assumption (2) it is easy to deduce that h is closed. Then

$$\|C\|\, l(\varphi) \geq h(\varphi) \geq \mu\, l(\varphi) \tag{7}$$

for all $\varphi \in D(h)$ where $\|C\|$ is the essential supremum of the norms of the matrices $C(x)$ and l is the quadratic form of the Laplacian, $l(\varphi) = \|\Delta^{1/2}\varphi\|_2^2$. It follows immediately that h is closed. More is true.

Proposition 2.1 *The form* (6) *is closed on* $D(h) = L_{2;1}(\mathbf{R}^d)$ *if and only if* h *is strongly elliptic, i.e.,* $h(\varphi) \geq \mu\, l(\varphi)$ *for some* $\mu > 0$ *and all* $\varphi \in D(h)$.

Proof Strong ellipticity implies that h is closed by the foregoing comparison (7). Conversely, if the form is closed then $D(h)$ is a Hilbert space under the norm $|||\varphi|||^2 = h(\varphi) + \|\varphi\|_2^2$, by [Kat], Theorem VI.1.11. Alternatively, $D(\Delta^{1/2})$ is a Hilbert space with the graph norm. Moreover, $D(\Delta^{1/2}) = L_{2;1}(\mathbf{R}^d) = D(h)$ as sets and both the spaces $D(h)$ and $D(\Delta^{1/2})$ are continuously embedded in $L_2(\mathbf{R}^d)$. Hence, by the closed graph theorem, there is a $\mu > 0$ such that $|||\varphi|||^2 \geq \mu\|(I+\Delta)^{1/2}\varphi\|_2^2$ for all $\varphi \in L_{2;1}(\mathbf{R}^d)$. Then it follows that

$$h(\varphi) \geq \mu\, l(\varphi) - (1-\mu)\, \|\varphi\|_2^2$$

for all $\varphi \in D(h)$. Now one can evaluate this inequality with φ replaced by φ_k where $\varphi_k(x) = e^{ikx.\xi}\varphi(x)$ with $k \in \mathbf{R}$, $\xi \in \mathbf{R}^d$ and $\varphi \in C_c^\infty(\mathbf{R}^d)$. Then using $\|\varphi_k\|_2 = \|\varphi\|_2$ one calculates that

$$\int_{\mathbf{R}^d} dx\, |\varphi(x)|^2\, (\xi, C(x)\xi) = \lim_{k\to\infty} k^{-2} h(\varphi_k) \geq \lim_{k\to\infty} k^{-2}\mu\, l(\varphi_k) = \mu\, |\xi|^2\, \|\varphi\|_2^2 \quad .$$

Therefore $C \geq \mu\, I$ almost everywhere and $h \geq \mu\, l$. □

Remark 2.2 This argument establishes that the subellipticity condition (5) with $\gamma = 1$ is equivalent to the strong ellipticity condition (4), i.e., if $\gamma = 1$ one can choose $\nu = 0$.

In general the form h is not closable. Nevertheless there are various useful criteria for closability.

First the above comparison argument gives a general criterion for closability. If k is a closable form with $D(k) = D(h)$ and one has estimates

$$a_1\, k(\varphi) \le h(\varphi) \le a_2\, k(\varphi)$$

for some $a_1, a_2 > 0$ and all $\varphi \in D(h)$ then h is closable. This reasoning can be applied to some degenerate operators. For example, if h is the form of an elliptic operator and $\mu_m(x), \mu_M(x)$ denote the smallest and largest eigenvalues of the coefficient matrix $C(x)$ for all $x \in \mathbf{R}^d$ and if $\mu_M \le a\,\mu_m$ almost everywhere for some constant $a > 0$, then the closability of h is equivalent to closability of the form $k(\varphi) = \sum_{i=1}^d (\partial_i\varphi, \mu_m \partial_i\varphi)$ with $D(k) = D(h)$.

Secondly, if the coefficients c_{ij} are once-continuously differentiable then h is closable since H can be identified as a symmetric operator $H = -\sum_{i,j=1}^d c_{ij}\partial_i\partial_j - \sum_{j=1}^d (\sum_{i=1}^d \partial_i c_{ij})\partial_j$ with domain $D(H) = L_{2;2}(\mathbf{R}^d) = \bigcap_{i,j=1}^d D(\partial_i\partial_j)$. Then $h(\varphi) = (\varphi, H\varphi)$ for all $\varphi \in D(H)$ and the form h is closable by the Friederich's extension method (see, [Kat], Section VI.2.3).

Thirdly, in one-dimension a complete characterization of closability is given in [FOT], pages 105–107. This observation allows one to construct examples of non-closable elliptic h in higher dimensions. Moreover the argument gives a sufficient condition for closability in higher dimensions which covers many situations of degeneracy.

Proposition 2.3 *Let $\mu_m(x)$ be the smallest eigenvalue of the coefficient matrix $C(x)$ for all $x \in \mathbf{R}^d$. Suppose for almost every $x \in \mathbf{R}^d$ there exists a neighbourhood U of x such that $\mu_m > 0$ almost everywhere on U and $\int_U \mu_m^{-1} < \infty$. Then the form h is closable.*

Proof See [MaR], Section II.2b. □

Many of the examples which we subsequently consider are covered by the next corollary.

Corollary 2.4 *If μ_m is continuous and has a discrete set of zeros then the form h is closable.*

In the general situation, when it is unclear if h is closable, we adopt a different approach to the definition of the elliptic operator. We will define it by an approximation method akin to the viscosity method of partial differential equations.

Define h_ε for each $\varepsilon \in \langle 0, 1]$ by $D(h_\varepsilon) = D(h) = D(l)$ and

$$h_\varepsilon(\varphi) = h(\varphi) + \varepsilon\, l(\varphi) \quad .$$

Then h_ε is the closed form associated with the strongly elliptic operator with coefficients $C_\varepsilon = C + \varepsilon I$. But $\varepsilon \mapsto h_\varepsilon(\varphi)$ decreases monotonically as ε decreases for each $\varphi \in D(h)$. Therefore it follows from a result of Kato, [Kat] Theorem VIII.3.11, that the H_ε converge in the strong resolvent sense, as $\varepsilon \to 0$, to a positive self-adjoint operator H_0 which we refer to as the **viscosity operator** with coefficients $C = (c_{ij})$. This procedure gives a precise meaning to the operator H formally given by (1).

In the following we frequently have to compare two forms and two self-adjoint operators. If k_1 and k_2 are two symmetric forms with domains $D(k_1)$ and $D(k_2)$ in the same Hilbert space, then we write $k_1 \leq k_2$ if $D(k_1) \supseteq D(k_2)$ and $k_1(\varphi) \leq k_2(\varphi)$ for all $\varphi \in D(k_2)$. Next, there is a one-one correspondence between lower bounded self-adjoint operators and closed, densely defined, symmetric, lower bounded quadratic forms. Hence, if, in addition, k_1 and k_2 are lower bounded, closed and densely defined and K_1 and K_2 are the associated self-adjoint operators then we write $K_1 \leq K_2$ in the sense of quadratic forms if $k_1 \leq k_2$.

Let h_0 denote the form associated with H_0, i.e., $D(h_0) = D(H_0^{1/2})$ and $h_0(\varphi) = \|H_0^{1/2}\varphi\|_2^2$. Although the construction of the form h_0 might appear arbitrary it does have an interesting property of universality.

Proposition 2.5 *The following are valid.*

I. *The viscosity form h_0 is the largest positive, symmetric, closed, quadratic form k with $k \leq h$.*

II. *$h_0(\varphi) = h(\varphi)$ for all $\varphi \in D(h)$ if and only if h is closable and then $h_0 = \overline{h}$, the closure of h.*

Proof Simon [Sim2] defines the regular part of a general positive symmetric densely-defined quadratic form as the largest closable symmetric quadratic form k with $k \leq h$. Therefore, with this terminology, the first statement states that h_0 is the closure of the regular part of h. Then the Statement I follows directly from Theorem 3.2 of [Sim2].

If, however, $h_0(\varphi) = h(\varphi)$ for all $\varphi \in D(h)$, then h_0 is a closed extension of h. Hence h is closable. Conversely if h is closable then h equals the regular part of h and $h_0 = \overline{h}$ by Statement I. □

One implication of Statement I of Proposition 2.5 is that h_0 is independent of the particular approximation technique we have used, i.e., the addition of a small multiple of the Laplacian. The same limit would be obtained if one were to use a multiple of the square of the Laplacian. The latter would correspond more closely to a viscosity term.

Quadratic forms play a significant role in convex analysis [EkT] and convergence theory [Bra] [Mas] but the emphasis is rather different to that of operator theory. In these areas of application properties of lower semicontinuity are important. If h is a positive symmetric quadratic form defined on a dense subspace $D(h)$ of the Hilbert space $\mathcal{H}$ and if one extends h to $\mathcal{H}$ by setting $h(\varphi) = \infty$ if $\varphi \notin D(h)$ then h is closed if and only if the extension is lower semicontinuous [Sim1] [Kat], Lemma VIII.3.14a. In general the lower semicontinuous envelope of the extension, which is variously called the lower semi continuous regularization of h [EkT] page 10 or the relaxed form [Mas] page 28, determines a closed quadratic form. The latter is the closure of the form h, if h is closable, or is the closure of the regular part, in Simon's terminology, if h is not closable. The regularization, or relaxation, has been used in a variety of applications to nonlinear phenomena and discontinuous media (see, for example, [Bra] [EkT] [Jos] [Mas] [Mos] and references therein). Mosco gives examples, on pages 414–416, of relaxed forms which can be traced back to the basic example given in Beurling and Deny's paper [BeD].

The viscosity operator H_0 generates a self-adjoint contraction semigroup $S^{(0)}$ on $L_2(\mathbf{R}^d)$. Then since H_0 is defined as the strong resolvent limit of the strongly elliptic operators H_ε associated with the closed forms h_ε it follows that the semigroup $S^{(0)}$ is the strong limit of the self-adjoint contraction semigroups $S^{(\varepsilon)}$ generated by the H_ε. In particular

$S_t^{(\varepsilon)}$ converges strongly, on $L_2(\mathbf{R}^d)$, to $S_t^{(0)}$ and the convergence is uniform for t in finite intervals. Note that each h_ε is a Dirichlet form, i.e., it satisfies the Beurling–Deny criteria (see, for example, [RSe3], Appendix to Section XIII.12, or [Dav2], Section 1.3). Specifically a positive, symmetric, closed, quadratic form h on L_2 is a Dirichlet form if it satisfies the following two conditions:

1. $\varphi \in D(h)$ implies $|\varphi| \in D(h)$ and $h(|\varphi|) \le h(\varphi)$,
2. $\varphi \in D(h)$ implies $\varphi \wedge \mathbb{1} \in D(h)$ and $h(\varphi \wedge \mathbb{1}) \le h(\varphi)$.

(For a full description of the theory of Dirichlet forms see [BoH] [FOT] [Sil].)

The primary result of the Beurling–Deny theory is that h is a Dirichlet form if and only if the semigroup S generated by the corresponding operator H on L_2 is positivity preserving and extends from $L_2 \cap L_p$ to a contraction semigroup on L_p for all $p \in [1,\infty]$. Therefore the semigroups $S^{(\varepsilon)}$ are positivity preserving and extend to positivity preserving contraction semigroups, also denoted by $S^{(\varepsilon)}$, on each of the spaces $L_p(\mathbf{R}^d)$ with $p \in [1,\infty]$. Since $S^{(\varepsilon)}$ converges strongly to $S^{(0)}$ on $L_2(\mathbf{R}^d)$ it follows that $S^{(0)}$ is positivity preserving. It also extends to a contraction semigroup on the L_p-spaces by observing that

$$|(\varphi, S_t^{(\varepsilon)}\psi)| \le \|\varphi\|_p \, \|\psi\|_q$$

for all $\varphi \in L_2 \cap L_p$ and $\psi \in L_2 \cap L_q$ where p and q are conjugate exponents. Then similar estimates follow for $S_t^{(0)}$ by taking the limit $\varepsilon \to 0$. Therefore $S^{(0)}$ extends to a contraction semigroup on all the L_p-spaces by a density argument. The resulting extensions are obviously positivity preserving and so h_0 must be a Dirichlet form.

Finally we note that as a consequence of positivity and contractivity the viscosity semigroup $S^{(0)}$ satisfies the Markov property

$$0 \le S_t^{(0)}\mathbb{1} \le \mathbb{1} \tag{8}$$

for all $t > 0$ on $L_\infty(\mathbf{R}^d)$. In the next section we will, however, prove that $S_t^{(0)}\mathbb{1} = \mathbb{1}$. This stronger property is often referred to as conservation of probability or stochastic completeness. It is the property that motivated the work of Gaffney [Gaf].

3 L_2 off-diagonal estimates

One may associate with the coefficients $C = (c_{ij})$ a 'distance' $d_C \colon \mathbf{R}^d \times \mathbf{R}^d \to [0,\infty]$ by setting

$$d_C(x\,;y) = \sup_{\psi \in \mathcal{D}} |\psi(x) - \psi(y)| \tag{9}$$

for all $x, y \in \mathbf{R}^d$, where

$$\mathcal{D} = \{\psi \in C_c^\infty(\mathbf{R}^d) : \psi \text{ real and } \Big\|\sum_{i,j=1}^d c_{ij}\,(\partial_i\psi)\,(\partial_j\psi)\Big\|_\infty \le 1\} \quad . \tag{10}$$

If C is strongly elliptic then it follows from the bounds $\|C\|\,I \ge C \ge \mu\,I$ that

$$\|C\|^{-1/2}\,|x-y| \le d_C(x\,;y) \le \mu^{-1/2}\,|x-y|$$

for all $x, y \in \mathbf{R}^d$, i.e., d_C is a proper distance and it is equivalent to the Euclidean distance. For degenerate operators, however, d_C is a pseudodistance, i.e., it has the metric properties of a distance but it can take the value infinity. Nevertheless, for brevity we will refer to it as a distance.

There are a variety of other methods of associating a distance with C especially if the coefficients are continuous. Then one may adopt one of several equivalent 'shortest path' definitions (see [JeS] for a survey and comparison of various possibilities). A definition of the foregoing nature was introduced by Biroli and Mosco [BiM] in the general context of Dirichlet forms and this was crucial for the extension of many concepts of elliptic operator theory to this setting [Stu2] [Stu3]. In the case of degenerate C it is not evident that (9) is the most appropriate definition (see Section 6) but it is adequate for many purposes.

Our immediate purpose is to examine a general type of Gaussian bound on $L_2(\mathbf{R}^d)$ which originated in the work of Gaffney [Gaf]. Bounds of this type have subsequently been considered by various authors (see, for example, [Aus2] [CGT] [Dav3] [Gri] [Stu2] [Stu4]). The bounds are variously called integrated Gaussian estimates, an integrated maximum principle or L_2 off-diagonal bounds.

For all $x \in \mathbf{R}^d$ and $r > 0$ set $B_C(x\,;r) = \{y \in X : d_C(x\,;y) < r\}$. In the sequel we fix $x_1, x_2 \in \mathbf{R}^d$ and $r_1, r_2 > 0$ and consistently use the notation $B_1 = B_C(x_1\,;r_1)$ and $B_2 = B_C(x_2\,;r_2)$ for balls and set

$$\tilde{d}_C(B_1\,;B_2) = (d_C(x_1\,;x_2) - r_1 - r_2) \vee 0 \quad .$$

Note that it follows from the triangle inequality that

$$\tilde{d}_C(B_1\,;B_2) \le d_C(B_1\,;B_2) = \inf_{x\in B_1} \inf_{y\in B_2} d_C(x\,;y) \quad .$$

The subsequent proof of L_2 off-diagonal bounds for the viscosity semigroup $S^{(0)}$ follows the arguments of Davies [Dav3]. Care has to be taken since the distance d_C can take the value infinity. We adopt the convention $e^{-\infty} = 0$.

Proposition 3.1 *The viscosity semigroup $S^{(0)}$ satisfies*

$$|(\varphi_1, S_t^{(0)}\varphi_2)| \le e^{-\tilde{d}_C(B_1;B_2)^2/(4t)}\, \|\varphi_1\|_2\, \|\varphi_2\|_2 \tag{11}$$

$\varphi_1 \in L_2(B_1)$, $\varphi_2 \in L_2(B_2)$ and $t > 0$.

Proof If $d_C(x_1\,;x_2) - r_1 - r_2 \le 0$ then (11) follows from the contractivity of $S^{(0)}$, so we may assume that $d_C(x_1\,;x_2) - r_1 - r_2 > 0$. In particular $d_C(x_1\,;x_2) \in \langle 0, \infty]$. Let $r \in \langle 0, \infty\rangle$ and suppose that $r < d_C(x_1\,;x_2)$ and $r - r_1 - r_2 > 0$. By definition of d_C there exists a $\psi \in \mathcal{D}$ such that $\psi(x_2) - \psi(x_1) > r$.

Consider the bounded multiplication operator U_ρ defined by $U_\rho\varphi = e^{-\rho\psi}\varphi$ for all $\rho \in \mathbf{R}$. If $\varphi \in D(h)$ it follows that $U_\rho\varphi \in D(h)$ and $\partial_i U_\rho\varphi = U_\rho(\partial_i - (\partial_i\psi))\varphi$. Moreover, if φ is real

$$\sum_{i,j=1}^d (\partial_i U_\rho\varphi, c_{ij}\partial_j U_\rho^{-1}\varphi) = h(\varphi) - \rho^2 \sum_{i,j=1}^d ((\partial_i\psi)\varphi, c_{ij}(\partial_j\psi)\varphi)$$
$$\ge h(\varphi) - \rho^2\, \|\varphi\|_2^2$$

since the terms linear in ρ cancel by reality and symmetry. Similarly, if H_ε are the strongly elliptic approximants, with the coefficients $c_{ij}+\varepsilon\delta_{ij}$, to the viscosity operator H_0 one has bounds

$$h_\varepsilon(U_\rho\varphi, U_\rho^{-1}\varphi) \geq -\rho^2(1+\varepsilon\|\nabla\psi\|^2)\|\varphi\|_2^2$$

for all real $\varphi\in D(h)=L_{2;1}$. Therefore using an obvious differential inequality one deduces that $\|U_\rho S_t^{(\varepsilon)}U_\rho^{-1}\|_{2\to2}\leq e^{\rho^2(1+\varepsilon\|\nabla\psi\|^2)t}$. Then by taking the the limit $\varepsilon\downarrow 0$ one concludes that

$$\|U_\rho S_t^{(0)}U_\rho^{-1}\|_{2\to2}\leq e^{\rho^2 t}$$

for all $\rho\in\mathbf{R}$ and $t>0$. The estimate is initially valid on the real L_2-space and then by polarization on the complex space.

Next, if $x\in B_1$ then $\psi(x)-\psi(x_1)\leq d_C(x\,;x_1)<r_1$. So $\rho\psi(x)\leq\rho(\psi(x_1)+r_1)$ for all $\rho>0$ and

$$\|U_\rho^{-1}\varphi_1\|_2=\|e^{\rho\psi}\varphi_1\|_2\leq e^{\rho(\psi(x_1)+r_1)}\,\|\varphi_1\|_2 \quad .$$

Alternatively, if $x\in B_2$ then $|\psi(x)-\psi(x_2)|\leq d_C(x\,;x_2)<r_2$. So

$$\psi(x)-\psi(x_1)=\psi(x_2)-\psi(x_1)-(\psi(x_2)-\psi(x))>r-r_2$$

and $\rho\psi(x)\geq\rho(\psi(x_1)+r-r_2)$ for all $\rho>0$. Therefore

$$\|U_\rho\varphi_2\|_2=\|e^{-\rho\psi}\varphi_2\|_2\leq e^{-\rho(\psi(x_1)+r-r_2)}\,\|\varphi_2\|_2 \quad .$$

Combining these estimates one deduces that

$$\begin{aligned}|(\varphi_1,S_t^{(0)}\varphi_2)| &= |(U_\rho^{-1}\varphi_1,(U_\rho S_t^{(0)}U_\rho^{-1})\,U_\rho\varphi_2)|\\ &\leq e^{-\rho(r-r_1-r_2)}e^{\rho^2 t}\|\varphi_1\|_2\,\|\varphi_2\|_2 \quad .\end{aligned}$$

Then setting $\rho=(2t)^{-1}(r-r_1-r_2)>0$ gives the bounds

$$|(\varphi_1,S_t^{(0)}\varphi_2)|\leq e^{-(r-r_1-r_2)^2/(4t)}\,\|\varphi_1\|_2\,\|\varphi_2\|_2$$

for all $t>0$. Since the estimate is valid for all $r\in\langle 0,\infty\rangle$ such that $r<d_C(x_1\,;x_2)$ and $r-r_1-r_2>0$ one can take the limit $r\uparrow d_C(x_1\,;x_2)$ and one obtains

$$|(\varphi_1,S_t^{(0)}\varphi_2)|\leq e^{-(d_C(x_1;x_2)-r_1-r_2)^2/(4t)}\,\|\varphi_1\|_2\,\|\varphi_2\|_2$$

for all $t>0$. □

Next we observe that the wave equation associated with H_0 has finite speed of propagation [CGT] [Mel] [Sik1].

Proposition 3.2 *If $\varphi_1\in L_2(B_1)$, $\varphi_2\in L_2(B_2)$ then*

$$(\varphi_1,\cos(tH_0^{1/2})\varphi_2)=0 \tag{12}$$

for all $t\leq\tilde d_C(B_1\,;B_2)$.

Proof This is in fact a corollary of Proposition 3.1 since the off-diagonal bounds are equivalent to the finite speed of propagation by the reasoning of [Sik2]. The principal idea is the following.

Lemma 3.3 *Let H be a positive self-adjoint operator on the Hilbert space $\mathcal{H}$. Fix $\varphi, \psi \in \mathcal{H}$ and $r \in \langle 0, \infty \rangle$. The following conditions are equivalent.*

I. $|(\psi, e^{-tH}\varphi)| \le e^{-r^2/(4t)} \, \|\psi\| \, \|\varphi\|$ *for all* $t > 0$.

II. *There is an* $a \ge 1$ *such that* $|(\psi, e^{-tH}\varphi)| \le a \, e^{-r^2/(4t)} \, \|\psi\| \, \|\varphi\|$ *for all* $t > 0$.

III. $(\psi, \cos(tH^{1/2})\varphi) = 0$ *for all* $t \le r$.

Proof Clearly Condition I implies Condition II. Next assume Condition II. Let $\mathbf{C}_+ = \{z \in \mathbf{C} : \mathrm{Re}\, z \ge 0, z \ne 0\}$ and set $S_z = e^{-zH}$ for all $z \in \mathbf{C}_+$. Define $u \colon \mathbf{C}_+ \to \mathbf{R}$ by

$$u(z) = (\psi, S_{z^{-1}}\varphi) \quad .$$

Then u is continuous, bounded on its domain of definition and an analytic function on $\{z \in \mathbf{C} : \mathrm{Re}\, z > 0\}$. Then

$$\sup_{t>0} |e^{r^2 t/4} u(t)| \le a \, \|\psi\| \, \|\varphi\| \quad .$$

Moreover, it follows from positivity and self-adjointness of H that

$$\sup_{z \in i\mathbf{R} \backslash \{0\}} |e^{r^2 z/4} u(z)| \le \|\psi\| \, \|\varphi\| \quad .$$

Hence, by the Phragmén-Lindelöf theorem for a quadrant (see [Mar] vol. II, Theorem 7.5, or [StW], Lemma 4.2, or [GeS], Section IV.7.2),

$$|e^{r^2 z/4} u(z)| \le a \, \|\psi\| \, \|\varphi\| \quad . \tag{13}$$

Consequently

$$|u(z)| \le a \, e^{-r^2 (\mathrm{Re}\, z)/4} \|\psi\| \, \|\varphi\| \tag{14}$$

for all z such that $\mathrm{Re}\, z > 0$. Now

$$(\psi, S_t\varphi) = (\pi t)^{-1/2} \int_0^\infty ds \, e^{-s^2/(4t)} (\psi, \cos(sH^{1/2})\varphi) \quad . \tag{15}$$

Hence changing variables one finds

$$t^{-1/2} u(4t) = \int_0^\infty ds \, e^{-st} \, w(s)$$

with $w(s) = (\pi s)^{-1/2} (\psi, \cos(sH)^{1/2}\varphi)$. Therefore $z \mapsto z^{-1/2} u(4z)$ is the Fourier–Laplace transform of the function $s \mapsto w(s)$. Then it follows from the bounds (14) and the Paley–Wiener theorem ([Hör2], Theorem 7.4.3) that w is supported in the half-line $[r^2, \infty\rangle$. Hence Condition III is valid.

Now assume Condition III. Then the integral relation (15) gives

$$\begin{aligned} |(\psi, S_t\varphi)| &\le (\pi t)^{-1/2} \int_0^\infty ds \, e^{-s^2/(4t)} |(\psi, \cos(sH^{1/2})\varphi)| \\ &\le (\pi t)^{-1/2} \int_r^\infty ds \, e^{-s^2/(4t)} \|\psi\| \, \|\varphi\| \le e^{-r^2/(4t)} \|\psi\| \, \|\varphi\| \end{aligned} \tag{16}$$

and so Condition I is valid. □

The statement of Proposition 3.2 follows immediately from Proposition 3.1 and Lemma 3.3 if $d_C(x_1\,; x_2) < \infty$ and by taking a limit $r \to \infty$ if the distance is infinite. □

The next lemma is a simple consequence of Proposition 3.2.

Lemma 3.4 *If Ψ is an even bounded Borel function with* $\operatorname{supp}\Psi \subseteq [-1,1]$ *then*

$$(\varphi_1, \widehat{\Psi}(rH_0^{1/2})\varphi_2) = 0$$

for all $r \leq \tilde{d}_C(B_1\,;B_2)$, $\varphi_1 \in L_2(B_1)$ and $\varphi_2 \in L_2(B_2)$, where $\widehat{\Psi}$ denotes the Fourier transform of Ψ.

Proof Since Ψ is even,

$$\widehat{\Psi}(rH_0^{1/2}) = (2\pi)^{-1}\int_{\mathbf{R}} dt\,\Psi(t)\cos(rtH_0^{1/2}) \quad .$$

But $\operatorname{supp}\Psi \subseteq [-1,1]$ and the statement of the lemma follows immediately from Proposition 3.2. □

The L_2 off-diagonal bounds can be extended to more general sets than balls for strongly elliptic operators or operators with continuous coefficients but this is not strictly relevant to the sequel. One can also derive off-diagonal bounds for general sets without assuming strong ellipticity or continuity of the coefficients if one uses the Euclidean distance. Then, however, the Gaussian factor changes.

Lemma 3.5 *If $\|C\| \neq 0$ then*

$$|(\varphi_1, S_t^{(0)}\varphi_2)| \leq e^{-d_e(V_1;V_2)^2/(4\|C\|t)}\,\|\varphi_1\|_2\,\|\varphi_2\|_2$$

for all non-empty measurable $V_1, V_2 \subset \mathbf{R}^d$ and all $\varphi_1 \in L_2(V_1)$, $\varphi_2 \in L_2(V_2)$, $t > 0$, where $d_e(V_1\,;V_2)$ is the Euclidean distance between V_1 and V_2.

Proof Set $N = d_e(V_1\,;V_2) + 1$ and define $\psi\colon \mathbf{R}^d \to \mathbf{R}$ by $\psi(x) = d_e(x\,;V_2) \wedge N$. Then ψ is bounded and $|\psi(x) - \psi(y)| \leq |x-y|$ for all $x, y \in \mathbf{R}^d$. Therefore ψ is partial differentiable, in the L_∞ sense, and $\sum_{i,j=1}^d (\partial_i\psi)\,c_{ij}\,(\partial_j\psi) \leq \|C\|\sum_{i=1}^d |\partial_i\psi|^2 \leq \|C\|$ almost everywhere.

Now for all $\rho \in \mathbf{R}$ define the bounded multiplication operator U_ρ by $U_\rho\varphi = e^{-\rho\psi}\varphi$. Then one computes as in the proof of Proposition 3.1 that $\|U_\rho S_t^{(0)} U_\rho^{-1}\|_{2\to 2} \leq e^{\|C\|\rho^2 t}$ for all $\rho \in \mathbf{R}$ and $t > 0$. Next,

$$|(\varphi_1, S_t^{(0)}\varphi_2)| = |(U_\rho^{-1}\varphi_1, (U_\rho S_t^{(0)} U_\rho^{-1})U_\rho\varphi_2)| \leq e^{\|C\|\rho^2 t}\|U_\rho^{-1}\varphi_1\|_2\,\|U_\rho\varphi_2\|_2 \quad .$$

But $U_\rho\varphi_2 = \varphi_2$ and if $\rho \leq 0$ then $\|U_\rho^{-1}\varphi_1\|_2 \leq e^{\rho(d_e(V_1;V_2)\wedge N)}\|\varphi_1\|_2 = e^{\rho d_e(V_1;V_2)}\|\varphi_1\|_2$. Hence choosing $\rho = -(2t)^{-1}d_e(V_1;V_2)$ establishes the lemma. □

The last lemma allows one to prove that $S^{(0)}$ conserves probability, i.e., the Dirichlet form h_0 is conservative in the terminology of [FOT], page 49.

Proposition 3.6 *The extension of $S^{(0)}$ to $L_\infty(\mathbf{R}^d)$ satisfies*

$$S_t^{(0)}\mathbb{1} = \mathbb{1}$$

for all $t > 0$.

Proof We may assume that $\|C\| \neq 0$. Let $\varphi \in C_c(\mathbf{R}^d)$. Let $R > 0$ and suppose that $\operatorname{supp}\varphi \subset B_R$ where B_R is the (Euclidean) ball of radius R centred at the origin. Let χ_R be a positive C^∞-function with $\chi_R(x) = 1$ if $|x| \leq 2R$ and $\chi_R(x) = 0$ if $|x| \geq 3R$. Then with $(\cdot,\cdot)$ denoting as usual the pairing between L_p and L_q one has

$$
\begin{aligned}
|(S_t^{(0)}\mathbb{1},\varphi) - (\mathbb{1},\varphi)| \leq |(S_t^{(0)}(\mathbb{1}-\chi_R),\varphi)| + |(S_t^{(0)}\chi_R,\varphi) - (S_t^{(\varepsilon)}\chi_R,\varphi)| \\
+ |(S_t^{(\varepsilon)}(\mathbb{1}-\chi_R),\varphi)|
\end{aligned}
$$

for all $\varepsilon > 0$, where we have used $S_t^{(\varepsilon)}\mathbb{1} = \mathbb{1}$. The latter equality follows from the strong ellipticity of H_ε (see, for example, [ElR1], page 145, proof of Theorem 4.6). Now Lemma 3.5 gives

$$
\begin{aligned}
|(S_t^{(0)}(\mathbb{1}-\chi_R),\varphi)| &\leq \sum_{n=2}^\infty |(\mathbb{1}_{B_{(n+1)R}\backslash B_{nR}}(\mathbb{1}-\chi_R), S_t^{(0)}\varphi)| \\
&\leq \sum_{n=2}^\infty e^{-d_e(B_{(n+1)R}\backslash B_{nR};B_R)^2(4\|C\|t)^{-1}}\|\mathbb{1}_{B_{(n+1)R}\backslash B_{nR}}\|_2\|\varphi\|_2 \\
&\leq \sum_{n=2}^\infty e^{-(n-1)^2R^2(4\|C\|t)^{-1}}|B_{(n+1)R}|^{1/2}\|\varphi\|_2 \\
&\leq \sum_{n=2}^\infty (n+1)^{d/2}e^{-(n-1)^2R^2(4\|C\|t)^{-1}}|B_R|^{1/2}\|\varphi\|_2 \quad .
\end{aligned}
$$

Similarly,

$$
|(S_t^{(\varepsilon)}(\mathbb{1}-\chi_R),\varphi)| \leq \sum_{n=2}^\infty (n+1)^{d/2}e^{-(n-1)^2R^2(4\|C\|t)^{-1}}|B_R|^{1/2}\|\varphi\|_2
$$

for all $\varepsilon > 0$. Hence

$$
\begin{aligned}
|(S_t^{(0)}\mathbb{1},\varphi) - (\mathbb{1},\varphi)| \leq |(S_t^{(0)}\chi_R,\varphi) - (S_t^{(\varepsilon)}\chi_R,\varphi)| \\
+ 2\sum_{n=2}^\infty (n+1)^{d/2}e^{-(n-1)^2R^2(4\|C\|t)^{-1}}|B_R|^{1/2}\|\varphi\|_2
\end{aligned}
$$

for all $\varepsilon > 0$ and $R > 0$ such that $\operatorname{supp}\varphi \subset B_R$. Since $S_t^{(\varepsilon)}$ converges strongly to $S_t^{(0)}$ as $\varepsilon \to 0$ the desired result follows by taking successive limits $\varepsilon \to 0$ and $R \to \infty$. □

The proof of the conservation property of Proposition 3.6 is partially based on the observation that it is valid for semigroups generated by strongly elliptic operators. But in the latter case the statement of the proposition extends to a wider class of functions by general functional analysis.

Corollary 3.7 *Let H be a strongly elliptic operator in divergence form with real measurable coefficients and Φ a function which is bounded and holomorphic in a strip $\{z \in \mathbf{C} : |\operatorname{Im} z| < 2\delta\}$ for some $\delta > 0$. Then $\Phi(H^{1/2})$ extends to a bounded operator on L_∞ and the extension, still denoted by $\Phi(H^{1/2})$, satisfies*

$$
\Phi(H^{1/2})\mathbb{1} = \Phi(0)\mathbb{1} \quad . \tag{17}
$$

Proof If S denotes the self-adjoint semigroup generated by H on $L_2(\mathbf{R}^d)$ then S extends to a positive contraction semigroup on the spaces $L_p(\mathbf{R}^d)$ for all $p \in [1,\infty]$, and $S_t 1\!\!1 = 1\!\!1$. Moreover the semigroup S on $L_2(\mathbf{R}^d)$ is holomorphic in the open right half-plane. But its kernel satisfies Gaussian bounds and these extend to the open right half-plane (see [Dav2] especially Theorem 3.4.8, or [ElR3] Theorem 1.1). Therefore the extension of S to the L_p-spaces is also holomorphic in the open right half-plane.

Next observe that the Poisson semigroup P generated by $H^{1/2}$ is given by

$$P_t = (4\pi)^{-1/2} \int_0^\infty ds\, t\, s^{-3/2} e^{-t^2/(4s)} S_s \quad .$$

Therefore P_t maps L_∞ into L_∞ and $P_t 1\!\!1 = 1\!\!1$ for all $t > 0$. Moreover it readily follows that P is holomorphic in the open right half-plane on each of the L_p-spaces. Then $(\lambda I + H^{1/2})^{-1}$ maps L_∞ into L_∞ for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$ and $(\lambda I + H^{1/2})^{-1} 1\!\!1 = \lambda^{-1} 1\!\!1$. In addition $z \mapsto (zI + H^{1/2})^{-1}$ is analytic on $\mathbf{C}\backslash\langle -\infty, 0]$. Hence $(\lambda I + H^{1/2})^{-1} 1\!\!1 = \lambda^{-1} 1\!\!1$ for all $\lambda \in \mathbf{C}\backslash\langle -\infty, 0]$.

If Γ is a contour in the complex plane from $\infty + i\delta$ to $\infty - i\delta$ contained in the set $\{z \in \mathbf{C} : |\operatorname{Im} z| < 2\delta\}\backslash[0,\infty\rangle$ then

$$(2\pi i)^{-1} \int_\Gamma dz\, \Phi(z)\, (zI - H^{1/2})^{-1}$$

is an operator which maps L_∞ into L_∞ and L_2 into L_2. Moreover, it equals the operator $\Phi(H^{1/2})$ defined on L_2 by spectral theory. Hence $\Phi(H^{1/2})$ extends to a bounded operator on L_∞. In addition

$$\Phi(H^{1/2})\, 1\!\!1 = (2\pi i)^{-1} \int_\Gamma dz\, \Phi(z)\, z^{-1}\, 1\!\!1 = \Phi(0)\, 1\!\!1$$

as desired. □

4 Subelliptic heat kernel estimates

The L_2 off-diagonal bounds derived in Proposition 3.1 are valid for all second-order elliptic operators in divergence form. Next we examine pointwise Gaussian upper bounds on the distribution kernel $K^{(0)}$ of the viscosity semigroup $S^{(0)}$ but for these we require a subellipticity estimate. We define H_0 to be subelliptic (of order γ) if there exist $\mu > 0$, $\nu \geq 0$ and $\gamma \in \langle 0, 1]$ such that

$$H_0 \geq \mu\, \Delta^\gamma - \nu\, I \quad . \tag{18}$$

in the sense of quadratic forms. This is equivalent to requiring that the Sobolev inequalities

$$\|H_0^{1/2}\varphi\|_2^2 + \nu\, \|\varphi\|_2^2 \geq \mu\, \|\Delta^{\gamma/2}\varphi\|_2^2$$

are satisfied for all $\varphi \in D(H_0^{1/2}) = D(h_0)$. Note that if $\gamma = 1$ then one may choose $\nu = 0$ and (18) reduces to the strong ellipticity condition $H_0 \geq \mu\,\Delta$ (see Remark 2.2). Note further that the order is not uniquely defined. If (18) is satisfied for one value of γ it is also satisfied for all smaller $\gamma \in \langle 0, 1\rangle$ since one has inequalities $\Delta^\alpha \leq a\,(\Delta^\beta + I)$ for $\beta > \alpha$.

In the sequel when we assume that H_0 is subelliptic then μ, ν and γ will always denote the parameters in the subellipticity condition (18).

The first result is an estimate for small t which follows by variation of standard arguments.

Proposition 4.1 *Assume the viscosity operator H_0 is subelliptic of order γ.*

There is an $a > 0$, depending only on γ and d, such that

$$\|S_t^{(0)}\|_{p\to q} \le a\, e^{\nu}\, (\mu(t\wedge 1))^{-d/(2r\gamma)} \tag{19}$$

for all $t > 0$ and $p, q \in [1,\infty]$ with $p \le q$ where $r^{-1} = p^{-1} - q^{-1}$.

Moreover, for each $\delta > 0$ there exists an $a > 0$, depending only on δ, γ and d, such that

$$K_t^{(0)}(x\,;y) \le a\, e^{2\nu}\, (\mu(t\wedge 1))^{-d/(2\gamma)}\, e^{-d_C(x;y)^2((4+\delta)t)^{-1}} \qquad (x,y)\text{-}a.e. \tag{20}$$

uniformly for all $t > 0$.

Proof The starting point is the fractional Nash inequality

$$\|\varphi\|_2^{2+4\gamma/d} \le c_1\, \|\Delta^{\gamma/2}\varphi\|_2^2\, \|\varphi\|_1^{4\gamma/d} \tag{21}$$

which is valid for all $\varphi \in L_1(\mathbf{R}^d) \cap D(\Delta^{\gamma/2})$ and a $c_1 > 0$, depending only on γ and d. This follows by a slight variation of Nash's original arguments (which he attributes to Stein, see [Nas] page 935) for strongly elliptic operators (see, for example, [Rob] page 169). The principal point is that if T denotes the self-adjoint semigroup generated by Δ^γ then the Hölder inequality gives

$$\|T_t\|_{1\to 2} = \|T_t\|_{2\to\infty} = \Big(\int_{\mathbf{R}^d} dp\, e^{-2tp^{2\gamma}}\Big)^{1/2} = t^{-d/(2\gamma)}\|T_1\|_{2\to\infty}$$

for all $t > 0$. Therefore

$$\begin{aligned}\|\varphi\|_2 &\le \|(I-T_t)\varphi\|_2 + \|T_t\|_{1\to 2}\|\varphi\|_1\\ &\le t\,\|\Delta^{\gamma/2}\varphi\|_2 + t^{d/(2\gamma)}\,\|T_1\|_{1\to 2}\|\varphi\|_1\end{aligned}$$

for all $t > 0$. Optimization with respect to t then yields (21).

Combination of the Nash inequality (21) and the subellipticity condition (18) gives the Nash inequality

$$\|\varphi\|_2^{2+4\gamma/d} \le c_1\, \mu^{-1}\Big(h_0(\varphi) + \nu\, \|\varphi\|_2^2\Big)\|\varphi\|_1^{4\gamma/d} \tag{22}$$

for all $\varphi \in L_1(\mathbf{R}^d) \cap D(h)$.

Next in order to avoid domain problems we use the approximants h_ε. Since $h_\varepsilon \ge h_0$ there are inequalities similar to (22) with h_0 replaced by h_ε. Then, following Nash (see, for example, [CKS] Theorem 2.1), one obtains bounds

$$\|S_t^{(\varepsilon)}\|_{1\to 2} = \|S_t^{(\varepsilon)}\|_{2\to\infty} \le c_2\, (\mu t)^{-d/(4\gamma)}\, e^{\nu t} \tag{23}$$

uniform for all $\varepsilon, t > 0$, where $c_2 = (cd/(4\gamma))^{d/(4\gamma)}$. Then it follows from the strong convergence that

$$\|S_t^{(0)}\varphi\|_2 = \lim_{\varepsilon\downarrow 0}\|S_t^{(\varepsilon)}\varphi\|_2 \le c_2\, (\mu t)^{-d/(4\gamma)}\, e^{\nu t}\|\varphi\|_1$$

for all $t > 0$ and $\varphi \in L_1 \cap L_2$. Since $S^{(0)}$ is a contraction semigroup it further follows that $\|S_t^{(0)}\varphi\|_2 \le \|S_1^{(0)}\varphi\|_2$ for all $t \ge 1$ and $\varphi \in L_1 \cap L_2$. Hence

$$\|S_t^{(0)}\|_{1\to 2} \le c_2\, e^{\nu}\, (\mu(t\wedge 1))^{-d/(4\gamma)}$$

for all $t > 0$. Then the bounds (19) follow with the aid of the contractivity of $S^{(0)}$ by interpolation.

One also has bounds analogous to (19) for the approximants $S^{(\varepsilon)}$ and so

$$\|K_t^{(\varepsilon)}\|_\infty = \|S_t^{(\varepsilon)}\|_{1\to\infty} \leq \|S_{t/2}^{(\varepsilon)}\|_{2\to\infty}^2 \leq c_3\,(\mu(t\wedge 1))^{-d/(2\gamma)}$$

uniformly for all $\varepsilon, t > 0$ with $K^{(\varepsilon)}$ the kernel of the approximating semigroups $S^{(\varepsilon)}$ and $c_3 = c_2^2\,e^{2\nu}\,2^{d/(2\gamma)}$. Now one can extend these latter bounds to the Davies' perturbation of $S^{(\varepsilon)}$ by Davies' method [Dav1] as elaborated by Fabes and Stroock [FaS] to obtain the Gaussian bounds

$$K_t^{(\varepsilon)}(x\,;y) \leq a\,e^{2\nu}\,(\mu(t\wedge 1))^{-d/(2\gamma)}\,e^{-d_{C_\varepsilon}(x;y)^2((4+\delta)t)^{-1}} \tag{24}$$

with a independent of ε, μ and ν.

Next one has the following convergence result for the kernels.

Lemma 4.2 *Assume the viscosity operator H_0 is subelliptic. Then the kernels $K_t^{(\varepsilon)}$ converge in the weak* sense on $L_\infty(\mathbf{R}^d\times\mathbf{R}^d)$, as $\varepsilon\to 0$, to the kernel $K^{(0)}$.*

Proof If $\varepsilon \in \langle 0, 1]$ then $\|C_\varepsilon\| \leq \|C\| + 1$. Therefore $d_{C_\varepsilon}(x\,;y) \geq (1+\|C\|)^{-1/2}|x-y|$ and it follows from (24) that there are $a, b > 0$, depending only on μ, ν, γ and $\|C\|$, such that

$$K_t^{(\varepsilon)}(x\,;y) \leq a\,(t\wedge 1)^{-d/(2\gamma)}\,e^{-b|x-y|^2t^{-1}} \tag{25}$$

uniformly for all $t > 0$, $x, y \in \mathbf{R}^d$ and $\varepsilon \in \langle 0, 1]$. The convergence of the $K^{(\varepsilon)}$ follows from these uniform upper bounds and the L_2-convergence of $S^{(\varepsilon)}$ to $S^{(0)}$ (see, for example, [ElR2] proof of Proposition 2.2). □

Moreover one has convergence of the distances.

Lemma 4.3 $\lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y) = \sup_{\varepsilon>0} d_{C_\varepsilon}(x\,;y) = d_C(x\,;y)$ *for all* $x, y \in \mathbf{R}^d$.

Proof If $\varepsilon_1 \leq \varepsilon_2$ then $d_{C_{\varepsilon_2}} \leq d_{C_{\varepsilon_1}} \leq d_C$ by the definition of the distances. So

$$\lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y) = \sup_{\varepsilon>0} d_{C_\varepsilon}(x\,;y) \leq d_C(x\,;y) \tag{26}$$

for all $x, y \in \mathbf{R}^d$. Fix $x, y \in \mathbf{R}^d$. Let $\psi \in C_c^\infty(\mathbf{R}^d)$ and suppose $\sum_{i,j=1}^d (\partial_i\psi)\,c_{ij}\,(\partial_j\psi) \leq 1$ almost everywhere. Then $\sum_{i,j=1}^d (\partial_i\psi)\,(c_{ij}+\varepsilon\delta_{ij})\,(\partial_j\psi) \leq 1+\varepsilon M$, with $M = \sum_{i=1}^d \|\partial_i\psi\|_\infty^2$, almost everywhere for all $\varepsilon > 0$. If $\psi_\varepsilon = (1+\varepsilon M)^{-1/2}\psi$ then $\psi_\varepsilon \in C_c^\infty(\mathbf{R}^d)$ and $\sum_{i,j=1}^d (\partial_i\psi_\varepsilon)\,(c_{ij}+\varepsilon\delta_{ij})\,(\partial_j\psi_\varepsilon) \leq 1$ almost everywhere. So

$$d_{C_\varepsilon}(x\,;y) \geq |\psi_\varepsilon(x) - \psi_\varepsilon(y)| = (1+\varepsilon M)^{-1/2}\,|\psi(x)-\psi(y)|$$

for all $\varepsilon > 0$. Now take the limit $\varepsilon \to 0$. Then

$$\lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y) \geq |\psi(x)-\psi(y)| \quad .$$

But this implies that $\lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y) \geq d_C(x\,;y)$ and the lemma follows. □

The statement of Proposition 4.1 concerning the kernel follows from (24) in the limit $\varepsilon \to 0$ as a consequence of Lemmas 4.2 and 4.3. □

There are a number of alternative ways of passing from the semigroup estimates to the pointwise estimates on the kernel. Theorem 4 of [Sik2] is based on an argument which exploits the finite speed of propagation and which is applicable in the current context.

Note that if for some $x, y \in \mathbf{R}^d$ one has $d_C(x;y) = \infty$ and $K_t^{(0)}$ is continuous at (x, y) then $K_t^{(0)}(x\,;y) = 0$. Further the foregoing arguments give a bound on the kernel which does not decrease with t. We will see in Section 6 that this is the best one can hope for unless one has more information such as continuity or strict positivity of the kernel.

The Gaussian upper bounds in fact give information on lower bounds by a variation of standard arguments for strongly elliptic operators.

Corollary 4.4 *Assume that the viscosity operator H_0 is subelliptic. Let $r, t > 0$. Then there is an $a' > 0$ such that*

$$(\varphi, S_t^{(0)}\varphi) \geq a' \, \|\varphi\|_1^2 \tag{27}$$

for all positive $\varphi \in L_1(\mathbf{R}^d) \cap L_2(\mathbf{R}^d)$ with $\mathrm{diam}(\mathrm{supp}\,\varphi) \leq r$, where the diameter is with respect to the Euclidean distance. Hence if $K_t^{(0)}$ is continuous at $(x, x) \in \mathbf{R}^d \times \mathbf{R}^d$ then

$$K_t^{(0)}(x\,;x) \geq a' \quad .$$

The value of a' depends on H_0 only through the parameters μ, ν, γ and $\|C\|$.

Proof Since $S^{(0)}$ is self-adjoint it follows that $(\varphi, S_t^{(0)}\varphi) = \|S_{t/2}^{(0)}\varphi\|_2^2 \geq 0$. Hence

$$|(\varphi, S_t^{(0)}\psi)|^2 \leq (\varphi, S_t^{(0)}\varphi)\,(\psi, S_t^{(0)}\psi) \tag{28}$$

for all $\varphi, \psi \in L_2(\mathbf{R}^d)$. Next let $x_0 \in \mathbf{R}^d$ and let φ be a positive integrable function with support in the Euclidean ball $B_e(x_0\,;r) = \{y \in \mathbf{R}^d : |y - x_0| < r\}$. Further let $R > 2r$ and let ψ be the characteristic function of the ball $B_e(x_0\,;R)$. We evaluate (28) with this choice of φ and ψ.

First one has

$$(\psi, S_t^{(0)}\psi) \leq \|\psi\|_2^2 = V_e(R)$$

where $V_e(R)$ is the volume of $B_e(x_0\,;R)$.

Secondly, $S_t^{(0)}1\!\!1 = 1\!\!1$. Hence

$$(\varphi, S_t^{(0)}\psi) = (\varphi, 1\!\!1) - (\varphi, S_t^{(0)}(1\!\!1 - \psi)) \geq \|\varphi\|_1\Big(1 - \sup_{x \in B_e(x_0;r)} \int_{\{y:|y-x_0|\geq R\}} dy\, K_t^{(0)}(x\,;y)\Big) \quad .$$

Then since $K_t^{(\varepsilon)}$ satisfies the bounds (25) it follows from Lemma 4.2 that there are $a, b > 0$, depending only on μ, ν, γ and $\|C\|$, such that

$$K_t^{(0)}(x\,;y) \leq a\,(t \wedge 1)^{-d/(2\gamma)}\, e^{-b|x-y|^2 t^{-1}} \tag{29}$$

uniformly for all $t > 0$, $x, y \in \mathbf{R}^d$. Hence one can choose R sufficiently large that

$$(\varphi, S_t^{(0)}\psi) \geq 2^{-1}\|\varphi\|_1 \quad .$$

Thirdly, substituting these last two estimates in (28) one deduces that

$$(\varphi, S_t^{(0)}\varphi) \geq (4V_e(R))^{-1} \, \|\varphi\|_1^2 \quad .$$

It follows that (27) is valid with $a' = (4V_e(R))^{-1}$. The value of R is dictated by the Gaussian bounds (29) and hence depends on H_0 only through the parameters μ, ν, γ and $\|C\|$.

Finally suppose $K_t^{(0)}$ is continuous at a diagonal point, which we may take to be $(0,0)$. Then for $\lambda > 0$ replace φ in (27) by φ_λ where $\varphi_\lambda(x) = \lambda^{-d}\varphi(\lambda^{-1}x)$. It follows that $\|\varphi_\lambda\|_1 = \|\varphi\|_1$. Moreover,

$$\lim_{\lambda\to 0}(\varphi_\lambda, S_t^{(0)}\varphi_\lambda) = \lim_{\lambda\to 0}\int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy \, \varphi(x)\,\varphi(y)\, K_t^{(0)}(\lambda x\,;\lambda y) = \|\varphi\|_1^2\, K_t^{(0)}(0\,;0) \quad .$$

Therefore $K_t^{(0)}(0\,;0) \geq a'$. □

Remark 4.5 If the kernel $K_t^{(0)}$ is a continuous function on $\mathbf{R}^d \times \mathbf{R}^d$ it follows from the corollary that it is strictly positive on the diagonal, i.e.,

$$\inf_{x\in\mathbf{R}^d} K_t^{(0)}(x\,;x) \geq a' > 0 \quad .$$

If, however, $K_t^{(0)}$ is uniformly continuous then one has a stronger off-diagonal property. Explicitly, if

$$\lim_{x\to 0} \|L(x)K_t^{(0)} - K_t^{(0)}\|_\infty = 0$$

where $(L(z)K_t^{(0)})(x\,;y) = K_t^{(0)}(x-z\,;y)$ then it follows from Corollary 4.4 that there are $a', r > 0$ such that

$$K_t^{(0)}(x\,;y) \geq a' > 0$$

for all $x, y \in \mathbf{R}^d$ with $|x-y| < r$. Uniform continuity of the kernel in the first variable is of course equivalent to uniform continuity in the second variable, by symmetry, and separate uniform continuity is equivalent to joint uniform continuity.

If the kernel is uniformly continuous then subellipticity implies large time Gaussian bounds of a different character geometric character to the small time bounds of Proposition 4.1. The uniform continuity implies that the kernel decays as $t \to \infty$ with the rate of decay dictated by the dimension d independent of the order of subellipticity.

Theorem 4.6 *Let H_0 be the viscosity operator with coefficients $C = (c_{ij})$ and $K^{(0)}$ the distribution kernel of the contraction semigroup $S^{(0)}$ generated by H_0. Assume*

1. *H_0 is subelliptic.*
2. *There are $a, r > 0$ such that $K_1^{(0)}(x\,;y) \geq a$ for almost every $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$ with $|x-y| \leq r$.*

Then for all $\delta > 0$ there exists an $a' > 0$ such that

$$K_t^{(0)}(x\,;y) \leq a'\, t^{-d/2} e^{-d_C(x;y)^2((4+\delta)t)^{-1}} \quad , \tag{30}$$

(x,y) *almost everywhere, for all* $t \geq 1$.

Moreover, for all $R > 0$ *there is an* $a'' > 0$ *such that*

$$(\varphi, S_t^{(0)}\varphi) \geq a''\, t^{-d/2}\, \|\varphi\|_1^2 \tag{31}$$

for all $t \geq 1$ *and positive* $\varphi \in L_1(\mathbf{R}^d) \cap L_2(\mathbf{R}^d)$ *with* $\operatorname{diam}(\operatorname{supp}\varphi) \leq R$. *Hence if* $K_t^{(0)}$ *is continuous at* $(x,x) \in \mathbf{R}^d \times \mathbf{R}^d$ *then*

$$K_t^{(0)}(x\,;x) \geq a''\, t^{-d/2}$$

for all $t \geq 1$.

Remark 4.7 The local lower bounds of Condition 2 follow from Condition 1 if $K_t^{(0)}$ is uniformly continuous. This is a consequence of Remark 4.5. Nevertheless we show in Section 6 that subellipticity does not necessarily imply uniform continuity nor does it imply strict positivity of the kernel.

Proof The proof of the upper bounds is again based on Nash's original arguments as elaborated by Carlen, Kusuoka and Stroock [CKS]. In particular the following lemma is a version of an argument in Section 4 of [CKS].

Lemma 4.8 *Assume that the distribution kernel* $K^{(0)}$ *satisfies the local lower bounds of Condition* 2 *of Theorem* 4.6. *Then there exists a* $\rho > 0$ *such that*

$$H_0 \geq \rho\, \Delta(I+\Delta)^{-1} \quad .$$

Proof Using spectral theory and the conservation property of Proposition 3.6 one has

$$\begin{aligned}
h_0(\varphi) &\geq t^{-1}(\varphi, (I - S_t^{(0)})\varphi) \\
&= (2t)^{-1}\Big((S_t^{(0)}\mathbb{1}, |\varphi|^2) + (|\varphi|^2, S_t^{(0)}\mathbb{1}) - (\varphi, S_t^{(0)}\varphi) - (S_t^{(0)}\varphi, \varphi)\Big) \\
&= (2t)^{-1}\int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy\, K_t^{(0)}(x\,;y)|\varphi(x)-\varphi(y)|^2
\end{aligned}$$

for all $\varphi \in D(h_0)$ and $t > 0$. Next choose a smooth positive function ρ with support in $\langle -r, r\rangle$ such that $\rho \leq a$ and $\rho = a$ if $|x| \leq r/2$. Then it follows by assumption that

$$K_1^{(0)}(x\,;y) \geq \rho(|x-y|^2)$$

for all $x, y \in \mathbf{R}^d$. Combining these inequalities one finds

$$h_0(\varphi) \geq \int_{\mathbf{R}^d} d\xi\, |\widehat{\varphi}(\xi)|^2 \int_{\mathbf{R}^d} dx\, \rho(|x|^2)\,(1 - \cos\xi.x) \quad .$$

But by the choice of ρ one can find a $\sigma > 0$ such that

$$\int_{\mathbf{R}^d} dx\, \rho(|x|^2)\,(1-\cos\xi.x) \geq \sigma\,(|\xi|^2 \wedge 1) \geq \sigma\,|\xi|^2(1+|\xi|^2)^{-1}$$

for all $\xi \in \mathbf{R}^d$. Therefore $h_0(\varphi) \geq \sigma\, l((I+\Delta)^{-1/2}\varphi)$ for all $\varphi \in D(h_0)$. □

Lemma 4.8 implies that there exists a $\sigma > 0$ such that

$$h_0(\varphi) \geq \sigma \int_{\mathbf{R}^d} d\xi \, |\widehat{\varphi}(\xi)|^2 \, \big(|\xi|^2 \wedge 1\big) \tag{32}$$

for all $\varphi \in D(h)$. Assume that $\varphi \in D(h) \cap L_1$. It then follows by Fourier transformation, as in the proof of Corollary 4.9 in [CKS], that

$$\begin{aligned}
\|\varphi\|_2^2 &= \int_{\{\xi : |\xi| \leq R\}} d\xi \, |\widehat{\varphi}(\xi)|^2 + \int_{\{\xi : |\xi| \geq R\}} d\xi \, |\widehat{\varphi}(\xi)|^2 \\
&\leq c\, R^d \, \|\varphi\|_1^2 + \int_{\{\xi : |\xi| \geq R\}} d\xi \, (R^{-2}|\xi|^2 \wedge 1) \, |\widehat{\varphi}(\xi)|^2 \\
&\leq c\, R^d \, \|\varphi\|_1^2 + R^{-2} \int_{\mathbf{R}^d} d\xi \, (|\xi|^2 \wedge 1) \, |\widehat{\varphi}(\xi)|^2 \\
&\leq c\, R^d \, \|\varphi\|_1^2 + R^{-2}\sigma^{-1} h_\varepsilon(\varphi)
\end{aligned}$$

for all $R \in \langle 0, 1]$ and $\varepsilon > 0$ where the last inequality uses (32) and c is the volume of the Euclidean unit ball in $\mathbf{R}^d$. Then the Nash inequality

$$\|\varphi\|_2^{2+4/d} \leq c' \, h_\varepsilon(\varphi) \|\varphi\|_1^{4/d} \tag{33}$$

follows for all $\varphi \in D(h_\varepsilon) \cap L_1$ with $h_\varepsilon(\varphi) \leq \|\varphi\|_1^2$ by setting $R = (h_\varepsilon(\varphi)/\|\varphi\|_1^2)^{1/(d+2)}$. The inequality is uniform for $\varepsilon \in \langle 0, 1]$. Note that (33) is analogous to the earlier Nash inequality (22) but with $\gamma = 1$ and $\nu = 0$. In addition there is the important restriction $h_\varepsilon(\varphi) \leq \|\varphi\|_1^2$.

Next it follows from the contractivity of $S^{(\varepsilon)}$ on L_1 that

$$\|S_t^{(\varepsilon)}\|_{1\to\infty} \leq \|S_1^{(\varepsilon)}\|_{1\to\infty} \tag{34}$$

for all $t \geq 1$. In particular $t \mapsto \|S_t^{(\varepsilon)}\|_{1\to\infty}$ is uniformly bounded for $t \geq 1$. The conditions (33) and (34) correspond to the assumptions of Theorem 2.9 of [CKS]. Therefore the theorem establishes the large time estimates

$$\|S_t^{(\varepsilon)}\|_{1\to\infty} \leq a' \, t^{-d/2} \tag{35}$$

for all $t \geq 1$. These estimates are again uniform for $\varepsilon \in \langle 0, 1]$.

The estimates (35) convert to large time Gaussian bounds, with the distance associated with C_ε, by Davies perturbation theory as in Proposition 4.1, but with $\gamma = 1$ and $\nu = 0$. Specifically one deduces that for all $\delta > 0$ there exists an $a' > 0$ such that

$$K_t^{(\varepsilon)}(x\,;y) \leq a' \, t^{-d/2} \, e^{-d_{C_\varepsilon}(x;y)^2((4+\delta)t)^{-1}}$$

uniformly for all $t \geq 1$, $x, y \in \mathbf{R}^d$ and $\varepsilon \in \langle 0, 1]$. Finally, taking the limit $\varepsilon \to 0$, one obtains the upper bounds (30) on $K^{(0)}$ by using Lemmas 4.2 and 4.3.

The proof of the lower bounds is a repetition of the argument used to prove Corollary 4.4. Now one chooses ψ to be the characteristic function of the ball $B_e(x_0\,; R\,t^{1/2})$ and uses the upper bounds (30). Moreover one uses the lower bound $d_C(x\,;y) \geq \|C\|^{-1/2}|x-y|$ to express the estimate of $|(\varphi, S_t^{(0)}\psi)|$ in terms of Euclidean parameters. We omit the details. □

Theorem 4.6 implies that subellipticity and local positivity gives estimates

$$a' \, t^{-d/2} \leq \|K_t^{(0)}\|_\infty \leq a \, t^{-d/2}$$

for all $t \geq 1$. Thus the asymptotic behaviour of the kernel is determined by the Euclidean dimension d and is independent of the geometry related to the distance d_C. This confirms the conclusions of Kusuoka and Stroock [KuS]. In fact Theorem 4.6 can be applied directly to the class of operators covered in Kusuoka and Stroock's main application Theorems 3.20 and 3.24. It is possible to verify that their subellipticity assumption (3.21) implies our assumption (18) and one can also verify that the kernels associated with their operators are uniformly continuous. Hence the local lower bounds assumed in Theorem 4.6 follow from the subellipticity by Remarks 4.5 and 4.7. These verifications will be contained in a separate article [ElR4].

Although the bounds of Theorem 4.6 verify the asymptotic behaviour suggested by Kusuoka and Stroock they are weaker than the conclusions of these authors in two respects. First the estimates do not give a Gaussian lower bound. Secondly the estimates rely on an explicit assumption of uniform local positivity for small t of uniform continuity. These features are, however, related and neither can be improved without further assumption. This will be established by examples in Section 6. The problem arising in the case of degenerate operators is that the semigroup kernel is not necessarily strictly positive. In fact it can take the value zero on sets $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ of non-zero measure. This behaviour occurs if $\gamma \in \langle 0, 1/2]$. It is possible that the kernel is strictly positive whenever $\gamma \in \langle 1/2, 1\rangle$ and that the for this range the properties of the kernel resemble those found for strongly elliptic operators , i.e., for operators with $\gamma = 1$. Alternatively it could be relevant that on a large class manifolds for which the heat kernel satisfies Gaussian upper bounds the matching lower bounds are equivalent to Hölder continuity of the kernel (see [Cou] and the extensive list of references therein).

5 Subelliptic resolvent estimates

In this section we use the off-diagonal bounds of Section 3 to establish pointwise on-diagonal lower bounds for the kernel of a high power of the resolvent. The bounds are more efficient than the earlier bounds since they are position dependent. Moreover they can be inverted to give lower bounds on the Euclidean volume of the balls B_C defined by the quasidistance d_C.

First we derive a statement for strongly elliptic operators which will be applied to the approximants H_ε. We use the notation K_S for the distribution kernel of a bounded operator S.

Theorem 5.1 *For all $m \in \mathbf{N}$ with $4m > d$ there exists an $a > 0$ such that for any strongly elliptic operator H with measurable coefficients*

$$a \, |B_C(x\,;r)| \geq K_{(I+r^2H)^{-2m}}(x\,;x)^{-1} \geq \|(I + r^2H)^{-m}\|_{2\to\infty}^{-2}$$

for all $x \in \mathbf{R}^d$ and $r > 0$.

Note that $K_{(I+r^2H)^{-2m}}$ is continuous by standard estimates for strongly elliptic operators if $4m > d$. Therefore its on-diagonal value is well-defined.

The proof of Theorem 5.1 requires an extension of Lemma 3.4.

Lemma 5.2 *If H is strongly elliptic and $\Psi \in \mathcal{S}(\mathbf{R})$ is an even function with* $\operatorname{supp}\Psi \subseteq [-1,1]$ *then the kernel $K_{\widehat{\Psi}(rH^{1/2})}$ is continuous for all $r > 0$ and*

$$K_{\widehat{\Psi}(rH^{1/2})}(x\,;y) = 0$$

for all $x, y \in \mathbf{R}^d$ with $r < d_C(x\,;y)$.

Proof For all $m \in \mathbf{N}$ define $\Phi_m \in \mathcal{S}(\mathbf{R})$ by $\Phi_m(\lambda) = \widehat{\Psi}(\lambda)(1+\lambda^2)^m$. We first show that the distributional kernel $K_{\widehat{\Psi}(rH^{1/2})}$ of $\widehat{\Psi}(rH^{1/2})$ is continuous. Note that

$$\widehat{\Psi}(rH^{1/2}) = (I + r^2H)^{-m}\,\Phi_{2m}(rH^{1/2})\,(I+r^2H)^{-m} \quad .$$

Then $\Phi_{2m}(rH^{1/2})$ maps L_2 into L_2. Moreover, if $4m > d$ then the resolvent $(I + r^2H)^{-m}$ maps L_2 into L_∞ and L_1 into L_2. So $\widehat{\Psi}(rH^{1/2})$ maps L_1 into L_∞ and the distributional kernel $K_{\widehat{\Psi}(rH^{1/2})}$ is a bounded function. Since $\Phi_{2m} \in \mathcal{S}(\mathbf{R})$ the same applies to the kernel $K_{\Phi_{2m}(rH^{1/2})}$ of $\Phi_{2m}(rH^{1/2})$. But if $2m > d$ then the kernel $K_{(I+r^2H)^{-m}}$ of $(I+r^2H)^{-m}$ is continuous, $\sup_{x\in\mathbf{R}^d}\int_{\mathbf{R}^d} dy\,|K_{(I+r^2H)^{-m}}(x\,;y)| < \infty$ and there are $c, \nu > 0$ such that

$$\int_{\mathbf{R}^d} dy\,|K_{(I+r^2H)^{-m}}(x_1\,;y) - K_{(I+r^2H)^{-m}}(x_2\,;y)| \le c\,|x_1 - x_2|^\nu$$

for all $x_1, x_2 \in \mathbf{R}^d$ as a result of standard estimates for strongly elliptic operators. Since

$$K_{\widehat{\Psi}(rH^{1/2})}(x\,;y) = \int dz_1 \int dz_2\, K_{(I+r^2H)^{-m}}(x\,;z_1)\,K_{\Phi_{2m}(rH^{1/2})}(z_1\,;z_2)\,K_{(I+r^2H)^{-m}}(z_2\,;y)$$

for all $x, y \in \mathbf{R}^d$ it follows that $K_{\widehat{\Psi}(rH^{1/2})}$ is Hölder continuous and in particular continuous.

Let $x_1, x_2 \in \mathbf{R}^d$ and suppose that $d_C(x_1\,;x_2) > r$. Set $\varepsilon = 2^{-1}(d_C(x_1\,;x_2) - r)$. Then Lemma 3.4 states that

$$\int dx \int dy\,\overline{\varphi_1(x)}\,K_{\widehat{\Psi}(rH^{1/2})}(x\,;y)\,\varphi_2(y) = (\varphi_1, \widehat{\Psi}(rH^{1/2})\varphi_2) = 0$$

for all $\varphi_1 \in L_2(B_C(x_1\,;\varepsilon))$ and $\varphi_2 \in L_2(B_C(x_2\,;\varepsilon))$. Since H is strongly elliptic the metric d_C is equivalent to the Euclidean metric d_e on $\mathbf{R}^d$. Hence there is a $\delta > 0$ such that $B_e(x_1\,;\delta) \subset B_C(x_1\,;\varepsilon)$ and $B_e(x_2\,;\delta) \subset B_C(x_2\,;\varepsilon)$, where B_e denotes the Euclidean ball. Then

$$\int dx \int dy\,\overline{\varphi_1(x)}\,K_{\widehat{\Psi}(rH^{1/2})}(x\,;y)\,\varphi_2(y) = 0$$

for all $\varphi_1 \in L_2(B_e(x_1\,;\delta))$ and $\varphi_2 \in L_2(B_e(x_2\,;\delta))$. Since $K_{\widehat{\Psi}(rH^{1/2})}$ is continuous this implies that $K_{\widehat{\Psi}(rH^{1/2})}(y_1\,;y_2) = 0$ for all $y_1 \in B_e(x_1\,;\delta)$ and $y_2 \in B_e(x_2\,;\delta)$. In particular $K_{\widehat{\Psi}(rH^{1/2})}(x_1\,;x_2) = 0$. □

Proof of Theorem 5.1 Fix $\Psi \in \mathcal{S}(\mathbf{R})$ even with $\operatorname{supp}\Psi \subseteq [-1,1]$ and $\int \Psi = 1$.

It follows from Corollary 3.7, applied with $\Phi = \widehat{\Psi}$, that $\widehat{\Psi}(rH^{1/2})\,\mathbb{1} = \widehat{\Psi}(0)\,\mathbb{1} = \mathbb{1}$ for all $r > 0$. Let $x \in \mathbf{R}^d$ and $r > 0$. Then the support property of Lemma 5.2 and the Cauchy–Schwarz inequality imply that

$$1 = \int dy\,K_{\widehat{\Psi}(rH^{1/2})}(x\,;y) \le |B_C(x\,;r)|^{1/2}\Big(\int dy\,|K_{\widehat{\Psi}(rH^{1/2})}(x\,;y)|^2\Big)^{1/2} \quad . \tag{36}$$

Let $m \in \mathbf{N}$ with $4m > d$. Define $\Phi_m \in \mathcal{S}(\mathbf{R})$ by $\Phi_m(\lambda) = \widehat{\Psi}(\lambda)(1+\lambda^2)^m$ as before. Then

$$\widehat{\Psi}(rH^{1/2}) = (I + r^2 H)^{-m}\, \Phi_m(rH^{1/2})$$

and $\|\Phi_m(rH^{1/2})\|_{2\to 2} = \|\Phi_m\|_\infty$. Now if S is bounded from $L_2(\mathbf{R}^d)$ to $L_\infty(\mathbf{R}^d)$ and T is bounded from $L_2(\mathbf{R}^d)$ to $L_2(\mathbf{R}^d)$ then

$$\int dy\, |K_{ST}(x\,;y)|^2 \leq \|T\|_{2\to 2}^2 \int dy\, |K_S(x\,;y)|^2 \quad .$$

Therefore applying this estimate to (36) with $S = (I + r^2 H)^{-m}$ and $T = \Phi_m(rH^{1/2})$ one finds

$$\begin{aligned} 1 &\leq |B_C(x\,;r)|\, \|\Phi_m\|_\infty^2 \int dy\, |K_{(I+r^2H)^{-m}}(x\,;y)|^2 \\ &= |B_C(x\,;r)|\, \|\Phi_m\|_\infty^2\, K_{(I+r^2H)^{-2m}}(x\,;x) \leq |B_C(x\,;r)|\, \|\Phi_m\|_\infty^2\, \|(I+r^2H)^{-m}\|_{2\to\infty}^2 \end{aligned}$$

where the second relation uses $K_S * K_T = K_{ST}$. □

Note that the statement of the proposition could be inverted to give

$$K_{(I+tH)^{-2m}}(x\,;x) \geq a_m^{-1}\, |B_C(x\,;t^{1/2})|^{-1} \quad ,$$

i.e., one has an on-diagonal lower bound for the kernel with the anticipated spatial dependence.

Next we consider lower bounds on the volume of the balls associated with a subelliptic operator. In the subelliptic situation a new phemonenon of separation occurs. This will be discussed in detail in Section 6. It is possible to have subspaces $L_2(\Omega)$ of $L_2(\mathbf{R}^d)$ which are invariant under $S_t^{(0)}$ for all $t > 0$. The following result is adapted to this situation.

Theorem 5.3 *Assume the viscosity operator H_0 is subelliptic of order γ. Further assume that there is a non-empty open subset $\Omega \subseteq \mathbf{R}^d$ such that $S_t^{(0)} L_2(\Omega) \subseteq L_2(\Omega)$.*

Then there are $a, R > 0$ such that

$$|B_C(x\,;r) \cap \Omega| \geq a\, r^{d/\gamma} \tag{37}$$

for all $x \in \Omega$ and $r \in \langle 0, R]$. The values of a and R depend only on the subellipticity parameters μ, ν and γ and are independent of Ω.

Proof Let $x \in \Omega$ and $r > 0$. For all $\varepsilon > 0$ let H_ε denote the strongly elliptic approximants to H_0 and let $\Psi \in \mathcal{S}(\mathbf{R})$ be even with $\operatorname{supp} \Psi \subseteq [-1, 1]$ and $\int \Psi = 1$. Since $\widehat{\Psi}(0) = 1$ and $\widehat{\Psi}(rH_\varepsilon^{1/2})$ is symmetric it follows from Corollary 3.7, applied with $\Phi = \widehat{\Psi}$, that

$$(\mathbb{1}, \varphi) = (\mathbb{1}, \widehat{\Psi}(rH_\varepsilon^{1/2})\varphi)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$.

Next since Ω is open there exists an $s_0 \in \langle 0, 1]$ such that $B_{C_1}(x\,;s_0) \subset \Omega$. Let $\delta \in \langle 0, 1]$ and $s \in \langle 0, s_0]$. Then $B_{C_\delta}(x\,;s) \subseteq B_{C_1}(x\,;s_0) \subset \Omega$. Let $\varepsilon \in \langle 0, \delta]$. If $x_0, y_0 \in \mathbf{R}^d$ and $d_{C_\delta}(x_0\,;y_0) > r$ then $d_{C_\varepsilon}(x_0\,;y_0) \geq d_{C_\delta}(x_0\,;y_0) > r$ and $K_{\widehat{\Psi}(rH_\varepsilon^{1/2})}(x_0\,;y_0) = 0$ by

Lemma 5.2. But there exists a positive, non-zero $\varphi \in C_c^\infty(\mathbf{R}^d)$ with $\operatorname{supp}\varphi \subset B_{C_\delta}(x\,;s)$. Then $\operatorname{supp}\widehat{\Psi}(rH_\varepsilon^{1/2})\varphi \subseteq B$ where for brevity we have set $B = B_{C_\delta}(x\,;r+s)$. Therefore

$$(1\!\!1, \varphi) = (1\!\!1_B, \widehat{\Psi}(rH_\varepsilon^{1/2})\varphi) \quad .$$

Since the ball B is relatively compact the characteristic function $1\!\!1_B$ is an L_2-function and since H_ε converges in the strong resolvent sense to H_0 it follows that $\widehat{\Psi}(rH_\varepsilon^{1/2})$ converges strongly to $\widehat{\Psi}(rH_0^{1/2})$ on $L_2(\mathbf{R}^d)$ by [ReS1] Theorem VIII.20. Therefore one deduces that

$$(1\!\!1, \varphi) = (1\!\!1_B, \widehat{\Psi}(rH_0^{1/2})\varphi) \quad .$$

Since $\varphi \in L_2(\Omega)$ and $S_t^{(0)}$ leaves $L_2(\Omega)$ invariant it follows that $\widehat{\Psi}(rH_0^{1/2})\varphi \in L_2(\Omega)$. Consequently

$$\|\varphi\|_1 = (1\!\!1, \varphi) = (1\!\!1_{B\cap\Omega}, \widehat{\Psi}(rH_0^{1/2})\varphi) \le \|1\!\!1_{B\cap\Omega}\|_2 \, \|\widehat{\Psi}(rH_0^{1/2})\|_{1\to 2} \, \|\varphi\|_1$$

from which one concludes that

$$1 \le |B_{C_\delta}(x\,;r+s)\cap\Omega|^{1/2} \, \|\widehat{\Psi}(rH_0^{1/2})\|_{1\to 2}$$

for all $\delta \in \langle 0,1]$ and $s \in \langle 0,s_0]$. But $|B_{C_\delta}(x\,;r+s)\cap\Omega|$ decreases as δ and s decrease to zero and

$$B_{C_\delta}(x\,;r+s) \subset B_{C_1}(x\,;r+1)$$

for all $\delta \in \langle 0,1]$ and $s \in \langle 0,s_0]$. Moreover $|B_{C_1}(x\,;r+1)| < \infty$. Therefore

$$1 \le |B_C(x\,;r)\cap\Omega|^{1/2} \, \|\widehat{\Psi}(rH_0^{1/2})\|_{1\to 2} \tag{38}$$

by Lemma 4.3.

Next let $m \in \mathbf{N}$. If $\Phi_m(\lambda) = \widehat{\Psi}(\lambda)(1+\lambda^2)^m$ as before then one has

$$\|\widehat{\Psi}(rH_\varepsilon^{1/2})\varphi\|_2 \le \|\Phi_m\|_\infty \|(I + r^2H_\varepsilon)^{-m}\varphi\|_2$$

for all $\varphi \in L_2(\Omega)$, $m \in \mathbf{N}$ and $\varepsilon > 0$ as in the proof of Theorem 5.1. Then in the limit $\varepsilon \to 0$ one deduces that

$$\|\widehat{\Psi}(rH_0^{1/2})\|_{1\to 2} \le \|\Phi_m\|_\infty \|(I + r^2H_0)^{-m}\|_{1\to 2} \tag{39}$$

for all $m \in \mathbf{N}$.

Finally it follows from the proof of Proposition 4.1 that there is an $a > 0$, depending only on γ and d, such that

$$\|S_t^{(0)}\|_{1\to 2} \le a\,(\mu t)^{-d/(4\gamma)} e^{\nu t}$$

for all $t > 0$. Then one estimates

$$\begin{aligned}\|(I + r^2H_0)^{-m}\|_{1\to 2} &\le ((m-1)!)^{-1} \int_0^\infty dt\, e^{-t} t^{m-1} \|S_{r^2t}^{(0)}\|_{1\to 2} \\ &\le a\,((m-1)!)^{-1} \int_0^\infty dt\, e^{-t} t^{m-1} (\mu r^2 t)^{-d/(4\gamma)} e^{\nu r^2 t}\end{aligned}$$

and the integral is finite if $r^2 < \nu^{-1}$ and $m > d/(4\gamma)$. Now fix $m \in \mathbf{N}$ with $m > d/(4\gamma)$ and set $R = (2\nu)^{-1/2}$. Then there is an $a' > 0$ such that

$$\|(I + r^2 H_0)^{-m}\|_{1\to 2} \le a'\, r^{-d/(2\gamma)} \tag{40}$$

uniformly for all $r \in \langle 0, R]$. Therefore the theorem follows by a combination of (38), (39) and (40). □

The statement of the theorem is related to Theorem 1 of Fefferman and Phong [FeP]. The latter result establishes for operators with smooth coefficients that subellipticity gives a local comparison, $d_C(x\,;y) \le a\,|x-y|^\gamma$ for $x, y \in \mathbf{R}^d$ with $|x|, |y| \le 1$, of the distance d_C and the subelliptic distance and the Euclidean distance. The statement of Theorem 5.3 is of a similar nature. It is weaker insofar it only compares the volume of balls but it is stronger insofar it is global and valid for operators with measurable coefficients. In fact we next show that it gives the Fefferman–Phong result in one-dimension without any smoothness requirements.

In one-dimension H is formally given by $H = -d\,c\,d$, with $d = d/dx$ and $0 \le c \in L_\infty(\mathbf{R})$, and the strongly elliptic approximants by $H_\varepsilon = -d\,(c+\varepsilon)\,d$. The distance d_{C_ε} is easily computed to be

$$d_{C_\varepsilon}(x\,;y) = \int_x^y dz\,(c(z)+\varepsilon)^{-1/2} \tag{41}$$

for all $x, y \in \mathbf{R}$ with $y > x$ and $d_C(x\,;y) = \lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y)$ by Lemma 4.3. Now the volume estimates of Theorem 5.3 allow one to deduce that d_C is finite-valued.

Theorem 5.4 *Assume $d = 1$ and H_0 is subelliptic of order γ. Then $c(x) > 0$ for almost all $x \in \mathbf{R}$ and*

$$d_C(x\,;y) = \int_x^y dz\, c(z)^{-1/2} \tag{42}$$

for all $x, y \in \mathbf{R}$ with $x < y$. Moreover, there exist $a_1, a_2 > 0$ such that

$$a_1\,|x-y| \le d_C(x\,;y) \le a_2\,(|x-y|^\gamma \vee |x-y|) \tag{43}$$

for all $x, y \in \mathbf{R}$. In particular d_C is finite-valued.

Proof The proof is based upon the volume estimates of Theorem 5.3 applied with $\Omega = \mathbf{R}$. These are valid since we assume H_0 to be subelliptic.

First one has $d_C(x\,;y) \ge \|c\|_\infty^{-1/2}\,|x-y|$. Secondly, since $B_{C_\varepsilon}(x\,;r) \supseteq B_C(x\,;r)$ the volume bounds give

$$|B_{C_\varepsilon}(x\,;r)| \ge a\,r^{1/\gamma}$$

for all $r \in \langle 0, R]$ uniformly for $\varepsilon > 0$. Now $d_{C_\varepsilon}(x\,;y) < \infty$ for all $x, y \in \mathbf{R}$. But, assuming always that $x < y$, there is a $z \in \langle x, y\rangle$ such that $d_{C_\varepsilon}(x\,;z) = d_{C_\varepsilon}(z\,;y)$. Then $B_{C_\varepsilon}(z\,;r) = \langle x, y\rangle$ where $r = 2^{-1} d_{C_\varepsilon}(x\,;y)$. So if $d_{C_\varepsilon}(x\,;y) \le 2R$ then

$$|y-x| = |B_{C_\varepsilon}(z\,;r)| \ge a\,r^{1/\gamma} = a\,(2^{-1} d_{C_\varepsilon}(x\,;y))^{1/\gamma}$$

and

$$d_{C_\varepsilon}(x\,;y) \le a'\,|x-y|^\gamma \tag{44}$$

where $a' = 2a^{-\gamma}$ is independent of ε.

Now let $x, y \in \mathbf{R}$ and suppose $2a'|x-y|^\gamma \le R$. We claim that

$$d_{C_\varepsilon}(x\,;y) \le 2a'\,|x-y|^\gamma \tag{45}$$

for all $\varepsilon > 0$. Indeed, if $\varepsilon \in \langle 1, \infty\rangle$ is large then $d_{C_\varepsilon}(x\,;y) \le \varepsilon^{-1/2}|x-y| \le R$. Hence $d_{C_\varepsilon}(x\,;y) \le a'\,|x-y|^\gamma$ by (44). If (45) is not valid for all $\varepsilon > 0$ then set

$$\varepsilon_0 = \sup\{\varepsilon > 0 : d_{C_\varepsilon}(x\,;y) > 2a'\,|x-y|^\gamma\} \quad .$$

By (41) the function $\varepsilon \mapsto d_{C_\varepsilon}(x\,;y)$ is continuous and decreasing. Therefore $d_{C_{\varepsilon_0}}(x\,;y) = 2a'\,|x-y|^\gamma \le R$ and $d_{C_{\varepsilon_0}}(x\,;y) \le a'\,|x-y|^\gamma$ by (44). This is a contradiction.

So $d_{C_\varepsilon}(x\,;y) \le 2a'\,|x-y|^\gamma$ for all $x, y \in \mathbf{R}$ and $\varepsilon > 0$ if $2a'|x-y|^\gamma \le R$. Then

$$d_C(x\,;y) = \sup_{\varepsilon>0} d_{C_\varepsilon}(x\,;y) \le 2a'\,|x-y|^\gamma$$

for all $x, y \in \mathbf{R}$ with $2a'\,|x-y|^\gamma \le R$ by Lemma 4.3. Hence by the triangle inequality there is an $a'' > 0$ such that $d_C(x\,;y) \le a''\,|x-y|^\gamma$ for all $x, y \in \mathbf{R}$ with $|x-y| \le 1$. Therefore, if $|x-y| \ge 1$ it follows again from the triangle inequality that $d_C(x\,;y) \le 2a''\,|x-y|$. Combining these bounds one deduces that

$$d_C(x\,;y) \le 2a''(|x-y|^\gamma \vee |x-y|)$$

for all $x, y \in \mathbf{R}$. This proves the upper bounds of (43) with $a_2 = 2a''$.

Next since $d_{C_\varepsilon}(x\,;y)$ is given by (41) and $\lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y) = d_C(x\,;y)$ by Lemma 4.3. But d_C is finite valued and so it follows that $c(x) > 0$ for almost every $x \in \mathbf{R}$. But then

$$d_C(x\,;y) = \lim_{\varepsilon\to 0} d_{C_\varepsilon}(x\,;y) = \int_x^y dz\, c(z)^{-1/2}$$

for all $x, y \in \mathbf{R}$ with $x < y$. □

A much stronger conclusion is valid if the coefficient c is twice-differentiable.

Proposition 5.5 *Let $d = 1$. Assume $c \ge 0$, $c \in C_b^2(\mathbf{R})$ and the corresponding viscosity operator H_0 is subelliptic. Then H_0 is strongly elliptic.*

Proof It follows from the above argument $c > 0$ almost everywhere and one has

$$c(y) - 2c(x) \le c(y) + c(2x-y) - 2c(x) = \int_0^{y-x} dt \int_{x-t}^{x+t} ds\, c''(s) \le \|c''\|_\infty (y-x)^2$$

for all $x, y \in \mathbf{R}$. Now let $x \in \mathbf{R}$. Then for all $y \in \langle x, x+1]$ one has by Cauchy–Schwarz

$$\begin{aligned}
y - x &= \int_x^y c^{-1/4}\, c^{1/4} \\
&\le \Big(\int_x^y c^{-1/2}\Big)^{1/2} \Big(\int_x^y c^{1/2}\Big)^{1/2} \\
&\le d_C(x\,;y)^{1/2} \Big(\Big(\int_x^y \mathbb{1}\Big)^{1/2} \Big(\int_x^y c\Big)^{1/2}\Big)^{1/2} \\
&\le a_2^{1/2} |y-x|^{\gamma/2}\, |y-x|^{1/4} \Big(\int_x^y c\Big)^{1/4}
\end{aligned}$$

where $a_2 > 0$ is as in (43). Therefore

$$\begin{aligned} a_2^{-2}|y-x|^{3-2\gamma} &\leq \int_x^y dt\, c(t) \\ &\leq \int_x^y dt\Big(2c(x) + \|c''\|_\infty\, (t-x)^2\Big) \\ &= 2c(x)\,|y-x| + 3^{-1}\|c''\|_\infty\,|y-x|^3 \quad . \end{aligned}$$

Rearranging gives

$$c(x) \geq 2^{-1}|y-x|^{2-2\gamma}\Big(a_2^{-2} - 3^{-1}\|c''\|_\infty\,|y-x|^{2\gamma}\Big) \quad .$$

Now choose $y = x + (3/(2a_2^2\|c''\|_\infty + 3))^{1/(2\gamma)}$. Then

$$c(x) \geq (4a_2^2)^{-1}(3/(2a_2^2\|c''\|_\infty + 3))^{(1-\gamma)/\gamma}$$

and H_0 is strongly elliptic. □

In the context of operators with smooth coefficients Fefferman and Phong derived a converse statement that local comparability of d_C and the Euclidean distance, i.e., estimates of the form (43), imply subellipticity. But no such general statement is possible for operators with measurable coefficients. We will give counterexamples in Section 6. The examples even have coefficients in $C_b^{2\gamma}$ for $\gamma \in \langle 0,1\rangle$. But we conclude this section with some simple examples of subelliptic operators.

Example 5.6 Let $(c_{ij}) \geq (c\,\delta_{ij})$ with

$$c(x) = \Big(\frac{|x|^2}{1+|x|^2}\Big)^\delta$$

and $\delta \in [0,1\rangle$. If $\delta < d/2$ then we prove that the corresponding H_0 is subelliptic of order $1-\delta$.

First suppose $d \geq 3$. Then one has the elementary quadratic form inequality $\Delta \geq \sigma\,|x|^{-2}$ with $\sigma = (d-2)^2/4$ (see, for example, [Kat] Remark VI.4.9a and (VI.4.24), or [ReS2] Lemma on page 169). The inequality immediately implies that $c \geq a_0\,(I+\Delta)^{-\delta}$ as quadratic forms for a suitable $a_0 > 0$. Then it follows from Proposition 2.5 that

$$H_0 \geq a_0\,\Delta(I+\Delta)^{-\delta} = a_0\,\Delta^{1-\delta}(\Delta(I+\Delta)^{-1})^\delta \quad . \tag{46}$$

But since $\delta \in [0,1\rangle$ one has

$$H_0 \geq a_0\,(\varepsilon(1+\varepsilon)^{-1})^\delta \int_\varepsilon^\infty dE_\Delta(\lambda)\,\lambda^{1-\delta} \geq a_0\,(\varepsilon(1+\varepsilon)^{-1})^\delta\,\Big(\Delta^{1-\delta} - \varepsilon^{1-\delta}\,I\Big)$$

for all $\varepsilon > 0$ where E_Δ denotes the spectral family of Δ. Thus the operator H_0 is subelliptic of order $1-\delta$.

A similar conclusion holds for $d = 2$ with $\delta \in [0,1\rangle$ and $d = 1$ with $\delta \in [0,1/2\rangle$ by the following fractional version of the foregoing argument.

It follows by a general result of Strichartz on multipliers on Sobolev spaces [Stri], Theorem 3.6, that $|x|^{-\delta}\Delta^{-\delta/2}$ is bounded on $L_2(\mathbf{R}^d)$ if $\delta \in [0, d/2\rangle$ (see [ReS2], Chapter IX, Exercise 39(b)). Therefore $\Delta^{-\delta/2}|x|^{-2\delta}\Delta^{-\delta/2}$ is bounded and this means that there is a $\sigma > 0$ such that $\Delta^\delta \geq \sigma\,|x|^{-2\delta}$ in the sense of quadratic forms. This is a fractional version of the foregoing estimate and it again gives a bound $c \geq a_0\,(I + \Delta)^{-\delta}$ as forms for a suitable $a_0 > 0$. Hence the estimate (46) is now valid under the restriction $\delta \in [0, d/2\rangle$ and the subellipticity estimate (18) again follows for $\delta < 1 \wedge (d/2)$ by the foregoing spectral argument. Thus if $d = 2$ the estimate is valid for all $\delta \in [0, 1\rangle$ but if $d = 1$ it is only established for $\delta \in [0, 1/2\rangle$. The situation for $d = 1$ and $\delta \in [1/2, 1\rangle$ is more complicated. It will be discussed in detail in Examples 6.7 and 6.8. □

Although the coefficients in these examples are only degenerate at the single point $x = 0$ it is easy to construct examples with a finite number of degeneracies with different orders of degeneracy.

Example 5.7 Let $x_1, \ldots, x_n$ be distinct points in $\mathbf{R}^d$ and $\delta_1, \ldots, \delta_n \in \langle 0, 1\rangle$. Set $\underline{\delta} = \min \delta_i$, $\overline{\delta} = \max \delta_i$ and define $d\colon \mathbf{R}^d \to [0, \infty\rangle$ by $d(x) = \min_{1\leq i\leq n} |x - x_i|^{\delta_i/\overline{\delta}}$. Now if $d \geq 3$ it follows from the bounds $\Delta \geq \sigma\,|x|^{-2}$ and translation invariance that $\Delta \geq \sigma\,|x - x_i|^{-2}$. Therefore one finds straightforwardly that there is a $\sigma' > 0$ such that

$$d(x)^{-2} = \max_{1\leq i\leq n} |x - x_i|^{-2\delta_i/\overline{\delta}} \leq \sum_{i=1}^n |x - x_i|^{-2\delta_i/\overline{\delta}} \leq \sum_{i=1}^n \sigma^{-\delta_i/\overline{\delta}}\Delta^{\delta_i/\overline{\delta}} \leq \sigma'(\Delta + \Delta^{\underline{\delta}/\overline{\delta}}) \quad .$$

Now consider operators with $(c_{ij}) \geq c\,\delta_{ij}$ where

$$c(x) = \Big(\frac{d(x)^2}{1 + d(x)^2}\Big)^\delta$$

and $\delta \in [0, 1\rangle$. It follows as above that there are $a, a' > 0$ such that

$$H_0 \geq a\,\Delta(I + \Delta + \Delta^{\underline{\delta}/\overline{\delta}})^{-\delta} \geq a'\,\Delta(I + \Delta)^{-\delta} \quad .$$

Then by spectral theory there are $\mu, \nu > 0$ such that

$$H_0 \geq \mu\,\Delta^{1-\delta} - \nu\,I$$

and H_0 is subelliptic of order $1 - \delta$. Similarly if $d \leq 2$ the fractional bounds $\Delta^\delta \geq \sigma\,|x|^{-2\delta}$ imply that $(\Delta + \Delta^{\underline{\delta}/\overline{\delta}})^\delta \geq \sigma'\,d(x)^{-2\delta}$. Hence one can establish that H_0 satisfies the subelliptic condition whenever $\delta < 1 \wedge d/2$. In the next section (see Examples 6.7 and 6.8) we return to the discussion of the situation for $d = 1$ and $\delta \in [1/2, 1\rangle$. □

6 Separation properties

The foregoing properties of elliptic and subelliptic operators are direct analogues of similar properties of strongly elliptic operators. The principal difference is the replacement of the Euclidean distance by the distance d_C. But now we examine a phenomenon which has no analogue for strongly elliptic operators, the phenomenon of separation either partial or complete. Degeneracy of the coefficients can lead to the system factoring into independent subsystems, i.e, there is a complete separation. It is also possible to have an incomplete separation but we will not examine this behaviour. These phenomena do not require any particular pathological property of the coefficients and can occur even if the coefficients are nearly C^2 and the operator is subelliptic.

In the sequel we shall need the following simple lemma.

Lemma 6.1 *Let Ω be a measurable subset of $\mathbf{R}^d$. Let S be a bounded self-adjoint operator on $L_2(\mathbf{R}^d)$ which extends to a bounded operator on $L_p(\mathbf{R}^d)$ for all $p \in [1,\infty]$. Suppose S is positivity preserving. Then the following are equivalent.*

I. *There exists a $p \in [1,\infty]$ such that $SL_p(\Omega) \subseteq L_p(\Omega)$.*

II. *There exists a $c > 0$ such that $S\mathbb{1}_\Omega \leq c\,\mathbb{1}_\Omega$.*

If $S\mathbb{1} = \mathbb{1}$ then $S\mathbb{1}_\Omega = \mathbb{1}_\Omega$ in Statement II.

Proof If Statement I holds for some $p \in [1,\infty]$ then by a density argument it is valid for $p = 2$. Next $L_2(\Omega)$ is a closed subspace of $L_2(\mathbf{R}^d)$ and S is self-adjoint. Hence by a standard argument $L_2(\Omega)$ is invariant for S if and only if S commutes with the orthogonal projection on $L_2(\Omega)$

$$SM_\Omega = M_\Omega S \quad , \tag{47}$$

where $M_\Omega(\varphi) = \mathbb{1}_\Omega\varphi$. Then, by another density argument, (47) holds also on L_∞ and $S\mathbb{1}_\Omega = SM_\Omega\mathbb{1} = M_\Omega S\mathbb{1} = \mathbb{1}_\Omega S\mathbb{1} \leq \|S\mathbb{1}\|_\infty\mathbb{1}_\Omega$. Thus Statement II holds with $c = \|S\mathbb{1}\|_\infty$. In particular, if $S\mathbb{1} = \mathbb{1}$ then $S\mathbb{1}_\Omega = \mathbb{1}_\Omega S\mathbb{1} = \mathbb{1}_\Omega$. Finally, if Statement II is valid then for all real $\varphi \in L_2(\Omega)$ one has

$$|S\varphi| \leq S|\varphi| \leq S(\|\varphi\|_\infty\mathbb{1}_\Omega) \leq c\,\|\varphi\|_\infty\mathbb{1}_\Omega$$

since S is positivity preserving. Therefore $S\varphi \in L_\infty(\Omega)$. Then it easily follows that Statement I is valid for $p = \infty$. This completes the proof of the lemma. □

Corollary 6.2 *Let S be a bounded self-adjoint operator on $L_2(\mathbf{R}^d)$ which extends to a bounded operator on $L_p(\mathbf{R}^d)$ for all $p \in [1,\infty]$. Suppose S is positivity preserving. Then*

$$\{\Omega \subseteq \mathbf{R}^d : \Omega \text{ is measurable and } SL_2(\Omega) \subseteq L_2(\Omega)\}$$

is a σ-algebra.

Separation of a semigroup of operators associated to a Dirichlet form can be characterized in terms of the Dirichlet form.

Lemma 6.3 *Let k be a Dirichlet form and T the associated semigroup. Let Ω be a measurable subset of $\mathbf{R}^d$. The following are equivalent.*

I. *$T_tL_2(\Omega) \subseteq L_2(\Omega)$ for all $t > 0$.*

II. *For all $\varphi \in D(k)$ one has $\varphi\mathbb{1}_\Omega \in D(k)$ and*

$$k(\varphi) = k(\varphi\mathbb{1}_\Omega) + k(\varphi\mathbb{1}_{\Omega^c}) \quad . \tag{48}$$

Proof I⇒II. Let $\varphi \in D(k)$. Then

$$t^{-1}(\varphi, (I - T_t)\varphi) = t^{-1}(\varphi 1\!\!1_\Omega, (I - T_t)\varphi 1\!\!1_\Omega) + t^{-1}(\varphi 1\!\!1_{\Omega^c}, (I - T_t)\varphi 1\!\!1_{\Omega^c}) \tag{49}$$

for all $t > 0$. Moreover, both terms on the right hand side of (49) are positive. Hence

$$\sup_{t>0} t^{-1}(\varphi 1\!\!1_\Omega, (I - T_t)\varphi 1\!\!1_\Omega) \le \sup_{t>0} t^{-1}(\varphi, (I - T_t)\varphi) = k(\varphi) < \infty$$

and $\varphi 1\!\!1_\Omega \in D(k)$. Then (48) follows by taking the limit $t \downarrow 0$ in (49).

II⇒I. Define the quadratic forms k_Ω on $L_2(\Omega)$ with form domain

$$D(k_\Omega) = \{\varphi 1\!\!1_\Omega : \varphi \in D(k)\}$$

and $k_\Omega(\varphi) = k(\varphi)$ for all $\varphi \in D(k_\Omega)$. Define similarly the form k_{Ω^c} on $L_2(\Omega^c)$. Let H_Ω and H_{Ω^c} be the associated self-adjoint operators. Then it follows from (48) that $H = H_\Omega \oplus H_{\Omega^c}$, where H is the operator associated to k. Then

$$T_t = e^{-tH_\Omega} \oplus e^{-tH^c_\Omega}$$

for all $t > 0$ and Statement I follows. □

The method to prove separation is contained in the following lemma, which assumes the existence of suitable cut-off functions. We shall give several examples after the lemma.

Lemma 6.4 *Let Ω be a measurable subset of $\mathbf{R}^d$. Let H_0 be a viscosity operator with coefficients c_{ij} on $L_2(\mathbf{R}^d)$. Suppose there exist $\chi_1, \chi_2, \ldots \in L_{\infty;1}$ such that $0 \le \chi_n \le 1$ for all $n \in \mathbf{N}$, $\lim_{n\to\infty} \chi_n = 1\!\!1_\Omega$ almost everywhere and*

$$\lim_{n\to\infty} \int_W \sum_{i,j=1}^d c_{ij}\,(\partial_i \chi_n)\,(\partial_j \chi_n) = 0$$

for any compact subset W of $\mathbf{R}^d$. Then $S_t^{(0)} L_2(\Omega^c) \subseteq L_2(\Omega^c)$ for all $t > 0$.

Proof It suffices to prove that $S_t^{(0)}(L_2(\Omega^c) \cap L_\infty(\Omega^c)) \subseteq L_2(\Omega^c)$ for all $t > 0$. Let $\varphi \in L_2(\Omega^c) \cap L_\infty(\Omega^c)$. Define $X \colon \langle 0, \infty\rangle \to [0, \infty\rangle$ by

$$X(t) = \int_\Omega |S_t^{(0)}\varphi|^2 = (S_t^{(0)}\varphi, 1\!\!1_\Omega S_t^{(0)}\varphi) \quad .$$

Then $\lim_{t\downarrow 0} X(t) = 0$. Moreover, X is differentiable and

$$X'(t) = -2(H_0 S_t^{(0)}\varphi, 1\!\!1_\Omega S_t^{(0)}\varphi)$$

for all $t > 0$. Since $X \ge 0$ it suffices to show that $X'(t) \le 0$ for all $t > 0$.

Fix $t > 0$ and $\tau \in C_c^\infty(\mathbf{R}^d)$ such that $\tau(x) = 1$ for all $x \in B_e(0\,;1)$ and $0 \le \tau \le 1$. Define $\tau_R \in C_c^\infty(\mathbf{R}^d)$ for all $R > 0$ by $\tau_R(x) = \tau(R^{-1}x)$. Set $\chi_{n,R} = \chi_n \tau_R$. Then $\chi_{n,R} \in L_{2;1}(\mathbf{R}^d) \cap L_{\infty;1}(\mathbf{R}^d)$ for all $n \in \mathbf{N}$ and $R > 0$.

One has

$$X'(t) = \lim_{R\to\infty} \lim_{n\to\infty} Y_{n,R}$$

where

$$Y_{n,R} = -2(H_0 S_t^{(0)}\varphi, \chi_{n,R}^2 \, S_t^{(0)}\varphi)$$

for all $n \in \mathbf{N}$ and $R \geq 1$. Let $n \in \mathbf{N}$ and $R \geq 1$. Since H_ε converges in the strong resolvent sense to H_0 it follows that $\Psi(H_\varepsilon)$ converges strongly to $\Psi(H_0)$ for each bounded continuous function Ψ on $\mathbf{R}$ by [ReS1], Theorem VIII.20(b). Therefore

$$Y_{n,R} = \lim_{\varepsilon \to 0} -2(H_\varepsilon S_t^{(\varepsilon)}\varphi, \chi_{n,R}^2 \, S_t^{(\varepsilon)}\varphi) \quad .$$

But integrating by parts one finds

$$\begin{aligned} -2(H_\varepsilon S_t^{(\varepsilon)}\varphi, \chi_{n,R}^2 \, S_t^{(\varepsilon)}\varphi) &= -2\sum_{i,j}((\partial_i S_t^{(\varepsilon)}\varphi), c_{ij}^{(\varepsilon)} \, \chi_{n,R}^2 \, (\partial_j S_t^{(\varepsilon)}\varphi)) \\ &\qquad - 4\sum_{i,j}((\partial_i S_t^{(\varepsilon)}\varphi), c_{ij}^{(\varepsilon)} \, \chi_{n,R} \, (\partial_j \chi_{n,R}) \, (S_t^{(\varepsilon)}\varphi)) \\ &\leq 2\sum_{i,j}(S_t^{(\varepsilon)}\varphi, c_{ij}^{(\varepsilon)} \, (\partial_i \chi_{n,R}) \, (\partial_j \chi_{n,R}) \, S_t^{(\varepsilon)}\varphi) \quad . \end{aligned}$$

Since $\chi_{n,R} \in L_{\infty;1}$ one can again use strong resolvent convergence of the approximants to deduce that

$$\begin{aligned} Y_{n,R} &\leq 2\sum_{i,j}(S_t^{(0)}\varphi, c_{ij}(\partial_i \chi_{n,R}) \, (\partial_j \chi_{n,R}) \, S_t^{(0)}\varphi) \\ &\leq 4\sum_{i,j}(S_t^{(0)}\varphi, \tau_R^2 \, c_{ij}(\partial_i \chi_n) \, (\partial_j \chi_n) \, S_t^{(0)}\varphi) \\ &\qquad + 4\sum_{i,j}(S_t^{(0)}\varphi, \chi_n^2 \, c_{ij}(\partial_i \tau_R) \, (\partial_j \tau_R) \, S_t^{(0)}\varphi) \\ &\leq 4\|S_t^{(0)}\varphi\|_\infty^2 \int \sum_{i,j} \tau_R^2 \, c_{ij}(\partial_i \chi_n) \, (\partial_j \chi_n) + a_1 \int_{B_e(0\,;R)^c} |S_t^{(0)}\varphi|^2 \quad , \end{aligned}$$

where $a_1 = 4\|C\|_\infty \sum_{i=1}^d \|\partial_i \tau\|_\infty^2$ and we used $R \geq 1$. Since $\operatorname{supp} \tau_R$ is compact one can use the assumption on the χ_n and take the limit $n \to \infty$. Next take the limit $R \to \infty$. Then the last term also tends to zero. Hence $X'(t) = \lim_{R\to\infty} \lim_{n\to\infty} Y_{n,R} \leq 0$ for all $t > 0$. □

The basic mechanism which leads to separation is one-dimensional. Therefore we begin by analyzing the one-dimensional situation in detail. Subsequently we examine higher dimensions and describe different aspects that can occur.

In one-dimension there is one positive coefficient c. Suppose that H_0 is subelliptic. Then $c > 0$ almost everywhere, $c^{-1/2}$ is integrable and d_C is explicitly given by (42) by Theorem 5.4. Now d_C is finite-valued and the L_2 off-diagonal bounds of Proposition 3.1 are particularly simple. Then bounded open intervals are balls with a unique centre and radius and $\tilde{d}_C(I_1\,;I_2) = d_C(I_1\,;I_2)$ for each pair of bounded open intervals I_1, I_2 because of the relation (42). Hence Proposition 3.1 gives

$$|(\varphi_1, S_t^{(0)}\varphi_2)| \leq e^{-d_C(I_1;I_2)/(4t)} \|\varphi_1\|_2 \, \|\varphi_2\|_2 \tag{50}$$

for all $t > 0$ and $\varphi_1 \in L_2(I_1)$, $\varphi_2 \in L_2(I_2)$ and all bounded open intervals I_1, I_2. But a much stronger estimate is valid if the system separates into several disjoint subsystems. The next proposition gives a criterion for the separation into two subsystems. Recall that an operator S is positivity improving on L_2 if $S\varphi > 0$ almost everywhere whenever $\varphi \geq 0$ with $\varphi \neq 0$.

Proposition 6.5 *Let $x_0 \in \mathbf{R}$, $c \in L_\infty(\mathbf{R})$ and $\alpha > 0$. Assume $c \geq 0$ almost everywhere, $c(x) > 0$ for all $x \in \langle x_0, x_0 + \alpha]$, the function c^{-1} is bounded on $\langle x_0 + \varepsilon, x_0 + \alpha]$ for all $\varepsilon \in \langle 0, \alpha\rangle$ and that $\int_{x_0}^{x_0+\alpha} c^{-1} = \infty$. Then*

$$S_t^{(0)} L_2(-\infty, x_0) \subseteq L_2(-\infty, x_0) \qquad \text{and} \qquad S_t^{(0)} L_2(x_0, \infty) \subseteq L_2(x_0, \infty)$$

for all $t > 0$. In particular the operator $S_t^{(0)}$ is not positivity improving for any $t > 0$.

Proof We may assume that $x_0 = 0$ and $\alpha = 1$. Secondly, for all $n \in \mathbf{N}$ define $\chi_n : \mathbf{R} \to [0, 1]$ by

$$\chi_n(x) = \begin{cases} 1 & \text{if } x \leq n^{-1} \quad , \\ \eta_n^{-1}\, \eta(x) & \text{if } x \in \langle n^{-1}, 1\rangle \quad , \\ 0 & \text{if } x \geq 1 \quad , \end{cases}$$

where

$$\eta(x) = \int_x^1 c^{-1} \qquad \text{and} \qquad \eta_n = \eta(n^{-1}) \quad .$$

Note that χ_n is absolutely continuous and decreasing. Moreover, $\lim_{n\to\infty} \chi_n = \mathbb{1}_{\langle -\infty, 0]}$ pointwise and $\chi_n' = -\eta_n^{-1}\, c^{-1}\, \mathbb{1}_{\langle n^{-1}, 1\rangle} \in L_\infty(\mathbf{R})$ for all $n \in \mathbf{N}$. Hence

$$\|c\, (\chi_n')^2\|_1 = \int_{1/n}^1 \eta_n^{-2}\, c^{-1} = \eta_n^{-1}$$

for all $n \in \mathbf{N}$ and the proposition follows from Lemma 6.4. □

Of course the conclusion of the proposition is also valid if the conditions on c on the right of x_0 are replaced by similar conditions on the left of x_0, i.e., if $c(x) > 0$ for all $x \in [x_0 - \alpha, x_0\rangle$, the function c^{-1} is bounded on the subsets $[x_0 - \alpha, x_0 - \varepsilon\rangle$ for all $\varepsilon \in \langle 0, \alpha\rangle$ and that $\int_{x_0-\alpha}^{x_0} c^{-1} = \infty$. Similar remarks are valid in other situations such as Propositions 6.6, 6.10 and 6.12. We will not repeat this remark.

It is clear from the Proposition 6.5 and Corollary 6.2 that if the coefficient c has several zeros of the appropriate type then the system can split into several pieces. For example one has the following.

Proposition 6.6 *Let $x_1, x_2 \in \mathbf{R}$ with $x_1 < x_2$ and let $c \in L_\infty(\mathbf{R})$ with $c \geq 0$ almost everywhere. Suppose there exists an $\alpha > 0$ such that for each $k \in \{1, 2\}$ one has $c(x) > 0$ for all $x \in \langle x_k, x_k + \alpha]$ and the function c^{-1} is bounded on $\langle x_k + \varepsilon, x_k + \alpha]$ for all $\varepsilon \in \langle 0, \alpha\rangle$. Further assume that $\int_{x_k}^{x_k+\alpha} c^{-1} = \infty$. Then $S^{(0)}$ leaves the subspaces $L_2(-\infty, x_1)$, $L_2(x_1, x_2)$ and $L_2(x_2, \infty)$ invariant and the semigroup is a direct sum of its restrictions to the subspaces. Each such restriction is a positive contraction semigroup which extends to a contraction semigroup on each of the L_p-spaces and which is conservative on the L_∞-spaces.*

Since $L_2(-\infty, x_0)$ is the orthogonal component of $L_2(x_0, \infty)$ in $L_2(\mathbf{R})$ one has a direct sum decomposition $L_2(\mathbf{R}) = L_2(-\infty, x_0) \oplus L_2(x_0, \infty)$. Then Proposition 6.5 establishes that S_t leaves the two subspaces invariant. Hence one has a direct sum decomposition $S_t = S_t^{-(0)} \oplus S_t^{+(0)}$ where $S^{\pm(0)}$ denote the restrictions of $S^{(0)}$ to the appropriate subspaces. Then it follows straightforwardly that $H_0 = H_{-0} \oplus H_{+0}$ where $H_{\pm 0}$ denote the generators of $S^{\pm(0)}$. In particular it follows from these observations that the action of $S^{(0)}$ is not ergodic and although the semigroup is positivity preserving it is not positivity improving.

The decomposition of $S^{(0)}$ implies that the L_2 off-diagonal bounds (50) can be strengthened since

$$|(\varphi_1, S_t^{(0)} \varphi_2)| = 0$$

for all $t > 0$ and $\varphi_1 \in L_2(I_1)$, $\varphi_2 \in L_2(I_2)$ with $I_1 \subseteq \langle -\infty, x_0]$ and $I_2 \subseteq [x_0, \infty\rangle$. This additional statement can be incorporated into (50) by replacing $d_C(I_1\,;I_2)$ by a set-theoretic 'distance' by the method of Sturm [Stu4], page 237. The definition of the 'distance' is superficially similar to the d_C-definition but it is specifically adapted to the domain of the Dirichlet form h_0. In particular if separation takes place the 'distance' between sets in different components is infinity.

The separation phenomenon allows us to complete the discussion of Example 5.6 for $d = 1$ and $\delta \in [1/2, 1\rangle$.

Example 6.7 Let $\delta \in \langle 1/2, 1\rangle$ and consider the one dimensional operator $H = -d\,c\,d$ with

$$c(x) = \Big(\frac{|x|^2}{1 + |x|^2}\Big)^\delta \quad .$$

We shall prove that H_0 is a subelliptic operator of order $1 - \delta$.

Let d denote the closed operator of differentiation on $L_2(\mathbf{R})$ and $d_\pm$ the corresponding operators on $L_2(\mathbf{R}_\pm)$ with domain $D(d_\pm) = \mathring{W}^{1,2}(\mathbf{R}_\pm)$, so with Dirichlet boundary conditions at the origin, where $\mathbf{R}_- = \langle -\infty, 0\rangle$ and $\mathbf{R}_+ = \langle 0, \infty\rangle$. Then the adjoints $d_\pm^*$ are the operators of differentiation on $L_2(\mathbf{R}_\pm)$ with domain $D(d_\pm^*) = W^{1,2}(\mathbf{R}_\pm)$, so with no boundary condition. Next define the form $h_{\pm,\varepsilon}$ on $L_2(\mathbf{R}_\pm)$ by

$$h_{\pm,\varepsilon}(\varphi_\pm) = (d_\pm^* \varphi_\pm, (c + \varepsilon) d_\pm^* \varphi_\pm)$$

and domain $D(h_{\pm,\varepsilon}) = D(d_\pm^*)$. Moreover, set $h_\varepsilon^{(N)}$ on $L_2(\mathbf{R}) = L_2(\mathbf{R}_-) \oplus L_2(\mathbf{R}_+)$ by

$$h_\varepsilon^{(N)}(\varphi_- \oplus \varphi_+) = h_{-,\varepsilon}(\varphi_-) + h_{+,\varepsilon}(\varphi_+)$$

and domain $D(h_\varepsilon^{(N)}) = D(h_{-,\varepsilon}) \oplus D(h_{+,\varepsilon})$. Now

$$h_\varepsilon(\varphi) = (d\varphi, (c + \varepsilon) d\varphi)$$

for all $\varepsilon > 0$ with $D(h_\varepsilon) = D(d)$. Then $h_\varepsilon^{(N)} \supseteq h_\varepsilon$ and $h_\varepsilon \geq h_\varepsilon^{(N)}$. Thus $H_\varepsilon \geq H_{-,\varepsilon} \oplus H_{+,\varepsilon}$ in the sense of quadratic forms where $H_{\pm,\varepsilon}$ are the positive self-adjoint operators associated with the forms $h_{\pm,\varepsilon}$. Then by strong resolvent convergence one has $H_0 \geq H_{-,0} \oplus H_{+,0}$ where $H_{\pm,0}$ are the viscosity operators associated with the $H_{\pm,\varepsilon}$. Therefore the problem of deriving subellipticity estimates on H_0 is reduced to deriving estimates on the operators $H_{\pm,0}$ on the subspaces $L_2(\mathbf{R}_\pm)$.

Let $l_\pm^D$ and $l_\pm^N$ be the forms on $L_2(\mathbf{R}_\pm)$ with domain $D(l_\pm^D) = D(d_\pm)$ and $D(l_\pm^N) = D(d_\pm^*)$ given by $l_\pm^D(\varphi) = \|d_\pm \varphi\|_2$ and $l_\pm^N(\varphi) = \|d_\pm^* \varphi\|_2$. Then the self-adjoint operators $\Delta_{\pm D}$ and

$\Delta_{\pm N}$ associated with $l^D_\pm$ and $l^N_\pm$ are called the Laplacians with Dirichlet and Neumann boundary conditions on $L_2(\mathbf{R}_\pm)$. Then $\Delta_{\pm D} \geq (4x^2)^{-1}$ in the sense of quadratic forms. This is the one-dimensional version of the estimate $\Delta \geq \sigma x^{-2}$ used in higher dimensions in Example 5.6. It is a special case of the Hardy inequality (see, for example, [Dav2], Lemma 1.5.1). Therefore $c \geq a_0\,(I + \Delta_{\pm D})^{-\delta}$, with $a_0 > 0$ and

$$h_{\pm,\varepsilon}(\varphi) \geq a_0\,(d_\pm^{\,*}\varphi, (I + \Delta_{\pm D})^{-\delta} d_\pm^{\,*}\varphi)$$

for all $\varphi \in D(d_\pm^{\,*})$. But

$$d_\pm^{\,*}(I + \Delta_{\pm N})^{-\delta} \supseteq (I + \Delta_{\pm D})^{-\delta} d_\pm^{\,*}$$

and so one has

$$h_{\pm,\varepsilon}(\varphi) \geq a_0\,(d_\pm^{\,*}\varphi, d_\pm^{\,*}(I + \Delta_{\pm N})^{-\delta}\varphi) = a_0\,(\Delta_{\pm N}^{1/2}\varphi, \Delta_{\pm N}^{1/2}(I + \Delta_{\pm N})^{-\delta}\varphi)$$

for all $\varphi \in D(d_\pm^{\,*})$. Therefore

$$H_{\pm,\varepsilon} \geq a_0\,\Delta_{\pm N}(I + \Delta_{\pm N})^{-\delta} = a_0\,\Delta_{\pm N}^{1-\delta}(\Delta_{\pm N}(I + \Delta_{\pm N})^{-1})^{\delta}$$

in the sense of quadratic forms. Then

$$H_{\pm,0} \geq a_0\,\Delta_{\pm N}^{1-\delta}(\Delta_{\pm N}(I + \Delta_{\pm N})^{-1})^{\delta}$$

by strong resolvent convergence.

Now let $\Delta_N = \Delta_{-N} \oplus \Delta_{+N}$. We call Δ_N the Laplacian on $L_2(\mathbf{R})$ with Neumann boundary conditions at the origin. Then

$$H_0 \geq H_{-,0} \oplus H_{+,0} \geq a_0\,\Delta_N^{1-\delta}(\Delta_N(I + \Delta_N)^{-1})^{\delta}$$

and by spectral theory there are $\mu, \nu > 0$ such that

$$H_0 \geq \mu\,\Delta_N^{1-\delta} - \nu\,I \quad , \tag{51}$$

i.e., the viscosity operator H_0 satisfies the subellipticity condition relative to the Neumann Laplacian. Up to now the estimates are in fact valid for all $\delta \in [0, 1]$.

Set $\gamma = 1 - \delta$. Then $\gamma < 1/2$. We strengthen the estimate (51) by use of standard properties of Sobolev spaces. Since $\gamma < 1/2$ it follows by straightforward argument involving sin and cos expansions that $D(\Delta_{\pm D}^{\gamma/2}) = (\Delta_{\pm N}^{\gamma/2})$. Hence there exists an $a_1 > 0$ such that $a_1\,\|\Delta_{\pm D}^{\gamma/2}\varphi\|_2 \leq \|\Delta_{\pm N}^{\gamma/2}\varphi\|_2 + \|\varphi\|_2$ for all $\varphi \in (\Delta_{\pm N}^{\gamma/2})$. Then by dilation invariance one deduces that

$$a_1\,\|\Delta_{\pm D}^{\gamma/2}\varphi\|_2 \leq \|\Delta_{\pm N}^{\gamma/2}\varphi\|_2$$

for all $\varphi \in (\Delta_{\pm N}^{\gamma/2})$. Hence

$$a_1^2\,\Delta_{\pm D}^{\gamma} \leq \Delta_{\pm N}^{\gamma} \tag{52}$$

in the sense of quadratic forms.

Set $\Delta_D = \Delta_{-D} \oplus \Delta_{+D}$. Then it follows from (52) that there is an $a_2 > 0$ such that $a_2\,\Delta_D^{\gamma} \leq \Delta_N^{\gamma}$ in the sense of quadratic forms. Recall that Δ denotes the Laplacian on $L_2(\mathbf{R})$. Then $\Delta \leq \Delta_D$ and therefore $\Delta^\gamma \leq \Delta_D^\gamma$. Hence $a_2\Delta^\gamma \leq \Delta_N^\gamma$. Consequently (51) gives

$$H_0 \geq a_2\,\mu\,\Delta^{1-\delta} - \nu I$$

i.e., H_0 is subelliptic of order $1 - \delta$. □

Finally we consider the intermediate case $\delta = 1/2$.

Example 6.8 Consider the one dimensional operator $H = -d\,c\,d$ with

$$c(x) = \Big(\frac{|x|^2}{1+|x|^2}\Big)^{1/2} \quad .$$

We shall prove that H_0 is not a subelliptic operator of order 1/2.

Since $c \in L_{\infty;1}(\mathbf{R})$ the form h is closable and $h_0 = \overline{h}$. Let $\varphi \in C_c^\infty(\mathbf{R})$ be such that $\varphi|_{[-1,1]} = 1$. For all $n \in \mathbf{N}$ set $\varphi_n = \chi_n \varphi$, where χ_n is as in the proof of Proposition 6.5. Then $\lim_{n\to\infty} \varphi_n = \varphi_-$ pointwise, where $\varphi_- = \varphi \mathbb{1}_{\langle -\infty,0]}$. Then it follows from a modification of the calculations used to prove Proposition 6.5 that $\lim_{n\to\infty} h(\varphi_n) = \int_{-\infty}^0 c\,|\varphi'|^2$.

Next, $\varphi_-, \varphi_n \in L_1$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \varphi_n = \varphi_-$ in $L_1(\mathbf{R})$. Hence $\lim_{n\to\infty} \hat\varphi_n(p) = \hat\varphi_-(p)$ for all $p \in \mathbf{R}$. Then

$$\int_1^\infty dp\,|p|\,|\hat\varphi_-(p)|^2 \leq \liminf_{n\to\infty} \int_1^\infty dp\,|p|\,|\hat\varphi_n(p)|^2 \leq \liminf_{n\to\infty} \|\Delta^{1/4}\varphi_n\|_2^2 \quad .$$

Since φ' vanishes in a neighbourhood of 0 one calculates that for all $p \neq 0$

$$\hat\varphi_-(p) = i\,p^{-1}\,\varphi(0) - i\,p^{-1}(\varphi'|_{\langle -\infty,0\rangle})\hat{\ }(p) = i\,p^{-1}\,\varphi(0) - p^{-2}(\varphi''|_{\langle -\infty,0\rangle})\hat{\ }(p) \quad .$$

But $\varphi''|_{\langle -\infty,0\rangle} \in \mathcal{S}(\mathbf{R})$ and therefore its Fourier transform $(\varphi''|_{\langle -\infty,0\rangle})\hat{\ }$ is bounded. Hence $\int_1^\infty dp\,|p|\,|\hat\varphi_-(p)|^2 = \infty$. So $\liminf_{n\to\infty} \|\Delta^{1/4}\varphi_n\|_2^2 = \infty$. Since $\sup_n h(\varphi_n) < \infty$ and $\|\varphi_n\|_2 \leq \|\varphi\|_2$ for all $n \in \mathbf{N}$ there are no $\mu, \nu > 0$ such that

$$h(\psi) \geq \mu \|\Delta^{1/4}\psi\|_2^2 - \nu\|\psi\|_2^2$$

for all $\psi \in D(h) = L_{2;1} = D(\Delta^{1/2})$. Thus H_0 is not subelliptic of order 1/2. □

One can now extend the conclusions of Example 5.7 to $d = 1$ and $\delta \in [1/2, 1\rangle$. For simplicity we only describe the case of two zeros with the same order.

Example 6.9 Let $\delta \in [1/2,1\rangle$ and $x_1 < x_2$. Define $d\colon \mathbf{R} \to [0,\infty\rangle$ by $d(x) = |x - x_1| \wedge |x - x_2|$. Set

$$c(x) = \Big(\frac{d(x)^2}{1+d(x)^2}\Big)^{\delta} \quad .$$

Then separation takes place at both x_1 and x_2 by Proposition 6.6. Thus one has a decomposition of the semigroup $S^{(0)}$ as a direct sum of its components on the invariant subspaces $L_2(-\infty, x_1)$, $L_2(x_1, x_2)$ and $L_2(x_2, \infty)$. Moreover, it follows by a slight variation of the arguments in Example 6.7 that one has subelliptic estimates

$$H_0 \geq \mu\,\Delta_N^{1-\delta} - \nu\,I$$

where Δ_N is the direct sum of the Laplacians on the invariant subspaces with Neumann boundary conditions at the endpoints x_1 and x_2. Then, arguing as before, one has the full subelliptic estimate

$$H_0 \geq \mu'\,\Delta^{1-\delta} - \nu\,I \quad ,$$

with Δ the Laplacian on $L_2(\mathbf{R})$, if $\delta \in \langle 1/2, 1\rangle$ but no such estimate is valid if $\delta = 1/2$. □

The Examples 5.6, 6.7, 6.8 and 6.9 show that the properties of subelliptic operators are quite different to those of strongly elliptic operators. If formally $H = -d\,c\,d$ and H_0 is the viscosity operator then H is strongly elliptic if $c \in C_b^2(\mathbf{R})$ by Proposition 5.5. If $\delta \in [1/2, 1\rangle$ and

$$c_\delta(x) = \Big(\frac{|x|^2}{1+|x|^2}\Big)^\delta$$

then c_δ is differentiable and $c'_\delta \in C^{2\delta-1}(\mathbf{R})$, with obvious modifications if $\delta = 1/2$. If H_0 is the viscosity operator with coefficient c_δ then H_0 is subelliptic by Example 6.7 if $\delta > 1/2$. Hence the viscosity semigroup $S^{(0)}$ has a bounded kernel $K^{(0)}$ by Proposition 4.1. But separation takes place by Proposition 6.5. Therefore

- for all $t > 0$ the operator $S_t^{(0)}$ is not positivity improving (Proposition 6.5)

and in particular

- for all $t > 0$ the kernel $K_t^{(0)}$ is not strictly positive.

On the other hand, for strongly elliptic operators with real measurable bounded coefficients the kernel is always strictly positive and Hölder continuous by Nash and De Giorgi. Now, however,

- for all $t > 0$ the kernel $K_t^{(0)}$ is not continuous.

Separation implies that $K_t^{(0)}(-x\,;y) = 0$ whenever $x, y > 0$. Hence if $K_t^{(0)}$ is continuous then $K_t^{(0)}(0\,;0) = \lim_{x\downarrow 0} K_t^{(0)}(-x\,;x) = 0$, in contradiction to Corollary 4.4.

The example also shows that Fefferman and Phong's geometric criterion for subellipticity, [FeP] Theorem 1, is sensitive to smoothness of the coefficients. Theorem 1 in [FeP] states that if $c_{ij} \in C^\infty(\mathbf{R}^d)$ with $C \geq 0$ everywhere and if $\delta \in \langle 0, 1]$ then H is subelliptic of order δ if and only if

$$\text{for all } x \in \mathbf{R}^d \text{ there is an } a > 0 \text{ such that } d_C(x\,;y) \leq a\,|x-y|^\delta \text{ for all } y \in B_e(x\,;1) \tag{53}$$

In Theorem 5.4 we showed in one-dimension, for operators with measurable coefficients, that subellipticity of order δ implies (53) and, in addition, that one can choose a independent of x. The converse is not valid in general since

- if $\delta = 1/2$ then (53) is valid with $\delta = 1/2$, but H_0 is not subelliptic of order 1/2

by Example 6.8.

This shows that the claim in [Stu4], Theorem 4.3.(ii), requires additional assumptions.

Kusuoka and Stroock [KuS] proved under suitable conditions that the kernel satisfies large time Euclidean Gaussian bounds involving the dimension d of the underlying space $\mathbf{R}^d$, But if $\delta \in \langle 1/2, 1\rangle$ and H_0 is the operator in Example 6.9 then H_0 is subelliptic but

- there are no $a, b > 0$ such that $K_t^{(0)}(x\,;y) \leq a\,t^{-d/2} e^{-b|x-y|^2 t^{-1}}$ for all $x, y \in \mathbf{R}^d$ and $t \geq 1$.

In fact

- there is no $a > 0$ such that $\|K_t^{(0)}\|_\infty \leq a\,t^{-d/2}$ for all $t \geq 1$.

In the notation of Example 6.9 one has $S_t^{(0)} \mathbb{1}_{[x_1,x_2]} = \mathbb{1}_{[x_1,x_2]}$. Hence

$$1 = \|S_t^{(0)} \mathbb{1}_{[x_1,x_2]}\|_\infty \leq \|S_t^{(0)}\|_{1\to\infty} \|\mathbb{1}_{[x_1,x_2]}\|_1 = \|K_t^{(0)}\|_\infty \|\mathbb{1}_{[x_1,x_2]}\|_1$$

for all $t > 0$ and $\|K_t^{(0)}\|_\infty \geq (x_2 - x_1)^{-1}$. This shows that the subellipticity condition in Theorem 4.6 is not sufficient to prove (30) and something more is necessary.

It is also evident that the kernel cannot satisfy Gaussian lower bounds because the kernel is not strictly positive.

Although the foregoing description of separation is restricted to one-dimension one can use the mechanism of Proposition 6.5 to construct examples in higher dimensions for which one obtains separation. As a first illustration we consider the separation of a compact subset which we take to be the Euclidean ball $B_e(0\,;1) = \{x \in \mathbf{R}^d : |x| < 1\}$.

Proposition 6.10 *Let $c \in L_\infty([0,\infty\rangle)$ with $c \geq 0$ almost everywhere. Assume $c(x) > 0$ for all $x \in \langle 1, 1+\alpha]$ and the function c^{-1} is bounded on $\langle 1+\varepsilon, 1+\alpha]$ for all $\varepsilon \in \langle 0, \alpha\rangle$ and that $\int_1^{1+\alpha} c^{-1} = \infty$. Define $c_{ij} \colon \mathbf{R}^d \to \mathbf{R}$ by $c_{ij}(x) = c(|x|)\,\delta_{ij}$. Let H_0 be the viscosity operator with coefficients c_{ij}. Then $S_t^{(0)} L_2(B_e(0\,;1)) \subseteq L_2(B_e(0\,;1))$ for all $t > 0$.*

Proof We may assume that $\int_1^2 c^{-1} = \infty$. For all $n \in \mathbf{N}$ define $\chi_n \colon \mathbf{R}^d \to [0,1]$ by

$$\chi_n(x) = \begin{cases} 1 & \text{if } |x| \leq 1 + n^{-1} \quad , \\ \eta_n^{-1}\,\eta(|x|) & \text{if } |x| \in \langle 1+n^{-1}, 2\rangle \quad , \\ 0 & \text{if } |x| \geq 2 \quad , \end{cases}$$

where

$$\eta(r) = \int_r^2 c^{-1} \qquad \text{and} \qquad \eta_n = \eta(1+n^{-1}) \quad .$$

Then $\lim_{n\to\infty} \chi_n = \mathbb{1}_{\overline{B_e(0\,;1)}}$ pointwise and $\chi_n \in L_{\infty;1}$ for all n. Next,

$$\sum_{i,j=1}^d c_{ij}(x)\,(\partial_i \chi_n)(x)\,(\partial_j \chi_n)(x) = \eta_n^{-2}\,c(|x|)^{-1} \mathbb{1}_{\langle 1+n^{-1},2\rangle}(|x|)$$

for almost every $x \in \mathbf{R}^d$. Therefore

$$\int_{\mathbf{R}^d} \sum_{i,j=1}^d c_{ij}\,(\partial_i \chi_n)\,(\partial_j \chi_n) = |B_e(0\,;1)|\,\eta_n^{-2} \int_{1+n^{-1}}^2 dr\, r^{d-1}\,c(r)^{-1} \leq |B_e(0\,;1)|\,\eta_n^{-1}$$

for all $n \in \mathbf{N}$ and the proposition follows from Lemma 6.4. □

One can use the reasoning in Examples 5.6 and 5.7 to obtain explicit examples of this form of separation.

Example 6.11 Define $d \colon \mathbf{R}^d \to \mathbf{R}$ by $d(x) = \min\{|x-y| : |y| = 1\}$. Let $\delta \in [1/2, 1\rangle$. Assume $(c_{ij}) \geq (c\,\delta_{ij})$ where

$$c(x) = \Big(\frac{d(x)^2}{1+d(x)^2}\Big)^\delta \quad .$$

Then separation takes place and the system factors into a component in the unit ball $B_e(0\,;1)$ and a component in its complement. The subellipticity properties of H_0 are similar to the one-dimensional examples. The operator is subelliptic with respect to the operator formed as the direct sum of the Laplacian in the interior of the ball, with Neumann boundary conditions on the boundary, and the Laplacian in the exterior, with the same boundary conditions, by [Dav2] Theorem 1.5.4. Moreover, the operator is subelliptic with respect to the full Laplacian on $\mathbf{R}^d$ if $\delta \in \langle 1/2, 1\rangle$ but not if $\delta = 1/2$.

Since $1\!\!1_{\overline{B_e(0;1)}} \in L_1(\mathbf{R}^d)$ one deduces again that $\|K_t^{(0)}\|_\infty \geq |B_e(0\,;1)|^{-1}$ for all $t > 0$. Therefore it is not possible to have bounds $\|K_t^{(0)}\|_\infty \leq a\, t^{-d/2}$ uniformly for all $t \geq 1$. $\square$

One can also construct examples in higher dimensions where the system separates across an infinitely extended surface. Note that we assume no bounds on the derivatives of the function Φ in the next proposition.

Proposition 6.12 *Let $\Phi \in C^1(\mathbf{R}^{d-1})$. Define*

$$\Omega = \{(y,z) \in \mathbf{R}^{d-1} \times \mathbf{R} : z \leq \Phi(y)\}$$

Let $c \in L_\infty(\mathbf{R})$ with $c \geq 0$ almost everywhere and $\alpha > 0$. Assume $c(x) > 0$ for all $x \in \langle 0, \alpha]$ and the function c^{-1} is bounded on the subsets $\langle \varepsilon, \alpha]$ for all $\varepsilon \in \langle 0, \alpha\rangle$ and that $\int_0^\alpha c^{-1} = \infty$. Define $c_{ij} \colon \mathbf{R}^{d-1} \times \mathbf{R} \to \mathbf{R}$ by $c_{ij}(y,z) = c(z - \Phi(y))\,\delta_{ij}$. Then

$$S_t^{(0)} L_2(\Omega) \subseteq L_2(\Omega)$$

for all $t > 0$.

Proof We may assume that $\alpha = 1$. Let χ_n denote the functions introduced in the proof of Proposition 6.5 and define $\tilde{\chi}_n \in L_{\infty;1}(\mathbf{R}^{d-1} \times \mathbf{R})$ by $\tilde{\chi}_n(y,z) = \chi_n(z - \Phi(y))$. Then $\lim_{n\to\infty} \|c\,(\chi_n')^2\|_1 = 0$. Moreover, $\lim_{n\to\infty} \tilde{\chi}_n = 1\!\!1_\Omega$ pointwise. For almost every $(y,z) \in \mathbf{R}^{d-1} \times \mathbf{R}$ one has

$$\sum_{i,j=1}^d c_{ij}(y,z)\,(\partial_i \chi_n)(y,z)\,(\partial_j \chi_n)(y,z) = c(z - \Phi(y))\,|\chi_n'(z - \Phi(y))|^2\,(1 + |(\nabla\Phi)(y)|^2) \quad .$$

Since $\lim_{n\to\infty} \|c\,(\chi_n')^2\|_1 = 0$ it follows that

$$\lim_{n\to\infty} \int_W \sum_{i,j=1}^d c_{ij}\,(\partial_i \chi_n)\,(\partial_j \chi_n) = 0$$

for any compact subset W of $\mathbf{R}^d$. Then the proposition is a consequence of Lemmas 6.4 and 6.1. $\square$

Although Proposition 6.12 only deals with separation by one surface one may easily extend the reasoning to separation across several surfaces. An interesting situation occurs if one has two disjoint surfaces such as $z = \pm\Phi(y)$ where $\Phi > 0$ and $\lim_{|y|\to\infty} \Phi(y) = 0$. Then the system splits into three components $\Omega_\pm = \{(y,z) \in \mathbf{R}^{d-1} \times \mathbf{R} : \pm z > \Phi(y)\}$ and $\Omega_0 = \{(y,z) \in \mathbf{R}^{d-1} \times \mathbf{R} : |z| \leq \Phi(y)\}$. All three components are unbounded but the Ω_0 component can have finite volume and the corresponding operator $H_0|_{\Omega_0}$ can have compact resolvent. Then again $\|K_t^{(0)}\|_\infty \geq |\Omega_0|^{-1}$ and one cannot have Euclidean Gaussian bounds. In this situation it is also not possible to have a small t power behaviour of the semigroup.

Example 6.13 Let $\Phi \in C^1(\mathbf{R}^{d-1})$ with $\Phi > 0$ and $\lim_{|y|\to\infty} \Phi(y) = 0$. Set $\Omega_\pm = \{(y,z) \in \mathbf{R}^{d-1} \times \mathbf{R} : \pm z > \Phi(y)\}$ and $\Omega_0 = \{(y,z) \in \mathbf{R}^{d-1} \times \mathbf{R} : |z| \leq \Phi(y)\}$. Let $\delta \in [1/2, 1\rangle$. Define $c_{ij}\colon \mathbf{R}^{d-1} \times \mathbf{R} \to \mathbf{R}$ by

$$c_{ij}(y,z) = \Big(\frac{(|z| - \Phi(y))^2}{1 + (|z| - \Phi(y))^2} \Big)^\delta \delta_{ij} \quad .$$

Then it follows as in the proof of Proposition 6.12 and by Corollary 6.2 that one has separation into the three components $\Omega_\pm$ and Ω_0. Now $S^{(0)}$ leaves $L_2(\Omega_0)$ invariant and $S^{(0)} 1\!\!1_{\Omega_0} = 1\!\!1_{\Omega_0}$ for all $t > 0$. Let $a, \alpha > 0$ and $\omega \geq 0$. Suppose

$$\|S^{(0)}_t\|_{1\to 2} \leq a\, t^{-\alpha} e^{\omega t} \tag{54}$$

for all $t > 0$. Fix $m \in \mathbf{N}$ with $m > \alpha$. Then there are $a_1, R > 0$ such that

$$\|(I + r^2 H_0)^{-m}\|_{1\to 2} \leq a_1\, r^{-2\alpha}$$

for all $r \in \langle 0, R]$. (See the proof of Theorem 5.3.) Therefore the estimate (38) gives

$$|B_C((y,z)\,;r) \cap \Omega_0| \geq a_1^{-1}\, r^{2\alpha}$$

for $(y,z) \in \Omega_0$ and all $r \in \langle 0, R]$. But with r fixed it follows that

$$\lim_{|y|\to\infty} |B_C((y,0)\,;r) \cap \Omega_0| = 0 \quad .$$

This is a contradiction. Therefore the bounds (54) are not possible. In particular H_0 cannot be subelliptic. □

Note that the last example differs in character from the earlier ones insofar one has $c_{ij}(y, \Phi(y)) = 0$ for all $y \in \mathbf{R}^{d-1}$, i.e., the coefficients are zero on an unbounded hypersurface.

Acknowledgements

This work was supported by the Australian Research Council's Discovery Grant Program. Parts of the work were carried out whilst the first author was visiting the Centre of Mathematics and its Applications at the ANU, with ARC support, and the second author was visiting the Eindhoven University of Technology. The third and fourth authors participated in the collaboration with the support of the ARC whilst on leave at the ANU .

All four authors thank Barry Simon and Alan McIntosh for helpful information regarding the low-dimensional examples in Sections 4 and 6.

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