A note on the evaluation of the mean value scheme for closed multichain queueing networks

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Abstract

This note deals with an algorithm for the evaluation of the mean value scheme in closed multichain queueing networks. The mean value scheme is a result of the Mean Value Analysis which has proved to be a very useful tool in the evaluation of queueing networks.

Several algorithms to evaluate the schemes have been proposed. We will present an algorithm which we believe to be new in a sense that it does not involve a nested enumeration of population vectors.

1. Introduction

In this note we will present an elegant algorithm to evaluate mean residence times, throughputs and mean queue lengths in a closed multichain queueing network. The algorithm will be based on the recently developed Mean Value Analysis, which has been described for instance in Reiser and Lavenberg [1980]. We restrict our network analysis to a very simple multichain queueing system, as it is our main purpose to show a new enumeration to implement the recursion in the mean value scheme.

In section 2 we will introduce the queueing network and describe the mean value scheme to evaluate the mean values. The algorithm will be developed in section 3. A few concluding remarks constitute section 4.

2. The mean value scheme

Consider a network with N single server FIFO queues and R closed chains. At queue n, $n = 1,2,\ldots,N$, all customers have independent exponential service times with common mean w_n. A closed chain r, $r = 1,2,\ldots,R$, has a Markov routing given by an irreducible stochastic matrix P^r and a fixed number of customers K_r. For reasons of presentation we will restrict ourselves to chains with one customer class.

Mean residence times, throughputs and mean queue lengths in such a network may be evaluated using recursive schemes based on the Mean Value Analysis. This analysis is based on Little's formula and an arrival theorem which states that a customer of a closed chain sees the system at a jump moment as if in equilibrium with one customer of his own chain removed. We will give the mean value scheme without any discussion. For an introduction to the Mean Value Analysis of closed multichain queueing networks we refer to Reiser and Lavenberg [1980] and for a proof of the arrival theorem to Sevcik and Mitrani [1981].

Let us introduce some notations. For n = 1,2,...,N and r = 1,2,...,R we define the following mean values:

S_{nr} : mean residence time of a chain r customer at queue n

Λ_{nr} : throughput of chain r customers at queue n

Q_{nr} : mean number of chain r customers at queue n

Q_n : mean number of customers at queue n .

Furthermore, the population vector K is defined as $K = (K_1,\dots,K_R)$. The mean values depend on K and will be denoted as $S_{nr}(K)$, $\Lambda_{nr}(K)$, $Q_{nr}(K)$ and $Q_n(K)$. For chain r, r = 1,2,...,R, the auxiliary quantities ϑ_{nr} at the successive queues n, n = 1,2,...,N, are defined as the unique solution of the linear system,

$$\vartheta_{nr} = \sum_{m=1}^{N} \vartheta_{mr} P^{r}_{mn} \quad , n = 1,2,\dots,N \text{ and } \sum_{m=1}^{N} \vartheta_{mr} = 1 .$$

Introducing $x_{nr} = \vartheta_{nr} w_n$, $T_{nr}(K) = \vartheta_{nr} S_{nr}(K)$ and $\Lambda_r(K) = \Lambda_{nr}(K)/\vartheta_{nr}$, the mean value scheme to compute resursively the relevant mean values is given by the following three relations

(1) $$T_{nr}(K) = (Q_n(K-e_r) + 1)x_{nr}$$

(2) $$\Lambda_r(K) = K_r / \sum_{m=1}^{N} T_{mr}(K)$$

(3) $$Q_n(K) = \sum_{r=1}^{R} \Lambda_r(K) T_{nr}(K)$$

where the recursion starts with $Q_n(0) = 0$. These relations are equivalent to relations (3.1), (3.3) and (3.4) in Reiser and Lavenberg [1980].

3. The algorithm

The recursion defined by the mean value scheme runs through all vectors in the range (0,...,0) to $(K_1,\ldots,K_R)$. We define the set $S_K \subset \mathbb{Z}^R$, the set of all these vectors, as

$$S_K := \left\{(k_1,\ldots,k_R) \,\middle|\, k_r \in \left\{0,1,\ldots,K_r\right\} \quad , \quad r = 1,2,\ldots,R\right\} .$$

The problem one meets in constructing an algorithm to evaluate the recursive mean value scheme is how to construct a feasible enumeration of the set S_K. In Zahorjan and Wong [1981] some algorithms are given. We will develop an enumeration based on a mapping of the multidimensional set S_K to a one dimensional subset of $\mathbb{Z}$.

Let the integers $X_{K,r}$, $r = 1,2,\ldots,R+1$, be defined by

$$X_{K,1} = 1$$

$$X_{K,r} = (K_{r-1} + 1)X_{K,r-1} = \prod_{j=1}^{r-1} (K_j + 1) \quad , \quad r = 2,3,\ldots,R+1 \; .$$

Note that the number of elements in the set S_K is $X_{K,R+1}$. We next define the map $\varphi_K : S_K \to \{0,1,\ldots,X_{K,R+1} - 1\}$ by,

$$\varphi_K((k_1,\ldots,k_R)) = \sum_{r=1}^{R} k_r X_r \quad , \quad (k_1,\ldots,k_R) \in S_K \; .$$

The map φ_K will be used to construct a feasible enumeration of the set S_K. We thereby have to consider two problems. First is that to evaluate the mean values at a vector $K = (k_1,\ldots,k_R)$, we need the mean values at $k-e_1,\ldots,k-e_R$. So the enumeration is feasible if and only if the mean values at $k-e_1,\ldots,k-e_R$ are evaluated before the mean values at k. Apart from this feasibility problem we have the question of the inverse map φ_K^{-1} of φ_K, which is needed in the enumeration algorithm.

To evaluate the mean value scheme we propose the linear enumeration of the set $\{0,1,\dots,X_{K,R+1} - 1\}$. The following two lemmata are the basis of this enumeration. Lemma 1 shows that the map φ_K is one-to-one and provides us with an algorithm to obtain the inverse mapping φ_K^{-1}. Lemma 2 shows the feasibility of the enumeration.

<u>Lemma 1</u>

The map $\varphi_K : S_K \to \{0,1,\dots,X_{K,R+1} - 1\}$ is one-to-one.

<u>Proof</u>: For $m \in \{0,1,\dots,X_{K,R+1} - 1\}$ define $k^* \in \mathbb{Z}^R$ by the following scheme

$$k_R^* = \text{entier } (m/X_{K,R})$$

$$k_r^* = \text{entier } ((m - \sum_{\ell=r+1}^{R} k_\ell^* X_{K,\ell})/X_{K,r}) \ , \ r = R-1, R-2, \dots, 1 \ .$$

One may verify that $k^* \in S_K$ and that $\varphi_K(k^*) = m$. That the map φ_K is one-to-one follows from the observation that the number of elements in the set S_K equals the number of elements in the set $\{0,1,\dots,X_{K,R+1} - 1\}$. □

<u>Lemma 2</u>

For all $k \in S_K$ with $k_r > 0$ the following relation holds

$$\varphi_K(k) = \varphi_K(k - e_r) + X_r \ .$$

<u>Proof</u>:

$$\varphi_K(k) = \sum_{\ell=1}^{R} k_\ell X_{K,\ell} - X_r + X_r = \varphi_K(k - e_r) + X_r \ .$$ □

We now can give the algorithm in pseudo-pascal.

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for m = 0 step 1 until X_{K,R+1} - 1 do
begin
    (k_1,...,k_R) := φ_K^{-1}(m);
    for r := 1 step 1 until R do
     if k_r = 0 then ∀_n : Q_nr(m) = 0
                else ∀_n : T_nr(m) = (Q_n(m - X_r) + 1)x_nr;
                           Λ_r (m) = k_r / Σ_n T_nr(m);
                       ∀_n : Q_nr(m) = Λ_r(m)T_nr(m);
    ∀_n : Q_n(m) := Σ_r Q_nr(m)
end
```

4. Concluding remarks

We have presented an elegant enumeration algorithm to evaluate the recursive mean value scheme. We do not claim that it is a very efficient algorithm. Especially with respect to the storage requirements the algorithm is not too friendly. However, it is so nice in its simplicity of implementation that we thought it worthwhile to bring it to your attention.

5. References

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