

## Importance sampling simulation for a germ-grain model

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# 1 Introduction

A germ-grain model may provide a good description for a very irregular pattern observed in microscopy materials science, biology and analysis of images. Perhaps the best known model is the Boolean model (Matheron [7]) formalizing a configuration of independent, randomly placed particles. A Boolean model is formed by placing random balls centered at the points of a Poisson process and taking the union of these balls. The points of the Poisson process are sometimes called the germs and the associated balls the grains. In a natural generalization of the Boolean model the Poisson process of grains is replaced by a general point process and balls by any compact sets or even more general objects. If we take these objects as a mark at the point of point process of germs, then such a marked point process  $N(\cdot, \cdot)$  we will call a *marked point process (abbreviated by m.p.p.) driving the germ-grain model*.

The simulation of a Boolean model within a compact set  $T$  falls into the following stages. First, the number of points is determined by simulating a Poisson random variable  $J$  with the parameter  $\lambda|T|$ , where  $\lambda$  is the intensity of the Poisson process and  $|T|$  the volume of the set  $T$ . Then  $J$  independent random points are simulated in  $T$  according to the Bernoulli process. Next, we generate  $J$  i.i.d. copies of radius  $r$ . Finally, the Boolean model is constructed by

$$\mathcal{A}(T) = \sum_{t_i \in T} (t_i \oplus B(r_i)) ,$$

where  $\oplus$  is Minkowski addition and  $B(R)$  is the ball with radius  $R$ .

We want to simulate the so-called rare event  $A$  for Boolean model  $\mathcal{A}(T)$ , or more generally for a germ-grain model. That is,  $\mathbb{P}_N(A)$  is "small" (typically of order  $10^{-6}$ ). Using the so-called Crude Monte Carlo (MC) method of simulation in this case is inefficient. Precisely, let  $n$  be a size of a sample and  $\mathbb{I}(A_1), \mathbb{I}(A_2), \dots, \mathbb{I}(A_n)$  replicants of  $\mathbb{I}(A)$ . Then we estimate  $p = \mathbb{P}_N(A)$  by

$$\hat{p} = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(A_i) .$$

By

$$\hat{\sigma}^2 = \hat{p}(1 - \hat{p})$$

we denote an empirical variance. According to the standard Central Limit Theorem

$$\sqrt{n}(\hat{p} - p) \xrightarrow{D} N(0, \sigma^2) ,$$

where  $\sigma^2 = p(1 - p)$ . Hence

$$\hat{p} \pm \frac{1.96}{\sqrt{n}} \hat{\sigma}$$

is an asymptotic 95% confidence interval. Note that although the absolute error  $\hat{\sigma}^2 := \hat{p}(1 - \hat{p})$  is small, the relative error is high:

$$\text{Re}(p) := \frac{\hat{\sigma}}{\hat{p}} \sim \frac{1}{\sqrt{\hat{p}}} \rightarrow +\infty, \quad \text{as } \hat{p} \rightarrow 0.$$

In other words, a confidence interval of width  $10^{-4}$  may look small, but if the point estimate  $\hat{p}$  is of order  $10^{-6}$ , then this estimation is in fact impossible.

Therefore we will use the Importance Sampling (IS) method. The main idea is to compute  $\mathbb{P}_N(A)$  by simulating a germ-grain model from a probability measure  $\mathbb{P}_{\hat{N}}$ , such

that  $\mathbb{P}_N$  is absolutely continuous with respect to it. By Theorem 2.1 this is the case when driving m.p.p.'s in both germ-grain models are absolutely continuous. We also find the Radon-Nikodym derivative  $L(\cdot)$ , that is

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{\tilde{N}}}(A) = L(A) .$$

Then

$$\mathbb{P}_N(A) = \mathbb{E}_{\tilde{N}}L(A)\mathbb{I}(A) , \quad (1.1)$$

where the expectation is with respect to measure  $\mathbb{P}_{\tilde{N}}$ . Hence to estimate  $\mathbb{P}_N(A)$  we generate  $(\mathbb{I}(A_1), L_1), (\mathbb{I}(A_2), L_2), \dots, (\mathbb{I}(A_n), L_n)$  from the measure  $\mathbb{P}_{\tilde{N}}$  and construct the estimator

$$\hat{p}_{IS} = \frac{1}{n} \sum_{i=1}^n L_i \mathbb{I}(A_i) . \quad (1.2)$$

The 95% confidence interval is then

$$\hat{p}_{IS} \pm \frac{1.96}{\sqrt{n}} \hat{\sigma}_{IS} , \quad (1.3)$$

where

$$\hat{\sigma}_{IS}^2 = \frac{1}{n} \sum_{i=1}^n L_i^2 \mathbb{I}(A_i) - \hat{p}_{IS}^2 . \quad (1.4)$$

We choose the measure  $\mathbb{P}_{\tilde{N}}$  (that is, the parameters of the new germ-grain model) in such a way that the event  $A$  is observed frequently. In other words, under a good choice of the parameters we decrease the relative error.

In applications sometimes we consider events on a random set  $\tau$ . This concept of localization plays a fundamental role in simulation. Our goal is to prepare an appropriate framework for it. Therefore, similarly like for a process on the real line, we introduce the notion of a set indexed martingale and a stopping set  $\tau$ . The likelihood process  $L(\cdot)$  is a mean one set indexed martingale. Then we can simulate an event  $A$  up to a stopping set  $\tau$ . Namely, we use Wald's formula:

$$\mathbb{P}(A) = \tilde{\mathbb{E}}[L(\tau); A] .$$

We analyze in detail the Boolean model, where the point process is a Poisson process and balls are i.i.d. distributed. We consider the event  $A$  such that balls form a chain (all circles in this chain are connected), which joins the origin with the border of box  $T$  with side length  $K$ . That is, we simulate the probability that the radius of the occupied component of the origin is greater than  $K$ . The problem of finding  $\mathbb{P}_N(A)$  is relevant in industry when we apply the electrodes to the plates of the dielectric materials. Because of the manufacturing process small holes arise in the electrodes. A chain of small holes crossing from origin to the border of the box means a diminished value of the capacitance. Typically the parameters of the model: size of  $T$ , intensity of Poisson process  $\lambda$  and distribution of radius of ball  $\nu$  are such that  $\mathbb{P}_N(A)$  is "small". We prove that the IS scheme for an appropriate choice of a new intensity of the Poisson process is logarithmic efficient which implies improvement. We also give some numerical results.

The paper is organized as follows. In Section 2 we find sufficient conditions under which the two germ-grain models are absolutely continuous. Section 3 deals with a wire frame model. In Section 4 we consider simulation on a general random stopping set. Finally, in Section 5 we analyze some numerical example.

## 2 Germ-grain model

We start with a formal definition of a germ-grain model. We will define a marked point process as a point process on a product space of locations and marks with the additional property that the marginal location process is itself a well-defined point process. By  $\mathcal{B}(X)$  we denote the Borel  $\sigma$ -field of  $X$ . Let  $(T, \mathcal{B}(T))$  and  $(E, \mathcal{B}(E))$  be subspaces of some vector Polish space  $(W, +)$  with binary operator  $+$ . Let  $N^E(\cdot)$  constitute a simple point process on  $T$ , that is integer valued random function such that  $N_t = 0$  or  $N_t = 1$  for any  $t \in T$ . Denote by  $\{t_n\}$  the points of  $N^E(\cdot)$ . Then  $N(\cdot, \cdot) = \{[t_n, m_n]\}$  is a marked point process, where  $m_n$  is the mark corresponding to  $t_n$ . In other words,

$$N(\omega, B, M) = \sum_{n \geq 1} \epsilon_{(t_n(\omega), m_n(\omega))}(B, M) \quad (2.1)$$

for  $B \in \mathcal{B}(T)$ ,  $M \in \mathcal{B}(E)$  and  $\epsilon_{t_n, m_n}(\cdot, \cdot)$  is a Dirac measure.

**Example 2.1.** In the case of the Boolean model,  $T$  and  $E$  are subspaces of  $\mathbb{R}^d$ . The space  $T$  is a location space and  $E$  is a mark space, that is a space of balls.

Let  $\mu(dt, dm)$  be a mean measure of  $N$ :

$$\mathbb{E}N(B, M) = \mu(B, M), \quad B \in \mathcal{B}(T), M \in \mathcal{B}(E) .$$

Similarly, let  $\lambda(dt)$  be a mean measure of point process  $N^E(\cdot)$ :

$$\mathbb{E}N^E(B) = \lambda(B), \quad B \in \mathcal{B}(T) .$$

It can be shown that  $\mu(dt, dm)$  is absolutely continuous with respect to  $\lambda(dt)$ , that is from the Radon-Nikodym theorem there exists density  $\nu_t(dm)$  such that

$$\mu(dt, dm) = \lambda(dt)\nu_t(dm) , \quad (2.2)$$

where  $\nu_t(dm)$  can be interpreted as the distribution of the mark of the point  $t$ .

**Example 2.2.** If  $N^E(\cdot)$  is a Cox process with i.i.d. marking, then  $\mu(dt, dm) = \lambda(dt)F(dm)$ , where  $F(\cdot)$  is a distribution of mark.

**Remark 2.1.** In the classical theory of marked processes on the real line, it is well known that under certain conditions on the probability space and filtration, the mean measure of a marked point process determines its distribution (Jacod [5]). As we shall see, this is not true for processes on general spaces. Consider the Poisson process  $N^E(\cdot) = N(\cdot, E)$  on  $T = [0, 1]^2$  with mean measure equal to Lebesgue measure  $\lambda(dt) = dt$ . However, the Lebesgue measure is also the mean measure of the following process. Let  $L_0, L_1, \dots$  be i.i.d. unit rate Poisson processes on  $[0, 1]$ . Denote the time of the  $k^{\text{th}}$  jump of  $L_i$ , by  $T_k^i$ . Now, let locations of the points be  $\{(T_i^0, T_k^i), i, k \geq 1\}$ .

Now, writing  $B \oplus M = \{b + m : b \in B, m \in M\}$  for the Minkowski addition of  $B$  and  $M$ , we define a germ-grain model by the union:

$$\mathcal{A}(N) = \cup_{n \geq 1} (t_n \oplus m_n) .$$

Denote  $S = T \oplus E$ .

Consider two marked points  $N$  and  $\tilde{N}$  on  $T \times E$  having the mean measures  $\mu$  and  $\tilde{\mu}$ , respectively. Let  $\mathbb{P}_N$  and  $\mathbb{P}_{\tilde{N}}$  be the distribution of the germ-grain model driven by m.p.p's  $N$  and  $\tilde{N}$ , respectively. Denote  $\mathcal{N} = \{n : \mathbb{P}(N(T \times E) = n) > 0\}$  and  $\tilde{\mathcal{N}} = \{n : \mathbb{P}(\tilde{N}(T \times E) = n) > 0\}$ . For  $n \in \mathcal{N}$  define

$$N_{|n}(B, M) := \mathbb{E}[N(B, M) | N(T, E) = n], \quad B \in \mathcal{B}(T), M \in \mathcal{B}(E).$$

Note, that  $N_{|n}$  is also a marked point process on  $T \times E$ . Let  $\mu_{|n}(dt, dm)$  be its mean measure. Similarly we define  $\tilde{N}_{|n}(\cdot, \cdot)$  and  $\tilde{\mu}_{|n}(\cdot, \cdot)$ .

**Proposition 2.1.** *Marked point process  $N(\cdot, \cdot)$  is absolutely continuous with respect to marked point process  $\tilde{N}(\cdot, \cdot)$  ( $N \ll \tilde{N}$ ) iff  $\mu_{|n} \ll \tilde{\mu}_{|n}$  for  $n \in \mathcal{N}$  and  $N(T \times E) \ll \tilde{N}(T \times E)$ .*

*Proof.* If  $N \ll \tilde{N}$ , then  $N(T \times E) \ll \tilde{N}(T \times E)$  and  $N_{|n} \ll \tilde{N}_{|n}$ , hence also  $\mu_{|n} \ll \tilde{\mu}_{|n}$ . We prove the converse implication. We use the notation  $(\mathbf{t}, \mathbf{m}) = (t_1, m_1, \dots, t_k, m_k)$  for the  $(T \times E)^k$ -valued vectors ( $k = 1, 2, \dots$ ). The  $k^{\text{th}}$  order factorial measure  $\alpha_{|n}^k$  of  $N_{|n}$  is a measure on  $(T \times E)^k$  defined by

$$\alpha_{|n}^k(d(\mathbf{t}, \mathbf{m})) = \mathbb{E}N_{|n}^k(d(\mathbf{t}, \mathbf{m})),$$

where

$$N_{|n}^k(d(\mathbf{t}, \mathbf{m})) = N_{|n}(d(t_1, m_1))(N_{|n} - \epsilon_{(t_1, m_1)})(d(t_2, m_2)) \dots (N_{|n} - \sum_{i=1}^{k-1} \epsilon_{(t_i, m_i)})(d(t_k, m_k)).$$

Thus

$$\int f(\mathbf{t}, \mathbf{m}) \alpha_{|n}^k(d(\mathbf{t}, \mathbf{m})) = \int \sum_{(\mathbf{t}, \mathbf{m})}^{\#} f(\mathbf{t}, \mathbf{m}) N_{|n}(d(\mathbf{t}, \mathbf{m})), \quad (2.3)$$

where the summation  $\sum^{\#}$  is over all  $n$ -tuples of distinct points in  $T$ . We prove that  $\alpha_{|n}^k$  is absolutely continuous with respect to  $\tilde{\alpha}_{|n}^k$  for each  $n \in \mathcal{N} \cap \tilde{\mathcal{N}}$  and  $k \leq n$ . We use induction. For  $k = 1$  the assertion is satisfied, since we have  $\alpha_{|n}^1(dt, dm) = \mu_{|n}(dt, dm)$  and  $\tilde{\alpha}_{|n}^1(dt, dm) = \tilde{\mu}_{|n}(dt, dm)$ . Assume that the assertion is satisfied for  $k - 1$ . Let

$$L_{k-1}(\mathbf{t}, \mathbf{m}) = \frac{d\alpha_{|n}^{k-1}}{d\tilde{\alpha}_{|n}^{k-1}}(\mathbf{t}, \mathbf{m}) \quad (2.4)$$

and

$$L(\mathbf{t}, \mathbf{m}) = \frac{d\mu_{|n}}{d\tilde{\mu}_{|n}}(\mathbf{t}, \mathbf{m}). \quad (2.5)$$

Then

$$\begin{aligned}
\alpha_{|n}^k(B_1 \times M_1, \dots, B_k \times M_k) &= \int_{T \times E} \int_{(T \times E)^{k-1}} \sum_{t_1, \dots, t_{k-1}}^{\#} \sum_{t_k \neq t_i, i \leq k-1} \\
&\quad \epsilon_{t_1, m_1}(B_1 \times M_1) \dots \epsilon_{t_k, m_k}(B_k \times M_k) \\
&\quad d\alpha_{|n}^{k-1}(t_1, m_1, \dots, t_{k-1}, m_{k-1}) dN_{|n}(t_k, m_k) \\
&= \int_{T \times E} \int_{(T \times E)^{k-1}} \sum_{t_1, \dots, t_{k-1}}^{\#} \sum_{t_k \neq t_i, i \leq k-1} \epsilon_{t_1, m_1}(B_1 \times M_1) \dots \epsilon_{t_k, m_k}(B_k \times M_k) \\
&\quad d\alpha_{|n}^{k-1}(t_1, m_1, \dots, t_{k-1}, m_{k-1}) d\mu_{|n}(t_k, m_k) \\
&= \int_{T \times E} \int_{(T \times E)^{k-1}} \sum_{t_1, \dots, t_{k-1}}^{\#} \sum_{t_k \neq t_i, i \leq k-1} L_{k-1}(t_1, m_1, \dots, t_{k-1}, m_{k-1}) L(t_k, m_k) \\
&\quad \epsilon_{t_1, m_1}(B_1 \times M_1) \dots \epsilon_{t_k, m_k}(B_k \times M_k) \\
&\quad d\tilde{\alpha}_{|n}^{k-1}(t_1, m_1, \dots, t_{k-1}, m_{k-1}) d\tilde{\mu}_{|n}(t_k, m_k) \\
&= \int_{(T \times E)^k} \sum_{t_1, \dots, t_k}^{\#} L_k(\mathbf{t}, \mathbf{m}) \epsilon_{t_1, m_1}(B_1 \times M_1) \dots \epsilon_{t_k, m_k}(B_k \times M_k) \tilde{\alpha}_{|n}^k(dt, dm),
\end{aligned}$$

where  $L_k(\mathbf{t}, \mathbf{m}) = L_{k-1}(t_1, m_1, \dots, t_{k-1}, m_{k-1})L(t_k, m_k)$ . Thus  $\alpha_{|n}^k \ll \tilde{\alpha}_{|n}^k$  for  $k \leq n$ . Suppose now that  $\tilde{N}(B \times M) = 0$ . That is,

$$\begin{aligned}
&\sum_{n \in \tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T \times E) = n) \int_{(T \times E)^n} \mathbf{1}_{B \times M} \tilde{N}_{|n}^n(d(\mathbf{t}, \mathbf{m})) \\
&= \sum_{n \in \tilde{\mathcal{N}}} \mathbb{P}(\tilde{N}(T \times E) = n) \int_{(B \times M)^n} \tilde{\alpha}_{|n}^n(d(\mathbf{t}, \mathbf{m})) = 0.
\end{aligned}$$

Hence all terms must be zero and for all  $n \in \tilde{\mathcal{N}} \supseteq \mathcal{N}$  we have

$$\int_{(B \times M)^n} \tilde{\alpha}_{|n}^n(d(\mathbf{t}, \mathbf{m})) = 0. \tag{2.6}$$

Thus from the above considerations for all  $n \in \mathcal{N}$

$$\int_{(B \times M)^n} \alpha_{|n}^n(d(\mathbf{t}, \mathbf{m})) = 0 \tag{2.7}$$

yielding  $N(B \times M) = 0$ . This completes the proof.  $\square$

**Remark 2.2.** For a marked Poisson process with intensity  $\lambda$  and a mark independent of a position with a distribution measure  $\nu(\cdot)$ , we have

$$\mathbb{P}(N(T \times E) = n) = \frac{(\lambda|T|)^n}{n!} e^{-\lambda|T|},$$

where  $|B|$  is the volume of a set  $B$ , and

$$\alpha_{|n}^n(d(\mathbf{t}, \mathbf{m})) = dt_1 \dots dt_n d\nu(m_1) \dots d\nu(m_n).$$

**Remark 2.3.** Note that from the assumptions of the Proposition 2.1 it follows that  $\mu$  is absolutely continuous with respect to  $\hat{\mu}$ . In fact, let  $B \times M$  be such that  $\hat{\mu}(B \times M) = 0$ . We have  $\mathcal{N} \subseteq \hat{\mathcal{N}}$ . Then

$$\sum_{n \in \hat{\mathcal{N}}} \mathbb{P}(\hat{N}(T \times E) = n) \hat{\mu}_{|n}(B \times M) = 0$$

and for  $n \in \hat{\mathcal{N}}$  we have  $\hat{\mu}_{|n}(B \times M) = 0$ . Hence also  $\mu_{|n}(B \times M) = 0$  for  $n \in \mathcal{N}$  and finally  $\mu(B \times M) = 0$ . The converse statement in general is not true. In fact, consider two points processes  $N^E(\cdot)$  and  $\hat{N}^E(\cdot)$  on  $T = [0, 1]^2$  given by  $N^E(\cdot) = \delta_{(1/2, 1/2)}(\cdot)$  and  $\hat{N}^E(\cdot) = \delta_{(1/2, 1/2)}(\cdot) + \delta_{(1/4, 1/4)}(\cdot)$ .

**Theorem 2.1.** *Let  $T$  and  $E$  be compact subsets of some vector Polish space  $W$ . If  $N \ll \hat{N}$ , then  $\mathbb{P}_N \ll \mathbb{P}_{\hat{N}}$ .*

**Remark 2.4.** The same assertion for a Boolean model driven by a Poisson process marked by i.i.d. balls was proved recently by Van Lieshout [13].

**Remark 2.5.** Note that if  $N$  and  $\hat{N}$  are Cox processes, then by Remark 2.2 the assumptions of Theorem 2.1 are satisfied.

*Proof.* Suppose that  $\mathbb{P}_{\hat{N}}(B) = 0$  for some  $B \in \mathcal{B}(S)$ . That is,

$$\begin{aligned} \sum_{n \in \hat{\mathcal{N}}} \mathbb{P}(\hat{N}(T \times E) = n) \int_{(T \times E)^n} \mathbb{1}_B \left( \sum_{i=1}^n (t_i \oplus m_i) \right) \hat{N}_{|n}^n(d(\mathbf{t}, \mathbf{m})) \\ = \sum_{n \in \hat{\mathcal{N}}} \mathbb{P}(\hat{N}(T \times E) = n) \int_{(T \times E)^n} \mathbb{1}_B \left( \sum_{i=1}^n (t_i \oplus m_i) \right) \hat{\alpha}_{|n}^n(d(\mathbf{t}, \mathbf{m})) = 0. \end{aligned}$$

Hence all terms must be zero and for all  $n \in \mathcal{N}$  we have

$$\int_{(T \times E)^n} \mathbb{1}_B \left( \sum_{i=1}^n (t_i \oplus m_i) \right) \hat{\alpha}_{|n}^n(d(\mathbf{t}, \mathbf{m})) = 0. \quad (2.8)$$

Thus from Proposition 2.1 for  $n \in \mathcal{N}$

$$\int_{(T \times E)^n} \mathbb{1}_B \left( \sum_{i=1}^n (t_i \oplus m_i) \right) \alpha_{|n}^n(d(\mathbf{t}, \mathbf{m})) = 0 \quad (2.9)$$

yielding  $\mathbb{P}_N(B) = 0$ . This completes the proof.  $\square$

The likelihood ratio  $d\mathbb{P}_{\hat{N}}/d\mathbb{P}_N$  is

$$L(A) := \frac{d\mathbb{P}_N}{d\mathbb{P}_{\hat{N}}}(A) = \frac{\sum_{n \in \mathcal{N}} \mathbb{P}(N(T \times E) = n) \int_{(T \times E)^n} \mathbb{1}_A \left( \sum_{i=1}^n (t_i \oplus m_i) \right) \alpha_{|n}^n(d(\mathbf{t}, \mathbf{m}))}{\sum_{n \in \hat{\mathcal{N}}} \mathbb{P}(\hat{N}(T \times E) = n) \int_{(T \times E)^n} \mathbb{1}_A \left( \sum_{i=1}^{k-1} (t_i \oplus m_i) \right) \hat{\alpha}_{|n}^n(d(\mathbf{t}, \mathbf{m}))}. \quad (2.10)$$

**Example 2.3.** [Poisson cluster process] Let  $N(\cdot, \cdot)$  be a Poisson process with the intensity  $\lambda$  marked by a point process  $N_i(\cdot)$  at position  $t_i$ , where  $N_i(\cdot)$  are conditionally independent given the realization of the parent Poisson process. Then  $\bar{N}(\cdot) = \sum_{i=1}^{\infty} (t_i \oplus N_i(\cdot))$  is a cluster Poisson process (see [6]). Assume that  $N_i(\cdot)$  are absolutely continuous with respect

to the unit Poisson process on  $S$  with the offspring intensity measure  $\nu(\cdot)$ . Let  $\bar{N}(\cdot)$  be the Poisson cluster process with intensity  $\bar{\lambda}$  and the offspring measure  $\bar{\nu}$ . Then

$$\frac{d\bar{N}}{dN}(A) = L(A) = \frac{\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda|T|} e^{|S|(1-n)} \sum_{\phi} \int_{S^n} \prod_{i=1}^n \nu(t_{\phi^{-1}(i)} - s_i) ds_1 \dots ds_n}{\sum_{n=1}^{\infty} \frac{\bar{\lambda}^n}{n!} e^{-\bar{\lambda}|T|} e^{|S|(1-n)} \sum_{\phi} \int_{S^n} \prod_{i=1}^n \bar{\nu}(t_{\phi^{-1}(i)} - s_i) ds_1 \dots ds_n}, \quad (2.11)$$

where the sum is over all ordered partitions  $\phi$  of  $A$ ; see Van Lieshout [12]. If each parent point has a single daughter point with displacement densities  $\nu(\cdot)$  and  $\bar{\nu}(\cdot)$ , then (2.11) reduces to

$$L(A) = e^{(\bar{\lambda}-\lambda)|T|} \left( \frac{\lambda}{\bar{\lambda}} \right)^{H(A)} \prod_{i=1}^{H(A)} \frac{\int_S \nu(t_i - s) ds}{\int_S \bar{\nu}(t_i - s) ds},$$

where  $H(A)$  denotes the number of points in configuration  $A$ .

Sometimes the expression (2.10) can be simplified. This is the case for a wire frame model.

### 3 Wire frame model

The germ-grain model in which there is one-to-one correspondence between the driving m.p.p. and the germ-grain model itself, we will call 'wire frame model'. The classical example is a Boolean disc model on  $\mathbb{R}^d$ , in which marks are discs centered at the location points  $\{t_i\}$  of the point process  $N^E(\cdot)$  with diameters  $\{r_i\}$ . That is,  $\mathcal{A}(N) = \bigcup_{i=1}^{\infty} S_i$ , where  $S_i = \{t \in \mathbb{R}^d : \|t - t_i\| = r_i\}$  and  $\|\cdot\|$  means Euclidean norm. Denote by  $\pi : T \oplus E \rightarrow T \times E$  the 1-1 mapping such that  $\pi(\mathcal{A}(N)) = N$ . Then from (2.10) we have

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{\bar{N}}}(A) = L(A) = \frac{dN}{d\bar{N}}(\pi(A)). \quad (3.1)$$

Consider the marked Cox processes  $N(\cdot, \cdot)$  and  $\bar{N}(\cdot, \cdot)$  with intensity measures  $\lambda(\cdot)$  and  $\bar{\lambda}(\cdot)$ , respectively, and with independent marking with measures  $\nu(\cdot)$  and  $\bar{\nu}(\cdot)$ . Let  $\lambda \ll \bar{\lambda}$  and  $\nu \ll \bar{\nu}$ . Then by Prop. 6.10, p. 233 of Karr [6],  $N \ll \bar{N}$  and hence also  $\mathbb{P}_N \ll \mathbb{P}_{\bar{N}}$ . In this case

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{\bar{N}}}(A) = \exp \left\{ \int_{\pi(A)} \log \left( \frac{d\lambda}{d\bar{\lambda}}(t) \right) d\bar{N}^E(t) \right\} \exp \left\{ \int_T \left( 1 - \frac{d\lambda}{d\bar{\lambda}}(t) \right) d\lambda(t) \right\} \prod_{i=1}^{H(\pi(A))} \frac{d\nu}{d\bar{\nu}}(t_i), \quad (3.2)$$

where  $H(\pi(A))$  is the number of points of the Cox process constructing a set  $A$ . In particular, if  $N(\cdot, \cdot)$  and  $\bar{N}(\cdot, \cdot)$  are marked Poisson processes, then

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{\bar{N}}}(A) = e^{(\bar{\lambda}-\lambda)|T|} \left( \frac{\lambda}{\bar{\lambda}} \right)^{H(\pi(A))} \prod_{i=1}^{H(\pi(A))} \frac{d\nu}{d\bar{\nu}}(t_i).$$

For example, if the diameters in both Boolean disc models have the same distribution, then

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{\bar{N}}}(A) = e^{(\bar{\lambda}-\lambda)|T|} \left( \frac{\lambda}{\bar{\lambda}} \right)^{H(\pi(A))}. \quad (3.3)$$

## 4 Random stopping

In this section, we will index m.p.p.'s  $N(\cdot, \cdot)$  and  $\tilde{N}(\cdot, \cdot)$  defined via (2.1) by a semilattice  $T$  of a compact subset of  $T \times E$ , where  $B \wedge C = B \cap C$  for  $B, C \in T \times E$ . Let  $W$  be a locally compact space. We assume that  $\emptyset \in T$ , that  $T$  is closed under arbitrary intersections, and that  $\overline{\cup_{B \in \mathcal{T}} B} = T$  ( $\overline{(\cdot)}$  denotes the closure of set). The probability space  $(\Omega, \mathcal{F}, \mathbb{P}_N)$  is equipped with a right-continuous filtration  $\{\mathcal{F}_B, B \in T\}$  that satisfies  $\mathcal{F}_B \subseteq \mathcal{F}_C$  for  $B \subseteq C$  and  $\mathcal{F}_\emptyset$  contains all the null sets. Let  $\mathcal{F} = \bigvee_B \mathcal{F}_B$ .

We will denote the class  $\mathcal{C}(u)$  of finite unions of sets in class  $\mathcal{C}$ . Let " $\subset$ " mean strict inclusion and  $(\cdot)^\circ$  denote the interior of a set. We make the following two assumptions:

**SHAPE** If  $B, C \in T$  and  $B, C \neq \emptyset$ , then  $B \cap C \neq \emptyset$ . Also, if  $B_1, \dots, B_k \in T$  and  $B \subseteq \sum_{i=1}^k B_i$ , then there exists index  $i$  ( $1 \leq i \leq k$ ) such that  $B \subseteq B_i$ .

**Separability from above** There exists an increasing sequence of finite subsemilattices  $T_n$  such that  $T_n \subseteq T$ ,  $\emptyset \in T_n$ ,  $\wedge_{T_n} = \cap$ ,  $\forall n$ , and sequence of functions  $g_n : T \rightarrow T_n(u) \cup \{T\}$  preserving all intersections, satisfying  $B \subseteq (g_n(B))^\circ \forall B \in T$ , such that  $g_n(B) \subseteq g_m(B)$  if  $n \geq m$ , and  $B = \bigcap_n g_n(B)$ ,  $\forall B \in T$ . Also, if  $B \neq \emptyset$ ,  $C \in T$  and  $B \subset C$ , then  $B \subset g_n(B) \cap C \forall n$ .

The condition SHAPE imposes a restriction on the geometric shapes of sets in  $T$ .

**Definition 4.1.**  $\tau : \Omega \rightarrow T$  is called a stopping set if, for any set  $B \in T$ ,  $\{\omega : B \subseteq \tau(\omega)\} \in \mathcal{F}_B$ ,  $\{\omega : \tau(\omega) = \emptyset\} \in \mathcal{F}_\emptyset$ .

Sometimes  $\tau$  is also called a simple stopping set.

**Example 4.1.** The classical example of set  $T$  and lattice  $\mathcal{T}$  fulfilling above conditions is the case  $T = E = [0, D]^d$  for some  $D > 0$  and  $\mathcal{T} = \{[0, t] \times [0, s] : t \in T, s \in E\}$ . For a filtration  $\{\mathcal{F}_B\}$  one may take  $\mathcal{F}_B = \cup_n \mathcal{F}_{g_n(B)}^\circ$ , where  $\mathcal{F}_B^\circ$  is generated by the driving m.p.p. on a set  $B \in T$ . As an example of stopping set  $\tau$  for a Boolean model one can consider the smallest (in a sense of inclusion) box  $[0, (n, n, \dots, n)]$  ( $n \in \mathbb{N}$ ) such that there exists a path through the balls joining origin with the corner  $(n, n, \dots, n)$  if it exists, otherwise  $\tau = T \times E$ . For more complicated examples, where e.g.  $W$  is a function space, see Merzbach [4].

Consider the wire frame models  $\mathbb{P}_N$  and  $\mathbb{P}_{\tilde{N}}$  such that  $\mathbb{P}_N \ll \mathbb{P}_{\tilde{N}}$ , that is (3.1) holds. Note that  $L(B)$  is a mean one martingale, that is  $\mathbb{E}[L(B)|\mathcal{F}_C] = L(C)$  for  $C \subseteq B$  and  $B, C \in T$ . Then from Th. 2.5 of Merzbach [3], we have Wald's formula

$$\mathbb{P}_N(A) = \mathbb{E}_{\tilde{N}}[L(\tau); A] \quad (4.1)$$

for  $A \in \mathcal{F}_\tau$ , where  $\mathcal{F}_\tau = \{F \in \mathcal{F} : F \cap \{\tau \subseteq B\} \in \mathcal{F}_B \forall B \in T\}$ . Then the IS estimator of  $\mathbb{P}_N(A)$  in a wire frame model is

$$p_{IS} = \sum_{i=1}^n L_i(\tau) \mathbb{I}(A_i) . \quad (4.2)$$

## 5 Radius of the occupied component of the origin

In this section we consider the Boolean disc model driven by the marked Poisson process  $N(\cdot, \cdot)$  with the intensity  $\lambda$  and marks being i.i.d. discs. Marked Poisson process  $\tilde{N}(\cdot, \cdot)$  has the intensity  $\tilde{\lambda}$  and marks being discs distributed like in a Boolean disc model governed by  $\mathbb{P}_N$ . We will consider a rare event  $A$  on  $T$  for which  $\lim_{\lambda \rightarrow 0} \mathbb{P}_N(A) = 0$ . By (3.3) we have

$$L(A) = e^{(\tilde{\lambda} - \lambda)|T|} \left( \frac{\lambda}{\tilde{\lambda}} \right)^{H(\pi(A))}.$$

We will give now an example of  $A$  for which the IS scheme work well, that is, it reduces the relative error.

**Definition 5.1.** *We say that the IS scheme is logarithmic efficient if*

$$\lim_{\lambda \rightarrow 0} \frac{\log \mathbf{Var} \hat{p}_{IS}}{\log p^2} \geq 1.$$

If

$$\lim_{\lambda \rightarrow 0} \frac{\mathbf{Var} \hat{p}_{IS}}{p} = 0,$$

then the IS scheme is an improvement over MC simulation.

Logarithmic efficiency implies improvement (see Asmussen [1]).

**Theorem 5.1.** *Let  $A$  be such an event that*

$$\lim_{\lambda \rightarrow 0} -\frac{1}{\beta(\lambda)} \log \mathbb{P}_\lambda(A) = 1 \tag{5.1}$$

for some positive scaling function  $\beta(\cdot)$ . Let  $\tilde{\lambda} > \lambda$ . If

$$\lim_{\lambda \rightarrow 0} \frac{\beta(\frac{\lambda^2}{\tilde{\lambda}})}{2\beta(\lambda)} \geq 1, \tag{5.2}$$

then IS scheme is logarithmic efficient. If  $\beta(\cdot)$  is strictly decreasing such that

$$\lim_{\lambda \rightarrow 0} \left( \beta\left(\frac{\lambda^2}{\tilde{\lambda}}\right) - \beta(\lambda) \right) = \infty, \tag{5.3}$$

then the IS scheme is an improvement.

*Proof.* Let  $\tilde{\lambda} = \frac{\lambda^2}{\tilde{\lambda}}$ . Denote by  $\mathbb{P}_\beta$  and  $\mathbb{E}_\beta$  the probability and the expectation with respect to the Poisson process with the intensity  $\beta$ , respectively. Note that

$$\begin{aligned} \mathbb{E}_{\tilde{\lambda}}[L^2(T); A] &= e^{(\tilde{\lambda} - \lambda)|T|} \mathbb{E}_\lambda \left[ \left( \frac{\lambda}{\tilde{\lambda}} \right)^{H(T)} ; A \right] \\ &= e^{(\tilde{\lambda} - \lambda)|T|} e^{(\tilde{\lambda} - \lambda)|T|} \mathbb{E}_{\tilde{\lambda}} \left[ \left( \frac{\lambda}{\tilde{\lambda}} \right)^{H(T)} \left( \frac{\lambda}{\tilde{\lambda}} \right)^{H(T)} ; A \right] \\ &= e^{\frac{1}{\tilde{\lambda}}(\tilde{\lambda} - \lambda)^2|T|} \mathbb{P}_{\tilde{\lambda}}(A). \end{aligned}$$

Thus

$$\lim_{\lambda \rightarrow 0} \log \mathbb{E}_{\tilde{\lambda}}[L^2(T); A] = \lim_{\lambda \rightarrow 0} \log \mathbb{P}_{\tilde{\lambda}}(A) = -\beta(\tilde{\lambda})$$

and  $\lim_{\lambda \rightarrow 0} \log \mathbb{P}_\lambda(A) = -\beta(\lambda)$ , which completes the proof in view of Definition 5.1.  $\square$

Assume from now, that the radius of discs is equal to  $r = 1$ . Let  $T = T(K) := [-K, K]^d \subseteq \mathbb{R}^d$  ( $d \geq 2$ ) for fixed  $K > 1$ . We shall apply the above theorem to the event  $A_0 := \{0 \longleftrightarrow \partial T(K)\}$  that there exists a path through balls of the Boolean model joining 0 with the surface  $\partial T$  of  $T$ :

$$\partial T(K) = \{t = (t_1, \dots, t_d) \in \mathbb{R}^d : \max_i |t_i| = K\} .$$

In other words, in the wire frame model, the origin is inside a disc which is connected through a chain of discs with the surface  $\partial T$ .

**Theorem 5.2.** *The IS scheme for  $A_0$  is logarithmic efficient.*

Before we give the proof we prove the following lemmas.

**Lemma 5.1.** *There exists a decreasing positive function  $\phi(\lambda)$  such that*

$$\lim_{\lambda \rightarrow 0} -\frac{1}{K\phi(\lambda)} \log \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) = 1 . \quad (5.4)$$

*Proof.* Using version BK and FKG inequalities for continuous percolation (see Th. 2.2 and Th. 2.3 of Meester and Roy [8]) we obtain (6.24) in Grimmett [2]. Then using the subadditive inequality limit theorem one can mimic the proof of Th. 6.10 of Grimmett [2] to obtain its version for the continuous percolation. That is, there exist strictly positive constants  $\rho$  and  $\sigma$ , independent of  $\lambda$ , and a decreasing positive function  $\phi(\lambda)$  such that

$$\rho K^{1-d} e^{-K\phi(\lambda)} \leq \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) \leq \sigma K^{d-1} e^{-K\phi(\lambda)}$$

for all  $K > 1$ . This completes the proof.  $\square$

**Lemma 5.2.** *Let  $0 \leq f(x) \leq 1$ . Then*

$$\lim_{\lambda \rightarrow 0} \frac{\phi(\lambda) \log(\lambda)}{\phi(\lambda f(\lambda)) \log(\lambda f(\lambda))} \geq 1 . \quad (5.5)$$

*Proof.* The main idea of the proof is to approximate the continuous problem by site percolation problems on a special lattice constructed by partitioning  $\mathbb{R}^d$  into small cubes. Let  $\kappa$  be a positive integer and  $\mathbb{Z}_\kappa^d = \frac{1}{\kappa} \mathbb{Z}^d$ . We partition  $\mathbb{R}^d$  into cubes whose centers are the points of  $\mathbb{Z}_\kappa^d$ , defining  $B_\kappa(t) = \times_{i=1}^d [t_i - \frac{1}{2\kappa}, t_i + \frac{1}{2\kappa}]$  for  $t \in \mathbb{Z}_\kappa^d$ . We turn  $\mathbb{Z}_\kappa^d$  into a lattice  $\mathcal{G}_\kappa$  by defining the adjacency relation  $\sim$  on  $\mathbb{Z}_\kappa^d$  with the rule that  $x \sim y$  iff there exist points  $u \in B_\kappa(x)$  and  $v \in B_\kappa(y)$  such that  $\rho(u, v) \leq 2$  where  $\rho(\cdot, \cdot)$  is the Euclidean distance. We shall consider site percolation on the ensuing lattice  $\mathcal{G}_\kappa$ . We declare a vertex  $x$  of  $\mathcal{G}_\kappa$  to be open if there exist one or more points of the Poisson process within the cube  $B_\kappa(x)$ , and closed otherwise. The states of different vertices are independent random variables and the probability  $p_\kappa(\lambda)$  that any given vertex is open is given by

$$p_\kappa(\lambda) = 1 - e^{-\lambda \kappa^{-d}} . \quad (5.6)$$

Let  $\gamma_\kappa = (1 + \frac{1}{\kappa} \sqrt{d})^d$ . From the rescaling property of a Boolean disc model and the considerations made by Meester and Roy [8], p. 60 or Grimmett [2], Sec. 12.10, for sufficiently small  $\lambda$  we have

$$\mathbb{P}_{p_\kappa(\lambda \gamma_\kappa)}^s(0 \longleftrightarrow \partial T(K)) \leq \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) \leq \mathbb{P}_{p_\kappa(\lambda)}^s(0 \longleftrightarrow \partial T(K)) , \quad (5.7)$$

where  $\mathbb{P}_{p_\kappa}^s(\cdot)$  is a law of site percolation on  $\mathcal{G}_\kappa$  defined by the adjacency relation  $\sim$  for  $p_\kappa$ . Thus by Th. 2.38 of Grimmett [2] applied to site percolation we have

$$\begin{aligned} & \frac{\log \mathbb{P}_{\lambda f(\lambda)}(0 \longleftrightarrow \partial T(K)) \log \lambda}{\log \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K)) \log \lambda f(\lambda)} \\ & \geq \frac{\log \mathbb{P}_{p_\kappa(\lambda f(\lambda))}(0 \longleftrightarrow \partial T(K)) \log \lambda}{\log \mathbb{P}_{p_\kappa(\lambda \gamma_\kappa)}(0 \longleftrightarrow \partial T(K)) \log \lambda f(\lambda)} \\ & \geq \frac{\log p_\kappa(\lambda f(\lambda)) \log \lambda}{\log p_\kappa(\lambda \gamma_\kappa) \log \lambda f(\lambda)}. \end{aligned}$$

Note that  $\lim_{\alpha \rightarrow 0} \frac{\log p_\kappa(\alpha)}{\log \alpha} = 1$ . This completes the proof.  $\square$

*Proof of Theorem 5.2* Let  $f(\lambda) = \frac{\lambda}{\tilde{\lambda}} < 1$ . Then from Lemmas 5.1 - 5.2 we have

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\log \mathbb{P}_{\tilde{\lambda}}(0 \longleftrightarrow \partial T(K))}{2 \log \mathbb{P}_\lambda(0 \longleftrightarrow \partial T(K))} &= \lim_{\lambda \rightarrow 0} \frac{\phi(\lambda f(\lambda))}{2\phi(\lambda)} \\ &\geq \lim_{\lambda \rightarrow 0} \frac{\log(\lambda f(\lambda))}{2 \log \lambda} = 1, \end{aligned}$$

which completes the proof of the theorem.  $\square$

We make also some numerical analysis using the IS scheme described in this section. We made 10000 simulations for  $K = 64$ ,  $d = 2$ ,  $r = 1$  and  $\tilde{\lambda} = 1, 7$ .

$\lambda$	$p_{\text{IS}}$
1.6	$7.5 * 10^{-6}$
1.5	$2.5 * 10^{-6}$
1.45	$1.2 * 10^{-6}$

Table 1: Simulation of the event  $0 \longleftrightarrow \partial T(K)$

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## References

- [1] Asmussen, S. (1999) *Stochastic simulation with a view towards stochastic processes*. MaPhySto Lecture Notes, no 2, University of Aarhus.
- [2] Grimmett, G. (1999) *Percolation*. Springer, Berlin.
- [3] Ivanoff, B.G. and Merzbach, E. (1995) Stopping and set-indexed local martingales. *Stoch. Proc. Appl.* **57** (1), 83–98.
- [4] Ivanoff, B.G. and Merzbach, E. (2000) *Set-indexed Martingales*. Chapman & Hall, Providence.

- [5] Jacod, J. (1975) Multivariate point processes: Predictable projection, Radon-Nikodym derivatives, representation of martingales. *Z. Wahrsch. verw. Gebiete* **31**, 235–253.
- [6] Karr, A.F. (1991) *Point Processes and their Statistical Inference*. 2nd ed., Probability: Pure and Applied, New York.
- [7] Matheron, G. (1975) *Random Sets and Integral Geometry*. John Wiley, Chichester.
- [8] Meester, R. and Roy, R. (1996) *Continuum Percolation*. Cambridge University Press, Cambridge.
- [9] Merzbach, E. and Nualart, D. (1986) A characterization of the spatial Poisson process and changing time. *Ann. Prob.* **14(4)**, 1380–1390.
- [10] Parthasarathy, K.R. (1967) *Probability measures on metric spaces*. Academic Press, New York.
- [11] Stoyan, D., Kendall, W.S. and Mecke, J. (1987) *Stochastic Geometry and its Applications*. John Wiley & Sons, Berlin.
- [12] Van Lieshout, M.N.M. (1995) *Stochastic geometry models in image analysis and spatial statistics*. CWI Tracts, Amsterdam.
- [13] Van Lieshout, M.N.M. (2002) On likelihood for Markov random sets and Boolean models. Manuscript.