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AN $H^1(\mathcal{P}^h)$-COERCIVE DISCONTINUOUS GALERKIN FORMULATION FOR THE POISSON PROBLEM: 1D ANALYSIS*

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Abstract. Coercivity of the bilinear form in a continuum variational problem is a fundamental property for finite-element discretizations: By the classical Lax–Milgram theorem, any conforming discretization of a coercive variational problem is stable; i.e., discrete approximations are well-posed and possess unique solutions, irrespective of the specifics of the underlying approximation space. Based on the prototypical one-dimensional Poisson problem, we establish in this work that most concurrent discontinuous Galerkin formulations for second-order elliptic problems represent instances of a generic conventional formulation and that this generic formulation is noncoercive. Consequently, all conventional discontinuous Galerkin formulations are a fortiori noncoercive, and typically their well-posedness is contingent on approximation-space-dependent stabilization parameters. Moreover, we present a new symmetric nonconventional discontinuous Galerkin formulation based on element Green's functions and the data local to the edges. We show that the new discontinuous Galerkin formulation is coercive on the broken Sobolev space $H^1(\mathcal{P}^h)$, viz., the space of functions that are elementwise in the H^1 Sobolev space. The coercivity of the new formulation is supported by calculations of discrete inf-sup constants, and numerical results are presented to illustrate the optimal convergence behavior in the energy-norm and in the $L_2(\Omega)$-norm.

Key words. finite element method, discontinuous Galerkin, elliptic problems, coercivity

AMS subject classifications. 65N30, 65N12

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1. Introduction.

The recent renewal of interest in discontinuous Galerkin (DG) methods for second-order elliptic boundary value problems can be attributed to twofold reasons. First, DG methods provide robust finite-element discretizations for hyperbolic conservation laws, as the interelement discontinuities enable an extension of Godunov's method for finite-volume methods. However, to extend these techniques to singularly perturbed elliptic problems, an appropriate treatment for the elliptic part of the operator is required. Second, the absence of interelement-continuity constraints renders DG methods ideally suited for hp adaptivity, e.g., based on a posteriori error estimation; see, for instance, [1, 8]. A comprehensive overview of the historical development of DG methods is provided in [7].

A framework for analyzing DG formulations for elliptic problems has recently been erected in [2]. Although the analysis in [2] clarifies basic properties of the different formulations, it does not seem to warrant a clear preference. The literature on DG methods for elliptic problems is dominated by formulations that possess edge terms composed of linear combinations of the jumps and averages of the test and trial functions and their normal derivatives. That is, denoting by u and v the test and trial functions, and by $[\![\cdot]\!]$ and $\{\cdot\}$ the jump and average of $(\cdot)$ at an interelement edge, these formulations contain terms conforming to $\{\partial_n u\}[\![v]\!]$, $[\![u]\!][\![v]\!]$, $[\![\partial_n u]\!][\![\partial_n v]\!]$, etc., where ∂_n represents the normal derivative. We refer to such formulations as

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conventional DG formulations and to the corresponding edge terms as *conventional edge terms*. Symmetric examples of such formulations are the *global element method* (GEM; see [10, 12]) and the *interior penalty* DG formulation (IPDG; cf. [2, 10, 12]). Nonsymmetric examples are the celebrated *Baumann and Oden* DG formulation (BODG [3]), the *stabilized* DG formulation (SDG [14]), the *nonsymmetric interior penalty* DG formulation (NIPDG [13]), and the family of formulations considered by Larson and Niklasson (LNDG [9]).

The essential deficiency of conventional DG formulations is that their bilinear form is not *strongly coercive* (and simultaneously continuous) on a continuum (infinite-dimensional) broken space, in contrast to the bilinear form in the classical continuous Galerkin (CG) formulation. For conciseness, we say that these methods are *noncoercive*. In particular, this implies that finite-element approximations can be ill-posed, despite well-posedness of the underlying continuum problem. Furthermore, a sequence of nested stable approximations need not converge monotonously, as the constants in the error-estimates are approximation-space-dependent, and cannot be bounded uniformly. Conventional DG formulations can be coercive on *discrete* approximation spaces. However, this generally requires stability parameters which increase unboundedly as the approximation space is refined. For example, for broken polynomial spaces the stability parameters are typically proportional to a monomial of the polynomial degree. Moreover, conventional DG formulations are in general subject to the assumption that the solution resides in $H^2(\Omega)$, whereas a formulation allowing solutions in $H^1(\Omega)$ would be more natural from the classical CG formulation perspective.[1]

Nonsymmetric conventional DG formulations can be well-posed without stability parameters. However, such formulations derive their well-posedness from *weak coercivity*. Moreover, for nonsymmetric formulations the error converges suboptimally in the $L_2(\Omega)$-norm for even-degree broken polynomial spaces. It has been conjectured that this behavior emanates from the nonsymmetry of the formulation; see [3, 9, 10].

Alternatively, DG formulations can be constructed by introducing *nonconventional edge terms*. We remark that the support of such terms is not necessarily restricted to the edges. Examples of such DG formulations are the *lift-operator-based* schemes in [2]. These include, among others, the *Bassi and Rebay* DG formulation (BRDG [4, 5]) and the *local* DG formulation (LDG [6]). However, lift-operators are explicitly defined using a discrete (finite-element) space. As a consequence, the continuum formulation with lift-operators, although consistent at a discrete level, is inconsistent at the continuum level. Therefore, for each approximation space, the edge-traces need to be lifted accordingly and the extension to a consistent continuum formulation is nonobvious.

A recent example of another nonconventional formulation is the discontinuous finite-element formulation based on second-order derivatives in [15]. This formulation resembles a least-squares form. However, it is based on second-order derivatives, thereby implicitly restricting the admissible functions to $H^2(\Omega)$ and, moreover, it is unknown if the bilinear form is simultaneously coercive and continuous.

In this paper, we first establish on the basis of the prototypical one-dimensional Poisson problem that a conventional DG formulation with a coercive bilinear form is *nonexistent*. We then present a new nonconventional symmetric DG formulation based on *element Green's functions* and the data local to the edges. The essential advantage of our new DG formulation is that it is *coercive* on the (infinite-dimensional)

[1] More precisely, only $H^{3/2+\epsilon}(\Omega)$ regularity is required. This ensures that the edge terms in conventional DG formulations are well defined.

broken Sobolev space $H^1(\mathcal{P}^{\mathsf{h}})$, the space of functions that are elementwise in the H^1 Sobolev space. On account of its coercivity, approximations of the new formulation inherit their well-posedness from the continuum formulation; i.e., well-posedness of the approximation problem is ensured for any approximation space and, in particular, for the usual broken polynomial spaces. Furthermore, optimal error estimates hold with constants that can be bounded uniformly independent of the specifics of the approximation space. Finally, we demonstrate that the new DG formulation is equivalent with the classical CG formulation, thus allowing solutions in $H^1(\Omega)$.

The contents of this paper are arranged as follows: section 2 presents the elliptic model problem, viz., the Poisson problem. Furthermore, mathematical preliminaries for DG formulations of the Poisson problem are given. Section 3 reviews elementary existence and uniqueness theorems for linear variational problems, to establish the differences between coercivity and weak coercivity, and to furnish the basis for our analysis in the ensuing sections. In section 4 we present the generic conventional DG formulation, and we prove its noncoerciveness. In section 5 we introduce the new DG formulation and we demonstrate its coercivity. Furthermore, we establish its equivalence with the classical CG formulation. Numerical results are presented in section 6. The coercivity of the new formulation is supported by calculations of discrete inf-sup constants. Moreover, the convergence behavior in the energy-norm and in the $L_2(\Omega)$-norm is investigated. Finally, section 7 contains concluding remarks.

2. Problem statement. In this work, we shall restrict ourselves to the simplest prototypical model problem for second-order elliptic boundary value problems, viz., the linear one-dimensional *Poisson* problem.

2.1. Poisson problem. Let $\Omega \subset \mathbb{R}$ be a bounded open interval. Its two-point boundary $\partial\Omega$ consists of two disjoint parts, $\Gamma_{\mathcal{D}}$ (nonempty) and $\Gamma_{\mathcal{N}}$ (possibly empty) on which Dirichlet and Neumann boundary conditions are imposed, respectively. The unit normal n at the boundary $\partial\Omega$ is defined to be outward with respect to the interval Ω.

Within this one-dimensional setting, we formulate the Poisson problem: Given an arbitrary $\bar{u} \in H^1(\Omega)$ with $\bar{u} = g_{\mathcal{D}}$ on $\Gamma_{\mathcal{D}}$,

$$\boxed{\begin{aligned}&\text{Find } u = \bar{u} + u_0 \in \bar{u} + H^1_{0,\mathcal{D}}(\Omega) : \\ &\qquad\qquad \mathscr{B}_{\mathrm{c}}(u_0, v) = \mathscr{L}_{\mathrm{c}}(v) \qquad \forall v \in H^1_{0,\mathcal{D}}(\Omega)\ ,\end{aligned}} \tag{2.1}$$

where the bilinear form $\mathscr{B}_{\mathrm{c}} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and the linear functional $\mathscr{L}_{\mathrm{c}} : H^1(\Omega) \to \mathbb{R}$ are defined as

$$\mathscr{B}_{\mathrm{c}}(u,v) := \int_\Omega \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x\ , \tag{2.2a}$$

$$\mathscr{L}_{\mathrm{c}}(v) := \int_\Omega f v\,\mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} v\big)_e - \mathscr{B}_{\mathrm{c}}(\bar{u}, v)\ . \tag{2.2b}$$

We define $H^1_{0,\mathcal{D}}(\Omega)$ to be the subspace of functions in the Sobolev space $H^1(\Omega)$ which vanish on the Dirichlet boundary $\Gamma_{\mathcal{D}}$, i.e.,

$$H^1_{0,\mathcal{D}}(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_{\mathcal{D}}\}\ .$$

For $f \in L_2(\Omega)$, Problem (2.1) possesses a unique solution $u \in H^2(\Omega)$ which, moreover, uniquely solves the boundary value problem

$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f \qquad \text{in } \Omega\,, \tag{2.3a}$$

$$u = g_{\mathcal{D}} \qquad \text{on } \Gamma_{\mathcal{D}}\,, \tag{2.3b}$$

$$\partial_n u = g_{\mathcal{N}} \qquad \text{on } \Gamma_{\mathcal{N}}\,, \tag{2.3c}$$

where $\partial_n(\cdot)$ denotes the normal derivative $\frac{\mathrm{d}}{\mathrm{d}n}(\cdot)$.

The variational problem (2.1) constitutes the classical CG formulation of the Poisson problem. It is *well-posed* (stable); i.e., there *exists* a *unique* solution $u \in H^1(\Omega)$ which, moreover, depends continuously on the auxiliary data. The well-posedness follows from the classical Lax–Milgram theorem on account of *coercivity* of the bilinear functional $\mathscr{B}_{\mathrm{c}}(\cdot,\cdot)$; see section 3. A *conforming approximation* to the *continuum* problem (2.1) is obtained by replacing $H^1_{0,\mathcal{D}}(\Omega)$ by a closed, generally finite-dimensional, *sub*space $\widehat{H}^1_{0,\mathcal{D}}(\Omega) \subset H^1_{0,\mathcal{D}}(\Omega)$. The corresponding approximate solution $\widehat{u} \in \bar{u} + \widehat{H}^1_{0,\mathcal{D}}(\Omega)$ can be extracted by solving the following *approximate* problem:

$$\begin{aligned}&\text{Find } \widehat{u} = \bar{u} + \widehat{u}_0 \in \bar{u} + \widehat{H}^1_{0,\mathcal{D}}(\Omega):\\ &\qquad\qquad \mathscr{B}_{\mathrm{c}}(\widehat{u}_0, v) = \mathscr{L}_{\mathrm{c}}(v) \qquad \forall v \in \widehat{H}^1_{0,\mathcal{D}}(\Omega)\,.\end{aligned} \tag{2.4}$$

As the coercivity of the underlying continuum problem transfers to the approximate problem, the approximate problem is automatically well-posed irrespective of the specifics of the approximation space $\widehat{H}^1_{0,\mathcal{D}}(\Omega)$. This is a particularly favorable property which enables, for example, subsequent stable approximations in an hp-adaptive finite element procedure. We emphasize that coercivity is generally lost in a DG formulation.

2.2. Broken Sobolev spaces. To facilitate the ensuing consideration of DG formulations, we introduce a *finite-element partition*. Let $\mathcal{P}^{\mathsf{h}} := \mathcal{P}^{\mathsf{h}}(\Omega)$ denote such a partition of the interval Ω; i.e., $\mathcal{P}^{\mathsf{h}}$ is a finite collection of open nonoverlapping subintervals (*elements*) K, such that

$$\Omega = \operatorname{int}\Bigg(\bigcup_{K\in\mathcal{P}^{\mathsf{h}}} \overline{K}\Bigg)\,.$$

The mesh parameter h associated with $\mathcal{P}^{\mathsf{h}}$ is defined as

$$\mathsf{h} := \max_{K\in\mathcal{P}^{\mathsf{h}}} h_K\,,$$

where h_K is the length of element K. The set of all (element) *edges*, $\Gamma := \Gamma(\mathcal{P}^{\mathsf{h}})$, can be divided into complementary subsets:

$$\Gamma = \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}} \cup \Gamma_{\mathcal{I}}\,,$$

where $\Gamma_{\mathcal{I}} := \Gamma_{\mathcal{I}}(\mathcal{P}^{\mathsf{h}})$ is the set of *interior* edges. We define a unit normal n_e at each edge $e \in \Gamma$. This normal coincides with the unit outward normal of Ω for *boundary* edges $e \in \partial\Omega = \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}}$ and we set $n_e := -1$ for interior edges $e \in \Gamma_{\mathcal{I}}$. For example, if e is an interior edge then we have $(\partial_n u)_e = \frac{\mathrm{d}u}{\mathrm{d}n}\big|_e = -\frac{\mathrm{d}u}{\mathrm{d}x}\big|_e$. We will further denote by $\mathcal{K}_e$ the set of elements sharing edge $e \in \Gamma$, that is,

$$\mathcal{K}_e := \big\{K \in \mathcal{P}^{\mathsf{h}} : \partial K \cap e = e\big\}\,.$$

Note that for boundary and interior edges e, the set $\mathcal{K}_e$ contains one element and two elements, respectively.

The functional setting of DG formulations is provided by the so-called (partition $\mathcal{P}^{\mathsf{h}}$ dependent) *broken Sobolev spaces* $H^m(\mathcal{P}^{\mathsf{h}})$ [10]. For any positive integer m, the broken Sobolev space $H^m(\mathcal{P}^{\mathsf{h}})$ is defined as

$$H^m(\mathcal{P}^{\mathsf{h}}) := \{v \in L_2(\Omega) : v\big|_K \in H^m(K)\ \forall K \in \mathcal{P}^{\mathsf{h}}\}\ ; \tag{2.5}$$

in other words, $H^m(\mathcal{P}^{\mathsf{h}})$ consists of functions for which the restriction to each element $K \in \mathcal{P}^{\mathsf{h}}$ is in $H^m(K)$. Equipped with the broken inner product

$$(u, v)_{H^m(\mathcal{P}^{\mathsf{h}})} := \sum_{K \in \mathcal{P}^{\mathsf{h}}} (u, v)_{H^m(K)}\ ,$$

$H^m(\mathcal{P}^{\mathsf{h}})$ is a Hilbert space. The corresponding norm will be denoted $\|\cdot\|_{H^m(\mathcal{P}^{\mathsf{h}})}$. Note that functions in a broken Sobolev space are generally discontinuous at the interior edges.

Functions in $H^1(\mathcal{P}^{\mathsf{h}})$ have traces on Γ. These are single-valued at boundary edges and double-valued at interior edges. To handle the traces, we introduce for each boundary edge $e \in \partial\Omega$ the usual boundary trace $(\cdot)_e$ as

$$u_e := \lim_{s \downarrow 0} u(e - n_e\, s)\ ,$$

and we introduce for each interior edge $e \in \Gamma_{\mathcal{I}}$ the $\pm$-trace, $(\cdot)_e^{\pm}$, as

$$u_e^{\pm} := \lim_{s \downarrow 0} u(e \pm s)\ .$$

Furthermore, we define the *average* $\{\cdot\}_e$ and the *jump* $[\![\cdot]\!]_e$ for each interior edge $e \in \Gamma_{\mathcal{I}}$ in the usual manner:

$$\begin{aligned} \{u\}_e &:= \tfrac{1}{2}(u_e^+ + u_e^-)\ , \\ [\![u]\!]_e &:= u_e^+ - u_e^-\ . \end{aligned}$$

These trace operators are bounded in $\mathbb{R}$ for functions in $H^1(\mathcal{P}^{\mathsf{h}})$; that is, trace inequalities hold.

2.3. DG formulations of the Poisson problem. Let $H := H(\mathcal{P}^{\mathsf{h}})$ be a broken space subordinate to the partition $\mathcal{P}^{\mathsf{h}}$ and $\widehat{H} := \widehat{H}(\mathcal{P}^{\mathsf{h}}) \subset H(\mathcal{P}^{\mathsf{h}})$ a finite-dimensional subspace. The generic form of a continuum DG formulation is given by the following abstract Galerkin variational problem:

$$\boxed{\begin{array}{l} \text{Find } u \in H : \\ \qquad\qquad \mathscr{B}(u, v) = \mathscr{L}(v) \qquad \forall v \in H\ . \end{array}} \tag{2.6}$$

Clearly, the continuum DG formulation should be consistent with the Poisson problem; i.e., the solution of (2.1) must comply with (2.6). The generic form of the corresponding approximate DG problem is given by

$$\boxed{\begin{array}{l} \text{Find } \widehat{u} \in \widehat{H} : \\ \qquad\qquad \mathscr{B}(\widehat{u}, v) = \mathscr{L}(v) \qquad \forall v \in \widehat{H}\ . \end{array}} \tag{2.7}$$

We will refer to the broken space associated with a particular DG problem as its *DG space*.

The conventional approach to constructing consistent DG formulations premises that $f \in L_2(\Omega)$ and, accordingly, $u \in H^2(\Omega) \subset H^2(\mathcal{P}^{\mathsf{h}}) \subset H$. Multiplication of (2.3a) with $v \in H$, integration on Ω, and elementwise integration by parts then yield

$$\sum_{K \in \mathcal{P}^{\mathsf{h}}} \int_K \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x - \sum_{e \in \Gamma_{\mathcal{I}}} \big(\partial_n u [\![v]\!]\big)_e - \sum_{e \in \Gamma_{\mathcal{D}}} \big((\partial_n u) v\big)_e = \int_\Omega f v \,\mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} v\big)_e \quad \forall v \in H \ .$$

For $u \in H^2(\Omega)$, $(\partial_n u)_e$ is well defined for $e \in \Gamma_{\mathcal{I}}$. However, for u in the DG space H, $(\partial_n u)_e$ is not uniquely defined at the interior edges. Therefore, $(\partial_n u)_e$ is conventionally replaced by $\{\partial_n u\}_e$. On account of $\{\partial_n u\}_e = (\partial_n u)_e$ for $u \in H^2(\Omega)$, this replacement preserves consistency. In addition, the bilinear form can be augmented with other products of edge values and/or edge derivatives, for instance, $\{\partial_n v\}_e [\![u]\!]_e$ for $e \in \Gamma_{\mathcal{I}}$. Most concurrent DG formulations are the result of such an augmentation and, accordingly, we will refer to such augmentations as *conventional edge terms*, and to the corresponding variational statements as *conventional* DG formulations. A precise definition is provided in section 4. Alternatively, the bilinear form can be endowed with other consistency-preserving edge terms, e.g., based on lift operators [4, 5, 6]. We collectively refer to such terms as *nonconventional edge terms*.

The above exposition furnishes the context for the problem considered in this paper. Our first objective is to establish that all conventional DG formulations are necessarily noncoercive, in contrast to the classical CG formulation. Conventional DG formulations are contingent on weak coercivity for their well-posedness. However, at variance with coercivity, weak coercivity does not transfer to subspaces and, consequently, well-posedness of the continuum DG formulation does not generally imply well-posedness of corresponding approximate DG problems. Moreover, we introduce a new nonconventional symmetric DG formulation based on *element Green's functions* that is coercive on the broken Sobolev space $H^1(\mathcal{P}^{\mathsf{h}})$.

3. Existence and uniqueness theorems for linear variational problems.

In this section we review elementary existence theorems pertaining to the well-posedness of linear variational problems. These theorems form the basis for our analysis in section 4 and section 5. Furthermore, a priori error estimates are given for Galerkin approximations.

Section 3.1 is concerned with the generalized Lax–Milgram theorem. This theorem provides the fundament for the classical Lax–Milgram theorem in section 3.2.

3.1. The generalized Lax–Milgram theorem.

The generalized Lax–Milgram theorem gives necessary and sufficient conditions for the well-posedness of a generic linear variational problem. Its proof can be found in [11, 16].[2]

THEOREM 1 (generalized Lax–Milgram). *Let H be a real Hilbert space with corresponding norm $\|\cdot\|_H$. Consider a continuous bilinear form $\mathscr{B} : H \times H \to \mathbb{R}$; i.e., there exists a positive constant c_b such that*

$$|\mathscr{B}(u,v)| \leq c_b \|u\|_H \|v\|_H \qquad \forall u, v \in H \ .$$

[2] The necessity of (3.1) is shown in [16].

If and only if $\mathscr{B}(\cdot,\cdot)$ is weakly coercive on $H \times H$, i.e., there exists a constant $\gamma > 0$ such that

$$\inf_{u\in H\setminus\{0\}} \sup_{v\in H\setminus\{0\}} \frac{\mathscr{B}(u,v)}{\|u\|_H \, \|v\|_H} \geq \gamma \, , \tag{3.1a}$$

$$\sup_{u\in H} \mathscr{B}(u,v) > 0 \qquad \forall v \in H \setminus \{0\} \, , \tag{3.1b}$$

then for every continuous linear functional $\mathscr{L} : H \to \mathbb{R}$, problem (2.6) *has a unique solution $u \in H$.*

Inequality (3.1a) is known as the *inf-sup condition*, and the supremum over all numbers γ in compliance with (3.1a) is referred to as the *inf-sup constant*.

Let $\widehat{H} \subset H$ be a closed subspace associated with an approximate variational problem. As a closed subspace of a Hilbert space is itself a Hilbert space, well-posedness of the approximate problem on $\widehat{H}$ is settled identically by Theorem 1 with H replaced by $\widehat{H}$. The corresponding inf-sup constant $\widehat{\gamma}$ then generally depends on the approximation space $\widehat{H}$, i.e., $\widehat{\gamma} := \widehat{\gamma}(\widehat{H})$. Moreover, if the approximate problem is well-posed, its solution $\widehat{u} \in \widehat{H}$ complies with the a priori estimate

$$\|u - \widehat{u}\|_H \leq (1+c_b/\widehat{\gamma}) \inf_{v\in\widehat{H}} \|u - v\|_H \, . \tag{3.2}$$

It is to be noted that weak coercivity on $H \times H$ does not imply weak coercivity on $\widehat{H} \times \widehat{H}$. Therefore, well-posedness of the continuum problem does not imply well-posedness of corresponding approximate problems. On account of the dependence of $\widehat{\gamma}$ on $\widehat{H}$, it moreover holds that if we consider a sequence of asymptotically dense nested approximation spaces $\widehat{H}^{(1)} \subset \widehat{H}^{(2)} \subset \cdots \subseteq H$, $\widehat{H}^{(m)} \to H$ as $m \to \infty$, then the corresponding approximations $\widehat{u}^{(m)}$ need not converge, or need not converge monotonously.

3.2. The classical Lax–Milgram theorem. A theorem on the well-posedness of linear Galerkin variational problems with more restrictive conditions and stronger implications is provided by the classical Lax–Milgram theorem (see, e.g., [11, 16]).

THEOREM 2 (classical Lax–Milgram). *Let $\mathscr{B} : H \times H \to \mathbb{R}$ be a continuous, (strongly) coercive bilinear form on H; i.e., there exists a positive constant κ such that*

$$|\mathscr{B}(u,u)| \geq \kappa \|u\|_H^2 \qquad \forall u \in H \, . \tag{3.3}$$

Then for every continuous linear functional $\mathscr{L} : H \to \mathbb{R}$, the variational problem (2.6) *possesses a unique solution $u \in H$.*

Coercivity on H is a sufficient condition for weak coercivity on $H \times H$. As coercivity transfers to subspaces $\widehat{H} \subset H$, it holds that well-posedness of the continuum problem implies well-posedness of approximate problems based on conforming subspaces. Moreover, the subspace approximation $\widehat{u} \in \widehat{H}$ satisfies the a priori estimate

$$\|u - \widehat{u}\|_H \leq c_b/\kappa \inf_{v\in\widehat{H}} \|u - v\|_H \, , \tag{3.4}$$

where c_b and κ denote the continuity and coercivity constant of the bilinear form on $H \times H$, respectively. It is to be noted that the constants in (3.4) are independent of the approximation space. Hence, if $\widehat{H}^{(2)} \subset H$ is a larger approximation space than $\widehat{H}^{(1)} \subset \widehat{H}^{(2)}$, then the error (measured in $\|\cdot\|_H$) of the corresponding approximate solution $\widehat{u}^{(2)} \in \widehat{H}^{(2)}$ is at most equal to that of the approximate solution $\widehat{u}^{(1)} \in \widehat{H}^{(1)}$. In particular, this implies that if we consider a sequence of asymptotically dense nested approximation spaces $\widehat{H}^{(1)} \subset \widehat{H}^{(2)} \subset \cdots \subseteq H$ and $\widehat{H}^{(m)} \to H$ as $m \to \infty$, then the error $\|u - \widehat{u}^{(m)}\|_H$ converges monotonously to 0 as m increases.

Let us consider an arbitrary variational continuum problem. Under the condition of coercivity of the bilinear form, conforming approximate problems are well-posed if the continuum problem is well-posed. The premise of coercivity not only provides a sufficient condition; it is also *necessary.*

PROPOSITION 3. *Consider a continuous linear functional $\mathscr{L} : H \to \mathbb{R}$ and continuous bilinear form $\mathscr{B} : H \times H \to \mathbb{R}$. If and only if $\mathscr{B}(\cdot,\cdot)$ is coercive on H, then well-posedness of the continuum problem* (2.6) *implies well-posedness of the approximate problem* (2.7).

Proof. (i) Forward implication: By Theorem 2, coercivity on H ensures that the continuum problem (2.6) and the approximate problem (2.7) are well-posed.

(ii) Reverse implication: We show the proof by contradiction. We assume that $\mathscr{B}(\cdot,\cdot)$ is weakly coercive on $H \times H$, but not coercive, and then construct a subspace $\widehat{H} \subset H$ in which the approximate problem is ill-posed. As H is a closed space, noncoercivity implies the existence of a $\bar{u} \in H \setminus \{0\}$ such that $\mathscr{B}(\bar{u},\bar{u}) = 0$. Taking the approximate space as the one-dimensional space $\widehat{H} = \operatorname{span}\{\bar{u}\}$, it follows that

$$\inf_{u \in \widehat{H}\setminus\{0\}} \sup_{v \in \widehat{H}\setminus\{0\}} \frac{\mathscr{B}(u,v)}{\|u\|_H \|v\|_H} = \frac{\mathscr{B}(\bar{u},\bar{u})}{\|\bar{u}\|_H^2} = 0\ ;$$

i.e., weak coercivity does not hold on $\widehat{H} \times \widehat{H}$. By Theorem 1 weak coercivity on $\widehat{H} \times \widehat{H}$ is necessary for well-posedness of the approximate problem. □

4. Conventional DG formulations.

This section is concerned with an analysis of the generic properties of conventional DG formulations. To this end, we introduce a generic consistent conventional DG formulation in section 4.1. Section 4.2 establishes the existence of well-posed conventional DG formulations. Section 4.3 proves that consistent conventional DG formulations are necessarily noncoercive.

4.1. Generic conventional DG formulation.

Consider the following bilinear form $\mathscr{B}_\Lambda(\cdot,\cdot)$ and linear functional $\mathscr{L}_\Lambda(\cdot)$:[3]

$$\mathscr{B}_\Lambda(u,v) := \sum_{K \in \mathcal{P}^h} \int_K \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \sum_{e \in \Gamma} \left(\boldsymbol{u}^\mathsf{T} \Lambda \boldsymbol{v}\right)_e , \tag{4.1a}$$

$$\mathscr{L}_\Lambda(v) := \int_\Omega f v\,\mathrm{d}x + \sum_{e \in \Gamma_\mathcal{D}} \left(g_\mathcal{D} \bar{\Lambda} \bar{\boldsymbol{v}}\right)_e + \sum_{e \in \Gamma_\mathcal{N}} \left(g_\mathcal{N} \bar{\Lambda} \bar{\boldsymbol{v}}\right)_e , \tag{4.1b}$$

[3]For notational transparency, in a composition of terms with a subscript $(\cdot)_e$, we suppress the subscript of the individual terms and append it to enclosing parentheses. For example, $(g_\mathcal{D}\bar{\Lambda}\bar{\boldsymbol{v}})_e$ is to be interpreted as $g_{\mathcal{D}e}\bar{\Lambda}_e\bar{\boldsymbol{v}}_e$. Moreover, we occasionally suppress the subscript $(\cdot)_e$ entirely if the dependence is apparent from the context.

where boldfaced variables, such as $\boldsymbol{v}$, and boldfaced overlined variables, such as $\bar{\boldsymbol{v}}$, denote (column-) vectors containing values at edge e according to

$$\boldsymbol{v}_e := \begin{cases} \left(h^{-\frac{1}{2}}\,[\![v]\!],\ h^{\frac{1}{2}}\,\{\partial_n v\},\ h^{\frac{1}{2}}\,[\![\partial_n v]\!],\ h^{-\frac{1}{2}}\,\{v\}\right)_e^{\mathsf{T}}\,, & e \in \Gamma_{\mathcal{I}}\,, \\ \left(h^{-\frac{1}{2}}\,v,\ h^{\frac{1}{2}}\,\partial_n v\right)_e^{\mathsf{T}}\,, & e \in \partial\Omega\,, \end{cases} \tag{4.2a}$$

$$\bar{\boldsymbol{v}}_e := \begin{cases} \left(h^{-1}\,v,\ \partial_n v\right)_e^{\mathsf{T}}\,, & e \in \Gamma_{\mathcal{D}}\,, \\ \left(v,\ h\,\partial_n v\right)_e^{\mathsf{T}}\,, & e \in \Gamma_{\mathcal{N}}\,. \end{cases} \tag{4.2b}$$

The matrices $\Lambda_e \in \mathbb{R}^{4\times4}$ ($e \in \Gamma_{\mathcal{I}}$), and $\Lambda_e \in \mathbb{R}^{2\times2}$, $\bar{\Lambda}_e \in \mathbb{R}^{1\times2}$ ($e \in \partial\Omega$) specify bilinear relations between edge values and edge derivatives of u and v. A conventional edge term can now be precisely defined as any term in the bilinear form conforming to $(\boldsymbol{u}^{\mathsf{T}}\Lambda\boldsymbol{v})_e$ for all $e \in \Gamma$, and any term in the linear form conforming to $(g_{\mathcal{D}}\bar{\Lambda}\bar{\boldsymbol{v}})_e$ for $e \in \Gamma_{\mathcal{D}}$ or $(g_{\mathcal{N}}\bar{\Lambda}\bar{\boldsymbol{v}})_e$ for $e \in \Gamma_{\mathcal{N}}$.

The constants $h : \Gamma \to \mathbb{R}$ in (4.2) are local mesh parameters introduced to minimize the mesh dependence of the matrices. Typically, for $e \in \Gamma_{\mathcal{I}}$, h_e is set to the average of the lengths of the elements sharing edge e and for $e \in \partial\Omega$ it is set to half the length of the element contiguous to edge e; i.e.,

$$h_e = \tfrac{1}{2} \sum_{K \in \mathcal{K}_e} h_K \qquad \forall e \in \Gamma\,. \tag{4.3}$$

To provide a functional setting for conventional DG formulations, we introduce the norm $\|\cdot\|_{H_\Lambda}$,

$$\|u\|_{H_\Lambda}^2 := \sum_{K \in \mathcal{P}^{\mathsf{h}}} |u|_{1,K}^2 + \sum_{e \in \Gamma} \left(\boldsymbol{u}^{\mathsf{T}} \mathrm{D}_\Lambda \boldsymbol{u}\right)_e\,, \tag{4.4}$$

where the seminorm $|\cdot|_{1,K}$ is defined by

$$|u|_{1,K}^2 := \int_K \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x\,,$$

and $\mathrm{D}_\Lambda (= \mathrm{D}_{\Lambda_e})$ is a diagonal matrix in $\mathbb{R}^{4\times4}$ for $e \in \Gamma_{\mathcal{I}}$ and in $\mathbb{R}^{2\times2}$ for $e \in \partial\Omega$ with diagonal entries

$$(\mathrm{D}_\Lambda)_{ii} := \textstyle\sum_j \left(\left|(\mathrm{S}_\Lambda)_{ij}\right| + \left|(\mathrm{A}_\Lambda)_{ij}\right|\right) \quad \text{with} \quad \mathrm{S}_\Lambda := \tfrac{1}{2}\left(\Lambda + \Lambda^{\mathsf{T}}\right), \quad \mathrm{A}_\Lambda := \tfrac{1}{2}\left(\Lambda - \Lambda^{\mathsf{T}}\right); \tag{4.5}$$

i.e., D_Λ is obtained from Λ by lumping the absolute values of its symmetric part S_Λ and its antisymmetric part A_Λ to the diagonal.[4] The matrices Λ and D_Λ are then related by

$$\begin{aligned} \left|\boldsymbol{u}^{\mathsf{T}}\Lambda\boldsymbol{v}\right| &= \left|\textstyle\sum_{i,j} \boldsymbol{u}_i (\mathrm{S}_{\Lambda ij} + \mathrm{A}_{\Lambda ij}) \boldsymbol{v}_j\right| \le \textstyle\sum_{i,j} |\boldsymbol{u}_i| \left(|\mathrm{S}_{\Lambda ij}| + |\mathrm{A}_{\Lambda ij}|\right) |\boldsymbol{v}_j| \\ &\le \sqrt{\textstyle\sum_{i,j} (|\mathrm{S}_{\Lambda ij}| + |\mathrm{A}_{\Lambda ij}|) \boldsymbol{u}_i^2} \sqrt{\textstyle\sum_{i,j} (|\mathrm{S}_{\Lambda ij}| + |\mathrm{A}_{\Lambda ij}|) \boldsymbol{v}_j^2} = \sqrt{\boldsymbol{u}^{\mathsf{T}} \mathrm{D}_\Lambda \boldsymbol{u}}\, \sqrt{\boldsymbol{v}^{\mathsf{T}} \mathrm{D}_\Lambda \boldsymbol{v}}\,. \end{aligned} \tag{4.6}$$

[4] Strictly speaking, $\|\cdot\|_{H_\Lambda}$ is a norm only if $(\mathrm{D}_\Lambda)_{11} > 0$ for $e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$. This implies that the bilinear form incorporates $[\![u]\!]_e$ and/or $[\![v]\!]_e$ on $\Gamma_{\mathcal{I}}$ and u_e and/or v_e on $\Gamma_{\mathcal{D}}$. However, in Proposition 4 it will be shown that any consistent formulation necessarily contains such terms. Hence, there is no loss of generality in proceeding under the assumption that $\|\cdot\|_{H_\Lambda}$ provides a norm.

The second inequality in (4.6) follows from the discrete Schwartz inequality, viz., $\sum |x_i y_i| \le \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$. We now define the space H_Λ as the completion of $H^2(\mathcal{P}^\mathsf{h})$ under norm $\|\cdot\|_{H_\Lambda}$:

$$\boxed{H_\Lambda := H_\Lambda(\mathcal{P}^\mathsf{h}) = \overline{H^2(\mathcal{P}^\mathsf{h})}^{\|\cdot\|_{H_\Lambda}} .} \tag{4.7}$$

The Hilbert space H_Λ defined in this manner provides the appropriate space for the generic conventional DG formulation:

$$\boxed{\begin{array}{l}\text{Find } u \in H_\Lambda : \\ \qquad\qquad \mathscr{B}_\Lambda(u,v) = \mathscr{L}_\Lambda(v) \qquad \forall v \in H_\Lambda .\end{array}} \tag{4.8}$$

The appropriateness of H_Λ is rigorously settled in section 4.2. Let us note that H_Λ is a generalization of the space used in [3].

4.2. Well-posedness results for the continuum formulation. Under certain conditions on the matrices Λ and $\bar{\Lambda}$, (4.8) provides a consistent, well-posed weak formulation of (2.3). The proposition below specifies necessary and sufficient conditions on the matrices Λ and $\bar{\Lambda}$ for consistency with (2.3).

PROPOSITION 4 (consistency of conventional DG). *If and only if the matrices* Λ, $\bar{\Lambda}$ *are of the form*

$$\Lambda_e = \begin{pmatrix} \alpha & \delta & \gamma^{\mathrm{u}} & \zeta^1 \\ -1 & 0 & 0 & 0 \\ \gamma^{\mathrm{l}} & \varepsilon & \beta & \zeta^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}_e \qquad \forall e \in \Gamma_{\mathcal{I}} , \tag{4.9a}$$

$$\Lambda_e = \begin{pmatrix} \alpha & \delta \\ -1 & 0 \end{pmatrix}_e , \quad \bar{\Lambda}_e = \begin{pmatrix} \alpha & \delta \end{pmatrix}_e \qquad \forall e \in \Gamma_{\mathcal{D}} , \tag{4.9b}$$

$$\Lambda_e = \begin{pmatrix} 0 & 0 \\ \varepsilon & \beta \end{pmatrix}_e , \quad \bar{\Lambda}_e = \begin{pmatrix} \varepsilon{+}1 & \beta \end{pmatrix}_e \qquad \forall e \in \Gamma_{\mathcal{N}} \tag{4.9c}$$

for certain fixed parameters $\alpha_e, \beta_e, \gamma_e^{\mathrm{u}}, \gamma_e^{\mathrm{l}}, \delta_e, \varepsilon_e, \zeta_e^1, \zeta_e^2 \in \mathbb{R}$ *(for all* $e \in \Gamma$*), then the corresponding conventional DG formulation* (4.8) *is consistent with* (2.3)*; i.e., the solution* $u \in H^2(\Omega) \subset H_\Lambda$ *of* (2.3) *complies with* (4.8).

Proof. (i) Forward implication: Let $u \in H^2(\Omega)$ solve (2.3). Multiplying (2.3a) by an arbitrary $v \in H_\Lambda$, integrating on Ω, and invoking integration by parts, elementwise, we obtain

$$\sum_{K \in \mathcal{P}^\mathsf{h}} \int_K \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{I}}} \big(\partial_n u[\![v]\!]\big)_e + \sum_{e \in \partial\Omega} \big((\partial_n u)v\big)_e . \tag{4.10}$$

From (4.1a) and (4.10) it follows that

$$\mathscr{B}_\Lambda(u,v) = \int_\Omega fv\,\mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{I}}} \big(\partial_n u[\![v]\!]\big)_e + \sum_{e \in \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}}} \big((\partial_n u)v\big)_e + \sum_{e \in \Gamma} \big(\boldsymbol{u}^\mathsf{T}\Lambda\boldsymbol{v}\big)_e . \tag{4.11}$$

The boundary conditions (2.3b) and (2.3c) imply that $u_e = (g_{\mathcal{D}})_e$ for $e \in \Gamma_{\mathcal{D}}$ and $(\partial_n u)_e = (g_{\mathcal{N}})_e$ for $e \in \Gamma_{\mathcal{N}}$. Moreover, on account of the C^1-continuity of functions in $H^2(\Omega)$, the solution u complies with $[\![u]\!]_e = [\![\partial_n u]\!]_e = 0$ and $\{\partial_n u\}_e = (\partial_n u)_e$ for $e \in \Gamma_{\mathcal{I}}$. Hence, upon replacing Λ in (4.11) with (4.9), we obtain

$$\mathscr{B}_\Lambda(u,v) = \int_\Omega fv \, \mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{D}}} \big(\alpha g_{\mathcal{D}} v/h + \delta g_{\mathcal{D}} \partial_n v\big)_e + \sum_{e\in\Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} v + \varepsilon g_{\mathcal{N}} v + \beta h g_{\mathcal{N}} \partial_n v\big)_e \,.$$

By (4.1b) and (4.9), for any $v \in H_\Lambda$,

$$\mathscr{L}_\Lambda(v) = \int_\Omega fv \, \mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{D}}} \big(\alpha g_{\mathcal{D}} v/h + \delta g_{\mathcal{D}} \partial_n v\big)_e + \sum_{e\in\Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} v + \varepsilon g_{\mathcal{N}} v + \beta h g_{\mathcal{N}} \partial_n v\big)_e \,,$$

and, hence, $\mathscr{B}_\Lambda(u,v) = \mathscr{L}_\Lambda(v)$ for all $v \in H_\Lambda$.

(ii) Reverse implication: By (4.1b), (4.8), and (4.11),

$$\sum_{e\in\Gamma_{\mathcal{I}}} \big(\partial_n u [\![v]\!]\big)_e + \sum_{e\in\partial\Omega} \big((\partial_n u) v\big)_e + \sum_{e\in\Gamma} \big(\boldsymbol{u}^\mathsf{T} \Lambda \boldsymbol{v}\big)_e = \sum_{e\in\Gamma_{\mathcal{D}}} \big(g_{\mathcal{D}} \bar{\Lambda} \bar{\boldsymbol{v}}\big)_e + \sum_{e\in\Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} \bar{\Lambda} \bar{\boldsymbol{v}}\big)_e \quad \forall v \in H_\Lambda \,.$$

Upon rearranging the summations, replacing $\boldsymbol{u}_e$ according to its definition (4.2), and invoking the boundary conditions in (2.3) and $[\![u]\!]_e = [\![\partial_n u]\!]_e = 0$, $\{\partial_n u\}_e = (\partial_n u)_e$, and $\{u\}_e = u_e$ for $e \in \Gamma_{\mathcal{I}}$, we obtain

(4.12)
$$\sum_{e\in\Gamma_{\mathcal{I}}} \Big(\partial_n u [\![v]\!] + (0,\ h^{\frac12}\, \partial_n u, 0,\ h^{-\frac12}\, u)\Lambda \boldsymbol{v}\Big)_e + \sum_{e\in\Gamma_{\mathcal{D}}} \Big((\partial_n u) v + (h^{-\frac12}\, g_{\mathcal{D}},\ h^{\frac12}\, \partial_n u)\Lambda \boldsymbol{v}\,\Big)_e + \sum_{e\in\Gamma_{\mathcal{N}}} \Big(g_{\mathcal{N}} v + (h^{-\frac12}\, u,\ h^{\frac12}\, g_{\mathcal{N}})\Lambda \boldsymbol{v}\Big)_e = \sum_{e\in\Gamma_{\mathcal{D}}} \big(g_{\mathcal{D}} \bar{\Lambda} \bar{\boldsymbol{v}}\big)_e + \sum_{e\in\Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} \bar{\Lambda} \bar{\boldsymbol{v}}\big)_e$$

for all $v \in H_\Lambda$. Selecting a $v \in H_\Lambda$ such that $[\![v]\!]_e = 1$ for some edge $e \in \Gamma_{\mathcal{I}}$ and such that all other edge terms vanish, we obtain the identity

$$\Big(\partial_n u + (\partial_n u)\Lambda_{21} + u\Lambda_{41}/h\Big)_e = 0 \,.$$

Therefore, $(\Lambda_{21})_e = -1$ and $(\Lambda_{41})_e = 0$. Similarly, by making appropriate choices for the test function $v \in H_\Lambda$ in (4.12), the precise form (4.9) can be established. ◻

To establish the conditions on the matrices $\Lambda, \bar{\Lambda}$ in (4.9) for well-posedness, we appeal to the generalized Lax–Milgram theorem, Theorem 1. In particular, we establish the conditions for continuity of $\mathscr{L}_\Lambda(\cdot)$, and for continuity and weak coercivity of $\mathscr{B}_\Lambda(\cdot,\cdot)$. Continuity of the bilinear form is in fact independent of the precise form of Λ. This is asserted by the following proposition.

PROPOSITION 5 (continuity of $\mathscr{B}_\Lambda$). *The bilinear form $\mathscr{B}_\Lambda(\cdot,\cdot)$ given in* (4.1a) *is continuous on H_Λ, i.e.,*

$$|\mathscr{B}_\Lambda(u,v)| \le c_b \|u\|_{H_\Lambda} \|v\|_{H_\Lambda} \qquad \forall u, v \in H_\Lambda \,,$$

with continuity constant $c_b = 1$.

Proof. First, note that

$$|\mathscr{B}_\Lambda(u,v)| \le \sum_{K\in\mathcal{P}^h} \int_K \left|\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\right| \mathrm{d}x + \sum_{e\in\Gamma} \big|\boldsymbol{u}^\mathsf{T} \Lambda \boldsymbol{v}\big|_e \,.$$

From the Schwarz inequality and (4.6) it follows that

$$|\mathscr{B}_\Lambda(u,v)| \leq \sum_{K\in\mathcal{P}^{\mathsf{h}}} |u|_{1,K}\,|v|_{1,K} + \sum_{e\in\Gamma}\left(\sqrt{\boldsymbol{u}^{\mathsf{T}}\mathrm{D}_\Lambda\boldsymbol{u}}\sqrt{\boldsymbol{v}^{\mathsf{T}}\mathrm{D}_\Lambda\boldsymbol{v}}\,\right)_e .$$

Application of the discrete Schwarz inequality then yields

$$|\mathscr{B}_\Lambda(u,v)| \leq \left(\sum_{K\in\mathcal{P}^{\mathsf{h}}} |u|^2_{1,K} + \sum_{e\in\Gamma}\left(\boldsymbol{u}^{\mathsf{T}}\mathrm{D}_\Lambda\boldsymbol{u}\right)_e\right)^{1/2}\left(\sum_{K\in\mathcal{P}^{\mathsf{h}}} |v|^2_{1,K} + \sum_{e\in\Gamma}\left(\boldsymbol{v}^{\mathsf{T}}\mathrm{D}_\Lambda\boldsymbol{v}\right)_e\right)^{1/2} . \qquad \square$$

In a similar manner it can be shown that for all $f \in [H_\Lambda]'$, $\mathscr{L}_\Lambda(\cdot)$ is a continuous functional on H_Λ. Hence, it remains to derive the conditions on the matrices Λ_e in (4.9) which yield $\mathscr{B}_\Lambda(\cdot,\cdot)$ weakly coercive on $H_\Lambda \times H_\Lambda$. Sufficient conditions for weak coercivity are established in Proposition 6. As the proof is rather elaborate, it is transferred to Appendix A.

PROPOSITION 6 (weak coercivity of $\mathscr{B}_\Lambda$). *If the parameters in the matrices* Λ_e *in* (4.9) *satisfy the algebraic conditions*

$$\text{(4.13a)}\qquad \left.\begin{array}{l} \alpha\in\mathbb{R},\\ \beta,\gamma^{\mathrm{u}},\gamma^{\mathrm{l}},\delta,\varepsilon\in\mathbb{R}:\\ \qquad \beta,\gamma^{\mathrm{u}},\gamma^{\mathrm{l}},\varepsilon=0 \ \wedge\ 4>|\delta|\neq 0,\\ \textit{or}\quad \delta\beta-\varepsilon\gamma^{\mathrm{u}}\neq 0\ \wedge\ 4>\tfrac{1}{2}|\delta+1|+\tfrac{1}{2}|\delta-1|+|\varepsilon|,\\ \zeta^1,\zeta^2=0 \end{array}\right\}\quad \forall e\in\Gamma_{\mathcal{I}}\,,$$

$$\text{(4.13b)}\qquad \alpha\in\mathbb{R},\quad 4>|\delta|\neq 0 \qquad\qquad \forall e\in\Gamma_{\mathcal{D}}\,,$$

$$\text{(4.13c)}\qquad \beta\in\mathbb{R},\quad \varepsilon=0 \qquad\qquad \forall e\in\Gamma_{\mathcal{N}}\,,$$

then the corresponding bilinear form $\mathscr{B}_\Lambda(\cdot,\cdot)$ *in* (4.1a) *is weakly coercive on* $H_\Lambda\times H_\Lambda$ *with an inf-sup constant* $\gamma_\Lambda > 0$.

Let us note that $\{\delta : 4 > |\delta|\} = \{\delta : 4 > \frac{1}{2}|\delta+1| + \frac{1}{2}|\delta-1|\}$. Proposition 6 generalizes the proof of weak coercivity of the BODG in [3] to any consistent conventional DG formulation. We remark that although the conditions in (4.13) are unrestrictive, they can in fact be further weakened.

In conclusion, by the generalized Lax–Milgram theorem, Theorem 1, if the matrices Λ, $\bar{\Lambda}$ conform to (4.9) and (4.13), then for every $f \in [H_\Lambda]'$ the corresponding conventional DG formulation (4.8) is well-posed and consistent with (2.3). In Table 1, we have summarized the parameter choices for several conventional DG formulations

TABLE 1
Parameters in the matrices Λ, $\bar{\Lambda}$ *in* (4.9) *for several conventional DG formulations.*

DG formulation		α	β	$\gamma^{\mathrm{u}},\gamma^{\mathrm{l}}$	δ	ε	ζ^1,ζ^2
GEM	[10, 12]	0	0	0	−1	0	0
IPDG	[2, 10, 12]	α	0	0	−1	0	0
BODG	[3]	0	0	0	1	0	0
NIPDG	[13]	α	0	0	1	0	0
SDG	[14]	0	β	0	1	0	0
LNDG	[9]	α	0	0	δ	0	0

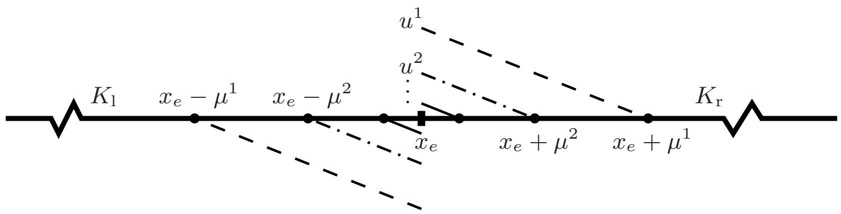

FIG. 1. *Example of a Cauchy sequence* $\{u^i\}$ *in* H_Λ *satisfying* (4.15).

that have appeared in the literature. It can be verified that all formulations, except LNDG, satisfy immediately the conditions in (4.13). LNDG requires the auxiliary condition $4 > |\delta| \neq 0$.

4.3. Noncoercivity of consistent conventional DG formulations. The exposition in section 3 motivates the pursuit of a consistent formulation that is coercive on H_Λ, rather than only weakly coercive. However, the proposition below asserts that a coercive consistent conventional DG formulation is nonexistent; i.e., of the DG formulations in compliance with (4.9), none is coercive.

PROPOSITION 7 (noncoercivity of $\mathscr{B}_\Lambda$). *The bilinear form* $\mathscr{B}_\Lambda(\cdot,\cdot)$ *in* (4.1a) *with* Λ *subject to the consistency requirement* (4.9) *is noncoercive on* H_Λ*; i.e., a positive constant* κ *such that*

$$|\mathscr{B}_\Lambda(u,u)| \geq \kappa \|u\|^2_{H_\Lambda} \qquad \forall u \in H_\Lambda \tag{4.14}$$

is nonexistent.

Proof. We show the existence of a Cauchy sequence $\{u^i\}$ in H_Λ such that $\mathscr{B}_\Lambda(u^i,u^i) \to 0$ and $\|u^i\|_{H_\Lambda} \to c \geq 1$ as $i \to \infty$. Consider an interior edge $e \in \Gamma_\mathcal{I}$ and the left and right elements $K_\mathrm{l}, K_\mathrm{r} \in \mathcal{K}_e$ contiguous to this edge.[5] The Cauchy sequence is chosen such that its elements $u^i \in H_\Lambda$ have local support ($\mathrm{supp}(u^i) \subset \overline{K_\mathrm{l} \cup K_\mathrm{r}}$ with strict inclusion) and, moreover,

$$|u^i|_{1,K_\mathrm{l}}, |u^i|_{1,K_\mathrm{r}} \to 0\ , \tag{4.15a}$$

$$\{u^i\}_e = 0\ ,\ [\![u^i]\!]_e \to 0\ , \tag{4.15b}$$

$$h_e^{\frac{1}{2}}\{u^i_n\}_e = 1\ ,\ [\![u^i_n]\!]_e = 0\ . \tag{4.15c}$$

An example of a sequence satisfying (4.15) is the sequence $u^1, u^2, \ldots$ depicted in Figure 1. The support of u^i is the closed interval in $\mathbb{R}$ with length $2\mu^i$ centered at e. The length of the support set forms a Cauchy sequence $\{\mu^i\}$ in $\mathbb{R}$ with limit $\lim_{i\to\infty} \mu^i = 0$. Moreover, within the support set, u^i is an asymmetric, piecewise linear function.

From the consistency conditions (4.9) on Λ_e and the properties (4.15) of the sequence $\{u^i\}$ it follows that

$$\begin{aligned}\mathscr{B}_\Lambda(u^i,u^i) &= |u^i|^2_{1,K_\mathrm{l}} + |u^i|^2_{1,K_\mathrm{r}} + \left(h^{-\frac{1}{2}}[\![u^i]\!]\ 1\ 0\ 0\right)_e \begin{pmatrix} \alpha & \delta & \gamma^\mathrm{u} & \zeta^1 \\ -1 & 0 & 0 & 0 \\ \gamma^\mathrm{l} & \varepsilon & \beta & \zeta^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}_e \begin{pmatrix} h^{-\frac{1}{2}}[\![u^i]\!] \\ 1 \\ 0 \\ 0 \end{pmatrix}_e \\ &= |u^i|^2_{1,K_\mathrm{l}} + |u^i|^2_{1,K_\mathrm{r}} + \left(\alpha[\![u^i]\!]^2/h + (\delta-1)h^{-\frac{1}{2}}[\![u^i]\!]\right)_e\ ,\end{aligned}$$

[5]If there are no interior edges ($\Gamma_\mathcal{I} = \emptyset$), a proof of noncoercivity can be established similarly by considering a Dirichlet boundary edge.

and, hence, $\mathscr{B}_\Lambda(u^i, u^i) \to 0$ as $i \to \infty$. Furthermore, the norm of u^i reduces to

$$\|u^i\|^2_{H_\Lambda} = |u^i|^2_{1,K_l} + |u^i|^2_{1,K_r} + \left(h^{-\frac{1}{2}}[\![u^i]\!]\ 1\ 0\ 0\right)_e \begin{pmatrix} (\mathrm{D}_\Lambda)_{11} & 0 & 0 & \cdots \\ 0 & (\mathrm{D}_\Lambda)_{22} & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}_e \begin{pmatrix} h^{-\frac{1}{2}}[\![u^i]\!] \\ 1 \\ 0 \\ 0 \end{pmatrix}_e$$
$$= |u^i|^2_{1,K_l} + |u^i|^2_{1,K_r} + \left((\mathrm{D}_\Lambda)_{11}[\![u^i]\!]^2/h + (\mathrm{D}_\Lambda)_{22}\right)_e .$$

Thus, as $i \to \infty$,

$$\|u^i\|^2_{H_\Lambda} \to (\mathrm{D}_\Lambda)_{22} = \left(\tfrac{1}{2}|\delta+1| + \tfrac{1}{2}|\delta-1| + |\varepsilon|\right)_e \geq 1 .$$

The identity follows by replacing $(\mathrm{D}_\Lambda)_{22}$ in accordance with (4.5) and (4.9a). ∎

5. A new symmetric DG formulation with $H^1(\mathcal{P}^\mathsf{h})$-coercivity.

In this section we present a new *nonconventional* coercive symmetric DG formulation based on element Green's functions. Section 5.1 presents the variational formulation. In section 5.2 we establish continuity properties of the corresponding bilinear and linear forms. Finally, in section 5.3 we demonstrate consistency and, most importantly, well-posedness on account of coercivity on $H^1(\mathcal{P}^\mathsf{h})$.

5.1. Weak formulation with element Green's functions.

By Proposition 7, a coercive DG formulation must contain nonconventional edge terms. Below, we present a formulation based on *element Green's functions*.

Consider an edge $e \in \Gamma_\mathcal{I} \cup \Gamma_\mathcal{D}$, and a contiguous element $K \in \mathcal{K}_e$. The two-point boundary of K is denoted by $\partial K = \{e, \bar{e}\}$. With the pair (K, e) we associate a function $\phi_{K,e} : K \to \mathbb{R}$ by the auxiliary boundary-value problem

$$-\frac{\mathrm{d}^2\phi_{K,e}}{\mathrm{d}x^2} = 0 \qquad \text{in } K, \tag{5.1a}$$

$$\phi_{K,e} = \begin{cases} -n_e\, n_K & \text{on } e , \\ 0 & \text{on } \bar{e} , \end{cases} \tag{5.1b}$$

where n_K is the unit outward normal of K. For each edge e and each $K \in \mathcal{K}_e$, the solutions of (5.1) are linear functions on K; see Figure 2. Specifically, $\phi_{K,e}$ corresponds to the element *Dirichlet-to-Neumann Green's function* for the one-dimensional Laplacian. To corroborate this assertion, we multiply (5.1a) with $u \in H^2(K)$ and integrate on K. Upon performing integration by parts twice, and invoking the boundary conditions (5.1b), we obtain

$$(\partial_n u)_e = \int_K -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\phi_{K,e}\,\mathrm{d}x - \sum_{\{e,\bar{e}\}} \left(u\,\frac{\mathrm{d}\phi_K}{\mathrm{d}x}\,n_K\right) ,$$

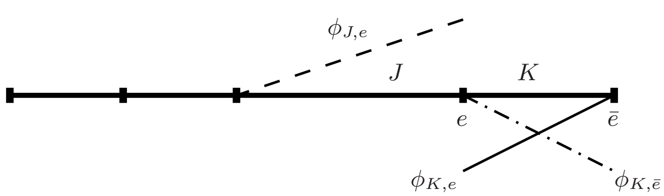

FIG. 2. *Several solutions of auxiliary problem* (5.1).

which shows that the "Neumann value" $\partial_n u$ at e is readily expressed in terms of the Laplacian and "Dirichlet" values of u at e and $\bar{e}$.

For each edge $e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$ we define the functionals $\Phi_e : H^1(\mathcal{P}^{\mathsf{h}}) \to \mathbb{R}$ and $\bar{\Phi}_e : [H^1_{0,\mathcal{D}}(\Omega)]' \to \mathbb{R}$ as

$$\Phi_e(u) := \sum_{K \in \mathcal{K}_e} \theta_{K,e} \int_K \frac{\mathrm{d}u}{\mathrm{d}x} \frac{\mathrm{d}\phi_{K,e}}{\mathrm{d}x} \, \mathrm{d}x \, , \tag{5.2a}$$

$$\bar{\Phi}_e(f) := \sum_{K \in \mathcal{K}_e} \theta_{K,e} \int_K f \phi_{K,e} \, \mathrm{d}x \, . \tag{5.2b}$$

The functionals constitute weighted combinations of contributions of elements that share edge e. The partition-dependent constants $\theta_{K,e} \in \mathbb{R}$ (for $K \in \mathcal{K}_e$) are defined as

$$\theta_{K,e} := h_K \Big/ \sum_{J \in \mathcal{K}_e} h_J \, . \tag{5.3}$$

Trivially, $\theta_{K,e} = 1$ for $e \in \Gamma_{\mathcal{D}}$, $K \in \mathcal{K}_e$. It is to be noted that the following partition-of-unity property holds:

$$\sum_{K \in \mathcal{K}_e} \theta_{K,e} = 1 \qquad \forall e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}} \, . \tag{5.4}$$

Equations (5.2)–(5.4) enable us to condense the new DG formulation into the following variational problem:

$$\boxed{\begin{array}{l} \text{Find } u \in H^1(\mathcal{P}^{\mathsf{h}}) : \\ \qquad\qquad \mathscr{B}_{\Phi}(u,v) = \mathscr{L}_{\Phi}(v) \qquad \forall v \in H^1(\mathcal{P}^{\mathsf{h}}) \, , \end{array}} \tag{5.5}$$

where

$$\begin{aligned} \mathscr{B}_{\Phi}(u,v) := {} & \sum_{K \in \mathcal{P}^{\mathsf{h}}} \int_K \frac{\mathrm{d}u}{\mathrm{d}x} \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{I}}} \Big(\alpha [\![u]\!] [\![v]\!] / h + [\![u]\!] \Phi(v) + \Phi(u) [\![v]\!] \Big)_e \\ & + \sum_{e \in \Gamma_{\mathcal{D}}} \Big(\alpha u v / h + u \Phi(v) + \Phi(u) v \Big)_e \, , \end{aligned} \tag{5.6a}$$

$$\begin{aligned} \mathscr{L}_{\Phi}(v) := {} & \int_{\Omega} f v \, \mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{I}}} \Big(\bar{\Phi}(f) [\![v]\!] \Big)_e \\ & + \sum_{e \in \Gamma_{\mathcal{D}}} \Big(\alpha g_{\mathcal{D}} v / h + g_{\mathcal{D}} \Phi(v) + \bar{\Phi}(f) v \Big)_e + \sum_{e \in \Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} v \big)_e \, . \end{aligned} \tag{5.6b}$$

Note that in a composition of terms with a subscript $(\cdot)_e$, we adhere to the standing notational convention that the subscript of the individual terms is suppressed and appended to the enclosing parenthesis instead.

Let us allude to the fact that the edge terms involving Φ and $\bar{\Phi}$ are nonconventional. The parameters $\alpha_e \in \mathbb{R}$ ($e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$) are associated with conventional edge terms, viz., jumps of u and v at edge e. The rationale for adding these terms is elucidated by the coercivity analysis in section 5.3. The local mesh parameter h_e can in principle be selected in a similar manner as in conventional DG formulations; cf. (4.3). In what follows, we stipulate only that $h_e \leq \frac{1}{2} \sum_{K \in \mathcal{K}_e} h_K$.

5.2. Continuity properties of $\mathscr{B}_\Phi$ and $\mathscr{L}_\Phi$. To facilitate the ensuing analysis, we equip $H^1(\mathcal{P}^{\mathsf{h}})$ with the energy norm $\|\!|\cdot|\!\|$ according to

$$\|\!|u|\!\|^2 := \sum_{K\in\mathcal{P}^{\mathsf{h}}} |u|_{1,K}^2 + \sum_{e\in\Gamma_{\mathcal{I}}} \big([\![u]\!]^2/h\big)_e + \sum_{e\in\Gamma_{\mathcal{D}}} \big(u^2/h\big)_e \,.$$

The norm $\|\!|\cdot|\!\|$ is equivalent to $\|\cdot\|_{H^1(\mathcal{P}^{\mathsf{h}})}$. We then have the following proposition.

PROPOSITION 8 (continuity of $\mathscr{B}_\Phi$). *The bilinear form $\mathscr{B}_\Phi(\cdot,\cdot)$ given in* (5.6a) *is continuous on $H^1(\mathcal{P}^{\mathsf{h}})$, i.e.,*

$$|\mathscr{B}_\Phi(u,v)| \le c_b \|\!|u|\!\|\|\!|v|\!\| \qquad \forall u,v\in H^1(\mathcal{P}^{\mathsf{h}})\,,$$

with, in particular, continuity constant $c_b = \max\big\{2, 1{+}\max_{e\in\Gamma_{\mathcal{I}}\cup\Gamma_{\mathcal{D}}}\alpha_e\big\}$.

Proof. First note that

$$|\mathscr{B}_\Phi(u,v)| \le \sum_{K\in\mathcal{P}^{\mathsf{h}}}\int_K \Big|\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\Big|\,\mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{I}}}\Big(\alpha\big|[\![u]\!][\![v]\!]\big|/h + \big|[\![u]\!]\Phi(v)\big| + \big|\Phi(u)[\![v]\!]\big|\Big)_e \\ + \sum_{e\in\Gamma_{\mathcal{D}}}\Big(\alpha|uv|/h + \big|u\Phi(v)\big| + \big|\Phi(u)v\big|\Big)_e\,.$$

Application of the Schwarz inequality to the first term and subsequent application of the discrete Schwarz inequality yield

$$|\mathscr{B}_\Phi(u,v)| \le \Bigg(\sum_{K\in\mathcal{P}^{\mathsf{h}}}|u|_{1,K}^2 + \sum_{e\in\Gamma_{\mathcal{I}}}\Big((1{+}\alpha)[\![u]\!]^2/h + h\,\Phi(u)^2\Big)_e \\ + \sum_{e\in\Gamma_{\mathcal{D}}}\Big((1{+}\alpha)u^2/h + h\,\Phi(u)^2\Big)_e\Bigg)^{1/2}\Big(\cdots\Big)^{1/2},$$

where the dots $(\cdots)$ represent an identical term with u replaced by v. Moreover, by consecutively applying the inequality

$$\big(\theta x + (1{-}\theta)y\big)^2 \le \theta x^2 + (1{-}\theta)y^2\,, \qquad x,y\in\mathbb{R}\,,\ 0\le\theta\le 1\,,$$

the Schwarz inequality, the identity $|\phi_{K,e}|_{1,K}^2 = 1/h_K$ for the $|\cdot|_{1,K}$ norm of $\phi_{K,e}$, definition (5.3), and $h_e \le \frac{1}{2}\sum_{K\in\mathcal{K}_e} h_K$, we derive the following important inequality:

$$\Phi_e(u)^2 = \Bigg(\sum_{K\in\mathcal{K}_e}\theta_{K,e}\int_K\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\phi_{K,e}}{\mathrm{d}x}\,\mathrm{d}x\Bigg)^2 \le \sum_{K\in\mathcal{K}_e}\theta_{K,e}\bigg(\int_K\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\phi_{K,e}}{\mathrm{d}x}\,\mathrm{d}x\bigg)^2 \\ \le \sum_{K\in\mathcal{K}_e}\theta_{K,e}|u|_{1,K}^2|\phi_{K,e}|_{1,K}^2 \le \sum_{K\in\mathcal{K}_e}\Big(\sum_{J\in\mathcal{K}_e}h_J\Big)^{-1}|u|_{1,K}^2 \le \frac{1}{2h_e}\sum_{K\in\mathcal{K}_e}|u|_{1,K}^2\,. \tag{5.7}$$

Therefore,

$$|\mathscr{B}_\Phi(u,v)| \leq \Bigg(\sum_{K\in\mathcal{P}^\mathsf{h}} |u|^2_{1,K} + \sum_{e\in\Gamma_\mathcal{I}} \Big((1+\alpha)[\![u]\!]^2/h + \tfrac{1}{2}\textstyle\sum_{K\in\mathcal{K}_e} |u|^2_{1,K} \Big)_e + \sum_{e\in\Gamma_\mathcal{D}} \Big((1+\alpha)u^2/h + \tfrac{1}{2}\textstyle\sum_{K\in\mathcal{K}_e} |u|^2_{1,K} \Big)_e \Bigg)^{1/2} \Big(\cdots \Big)^{1/2}$$

$$\leq \Bigg(2\sum_{K\in\mathcal{P}^\mathsf{h}} |u|^2_{1,K} + \sum_{e\in\Gamma_\mathcal{I}} \Big((1+\alpha)[\![u]\!]^2/h \Big)_e + \sum_{e\in\Gamma_\mathcal{D}} \Big((1+\alpha)u^2/h \Big)_e \Bigg)^{1/2} \Big(\cdots \Big)^{1/2} . \qquad \square$$

Before addressing the continuity of the linear form $\mathscr{L}_\Phi(\cdot)$, we introduce a function splitting in $H^1(\mathcal{P}^\mathsf{h})$. For any $v \in H^1(\mathcal{P}^\mathsf{h})$, we define its *discontinuous part* $v^\mathrm{d} := v^\mathrm{d}(v) \in H^1(\mathcal{P}^\mathsf{h})$ as

$$v^\mathrm{d} = \sum_{e\in\Gamma_\mathcal{I}} \Big([\![v]\!]_e \textstyle\sum_{K\in\mathcal{K}_e} \theta_{K,e}\mathfrak{E}_K(-\phi_{K,e}) \Big) + \displaystyle\sum_{e\in\Gamma_\mathcal{D}} \Big(v_e \textstyle\sum_{K\in\mathcal{K}_e} \theta_{K,e}\mathfrak{E}_K(-\phi_{K,e}) \Big) ,$$

where we have introduced the trivial-extension operators $\mathfrak{E}_K : H^1(K) \to H^1(\mathcal{P}^\mathsf{h})$,

$$\mathfrak{E}_K(\phi) = \begin{cases} \phi & \text{in } K , \\ 0 & \text{in } \Omega \setminus K . \end{cases}$$

Note that v^d is an elementwise linear function. The *continuous part* $v^\mathrm{c} := v^\mathrm{c}(v) \in H^1_{0,\mathcal{D}}(\Omega)$ is now defined as the completion of the splitting

$$v^\mathrm{c} = v - v^\mathrm{d} \qquad \forall v \in H^1(\mathcal{P}^\mathsf{h}) . \tag{5.8}$$

To corroborate that v^d and v^c indeed represent the discontinuous and the continuous parts of v, respectively, we note that

$$[\![v^\mathrm{d}]\!]_e = [\![v]\!]_e , \qquad [\![v^\mathrm{c}]\!]_e = 0 \qquad \forall e \in \Gamma_\mathcal{I} , \tag{5.9a}$$

$$v^\mathrm{d}_e = v_e , \qquad v^\mathrm{c}_e = 0 \qquad \forall e \in \Gamma_\mathcal{D} , \tag{5.9b}$$

$$v^\mathrm{d}_e = 0 , \qquad v^\mathrm{c}_e = v_e \qquad \forall e \in \Gamma_\mathcal{N} . \tag{5.9c}$$

In Figure 3, we illustrate for an example function v the corresponding v^d and v^c.

PROPOSITION 9 (continuity of $\mathscr{L}_\Phi$). *For $f \in [H^1_{0,\mathcal{D}}(\Omega)]'$, the linear functional $\mathscr{L}_\Phi(\cdot)$ in* (5.6b) *is continuous on $H^1(\mathcal{P}^\mathsf{h})$.*

Proof. First note that

$$\sum_{e\in\Gamma_\mathcal{I}} \Big(\bar{\Phi}(f)[\![v]\!] \Big)_e + \sum_{e\in\Gamma_\mathcal{D}} \Big(\bar{\Phi}(f)v \Big)_e = \sum_{e\in\Gamma_\mathcal{I}} \Big([\![v]\!]_e \textstyle\sum_{K\in\mathcal{K}_e} \theta_{K,e} \int_K f\phi_{K,e}\,\mathrm{d}x \Big) + \displaystyle\sum_{e\in\Gamma_\mathcal{D}} \Big(v_e \textstyle\sum_{K\in\mathcal{K}_e} \theta_{K,e} \int_K f\phi_{K,e}\,\mathrm{d}x \Big) = \displaystyle\int_\Omega f(-v^\mathrm{d})\,\mathrm{d}x . \tag{5.10}$$

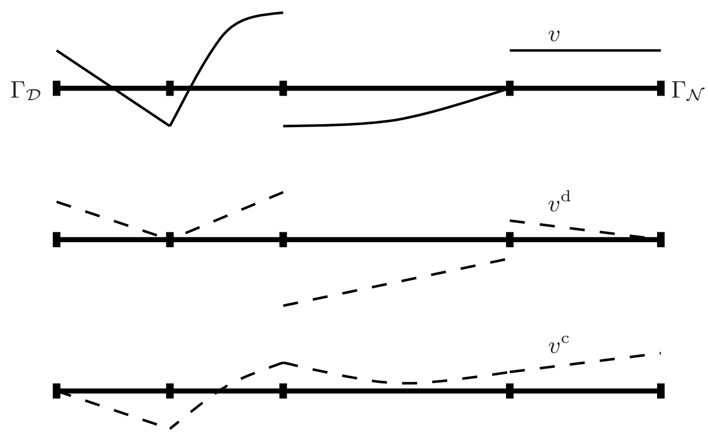

Fig. 3. *Illustration of v^{d} and v^{c} for an example function $v \in H^1(\mathcal{P}^{\mathsf{h}})$ on a domain for which the left boundary is $\Gamma_{\mathcal{D}}$ and the right boundary is $\Gamma_{\mathcal{N}}$.*

As $v - v^{\mathrm{d}} = v^{\mathrm{c}}$, we obtain for $\mathscr{L}_{\Phi}(v)$

$$\mathscr{L}_{\Phi}(v) = \int_{\Omega} f v^{\mathrm{c}}\,\mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{D}}} \Big(\alpha g_{\mathcal{D}} v/h + g_{\mathcal{D}} \Phi(v)\Big)_e + \sum_{e \in \Gamma_{\mathcal{N}}} \big(g_{\mathcal{N}} v\big)_e\,, \tag{5.11}$$

which can be bounded as follows

$$|\mathscr{L}_{\Phi}(v)| \le \left|\int_{\Omega} f v^{\mathrm{c}}\,\mathrm{d}x\right| + \sum_{e \in \Gamma_{\mathcal{D}}} \Big(|g_{\mathcal{D}}|\big(\alpha|v|/h + |\Phi(v)|\big)\Big)_e + \sum_{e \in \Gamma_{\mathcal{N}}} \big(|g_{\mathcal{N}}||v|\big)_e.$$

Since $v^{\mathrm{c}} \in H^1_{0,\mathcal{D}}(\Omega)$, the first term is bounded for $f \in [H^1_{0,\mathcal{D}}(\Omega)]'$. The other terms can also be bounded using (5.7) and the usual trace inequalities. ∎

5.3. Well-posedness results for the continuum formulation. At variance with conventional DG formulations, the new DG formulation is consistent with the more general Poisson problem (2.1).

Proposition 10 (consistency with classical CG formulation). *For all f in the dual space $[H^1_{0,\mathcal{D}}(\Omega)]'$, the DG formulation* (5.5) *is consistent with* (2.1)*, i.e., if $u \in H^1(\Omega)$ is the solution of* (2.1)*, then u satisfies* (5.5).

Proof. Let $u \in H^1(\Omega)$ solve (2.1) and let v be an arbitrary function in $H^1(\mathcal{P}^{\mathsf{h}})$. On account of $[\![u]\!]_e = 0$ for $e \in \Gamma_{\mathcal{I}}$ and $u = g_{\mathcal{D}}$ on $\Gamma_{\mathcal{D}}$, it follows from (5.6a) that

$$\mathscr{B}_{\Phi}(u, v) = \sum_{K \in \mathcal{P}^{\mathsf{h}}} \int_K \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{I}}} \Big(\Phi(u)[\![v]\!]\Big)_e + \sum_{e \in \Gamma_{\mathcal{D}}} \Big(\alpha g_{\mathcal{D}} v/h + g_{\mathcal{D}}\Phi(v) + \Phi(u)v\Big)_e\,.$$

Moreover, in analogy with (5.10), it holds that

$$\sum_{e \in \Gamma_{\mathcal{I}}} \Big(\Phi(u)[\![v]\!]\Big)_e + \sum_{e \in \Gamma_{\mathcal{D}}} \Big(\Phi(u)v\Big)_e = \sum_{K \in \mathcal{P}^{\mathsf{h}}} \int_K \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}(-v^{\mathrm{d}})}{\mathrm{d}x}\,\mathrm{d}x \quad \forall u, v \in H^1(\mathcal{P}^{\mathsf{h}})\,. \tag{5.12}$$

As $v - v^{\mathrm{d}} = v^{\mathrm{c}}$, we obtain

$$\mathscr{B}_{\Phi}(u,v) = \int_{\Omega} \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v^{\mathrm{c}}}{\mathrm{d}x}\,\mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{D}}} \Big(\alpha g_{\mathcal{D}} v/h + g_{\mathcal{D}}\Phi(v)\Big)_e \,,$$

where the sum of integrals is replaced by an integral over Ω, which is admissible because $u \in H^1(\Omega)$ and $v^{\mathrm{c}} \in H^1_{0,\mathcal{D}}(\Omega)$. Recalling from (2.1) that

$$\int_{\Omega} \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v^{\mathrm{c}}}{\mathrm{d}x}\,\mathrm{d}x = \int_{\Omega} f v^{\mathrm{c}}\,\mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{N}}} \left(g_{\mathcal{N}} v^{\mathrm{c}}\right)_e \,,$$

we finally obtain from (5.9c) that

$$\mathscr{B}_{\Phi}(u,v) = \int_{\Omega} f v^{\mathrm{c}}\,\mathrm{d}x + \sum_{e\in\Gamma_{\mathcal{D}}} \Big(\alpha g_{\mathcal{D}} v/h + g_{\mathcal{D}}\Phi(v)\Big)_e + \sum_{e\in\Gamma_{\mathcal{N}}} \left(g_{\mathcal{N}} v\right)_e \,.$$

Hence, $\mathscr{B}_{\Phi}(u,v)$ can be identified with $\mathscr{L}_{\Phi}(v)$ according to (5.11) for all $v \in H^1(\mathcal{P}^{\mathsf{h}})$. ☐

We remark that consistency can be established for any choice of $\theta_{K,e}$ in the operators Φ and $\bar{\Phi}$ in (5.2), provided that the partition-of-unity property (5.4) holds.

A fundamental property of the bilinear form $\mathscr{B}_{\Phi}(\cdot,\cdot)$ in (5.6a) is its *coercivity* on $H^1(\mathcal{P}^{\mathsf{h}})$.

PROPOSITION 11 (coercivity of $\mathscr{B}_{\Phi}$). *If the parameter $\alpha_e > 1$ for all $e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$, then the bilinear form $\mathscr{B}_{\Phi}(\cdot,\cdot)$ in* (5.6a) *is coercive on $H^1(\mathcal{P}^{\mathsf{h}})$, i.e.,*

$$|\mathscr{B}_{\Phi}(u,u)| \geq \kappa |\!|\!| u |\!|\!|^2 \qquad \forall u \in H^1(\mathcal{P}^{\mathsf{h}}) \,,$$

with, in particular, coercivity constant

$$\kappa = \min_{e\in\Gamma_{\mathcal{I}}\cup\Gamma_{\mathcal{D}}} \tfrac{1}{2}\Big((\alpha_e - 1) + 2 - \sqrt{(\alpha_e - 1)^2 + 4}\Big) \in (0,1). \tag{5.13}$$

Note that α_e can be chosen such that κ in (5.13) is bounded away from 0.

Proof. Consider an arbitrary $u \in H^1(\mathcal{P}^{\mathsf{h}})$. We show that there exists a κ in the interval $0 < \kappa < 1$ such that $\mathscr{B}_{\Phi}(u,u) - \kappa |\!|\!| u |\!|\!|^2 \geq 0$. First, we observe that

$$\begin{aligned}\mathscr{B}_{\Phi}(u,u) - \kappa |\!|\!| u |\!|\!|^2 = (1-\kappa) \sum_{K\in\mathcal{P}^{\mathsf{h}}} |u|^2_{1,K} + \sum_{e\in\Gamma_{\mathcal{I}}} \Big((\alpha-\kappa)[\![u]\!]^2/h + 2[\![u]\!]\Phi(u)\Big)_e \\ + \sum_{e\in\Gamma_{\mathcal{D}}} \Big((\alpha-\kappa)u^2/h + 2u\Phi(u)\Big)_e \,.\end{aligned}$$

Application of the Young inequality yields

$$\begin{aligned}\mathscr{B}_{\Phi}(u,u) - \kappa |\!|\!| u |\!|\!|^2 \geq (1-\kappa) &\sum_{K\in\mathcal{P}^{\mathsf{h}}} |u|^2_{1,K} \\ &+ \sum_{e\in\Gamma_{\mathcal{I}}} \left((\alpha-\kappa)[\![u]\!]^2/h - \frac{[\![u]\!]^2}{(1-\kappa)h} - (1-\kappa)h\,\Phi(u)^2\right)_e \\ &+ \sum_{e\in\Gamma_{\mathcal{D}}} \left((\alpha-\kappa)u^2/h \quad - \frac{u^2}{(1-\kappa)h} - (1-\kappa)h\,\Phi(u)^2\right)_e \,.\end{aligned}$$

We now invoke (5.7) to obtain

$$\begin{aligned}\mathscr{B}_\Phi(u,u) - \kappa \|u\|^2 \geq (1-\kappa) \sum_{K\in\mathcal{P}^{\mathsf{h}}} |u|^2_{1,K} \\ + \sum_{e\in\Gamma_{\mathcal{I}}} \Big(\big(\alpha-\kappa-\tfrac{1}{1-\kappa}\big) [\![u]\!]^2/h - \tfrac{1}{2}(1-\kappa) \textstyle\sum_{K\in\mathcal{K}_e} |u|^2_{1,K} \Big)_e \\ + \sum_{e\in\Gamma_{\mathcal{D}}} \Big(\big(\alpha-\kappa-\tfrac{1}{1-\kappa}\big) u^2/h \quad - \tfrac{1}{2}(1-\kappa) \textstyle\sum_{K\in\mathcal{K}_e} |u|^2_{1,K} \Big)_e .\end{aligned}$$

The summations over the elements cancel, except for the contributions of elements contiguous to Neumann boundaries, and, hence,

(5.14)
$$\begin{aligned}\mathscr{B}_\Phi(u,u) - \kappa \|u\|^2 \geq \sum_{e\in\Gamma_{\mathcal{I}}} \Big(\big(\alpha-\kappa-\tfrac{1}{1-\kappa}\big) [\![u]\!]^2/h \Big)_e + \sum_{e\in\Gamma_{\mathcal{D}}} \Big(\big(\alpha-\kappa-\tfrac{1}{1-\kappa}\big) u^2/h \Big)_e \\ + \tfrac{1}{2}(1-\kappa) \sum_{e\in\Gamma_{\mathcal{N}}} \sum_{K\in\mathcal{K}_e} |u|^2_{1,K} .\end{aligned}$$

If $\alpha_e > 1$ for all $e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$ and κ complies with (5.13), then $\alpha_e - \kappa - \frac{1}{1-\kappa} \geq 0$ for all $e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$. Furthermore, for $0 < \kappa < 1$ the final term in the right member of (5.14) is nonnegative and, therefore, $\mathscr{B}_\Phi(u,u) - \kappa \|u\|^2 \geq 0$. ∎

By the classical Lax–Milgram theorem, Theorem 2, we can now conclude that if $\alpha_e > 1$ for all $e \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$, then for all $f \in [H^1_{0,\mathcal{D}}(\Omega)]'$ the new DG formulation (5.5) is well-posed and consistent with (2.1). Moreover, by virtue of the coercivity of the new DG formulation, conforming approximations in $H^1(\mathcal{P}^{\mathsf{h}})$ inherit their well-posedness from the continuum formulation, and optimal error estimates hold with uniformly bounded constants.

6. Numerical experiments. In this section we present numerical results for the new DG formulation. First, we investigate the sharpness of the estimate of the coercivity constant (5.13) by means of discrete inf-sup calculations. Next, we illustrate the optimal convergence behavior of the new formulation in appropriate norms.

6.1. Discrete inf-sup calculations. The estimate of the coercivity constant κ in (5.13) represents a lower bound. That is, a $\bar{\kappa} > \kappa$ possibly exists such that $|\mathscr{B}_\Phi(u,u)| \geq \bar{\kappa} \|u\|^2$ for all $u \in H^1(\mathcal{P}^{\mathsf{h}})$. An upper bound to the coercivity constant can be determined by establishing the discrete coercivity constant, viz., the coercivity constant in a finite-dimensional subspace $\widehat{H} \subset H^1(\mathcal{P}^{\mathsf{h}})$, according to

$$\widehat{\kappa} := \widehat{\kappa}(\widehat{H}) = \inf_{u\in\widehat{H}\setminus\{0\}} \frac{\mathscr{B}_\Phi(u,u)}{\|u\|^2} .$$

For a symmetric bilinear form on a finite-dimensional subspace $\widehat{H}$, the coercivity constant coincides with the discrete inf-sup constant

$$\widehat{\gamma} := \widehat{\gamma}(\widehat{H}) = \inf_{u\in\widehat{H}\setminus\{0\}} \sup_{v\in\widehat{H}\setminus\{0\}} \frac{\mathscr{B}_\Phi(u,v)}{\|u\| \, \|v\|} ,$$

which can be determined numerically by means of the procedure in [10]. Note that the discrete coercivity constants pertaining to a sequence of nested subspaces $\widehat{H}^{(1)} \subset$

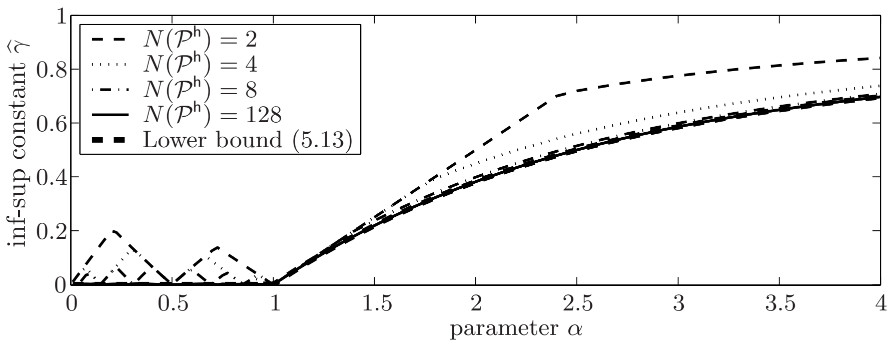

FIG. 4. *Discrete inf-sup constant $\widehat{\gamma}$ versus the parameter α for broken polynomial spaces on uniform partitions with $N(\mathcal{P}^{\mathsf{h}})$ elements. The inf-sup constant $\widehat{\gamma}$ is p independent.*

$\widehat{H}^{(2)} \subset \cdots \subseteq H^1(\mathcal{P}^{\mathsf{h}})$ form a nonincreasing sequence $\widehat{\kappa}^{(1)} \geq \widehat{\kappa}^{(2)} \geq \cdots \geq \bar{\kappa}$. Hence, the discrepancy between the discrete inf-sup constants corresponding to a sequence of nested subspaces and the estimate (5.13) provides a measure of the sharpness of the estimate.

To assess the sharpness of (5.13), we compute the discrete inf-sup constant of the bilinear form in the new DG formulation (5.5) for the Poisson problem on the open unit domain $\Omega = (0,1)$ with Dirichlet boundary conditions, i.e., $\partial\Omega = \Gamma_{\mathcal{D}} = \{0,1\}$. We restrict ourselves to uniform partitions $\mathcal{P}^{\mathsf{h}}$ of $N(\mathcal{P}^{\mathsf{h}})$ elements, and finite-dimensional approximation spaces consisting of broken polynomials with a uniform distribution of the polynomial degree p:

$$\widehat{H} = \mathbb{P}^{\mathsf{p}}(\mathcal{P}^{\mathsf{h}}) := \left\{u \in L_2(\Omega) : u|_K \in \mathbb{P}^{\mathsf{p}}(K)\ \forall K \in \mathcal{P}^{\mathsf{h}}\right\}.$$

Moreover, we use a uniform distribution of the parameter α_e ($=: \alpha$) for all $e \in \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{I}}$.

The results are displayed in Figure 4. In addition, the figure plots the lower bound κ according to (5.13). The numerical results convey that the computed inf-sup constants are independent of the polynomial degree p (results not displayed). They do, however, depend on $N(\mathcal{P}^{\mathsf{h}})$ and α. It appears that for large $N(\mathcal{P}^{\mathsf{h}})$ the discrete inf-sup constants indeed converge to the lower bound and, hence, the estimate of the coercivity constant κ in (5.13) is apparently sharp.

6.2. Error convergence behavior. We consider the new DG formulation for the Poisson problem (5.5) on the open unit domain $\Omega = (0,1)$ with homogeneous Dirichlet boundary conditions. The prescribed data f is selected such that the solution is $u(x) = \sin(\pi x)$. We consider uniform partitions $\mathcal{P}^{\mathsf{h}}$ with $N(\mathcal{P}^{\mathsf{h}})$ elements. The approximation spaces $\widehat{H}$ are the same as used in the inf-sup calculations above.

Figure 5 plots the error in the approximations. The figure indicates that the approximate solutions $\widehat{u}$ are pointwise exact at the interior and boundary edges. This behavior is characteristic for the classical CG method (2.4). Similarly, it can be proven that if $\mathbb{P}^1(\mathcal{P}^{\mathsf{h}}) \subset \widehat{H}$, i.e., if the approximation space contains the piecewise linear functions, then the DG approximation exhibits the same behavior. In Appendix B we elaborate the pointwise exactness for approximations to the new DG formulation (5.5). In particular, the pointwise exactness implies that

$$\widehat{u} \in \widehat{H} \cap \{u \in H^1(\Omega) : u = g_{\mathcal{D}} \text{ on } \Gamma_{\mathcal{D}}\}. \tag{6.1}$$

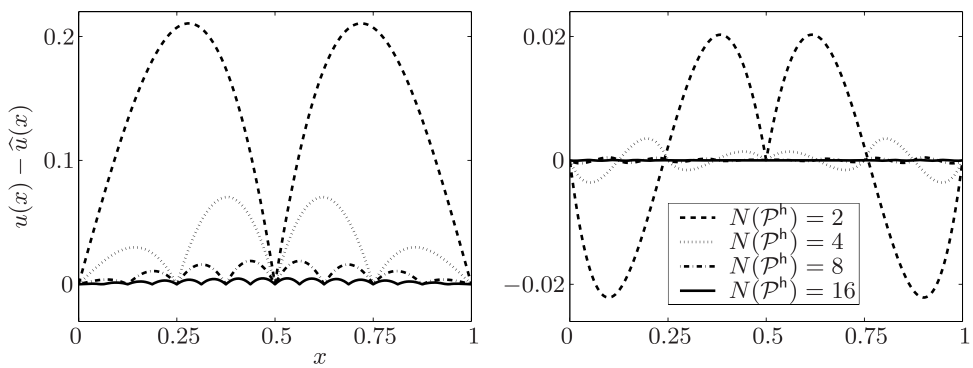

FIG. 5. *Pointwise error for broken polynomial spaces of order* $\mathsf{p} = 1$ *(left) and* $\mathsf{p} = 2$ *(right) on uniform partitions with* $N(\mathcal{P}^\mathsf{h})$ *elements.*

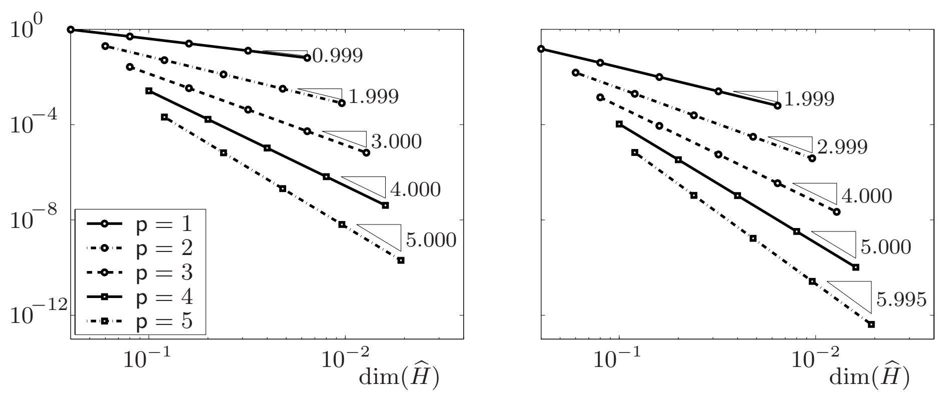

FIG. 6. *Error in the energy-norm (left),* $|||u - \hat{u}|||$*, and in the* $L^2(\Omega)$*-norm (right),* $\|u - \hat{u}\|_{L_2(\Omega)}$*, versus the dimension of the approximation space* $\dim(\widehat{H})$ *for broken polynomial spaces of order* $\mathsf{p} = 1, \ldots, 5$ *on uniform partitions.*

Moreover, $\widehat{u}$ is then identical to the approximate solution of the classical CG formulation (2.4) on $\widehat{H}^1_{0,\mathcal{D}}(\Omega) = \widehat{H} \cap H^1_{0,\mathcal{D}}(\Omega)$ with $\bar{u} \in \mathbb{P}^1(\Omega)$. Another implication is that the approximations are independent of the parameters α_e (provided that $\alpha_e > 1$ so that the approximate problem is well posed), because the terms associated with α_e vanish from the formulation; cf. Eqs. (5.5) and (5.6).

Figure 6 plots the energy-norm and the $L_2(\Omega)$-norm of the error versus the dimension of the approximation space, $\dim(\widehat{H}) := (\mathsf{p}+1)N(\mathcal{P}^\mathsf{h})$, for polynomial orders $\mathsf{p} = 1, 2, \ldots, 5$. The figures corroborate the optimal convergence behavior of the new DG formulation in both norms.

7. Conclusions. We established on the basis of the prototypical Poisson problem that most concurrent DG finite-element methods for second-order elliptic differential equations can be condensed into a generic conventional DG formulation. By means of this generic formulation, we showed that a coercive conventional DG formulation is nonexistent. Conventional DG formulations are contingent on weak coercivity for their well-posedness. However, as weak coercivity does not transfer to subspaces, well-posedness of the continuum problem does not generally imply well-posedness of approximate problems based on conforming subspaces.

We then presented a new nonconventional symmetric DG formulation that is coercive on the broken Sobolev space $H^1(\mathcal{P}^{\mathsf{h}})$. The new formulation is based on element Green's functions and the data local to the edges. On account of its coercivity, conforming approximations of the new formulation inherit their well-posedness from the continuum formulation, and optimal error estimates hold with approximation-space-independent constants. Furthermore, the new DG formulation is consistent with the classical CG formulation in that it admits solutions in $H^1(\mathcal{P}^{\mathsf{h}}) \supset H^1(\Omega)$, rather than $H^2(\mathcal{P}^{\mathsf{h}})$ which is common for conventional DG formulations.

We derived a lower bound for the coercivity constant of the bilinear form in the new formulation. The sharpness of this estimate was confirmed by means of numerical computations of discrete inf-sup constants. Furthermore, numerical experiments were conducted to assess the convergence behavior of the new formulation. The results corroborate that the formulation yields optimal convergence in the energy-norm and in the $L^2(\Omega)$-norm. Moreover, the results demonstrate that discrete approximations in subspaces that contain the piecewise linear functions are identical to classical CG approximations.

It is anticipated that the main attributes of the proposed DG formulation can be extended to higher-dimensional settings. Essentially, the Green's function provides a decomposition of the broken space into the continuous functions and their orthogonal complement. This decomposition can be used to construct a bilinear form that is both consistent and coercive. The generalization of the Green's function to higher dimensions is complex, but there is no fundamental obstacle that precludes such a generalization.

Appendix A. Proof of Proposition 6. The proof is supported by the following lemma.

LEMMA 12. *If there exist a linear continuous operator $v_{(\cdot)} : H_\Lambda \to H_\Lambda$ dependent only on the edge values $\boldsymbol{u}_e$, and a constant $c_1 > \frac{1}{2}$, such that*

$$\big(\Lambda(\boldsymbol{u}+\boldsymbol{v_u})\big)_e = c_1\big(\mathrm{D}_\Lambda \boldsymbol{u}\big)_e\,, \tag{A.1a}$$

$$\tfrac{1}{2}\sum_{K\in\mathcal{P}^{\mathsf{h}}} |v_u|^2_{1,K} \le \sum_{e\in\Gamma}\big(\boldsymbol{u}^{\mathsf{T}}\mathrm{D}_\Lambda \boldsymbol{u}\big)_e\,, \tag{A.1b}$$

$$\|v_u\|_{H_\Lambda} \le c_\Lambda \|u\|_{H_\Lambda} \tag{A.1c}$$

for all $e\in\Gamma$, then $\mathscr{B}_\Lambda(\cdot,\cdot)$ satisfies the inf-sup condition on $H_\Lambda \times H_\Lambda$.

Note that (A.1c) just expresses the continuity of the operator $v_{(\cdot)}$. Bold-faced variables and the matrix D_Λ are defined in (4.2) and (4.5), respectively.

Proof. By the Young inequality and (A.1a) and (A.1b) it holds that

$$\begin{aligned}\mathscr{B}_\Lambda(u,u+v_u) &= \sum_{K\in\mathcal{P}^{\mathsf{h}}}\Big(|u|^2_{1,K} + \textstyle\int_K u' v_u'\,\mathrm{d}x\Big) + \sum_{e\in\Gamma}\big(\boldsymbol{u}^{\mathsf{T}}\Lambda(\boldsymbol{u}+\boldsymbol{v_u})\big)_e \\ &\ge \sum_{K\in\mathcal{P}^{\mathsf{h}}}\Big((1-\tfrac{1}{2\epsilon})|u|^2_{1,K} - \tfrac{\epsilon}{2}|v_u|^2_{1,K}\Big) + c_1\sum_{e\in\Gamma}\big(\boldsymbol{u}^{\mathsf{T}}\mathrm{D}_\Lambda \boldsymbol{u}\big)_e \\ &\ge (1-\tfrac{1}{2\epsilon})\sum_{K\in\mathcal{P}^{\mathsf{h}}}|u|^2_{1,K} + (c_1-\epsilon)\sum_{e\in\Gamma}\big(\boldsymbol{u}^{\mathsf{T}}\mathrm{D}_\Lambda \boldsymbol{u}\big)_e\end{aligned}$$

for all $\epsilon > 0$. Recalling the definition of $\|\cdot\|_{H_\Lambda}$ according to (4.4), we note that for all $c_1 > \frac{1}{2}$ there exists an $\epsilon > \frac{1}{2}$ such that $\mathscr{B}_\Lambda(u,u+v_u) \ge (c_1-\epsilon)\|u\|^2_{H_\Lambda} > 0$. Using this in the inf-sup condition, we obtain

$$\sup_{v\in H_\Lambda\setminus\{0\}} \frac{\mathscr{B}_\Lambda(u,v)}{\|u\|_{H_\Lambda}\|v\|_{H_\Lambda}} \geq \frac{\mathscr{B}_\Lambda(u,u+v_u)}{\|u\|_{H_\Lambda}\|u+v_u\|_{H_\Lambda}} \geq \frac{(c_1-\epsilon)\|u\|^2_{H_\Lambda}}{\|u\|_{H_\Lambda}\big(\|u\|_{H_\Lambda}+\|v_u\|_{H_\Lambda}\big)}$$
$$\geq \frac{c_1-\epsilon}{1+c_\Lambda} > 0\,. \qquad \square$$

To prove that the inf-sup condition (3.1a) holds, we establish that under the conditions (4.13) there exists an operator $v_{(\cdot)} : H_\Lambda \to H_\Lambda$ in compliance with the premises of Lemma 12. The existence is verified by construction. Simple linear algebra conveys that if and only if the parameters in the matrices Λ_e in (4.9) satisfy

$$\left.\begin{aligned} \delta\beta-\varepsilon\gamma^{\mathrm{u}}\neq 0 \quad \text{or} \quad \beta,\gamma^{\mathrm{u}},\gamma^{\mathrm{l}},\delta,\varepsilon=0\\ \zeta^1,\zeta^2=0\end{aligned}\right\}\quad \forall e\in\Gamma_{\mathcal{I}}\,, \tag{A.2a}$$

$$\delta\neq 0 \qquad \forall e\in\Gamma_{\mathcal{D}}\,, \tag{A.2b}$$

$$\varepsilon=0 \qquad \forall e\in\Gamma_{\mathcal{N}}\,, \tag{A.2c}$$

then for each $u\in H_\Lambda$, (A.1a) admits a (nonunique) solution $(\boldsymbol{v}_u)_e$ for any $c_1\in\mathbb{R}$. Thus, (A.1a) yields the values of v_u at the edges $e\in\Gamma$. The kernel of the matrix Λ in (A.1a) accommodates arbitrary $\{v_u\}_e$ for $e\in\Gamma_{\mathcal{I}}$ and arbitrary $(v_u)_e$ for $e\in\Gamma_{\mathcal{N}}$. We set $(v_u)_e=0$ for $e\in\Gamma_{\mathcal{N}}$.

To facilitate the proof, we introduce an auxiliary operator $\bar{v}_{(\cdot)}$ from H_Λ to $\mathbb{P}^1(\mathcal{P}^{\mathsf{h}})$, viz., the space of piecewise linear functions on the partition $\mathcal{P}^{\mathsf{h}}$. The operator $\bar{v}_{(\cdot)}$ associates with each $u\in H_\Lambda$ the function $\bar{v}_u\in\mathbb{P}^1(\mathcal{P}^{\mathsf{h}})$ such that

$$\begin{aligned} [\![\bar{v}_u]\!]_e &= [\![v_u]\!]_e && \forall e\in\Gamma_{\mathcal{I}}\,,\\ \big(\bar{v}_u\big)_e &= \big(v_u\big)_e && \forall e\in\Gamma_{\mathcal{D}}\,,\\ \big(\bar{v}_u\big)_e &= 0 && \forall e\in\Gamma_{\mathcal{N}}\,,\end{aligned}$$

with $[\![v_u]\!]_e$ the previously determined jumps at edges. Specifically, we define $\bar{v}_u$ as

$$\lim_{K\ni x\to e}\big(\bar{v}_u\big|_K\big)(x) = \begin{cases} \frac{1}{2}n_e\, n_K h_K [\![v_u]\!]_e/h_e & \forall e\in\partial K\cap\Gamma_{\mathcal{I}}\,,\\ \frac{1}{2}h_K(v_u)_e/h_e & \forall e\in\partial K\cap\Gamma_{\mathcal{D}}\,,\\ 0 & \forall e\in\partial K\cap\Gamma_{\mathcal{N}}\,.\end{cases} \tag{A.3}$$

We can now define $v_{(\cdot)}$ as the map $u\to v_u$, where v_u is the limit of a Cauchy sequence $\{v_u^i\}$ in H_Λ with the properties

$$\begin{aligned} v_u^i\big|_K &\to \bar{v}_u\big|_K && \text{in } H^1(K) && \forall K\in\mathcal{P}^{\mathsf{h}}\,,\\ [\![\partial_n v_u^i]\!]_e &\to [\![\partial_n v_u]\!]_e && \text{in } \mathbb{R} && \forall e\in\Gamma_{\mathcal{I}}\,,\\ \{\partial_n v_u^i\}_e &\to \{\partial_n v_u\}_e && \text{in } \mathbb{R} && \forall e\in\Gamma_{\mathcal{I}}\,,\\ \big(\partial_n v_u^i\big)_e &\to \big(\partial_n v_u\big)_e && \text{in } \mathbb{R} && \forall e\in\partial\Omega\,,\end{aligned}$$

where $\{\partial_n v_u\}_e$ and $[\![\partial_n v_u]\!]_e$ refer to the previously determined average derivatives and derivative jumps at edges. Such a Cauchy sequence can be constructed in a similar manner as the sequence in the proof of Proposition 7. The operator $v_{(\cdot)}$ thus defined complies with (A.1a).

To ascertain that $v_{(\cdot)}$ satisfies (A.1b), we note that by (4.9) and (A.2), the second equation in the linear system (A.1a) yields

$$\text{(A.4a)}\qquad \big(h^{-\frac{1}{2}}[\![v_u]\!]\big)_e = -\big(h^{-\frac{1}{2}}[\![u]\!] + c_1(D_\Lambda)_{22}\, h^{\frac{1}{2}}\{\partial_n u\}\big)_e \qquad \forall e \in \Gamma_{\mathcal{I}}\ ,$$
$$\text{(A.4b)}\qquad \big(h^{-\frac{1}{2}}v_u\big)_e = -\big(h^{-\frac{1}{2}}u \quad + c_1(D_\Lambda)_{22}\, h^{\frac{1}{2}}\partial_n u \quad\big)_e \qquad \forall e \in \Gamma_{\mathcal{D}}\ ,$$

with, in particular,

$$\text{(A.5)}\qquad 1 \le \big((D_\Lambda)_{22}\big)_e = \begin{cases} \big(\frac{1}{2}|\delta+1| + \frac{1}{2}|\delta-1| + |\varepsilon|\big)_e & \forall e \in \Gamma_{\mathcal{I}}\ ,\\ \big(\frac{1}{2}|\delta+1| + \frac{1}{2}|\delta-1|\big)_e & \forall e \in \Gamma_{\mathcal{D}}\ . \end{cases}$$

Equations (A.3) and (A.4) yield

$$\begin{aligned} \left|\frac{\mathrm{d}\bar{v}_u}{\mathrm{d}x}\Big|_K\right|^2 &= \Big|\sum_{e\in\partial K} \big(\bar{v}_u|_K\, n_K\big)_e/h_K\Big|^2 \le 2\sum_{e\in\partial K}\big(\bar{v}_u|_K\big)_e^2/h_K^2\\ &\le \sum_{e\in\partial K\cap\Gamma_{\mathcal{I}}} \tfrac{1}{2}\Big([\![u]\!]/h + c_1(\mathrm{D}_\Lambda)_{22}\{\partial_n u\}\Big)_e^2 + \sum_{e\in\partial K\cap\Gamma_{\mathcal{D}}} \tfrac{1}{2}\Big(u/h + c_1(\mathrm{D}_\Lambda)_{22}\partial_n u\Big)_e^2 . \end{aligned}$$

From the relation $h_e = \frac{1}{2}\sum_{K\in\mathcal{K}_e} h_K$ in (4.3) it follows that

$$\begin{aligned} \tfrac{1}{2}\sum_{K\in\mathcal{P}^{\mathsf{h}}} |\bar{v}_u|_{1,K}^2 \le \tfrac{1}{2}\sum_{K\in\mathcal{P}^{\mathsf{h}}} h_K\bigg(&\textstyle\sum_{e\in\partial K\cap\Gamma_{\mathcal{I}}}\tfrac{1}{2}\Big([\![u]\!]/h + c_1(\mathrm{D}_\Lambda)_{22}\{\partial_n u\}\Big)_e^2\\ &+ \textstyle\sum_{e\in\partial K\cap\Gamma_{\mathcal{D}}}\tfrac{1}{2}\Big(u/h + c_1(\mathrm{D}_\Lambda)_{22}\partial_n u\Big)_e^2\bigg)\\ \le \sum_{e\in\Gamma_{\mathcal{I}}}&\Big([\![u]\!]^2/h + c_1^2(\mathrm{D}_\Lambda)_{22}^2 h\{\partial_n u\}^2\Big)_e\\ &+ \sum_{e\in\Gamma_{\mathcal{D}}}\Big(u^2/h + c_1^2(\mathrm{D}_\Lambda)_{22}^2 h(\partial_n u)^2\Big)_e\\ \le \max&\Big\{1, c_1^2 \max_{e\in\Gamma_{\mathcal{I}}\cup\Gamma_{\mathcal{D}}}\big((\mathrm{D}_\Lambda)_{22}\big)_e\Big\}\sum_{e\in\Gamma}\big(\boldsymbol{u}^\mathsf{T}\mathrm{D}_\Lambda\boldsymbol{u}\big)_e . \end{aligned}$$

Moreover, under the conditions (4.13) it holds that

$$\begin{aligned} \big(4 > \tfrac{1}{2}|\delta{+}1| + \tfrac{1}{2}|\delta{-}1| + |\varepsilon| \ \wedge\ \varepsilon \ne 0\big) \quad\text{or}\quad \big(4 > |\delta| \ne 0 \ \wedge\ \varepsilon = 0\big) &\qquad \forall e \in \Gamma_{\mathcal{I}}\ ,\\ 4 > |\delta| &\qquad \forall e \in \Gamma_{\mathcal{D}}\ . \end{aligned}$$

Inequality (A.5) then yields $1 \le \max_{e\in\Gamma_{\mathcal{I}}\cup\Gamma_{\mathcal{D}}}((\mathrm{D}_\Lambda)_{22})_e < 4$. As $v_u^i|_K \to \bar{v}_u^i|_K$ in $H^1(\mathcal{P}^{\mathsf{h}})$ as $i\to\infty$, there exists a $c_1 > \frac{1}{2}$ such that $c_1^2 \max_{e\in\Gamma_{\mathcal{I}}\cup\Gamma_{\mathcal{D}}}((\mathrm{D}_\Lambda)_{22})_e < 1$ and hence (A.1b) holds.

To establish (A.1c), we denote by Λ^- the (square) generalized inverse for the matrix Λ in equation (A.1a). We can then write

$$\boldsymbol{v}_{\boldsymbol{u}} = \Lambda^-(c_1\mathrm{D}_\Lambda - \Lambda)\boldsymbol{u}\ ,$$

and, thus,

$$\boldsymbol{v}_{\boldsymbol{u}}{}^\mathsf{T}\mathrm{D}_\Lambda\boldsymbol{v}_{\boldsymbol{u}} = \big\|\mathrm{D}_\Lambda^{\frac{1}{2}}\Lambda^-(c_1\mathrm{D}_\Lambda - \Lambda)\boldsymbol{u}\big\|^2 \le \underbrace{\big\|\mathrm{D}_\Lambda^{\frac{1}{2}}\Lambda^-(c_1\mathrm{D}_\Lambda - \Lambda)\mathrm{D}_\Lambda^{-\frac{1}{2}}\big\|^2}_{=:c_2}\big(\boldsymbol{u}^\mathsf{T}\mathrm{D}_\Lambda\boldsymbol{u}\big)\ ,$$

where $\|\cdot\|$ represents the usual Euclidian vector norm, and the corresponding matrix norm. Condition (A.1c) can then be verified straightforwardly:

$$\|v_u\|^2_{H_\Lambda} = \sum_{K\in\mathcal{P}^{\mathsf{h}}} |v_u|^2_{1,K} + \sum_{e\in\Gamma} \big(\boldsymbol{v_u}^{\mathsf{T}} \mathrm{D}_\Lambda \boldsymbol{v_u}\big)_e \le (2+c_2) \sum_{e\in\Gamma} \big(\boldsymbol{u}^{\mathsf{T}} \mathrm{D}_\Lambda \boldsymbol{u}\big)_e \le (2+c_2)\|u\|^2_{H_\Lambda} \ .$$

The second condition for weak coercivity of $\mathscr{B}_\Lambda(\cdot,\cdot)$, i.e.,

$$\sup_{u\in H_\Lambda} \mathscr{B}_\Lambda(u,v) > 0 \qquad \forall v \in H_\Lambda \setminus \{0\} \ ,$$

is easily established by means of the relation $\mathscr{B}_\Lambda(u,v) = \mathscr{B}_{\Lambda^{\mathsf{T}}}(v,u)$. Under the conditions in (4.13), we can construct an operator $u_{(\cdot)} : H_\Lambda \to H_\Lambda$, in a similar manner as the operator $v_{(\cdot)}$ above, such that

$$\sup_{u\in H_\Lambda} \mathscr{B}_{\Lambda^{\mathsf{T}}}(v,u) \ge \mathscr{B}_{\Lambda^{\mathsf{T}}}(v, v+u_v) > 0 \ .$$

Appendix B. Pointwise exactness of approximations. In this section we establish that the new DG formulation is pointwise exact on all edges $\Gamma = \Gamma_{\mathcal{I}} \cup \partial\Omega$ if the discrete approximation space contains the piecewise linear polynomials, i.e., $\mathbb{P}^1(\mathcal{P}^{\mathsf{h}}) \subseteq \widehat{H} \subset H^1(\mathcal{P}^{\mathsf{h}})$.

Let $\widehat{u} \in \widehat{H}$ be the solution of the approximation problem $\mathscr{B}_\Phi(\widehat{u}, v) = \mathscr{L}_\Phi(v)$ for all $v \in \widehat{H}$. First, we show that the jumps of $\widehat{u}$ are zero and that the Dirichlet boundary traces comply with the Dirichlet boundary condition, i.e.,

$$\text{(B.1)} \qquad [\![\widehat{u}]\!]_e = 0 \quad \forall e \in \Gamma_{\mathcal{I}} \ , \qquad \widehat{u} = g_{\mathcal{D}} \quad \text{on } \Gamma_{\mathcal{D}} \ .$$

Consider an arbitrary edge $\bar{e} \in \Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{D}}$. We construct a discontinuous test function $w = w(\bar{e}) \in \mathbb{P}^1(\mathcal{P}^{\mathsf{h}})$ such that $w^{\mathrm{c}} = 0$, $w^{\mathrm{d}} = w$ (cf. section 5.2 for the splitting $v = v^{\mathrm{c}} + v^{\mathrm{d}}$ into a continuous and a discontinuous part), and

$$\text{(B.2)} \qquad \begin{aligned} \alpha[\![w]\!]/h + \Phi(w) &= 0 && \forall e \in \Gamma_{\mathcal{I}} \setminus \{\bar{e}\} \ , \\ \alpha w/h + \Phi(w) &= 0 && \forall e \in \Gamma_{\mathcal{D}} \setminus \bar{e} \ , \\ \alpha[\![w]\!]/h + \Phi(w) &= 1 && \text{if } \bar{e} \in \Gamma_{\mathcal{I}} \ , \\ \alpha w/h + \Phi(w) &= 1 && \text{if } \bar{e} \in \Gamma_{\mathcal{D}} \ . \end{aligned}$$

It can be shown that the system of equations (B.2) admits a unique solution under the (sufficient) condition $\alpha_e > 1$. This condition is satisfied by assumption; see Proposition 11. As $w \in \mathbb{P}^1(\mathcal{P}^{\mathsf{h}}) \subset \widehat{H}$, it holds that $\mathscr{B}_\Phi(\widehat{u}, w) = \mathscr{L}_\Phi(w)$. From (5.11), (5.12), and $w = 0$ on $\Gamma_{\mathcal{N}}$, it follows that

$$\sum_{e\in\Gamma_{\mathcal{I}}} \Big(\big(\alpha[\![w]\!]/h + \Phi(w)\big) [\![\widehat{u}]\!] \Big)_e + \sum_{e\in\Gamma_{\mathcal{D}}} \Big(\big(\alpha w/h + \Phi(w)\big)\big(\widehat{u} - g_{\mathcal{D}}\big) \Big)_e = 0 \ .$$

Equation (B.1) now follows straightforwardly from the conditions (B.2).

We next establish that $\widehat{u}$ is exact on Neumann edges $\Gamma_{\mathcal{N}}$. Let $e_{\mathcal{N}}$ denote the Neumann edge and $e_{\mathcal{D}}$ the complementary Dirichlet edge. Further, let $\varphi_{\mathcal{N}} \in H^1_{0,\mathcal{D}}(\Omega)$ be the linear function which is $|\Omega|$ at $e_{\mathcal{N}}$ and which vanishes at $e_{\mathcal{D}}$. Using $\varphi_{\mathcal{N}}$ in (2.1), we obtain for the exact solution

$$\text{(B.3)} \qquad u(e_{\mathcal{N}}) = \int_\Omega f \varphi_{\mathcal{N}} \,\mathrm{d}x + g_{\mathcal{D}}(e_{\mathcal{D}}) + |\Omega| g_{\mathcal{N}}(e_{\mathcal{N}}) \ .$$

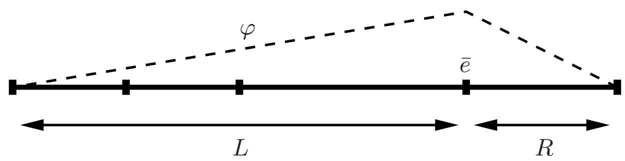

FIG. 7. *Global Green's function φ with respect to edge $\bar{e}$.*

Moreover, as $\varphi_{\mathcal{N}} \in \mathbb{P}^1(\mathcal{P}^{\mathsf{h}}) \subset \widehat{H}$, it holds that $\mathscr{B}_{\Phi}(\widehat{u}, \varphi_{\mathcal{N}}) = \mathscr{L}_{\Phi}(\varphi_{\mathcal{N}})$. On account of $[\![\varphi_{\mathcal{N}}]\!]_e = [\![\widehat{u}]\!]_e = 0$ for $e \in \Gamma_{\mathcal{I}}$, $\varphi_{\mathcal{N}}(e_{\mathcal{D}}) = 0$, and $\widehat{u}(e_{\mathcal{D}}) = g_{\mathcal{D}}(e_{\mathcal{D}})$, this implies

$$\sum_{K \in \mathcal{P}^{\mathsf{h}}} \int_K \frac{\mathrm{d}\widehat{u}}{\mathrm{d}x} \frac{\mathrm{d}\varphi_{\mathcal{N}}}{\mathrm{d}x} \, \mathrm{d}x = \int_{\Omega} f \varphi_{\mathcal{N}} \, \mathrm{d}x + |\Omega| g_{\mathcal{N}}(e_{\mathcal{N}}) \, .$$

The left side evaluates to $\widehat{u}(e_{\mathcal{N}}) - \widehat{u}(e_{\mathcal{D}})$, which is identical to $\widehat{u}(e_{\mathcal{N}}) - g_{\mathcal{D}}(e_{\mathcal{D}})$ by virtue of the previously established coincidence of $\widehat{u}(e_{\mathcal{D}})$ and $g_{\mathcal{D}}(e_{\mathcal{D}})$. We then conclude from (B.3) that $\widehat{u}(e_{\mathcal{N}}) = u(e_{\mathcal{N}})$.

Finally, we establish that $\widehat{u}$ is exact on interior edges $\Gamma_{\mathcal{I}}$. We consider an arbitrary edge $\bar{e} \in \Gamma_{\mathcal{I}}$ and define $L = L(\bar{e})$ and $R = R(\bar{e})$ to be the open subsets of Ω left and right of edge $\bar{e}$; see Figure 7. Furthermore, we define $\varphi = \varphi(\bar{e}) \in H^1_{0,\mathcal{D}}(\Omega)$ to be the global Green's function corresponding to $\bar{e}$, viz., a hat function for which the jump in the derivative at $\bar{e}$ equals -1. Inserting φ in (2.1), we obtain the following relation for the exact solution at edge $\bar{e}$:

$$u(\bar{e}) = \int_{\Omega} f \varphi \, \mathrm{d}x + \sum_{e \in \Gamma_{\mathcal{D}}} \vartheta(e) g_{\mathcal{D}}(e) + \sum_{e \in \Gamma_{\mathcal{N}}} \vartheta(e) u(e) \, , \tag{B.4}$$

where $\vartheta(e) := |R|/|\Omega|$ if e is a left edge, and $\vartheta(e) := |L|/|\Omega|$ if e is a right edge. Moreover, the identity $\mathscr{B}_{\Phi}(\widehat{u}, \varphi) = \mathscr{L}_{\Phi}(\varphi)$ yields

$$\sum_{K \in \mathcal{P}^{\mathsf{h}}} \int_K \frac{\mathrm{d}\widehat{u}}{\mathrm{d}x} \frac{\mathrm{d}\varphi}{\mathrm{d}x} \, \mathrm{d}x = \int_{\Omega} f \varphi \, \mathrm{d}x \, .$$

The left side evaluates to $\widehat{u}(e) - \sum_{e \in \partial\Omega} \vartheta(e) \widehat{u}(e)$. As $\widehat{u}$ is exact on the boundary edges, we finally conclude from (B.4) that $\widehat{u}(e) = u(e)$.

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