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**The uniform measure on a Galton-Watson tree
without the $X \log X$ condition**

Elie Aïdékon
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The uniform measure on a Galton–Watson tree without the XlogX condition

Elie Aïdékon¹

Summary. We consider a Galton–Watson tree with offspring distribution ν of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the n -th generation and taking the limit $n \rightarrow \infty$. In the case $E[\nu \ln(\nu)] < \infty$, this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to $\ln(m)$ ([2], [9]). When $E[\nu \ln(\nu)] = \infty$, we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres [10].

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d’enfants ν a une moyenne finie. La mesure uniforme sur la frontière de l’arbre s’obtient en chargeant chaque sommet de la n -ième génération avec une masse 1, puis en prenant la limite $n \rightarrow \infty$. Dans le cas $E[\nu \ln(\nu)] < \infty$, cette mesure a été très étudiée, et l’on sait que la dimension de Hausdorff de la mesure est égale à $\ln(m)$ ([2], [9]). Lorsque $E[\nu \ln(\nu)] = \infty$, nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres [10].

Keywords: Galton–Watson tree, Hausdorff dimension.

AMS subject classifications: 60J80, 28A78.

1 Introduction

Let \mathcal{T} be a Galton–Watson tree of root e , associated to the offspring distribution $q := (q_k, k \geq 0)$. We denote by GW the distribution of \mathcal{T} on the space of rooted trees, and ν a generic random variable on \mathbb{N} with distribution q . We suppose that $q_0 = 0$ and $m := \sum_{k \geq 0} kq_k \in (1, \infty)$:

¹Department of Mathematics and Computer science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. email: elie.aidekon@gmail.com

the tree has no leaf (hence survives forever) and is not degenerate. For any vertex u , we write $|u|$ for the height of vertex u ($|e| = 0$), $\nu(u)$ for the number of children of u , and Z_n is the population at height n . We define $S(\mathcal{T})$ as the set of all infinite self-avoiding paths of \mathcal{T} starting from the root and we define a metric on $S(\mathcal{T})$ by $d(r, r') := e^{-|r \wedge r'|}$ where $r \wedge r'$ is the highest vertex belonging to r and r' . The space $S(\mathcal{T})$ is called boundary of the tree, and elements of $S(\mathcal{T})$ are called rays.

When $E[\nu \ln(\nu)] < \infty$, it is well-known that the martingale $m^{-n}Z_n$ converges in L^1 and almost surely to a positive limit ([4]). Seneta [13] and Heyde [3] proved that in the general case (i.e allowing $E[\nu \ln(\nu)]$ to be infinite), there exist constants $(c_n)_{n \geq 0}$ such that

- (a) $W_\infty := \lim_{n \rightarrow \infty} \frac{Z_n}{c_n}$ exists a.s.
- (b) $W_\infty > 0$ a.s.
- (c) $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = m$.

In particular, for each vertex $u \in \mathcal{T}$, if $Z_k(u)$ stands for the number of descendants v of u such that $|v| = |u| + k$, we can define

$$W_\infty(u) := \lim_{k \rightarrow \infty} \frac{Z_k(u)}{c_k}$$

and we notice that $m^{-n} \sum_{|u|=n} W_\infty(u) = W_\infty(e)$.

Definition. The uniform measure (also called branching measure) is the unique Borel measure on $S(\mathcal{T})$ such that

$$\mathbf{UNIF}(\{r \in S(\mathcal{T}), r_n = u\}) := \frac{m^{-n}W_\infty(u)}{W_\infty(e)}$$

for any integer n and any vertex u of height n .

We observe that, for any vertex u of height n ,

$$\mathbf{UNIF}(\{r \in S(\mathcal{T}), r_n = u\}) = \lim_{k \rightarrow \infty} \frac{Z_k(u)}{Z_{n+k}}.$$

Therefore the uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of \mathbf{UNIF} , defined by

$$\dim(\mathbf{UNIF}) := \min\{\dim(E), \mathbf{UNIF}(E) = 1\}$$

where the minimum is taken over all subsets $E \subset S(\mathcal{T})$ and $\dim(E)$ is the Hausdorff dimension of set E . The case $E[\nu \ln(\nu)] < \infty$ has been well studied. In [2] and [9], it is shown that $\dim(\mathbf{UNIF}) = \ln(m)$ almost surely. A description of the multifractal spectrum is available in [5],[9],[12],[14]. The case $E[\nu \ln(\nu)] = \infty$ presented as Question 3.1 in [10] was left open. This case is proved to display an extreme behaviour.

Theorem 1.1. *If $E[\nu \ln(\nu)] = \infty$, then $\dim(\mathbf{UNIF}) = 0$ for GW-a.e tree \mathcal{T} .*

The drop in the dimension comes from bursts of offspring at some places of the tree \mathcal{T} . Namely, for \mathbf{UNIF} -a.e. ray r , the number of children of r_n will be greater than $(m - o(1))^n$ for infinitely many n . To prove it, we work with a particular measure Q , under which the distribution of the numbers of children of a uniformly chosen ray is more tractable. Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3.

2 A spine decomposition

For $k \geq 1$ and $s \in (0, 1)$, we call $\phi_k(s)$ the probability generating function of Z_k

$$\phi_k(s) := E[s^{Z_k}].$$

We denote by $\phi_k^{-1}(s)$ the inverse map on $(0, 1)$ and we let $s \in (0, 1)$. Then $M_n := \phi_n^{-1}(s)^{Z_n}$ defines a martingale and converges in L^1 to some $M_\infty > 0$ a.s ([3]). Therefore we can take in (a)

$$c_n := \frac{-1}{\ln(\phi_n^{-1}(s))}$$

which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(e)}$. For any vertex u at generation n , we define similarly

$$M_\infty(u) := \frac{1}{\phi_n^{-1}(s)} e^{-m^{-n}W_\infty(u)} = e^{1/c_n} e^{-m^{-n}W_\infty(u)}$$

which is the limit of the martingale $M_k(u) := e^{1/c_n} e^{-Z_k(u)/c_{n+k}}$. In [6], Lynch introduces the so-called derivative martingale

$$\partial M_n := e^{1/c_n} \frac{Z_n}{\phi_n'(\phi_n^{-1}(s))} M_n$$

and shows that the derivative martingale also converges almost surely and in L^1 (∂M_n is in fact bounded). Moreover the limit ∂M_∞ is positive almost surely. We deduce that the ratio $\phi'_n(\phi_n^{-1}(s))/c_n$ converges to some positive constant. In particular, it follows from (c) that

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{\phi'_{n+1}(\phi_{n+1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} = m.$$

We are interested in the probability measure Q on the space of rooted trees defined by

$$\frac{dQ}{d\text{GW}} := \partial M_\infty.$$

Let us describe this change of measure. We call a marked tree a couple (T, r) where T is a rooted tree and r a ray of the tree T . Let (\mathbb{T}, ξ) be a random variable in the space of all marked trees (equipped with some probability $\mathbb{P}(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level k and on the location of the ray at level k , (which we denote respectively by \mathbb{T}_k and ξ_k),

- the number of children of the vertices at generation k are independent
- the vertex ξ_k has a number $\nu(\xi_k)$ of children such that for any ℓ

$$(2.2) \quad \mathbb{P}(\nu(\xi_k) = \ell) = \tilde{q}_\ell^s := q_\ell \ell \exp\left(-\frac{\ell-1}{c_{k+1}}\right) \frac{\phi'_k(\phi_k^{-1}(s))}{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}$$

- the number of children of a vertex $u \neq \xi_k$ at generation k verifies for any ℓ

$$(2.3) \quad \mathbb{P}(\nu(u) = \ell) = \tilde{q}_\ell := q_\ell e^{1/c_k} \exp\left(-\frac{\ell}{c_{k+1}}\right)$$

- the vertex ξ_{k+1} is chosen uniformly among the children of ξ_k

As often in the literature, we will call the ray ξ the spine. We refer to [8], [7] for motivation on spine decompositions. In our case, we can see \mathbb{T} as a Galton-Watson tree in varying environment and with immigration. The fact that (2.2) and (2.3) define probabilities come from the equations (remember that by definition $e^{-1/c_k} = \phi_k^{-1}(s)$)

$$\begin{aligned} E [(\phi_{k+1}^{-1}(s))^\nu] &= \phi_k^{-1}(s), \\ E [\nu(\phi_{k+1}^{-1}(s))^{\nu-1}] &= \frac{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}{\phi'_k(\phi_k^{-1}(s))}. \end{aligned}$$

We mention that in [8], a similar decomposition was presented using the martingale $\frac{Z_n}{m^n}$. In this case, the offspring distribution of the spine is the size-biased distribution $(\frac{\ell q_\ell}{m})_{\ell \geq 0}$ whereas the other particles generate offspring according to the original distribution q . In particular, the offspring distributions do not depend on the generation. When $E[\nu \ln(\nu)] < \infty$, the process, which is a Galton–Watson process with immigration, has a distribution equivalent to GW. It is no longer true when $E[\nu \ln(\nu)] = \infty$, in which case the spine can give birth to a super-exponential number of children.

Proposition 2.1. *Under Q , the tree \mathcal{T} has the distribution of \mathbb{T} . Besides, for P -almost every tree \mathbb{T} , the distribution of ξ conditionally on \mathbb{T} is the uniform measure **UNIF**.*

Proof. For any tree T , we define T_n the tree T obtained by keeping only the n -first generations. Let T be a tree. We will prove by induction that, for any integer n and any vertex u at generation n ,

$$(2.4) \quad \mathbb{P}(\mathbb{T}_n = T_n, \xi_n = u) = \frac{\partial M_n}{Z_n} \text{GW}(\mathcal{T}_n = T_n).$$

For $n = 0$, it is straightforward since \mathbb{T}_0 and \mathcal{T}_0 are reduced to the root. We suppose that this is true for $n - 1$, and we prove it for n . Let \overleftarrow{u} denote the parent of u , and, for any vertex v at height $n - 1$, let $k(v)$ denote the number of children of v in the tree T . We have

$$\begin{aligned} & \mathbb{P}(\mathbb{T}_n = T_n, \xi_n = u \mid \mathbb{T}_{n-1} = T_{n-1}, \xi_{n-1} = \overleftarrow{u}) \\ &= \frac{1}{k(\overleftarrow{u})} \frac{\tilde{q}_{k(\overleftarrow{u})}^s}{\tilde{q}_{k(\overleftarrow{u})}} \prod_{|v|=n-1} \tilde{q}_{k(v)} \\ &= \frac{e^{1/c_n}}{e^{1/c_{n-1}}} \frac{\phi'_{n-1}(\phi_{n-1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} \frac{e^{Z_{n-1}/c_{n-1}}}{e^{Z_n/c_n}} \prod_{|v|=n-1} q_{k(v)} \\ &= \frac{e^{1/c_n}}{e^{1/c_{n-1}}} \frac{\phi'_{n-1}(\phi_{n-1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} \frac{e^{Z_{n-1}/c_{n-1}}}{e^{Z_n/c_n}} \text{GW}(\mathcal{T}_n = T_n \mid \mathcal{T}_{n-1} = T_{n-1}). \end{aligned}$$

We use the induction assumption to get

$$\mathbb{P}(\mathbb{T}_n = T_n, \xi_n = u) = e^{1/c_n} \frac{1}{\phi'_n(\phi_n^{-1}(s))} e^{-\frac{Z_n}{c_n}} \text{GW}(\mathcal{T}_n = T_n)$$

which proves (2.4). Summing over the n -th generation of T gives

$$\mathbb{P}(\mathbb{T}_n = T_n) = \partial M_n \text{GW}(\mathcal{T}_n = T_n) = Q(\mathcal{T} = T_n).$$

This computation also shows that $P(\xi_n = u | \mathbb{T}_n) = 1/Z_n$ which implies that ξ is uniformly distributed on the boundary $S(\mathbb{T})$. \square

Remark A. For u a vertex of \mathbb{T} at generation n , call $\mathbb{T}(u)$ the subtree rooted at u . A similar computation shows that if $u \notin \xi$, then the distribution P_u of $\mathbb{T}(u)$ (conditionally on \mathbb{T}_n and on ξ_n) verifies

$$\frac{dP_u}{dGW} = M_\infty(u).$$

3 Proof of Theorem 1.1

The following proposition shows that in the tree \mathcal{T} , there exist infinitely many times when the ball $\{r \in S(\mathcal{T}) : r_n = \xi_n\}$ has a 'big' weight.

Proposition 3.1. *Suppose that $E[\nu \ln(\nu)] = \infty$. Then we have P-a.s.*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m).$$

Proof. Let $1 < a < b < m$ and $n \geq 0$. We get from (2.2)

$$\begin{aligned} P(\nu(\xi_n) \in (a^n, b^n)) &= \frac{\phi'_n(\phi_n^{-1}(s))}{\phi'_{n+1}(\phi_{n+1}^{-1}(s))} E[\nu e^{-(\nu-1)/c_{n+1}}, \nu \in (a^n, b^n)] \\ &\geq \frac{\phi'_n(\phi_n^{-1}(s))}{\phi'_{n+1}(\phi_{n+1}^{-1}(s))} e^{-b^n/c_n} E[\nu, \nu \in (a^n, b^n)]. \end{aligned}$$

From (c) and (2.1), we deduce that for n large enough, we have

$$P(\nu(\xi_n) \in (a^n, b^n)) \geq \frac{1}{2m} E[\nu, \nu \in (a^n, b^n)].$$

Therefore, under the condition $E[\nu \ln(\nu)] = \infty$, we have

$$(3.1) \quad \sum_{n \geq 0} P(\nu(\xi_n) \in (a^n, b^n)) = \infty.$$

Let $H(\xi_n) := \{u \in \mathbb{T} : u \text{ child of } \xi_n, u \neq \xi_{n+1}\}$. By Remark A, we have

$$P\left(\sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \mid \mathbb{T}_{n+1}, \xi_{n+1}\right) = E_{GW} \left[\prod_{u \in \mathcal{H}} M_\infty(u), \sum_{u \in \mathcal{H}} W_\infty(u) \leq a^n \right]_{\mathcal{H} = H(\xi_n)}.$$

Since $M_\infty(u) \leq e^{1/c_n}$ for any $|u| = n$, we get

$$\mathbb{P} \left(\sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \mid \mathbb{T}_{n+1}, \xi_{n+1} \right) \leq e^{(\nu(\xi_n)-1)/c_n} \text{GW} \left(\sum_{u \in \mathcal{H}} W_\infty(u) \leq a^n \right)_{\mathcal{H}=H(\xi_n)}.$$

Let $(W_\infty^i, i \geq 1)$ be independent random variables distributed as $W_\infty(e)$ under GW. It follows that on the event $\{\nu(\xi_n) \in (a^n, b^n)\}$, we have

$$\mathbb{P} \left(\sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \mid \mathbb{T}_{n+1}, \xi_{n+1} \right) \leq e^{(b^n-1)/c_n} \text{GW} \left(\sum_{i=1}^{a^n} W_\infty^i \leq a^n \right) =: d_n.$$

We obtain that

$$\begin{aligned} \mathbb{P} \left(\sum_{u \in H(\xi_n)} W_\infty(u) > a^n \right) &\geq \mathbb{P} \left(\sum_{u \in H(\xi_n)} W_\infty(u) > a^n, \nu(\xi_n) \in (a^n, b^n) \right) \\ &\geq \mathbb{P}(\nu(\xi_n) \in (a^n, b^n)) (1 - d_n). \end{aligned}$$

By (c), $e^{(b^n-1)/c_n}$ goes to 1. Furthermore, we know from [13] that $E_{\text{GW}}[W_\infty(e)] = \infty$, which ensures by the law of large numbers that d_n goes to 0. By equation (3.1), we deduce that

$$\sum_{n \geq 0} \mathbb{P} \left(\sum_{u \in H(\xi_n)} W_\infty(u) > a^n \right) = \infty.$$

We use the Borel-Cantelli lemma to see that $\sum_{u \in H(\xi_n)} W_\infty(u) > a^n$ infinitely often. Since $W_\infty(\xi_n) \geq \frac{1}{m} \sum_{u \in H(\xi_n)} W_\infty(u)$, we get that $W_\infty(\xi_n) \geq a^n/m$ for infinitely many n , P-a.s. Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) \geq \ln(a).$$

Let a go to m to have the lower bound. Since $W_\infty(\xi_n) \leq \sum_{|u|=n} W_\infty(u) = m^n W_\infty(e)$, we have $\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) \leq \ln(m)$ hence Proposition 3.1. \square

We turn to the proof of the theorem.

Proof of Theorem 1.1. By Proposition 3.1, we have

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \right) = 1.$$

In particular, for P-a.e. \mathbb{T} ,

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \mid \mathbb{T} \right) = 1.$$

By Proposition 2.1, the distribution of ξ given \mathbb{T} is **UNIF**. Therefore, for P-a.e. \mathbb{T} ,

$$(3.2) \quad \mathbf{UNIF} \left(r \in S(\mathbb{T}) : \limsup_{n \rightarrow \infty} \frac{1}{n} \ln(W_\infty(r_n)) = \ln(m) \right) = 1.$$

Again by Proposition 2.1, the distribution of \mathbb{T} is the one of \mathcal{T} under Q . We deduce that (3.2) holds for Q -a.e. tree \mathcal{T} . Since Q and GW are equivalent, equation (3.2) holds for GW-a.e. tree \mathcal{T} . We call the Hölder exponent of **UNIF** at ray r^* the quantity

$$\text{Hö}(\mathbf{UNIF})(r^*) := \liminf_{n \rightarrow \infty} \frac{-1}{n} \ln(\mathbf{UNIF}(\{r \in S(\mathcal{T}) : r_n = r_n^*\})) .$$

By definition of **UNIF**, we can rewrite it

$$\text{Hö}(\mathbf{UNIF})(r^*) = \liminf_{n \rightarrow \infty} \frac{-1}{n} \ln(m^{-n} W_\infty(r_n^*) / W_\infty(e)) .$$

Therefore, for **UNIF**-a.e. ray r , $\text{Hö}(\mathbf{UNIF})(r) = 0$. By Theorem 14.15 of [11] (or § 14 of [1]), it implies that $\dim(\mathbf{UNIF}) = 0$ GW-almost surely. \square

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