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Multirate Numerical Integration for Parabolic PDEs

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Abstract. To solve PDE problems with different time scales that are localized in space, multirate time integration is examined. This technique enables one to use large time steps for slowly time-varying spatial regions, and small steps for rapidly varying ones. Multirate time stepping is coupled with the local uniform grid refinement and provides a robust and efficient method for the target problem class. We primarily consider implicit time stepping methods, suitable for parabolic problems. Numerical results are presented for a test problem.

Keywords: multirate time stepping, local time stepping, partial differential equations

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INTRODUCTION

There are usually two stages in the numerical solution of time-dependent partial differential equations (PDEs). The first stage is the spatial discretization in which the spatial derivatives of the PDE are discretized, for example with finite differences, finite volumes or finite element schemes. By discretizing the spatial operators, the PDE with its boundary conditions is converted into a system of ordinary differential equations (ODEs) in \mathbb{R}^m ,

$$w'(t) = F(t, w(t)), \quad w(0) = w_0, \quad (1)$$

called the semi-discrete system. Solutions of many problems have regions of high spatial activity or locally isolated internal regions with steep gradients where the solution is difficult to approximate. Using standard uniform fixed grid technique is computationally inefficient, since to get an accurate numerical approximation, it should have a very large number of grid points. Therefore non-uniform variable grids should be used.

Standard single-rate time integration methods for PDEs work with time steps that are varying in time but are constant over the spatial domain. There are, however, many problems of practical interest where the temporal variations have different time scales in different parts of the spatial domain. To exploit these local time scale variations, one needs multirate methods that use different, local time steps over the spatial domain [1, 2, 3, 4, 5]. A strategy for multirate time stepping for ordinary differential equations (ODEs) was presented in [1]. Results for two PDE problems discretized on a uniform spatial grid were discussed. In this paper we describe how the multirate time stepping can be used together with the non-uniform variable spatial grids for the solution of PDEs. We show that multirate time stepping can speed up the computation of the solution of PDEs also if adaptive non-uniform grids are used for their spatial discretization.

MULTIRATE TIME STEPPING

The multirate time stepping in this paper is based on the approach described in [1]. For a given global time step $\tau = t_n - t_{n-1}$, we first compute a tentative approximation at the time level t_n in all grid points. In those spatial regions where the error estimator indicates that smaller steps are needed, the computation is redone with halved step size $\frac{1}{2}\tau$. During the refinement stage, the values at the intermediate time levels in the grid points which are not refined might be needed. These values can be obtained by extrapolation, interpolation or by use of dense output built in the time integration method. The refinement is recursively continued until the error estimator is below a prescribed tolerance for all grid points. A schematic example, with space horizontally and time vertically, is presented in Figure 1. The intervals $[t_{n-1}, t_n]$ are called time slabs.

Proper interface treatment during the refinement step is very important for multirate time integration schemes. Use of interpolation and dense output of order lower than the order of the main time integration method can lead to order reduction.

For example, in [6] it was shown that the second-order trapezoidal rule with linear interpolation can lead to a first-order consistency for stiff problems. Another important point in connection with stiff problems is that the interpolation

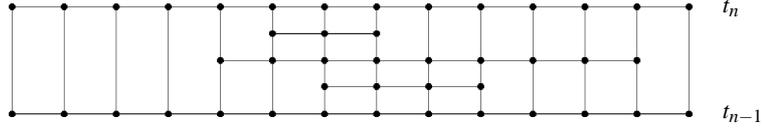


FIGURE 1. Multirate time stepping for a time interval $[t_{n-1}, t_n]$.

procedures which make explicit use of some function evaluations are inappropriate. In this case, the function resulting from a stiff problem can dramatically amplify the error of the numerical method.

NUMERICAL INTEGRATION METHOD

In this paper we use the two-stage second-order Rosenbrock ROS2 method [7] as our basic numerical integration method. To proceed from t_{n-1} to a new time level $t_n = t_{n-1} + \tau$, the method calculates

$$\begin{aligned}
 w_n &= w_{n-1} + \frac{3}{2}\bar{k}_1 + \frac{1}{2}\bar{k}_2, \\
 (I - \gamma\tau J)\bar{k}_1 &= \tau F(t_{n-1}, w_{n-1}) + \gamma\tau^2 F_t(t_{n-1}, w_{n-1}), \\
 (I - \gamma\tau J)\bar{k}_2 &= \tau F(t_n, w_{n-1} + \bar{k}_1) - \gamma\tau^2 F_t(t_{n-1}, w_{n-1}) - 2\bar{k}_1,
 \end{aligned} \tag{2}$$

where $J \approx F_w(t_{n-1}, w_{n-1})$. The method is linearly implicit: to compute the internal vectors \bar{k}_1 and \bar{k}_2 , a system of linear algebraic equations is to be solved. Method (2) is of order two for any choice of the parameter γ and for any choice of the matrix J . Furthermore, the method is A -stable for $\gamma \geq \frac{1}{4}$ and it is L -stable if $\gamma = 1 \pm \frac{1}{2}\sqrt{2}$. The ROS2 method also has a built in method which can be used for the error estimation [8].

SPATIAL GRID CONSTRUCTION

Due to the data structure simplicity we prefer binary space partition (BSP) grids. BSP grids are constructed by dividing any cell of an existing grid into 2^l equal size cells, where l is the dimension of the space. The resulting cells can be further partitioned by recursive application of the method.

At each time slab we make a prediction time step using the spatial grid from the previous time slab. Using a monitor function we decide where we want to refine the spatial grid (the points where the monitor value is larger than a prescribed tolerance $sptol$) and coarsen (the points where the monitor value is smaller than $\frac{1}{4}sptol$). For the experiments in this paper we use the curvature monitor. For 1D spatial grid for each grid point (i) this monitor is defined by

$$smon(i) = |\Delta x^2 u_{xx}(i)|. \tag{3}$$

For 2D case the formula for the monitor is

$$smon(i, j) = |\Delta x^2 u_{xx}(i, j)| + |\Delta y^2 u_{yy}(i, j)|. \tag{4}$$

We recursively continue with this procedure until we are happy with the value of the monitor in all spatial grid points. For the first time slab (where we do not have information from the previous time slab) we start with a coarse base uniform spatial grid with mesh size H .

With this procedure we get spatial grids similar to those used in the VLUGR approach [9]. The only difference is that in the VLUGR approach at each time step a coarse base spatial grid is used as a starting grid. The final spatial grid for a time step is obtained by recursive refinement of this coarse base grid. In our case we start with the coarse base grid for the first time slab and refine it until the value of the monitor function becomes smaller than a given tolerance. For the other time slabs we only update the spatial grid used in the previous time slab.

In this paper we use one-dimensional vertex centered grids for which the vertices of the cells are centered with respect to the grid points $\{x_j\}$. During the spatial grid refinement stage we consider all the points in which $|(h^*)^2 u_{xx}| > sptol$, where h^* is the size of the smaller adjacent spatial interval and u_{xx} is the numerical approximation of the second

derivative using central non-uniform vertex centered discretization [7]. Let us consider a point x_i in which we want to refine. If the neighbor intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ are of different size then we divide the largest of them into two smaller intervals. Otherwise, if the neighbor intervals are of the same size, we divide both of them into two smaller intervals.

At the grid coarsening stage we consider all the triplets (x_{i-1}, x_i, x_{i+1}) of consecutive points in which $|(2h^*)^2 u_{xx}(i)| < sptol$. Here we take $(2h^*)$ instead of h^* because $|(2h^*)^2 u_{xx}(i)|$ would be the value of the monitor function for the considered point after the coarsening. If

$$x_{i+1} - x_i = x_i - x_{i-1} = \frac{1}{2^k} H \quad \text{and} \quad 2^{k-1} \frac{x_i - x_0}{H} \in \mathbb{Z} \quad \text{with} \quad k \neq 0 \quad (5)$$

then we join these two intervals. These conditions assure us that we do not join two intervals which have different "fathers" in the refinement binary tree and that we do not get intervals larger than the mesh size of the base coarse uniform grid.

NUMERICAL EXPERIMENT

As a numerical experiment we consider the Allen-Cahn equation

$$u_t = \varepsilon u_{xx} + u(1 - u^2), \quad (6)$$

for $t > 0$, $-1 < x < 2$, with initial- and boundary conditions

$$u_x(-1, t) = 0, \quad u_x(2, t) = 0, \quad u(x, 0) = u_0(x), \quad (7)$$

where the initial profile is given by

$$u_0(x) = \begin{cases} \tanh((2x + 1.7)/(2\sqrt{\varepsilon})) & \text{for } -1 < x < -0.5, \\ \tanh((0.2 - x)/(2\sqrt{\varepsilon})) & \text{for } -0.5 \leq x < 0.28, \\ \tanh((x - 0.36)/(2\sqrt{\varepsilon})) & \text{for } 0.28 \leq x < 0.4865, \\ \tanh((0.613 - x)/(2\sqrt{\varepsilon})) & \text{for } 0.4865 \leq x < 0.7065, \\ \tanh((x - 0.8)/(2\sqrt{\varepsilon})) & \text{for } 0.7065 \leq x < 2. \end{cases} \quad (8)$$

The nonlinear reaction term in (6) has $u = 1$ and $u = -1$ as stable equilibrium states, whereas the zero solution is an unstable equilibrium. The solution of (6)–(8) starts with three ‘wells’, see Figure 2. The first well, on the left, persists during the integration interval. The second well is somewhat thinner than the others and it collapses at time $t \approx 41$, whereas the third well collapses at $t \approx 141$. For this problem we considered $\varepsilon = 9 \cdot 10^{-4}$ and we used an initial coarse space grid of 20 points. For the discretization of the spatial derivatives we used the second-order central non-uniform differences. A time-accurate numerical solution is shown in Figure 2.

To test the performance of the schemes, the time interval $[0, T]$ was considered with $T = 142$. At this output point, the solution is still changing in the third well; for larger times the solution becomes steady-state. In Table 1 the errors

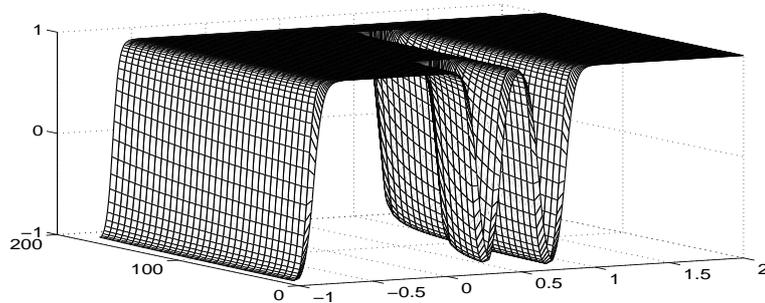


FIGURE 2. Evolution of the solution for problem (6)–(8).

(measured in the maximum norm with respect to an accurate reference solution computed for $sptol = 10^{-2}$) and the amount of work (number of space-time points) for different time tolerances and fixed $sptol = 10^{-2}$ are presented. For this problem there is a significant improvement in work with the multirate schemes compared to the single-rate scheme.

TABLE 1. Absolute maximal errors and amount of work with different tolerances for the Allen-Cahn problem

tol	Single-rate		Multirate	
	error	work	error	work
10^{-3}	$2.64 \cdot 10^{-1}$	164073	$3.26 \cdot 10^{-1}$	43192
$5 \cdot 10^{-4}$	$1.62 \cdot 10^{-1}$	492499	$1.93 \cdot 10^{-1}$	123684
10^{-4}	$4.62 \cdot 10^{-2}$	2960409	$5.30 \cdot 10^{-2}$	704436
$5 \cdot 10^{-5}$	$2.85 \cdot 10^{-2}$	3274746	$2.87 \cdot 10^{-2}$	828586
10^{-5}	$7.16 \cdot 10^{-3}$	4902552	$9.58 \cdot 10^{-3}$	1164722

CONCLUSIONS

In this paper we presented how multirate time stepping can be used together with adaptive non-uniform spatial grids for the solution of parabolic PDEs. Numerical experiment confirmed that the efficiency of time integration methods can be significantly improved by using large time steps for inactive components, without sacrificing accuracy.

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