Numerical Convergence of a Hermite interpolation based spatial spectral solver for 2D TE polarization

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Summary. We numerically investigate the convergence of simulations with a spatial spectral discretization based on Hermite interpolation. We use a test 2D transverse electric polarized scattering setup with an incident plane wave on a cylinder in the origin. The results are compared to analytical reference data and the errors are quantified for the most important simulation parameters.

1 Introduction

In this contribution we look into the convergence behaviour of the spatial spectral discretization developed in [3] for 2D TE problems. We examine the effects of the discretization parameters for a case for which we have an analytical solution, namely a dielectric cylinder and an incoming TE polarized wave. The approach is based on solving the following integral equation

$$E_z^i = E_z - \iint G(\vec{x}-\vec{x}')J(\vec{x}')dx_1'dx_2', \tag{1}$$

where $J(\vec{x})$ is the contrast current induced due to the incident TE plane wave $E_z^i(\vec{x})$ and is given by

$$J(\vec{x}) = j\omega\varepsilon_0\mu_0\chi(\vec{x})(E_z^i(\vec{x})+E_z^s(\vec{x})). \tag{2}$$

In this report we only consider the E field. We employ a Hermite interpolation [2] to discretize the problem. It allows easy Fourier and inverse Fourier transforms as well as addition and multiplication operations with $O(n\log(n))$ time or faster. Unlike in [2] we use a Hermite discretization in both dimensions while omitting the spectral path deformation. We demonstrate the impact of important simulation parameters on the accuracy of the computed electric field.

2 Formulation

To solve our system we discretize the contrast function in the spectral domain and transform it back to the spatial domain. We do this instead of direct spatial discretization because the sharp edge of the cylinder is hard to represent with polynomial interpolations while the spectral domain counterpart is of the form $J_1(ax)/x$ which is smooth and has only a removable singularity at the origin. For our (spatial) Green's function we have

$$G(\vec{x}) = j\frac{\pi}{2}H_0^{(1)}(|\vec{x}|). \tag{3}$$

This function has a singularity at $\vec{x}=0$ which is detrimental to the accuracy of the solver if we sample directly near it. We solve this by removing the singularity and treating it separately as follows

$$G(\vec{x}) = \tilde{G}(\vec{x}) + G^s(\vec{x}), \tag{4}$$

with

$$\tilde{G}(\vec{x}) = \frac{\pi}{2}jH_0(|\vec{x}|) + \exp(-B|\vec{x}|^2)\ln(|\vec{x}|) \tag{5}$$

for some steepness parameter B. We now have $\tilde{G}(0) = \ln(2)-\gamma$, with γ being the Euler Mascheroni constant, so it is not singular. For $G_s(\vec{x})$ we make an expansion

$$G_s(\vec{x}) = -\exp(-B|\vec{x}|^2)\ln(|\vec{x}|) \approx \sum_{n=1}^{N}\alpha_n f_n(\vec{x}) \tag{6}$$

in the spatial domain with basis functions $f_n(\vec{x}) = \exp(-\beta_n|\vec{x}|^2)$, based on [1]. These basis functions have simple analytical Fourier transforms

$$G_s(\vec{k}) \approx \sum_{n=1}^{N}\frac{\alpha_n}{2\beta_n}\exp(-\frac{k_0^2+k_1^2}{4\beta_n}) \tag{7}$$

which we can directly discretize in the spectral domain. We found a sufficiently accurate set of coefficients and basis functions to approximate G_s with 10 digits of precision. We discretize $\tilde{G}(\vec{x})$ in the spatial domain and transform it to the spectral domain. Then we directly discretize G_s in the spectral domain and add the two functions together to get our Green's function.

We use an iterative solver, GMRES, to solve for the total electric field E_z which we can directly compare to analytical reference data. We iterate until we reach a residual of 10^{-5}.

The discretization has the following simulation parameters that significantly influence the accuracy

1. The sampling distance Δ_x
2. The interpolation degree d
3. The discretization range r_d

which we vary to get the convergence behaviour of the solver. For $d=2$ this leads to piecewise linear approximations for our function while for $d \geq 3$

the piecewise functions are higher order polynomials. The cylinder is placed in the origin and has a radius of 4 and a dielectric contrast of $\varepsilon_r = 2$. The incident wave is a sinusoidal plane wave with $k_0 = 1$, from an angle of $\phi = \frac{\pi}{3}$ with respect to the x axis. An example of the resulting calculated field can be seen in Figure 1. The general shape of this field corresponds to our reference data with the largest source of errors being around the edge of the cylinder. We are primarily interested the far field. By investigating the errors in the far field instead of the near field we remove the extra variable of the range over which to test. Using reciprocity we get an expression for the far field

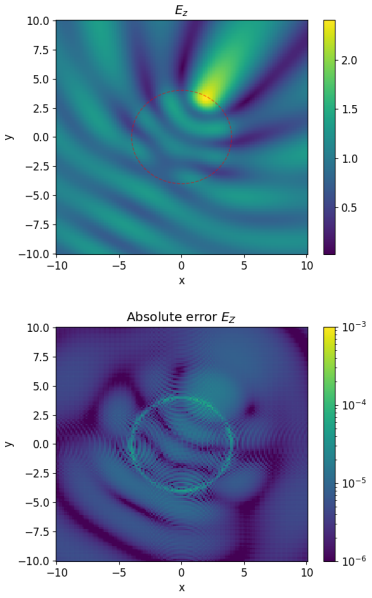

Fig. 1. (top)Resulting field as calculated by the solver (bottom) The absolute error as compared to the analytical solution. Both of these plots use $d = 5$, $r_x = 30$ and $\Delta_x = 0.1$

$$E_{z,ff}(\theta) \propto \iint_{\mathscr{D}} J(\vec{x}')e^{-j(\cos(\theta)x_1' + \sin(\theta)x_2')}dx_1'dx_2' \quad (8)$$

where θ is the angle of our observation point and the r dependence is omitted. We recognize this as a Fourier transform, this allows us to directly evaluate the far field by using the Fourier transform of the discretization and evaluating at the point $[k_0\cos(\theta), k_0\sin(\theta)]$. To quantify the error we take the relative L^2 norm given as

$$\varepsilon_{rel,ff} = \frac{||\bar{E}_{z,ff} - E_{z,ff}||_2}{||\bar{E}_{z,ff}||_2}. \quad (9)$$

With $\bar{E}_{z,ff}$ being our far field discrete reference data.

3 Results and conclusions

We plot the results for different degrees of interpolation for varying Δ_x and varying discretization ranges. We see the results in Figure 2.

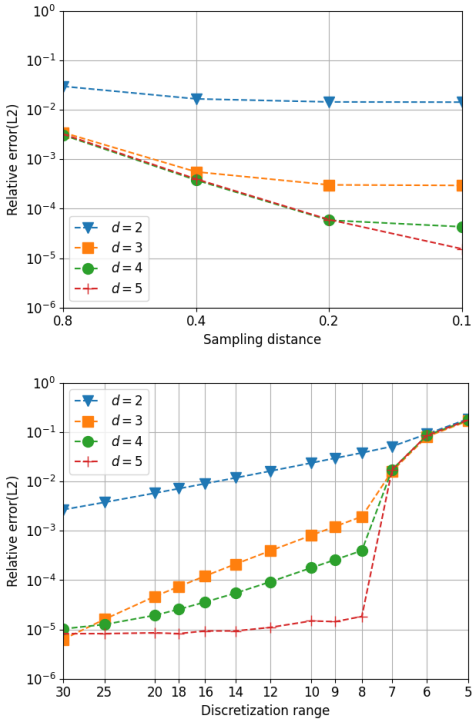

Fig. 2. (top) Far field errors vs sampling distance ($r_d = 15$). (bottom) Far field errors vs discretization range ($\Delta_x = 0.1$)

From these figures we can conclude that the simulation converges well for $d \geq 3$. We also see a jump in accuracy at $r_d = 8$, this is due to the convolution in (1) where G is used on twice the radius of the cylinder. It is of note that the 10^{-5} floor we reach is due to the cut-off residue in our iterative solver. The complexity of the multiplication of functions discretized by this discretization scales as $O(d^4(\frac{r_d}{\Delta_x})^2)$ for a 2D setup with equal sampling in both directions while the Fourier transforms scale as $O(d^3(\frac{r_d}{\Delta_x})^2\log(\frac{r_d}{\Delta_x}))$ for a single dimension. This makes increasing d particularly costly, nevertheless we see that increasing d yields a significant gain in accuracy.

References

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3. R.J. Dilz, Mark G.M.M. van Kraaij, and M.C. van Beurden. A 3d spatial spectral integral equation method for electromagnetic scattering from finite objects in a layered medium. *Optical and Quantum Electronics*, 50(5), May 2018.