Optimal morphs of planar orthogonal drawings

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OPTIMAL MORPHS OF PLANAR ORTHOGONAL DRAWINGS

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ABSTRACT. We describe an algorithm that morphs between two planar orthogonal drawings \( \Gamma_I \) and \( \Gamma_O \) of a graph \( G \), while preserving planarity and orthogonality. Necessarily drawings \( \Gamma_I \) and \( \Gamma_O \) must be equivalent, that is, there exists a homeomorphism of the plane that transforms \( \Gamma_I \) into \( \Gamma_O \). Our morph uses a linear number of linear morphs (linear interpolations between two drawings) and preserves linear complexity throughout the process, thereby answering an open question from Biedl et al. (ACM Transactions on Algorithms, 2013).

Our algorithm first unifies the two drawings to ensure an equal number of (virtual) bends on each edge. We then interpret bends as vertices which form obstacles for so-called wires: horizontal and vertical lines separating the vertices of \( \Gamma_O \). We can find corresponding wires in \( \Gamma_I \) that share topological properties with the wires in \( \Gamma_O \). The structural difference between the two drawings can be captured by the spirality \( s \) of the wires in \( \Gamma_I \), which guides our morph from \( \Gamma_I \) to \( \Gamma_O \). We prove that \( s = O(n) \) and that \( s + 1 \) linear morphs are always sufficient to morph between two planar orthogonal drawings, even for disconnected graphs.

1 Introduction

Continuous morphs of planar drawings have been studied for many years, starting as early as 1944, when Cairns [5] showed that there exists a planarity-preserving continuous morph between any two (compatible) triangulations that have the same outer triangle. These results were extended by Thomassen [9] in 1983, who gave a constructive proof of the fact that two compatible straight-line drawings of any planar graph can be morphed into each other while maintaining planarity. The resulting algorithm to compute such a morph takes exponential time (just as Cairns’ result). Thomassen also considered the orthogonal setting and showed how to morph between two rectilinear polygons with the same turn sequence. For planar straight-line drawings the question was settled by Alamdari et al. [1], following work by Angelini et al. [2]. They showed that \( O(n) \) uni-directional linear morphs are sufficient to morph between any compatible pair of planar straight-line drawings of a graph with \( n \) vertices while preserving planarity. Kleist et al. [7] have shown that the corresponding morph can be found in \( O(n^{1+\frac{\omega}{2}}) \) time, where \( \omega \) is the matrix multiplication exponent.

In this paper we consider the orthogonal setting, that is, we study planarity-preserving morphs between two planar orthogonal drawings \( \Gamma_I \) and \( \Gamma_O \) with maximum complexity \( n \), of

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a graph \( G \). Here the complexity of an orthogonal drawing is defined as the number of vertices and bends. All intermediate drawings must remain orthogonal. This immediately implies that the results of Alamdari et al. [1] do not apply, since they do not preserve orthogonality. Biedl et al. [4] described the first results in this setting, for so-called parallel drawings, where every edge has the same slope in both drawings. They showed how to morph between two parallel drawings using \( O(n) \) linear morphs while maintaining parallelity and planarity. More recently, Biedl et al. [3] showed how to morph between two planar orthogonal drawings using \( O(n^2) \) linear morphs, while preserving planarity, orthogonality, and linear complexity. In this paper we improve this bound further to \( O(n) \) linear morphs. This bound is tight, based on the lower bound for straight-line graphs proven by Alamdari et al. [1].

**Paper Outline.** In Section 2 we give preliminary definitions and explain how to create a unified graph \( G \): we add additional vertices to ensure \( \Gamma_I \) and \( \Gamma_O \) are orthogonal straight-line drawings of the unified graph \( G \). The complexity of \( \Gamma_I \) and \( \Gamma_O \) is still bounded by \( O(n) \) after the unification process.

Our main tool are so-called wires which are introduced in Section 3 (see also Figure 2). Wires capture the horizontal and vertical order of the vertices. Specifically, we consider a set of horizontal and vertical lines that separate the vertices of \( \Gamma_O \). If we consider the vertices of \( \Gamma_O \) as obstacles, then these wires define homotopy classes with respect to the vertices of \( G \) (for the embedding of \( G \) shared by \( \Gamma_I \) and \( \Gamma_O \)). These homotopy classes can be represented by orthogonal polylines (called wires) in \( \Gamma_I \) using orthogonal shortest and lowest paths as defined by Speckmann and Verbeek [8].

Intuitively our morph is simply straightening the wires in \( \Gamma_I \) using the spirality (the difference between the number of left and right turns) of the wires as a guiding principle. In Section 4 we show how this approach leads more or less directly to a linear number of linear morphs. However, the complexity of the intermediate drawings created by this algorithm might increase to \( \Theta(n^3) \).

In the remainder of the paper we show how to “batch” intermediate morphs. We argue solely based on sets of wires, hence the results apply to both connected and disconnected graphs. In particular, in Section 5 we show how to combine all intermediate morphs that act on segments of spirality \( s \) into one single linear morph. Hence we need only \( s \) linear morphs to morph from \( \Gamma_I \) to \( \Gamma_O \). In Section 5 we also show that each linear morph can be performed between two straight-line drawings, thereby reducing intermediate complexity and bounding the overall complexity of all intermediate drawings by \( \Theta(n^2) \). Finally, in Section 6 we further refine the approach to also maintain linear complexity, but it comes at the cost of a single additional linear morph. The final morph preserves planarity, orthogonality, and linear complexity while using only \( s + 1 \) linear morphs.

We implemented our algorithm and believe that the resulting morphs are natural and visually pleasing. A short movie is available online\(^1\). We suggest that the reader first considers the video to form an intuition of the constructed morphs.

\(^1\)See https://youtu.be/JhrgFQTiB5c.
2 Preliminaries

Orthogonal drawings. A drawing $\Gamma$ of a graph $G = (V,E)$ is a mapping from every vertex $v \in V$ to a unique point $\Gamma(v)$ in the Euclidean plane and from each edge $(u,v)$ to a simple curve in the plane starting at $\Gamma(u)$ and ending at $\Gamma(v)$. A drawing is planar if no two curves intersect in an internal point, and no vertices intersect a curve in an internal point. In a straight-line drawing every edge is represented by a single line-segment. A drawing is orthogonal if each edge is mapped to an orthogonal polyline consisting of horizontal and vertical segments meeting at bends. The complexity of an orthogonal drawing is the number of vertices and bends. Two planar drawings $\Gamma$ and $\Gamma'$ are equivalent if there exists a homeomorphism of the plane that transforms $\Gamma$ into $\Gamma'$.

A linear morph of two straight-line drawings $\Gamma$ and $\Gamma'$ can be described by a continuous linear interpolation of all vertices and bends, which are connected by straight segments. For each $0 \leq t \leq 1$ there exists an intermediate drawing $\Gamma_t$ where each vertex $v$ is drawn at $\Gamma_t(v) = (1-t)\Gamma_v + t\Gamma'_v$ ($\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma'$). A linear morph maintains planarity (orthogonality, linear complexity), if every intermediate drawing $\Gamma_t$ is planar (orthogonal, of linear complexity).

Slides. Biedl et al. [3] introduced slides as a particular type of linear morph that operates on the segments of the drawing. A zigzag consists of three consecutive segments with endpoints $\alpha, \beta, \gamma$ and $\delta$, where $\beta$ forms a left turn and $\gamma$ a right turn or vice versa (see Figure 1(a)). Assume for now that $\beta$ is a left turn and $\gamma$ a right turn. Let $\mathcal{V}$ be the set of all vertices and bends of the drawing that are right of or on the ray originating at $\beta$ and passing through $\alpha$, or strictly left of the ray originating at $\gamma$ and passing through $\delta$. We exclude $\gamma$ from $\mathcal{V}$. The corresponding region is shaded in Figure 1. Note that there are no bends and vertices on the vertical segment itself as the drawing is planar. A zigzag-eliminating slide is a linear morph that straightens a zigzag by shifting all vertices and bends in $\mathcal{V}$ by the vector $\gamma - \beta$ (see Figure 1(b)). A zigzag-eliminating slide is a particular linear morph that by construction maintains planarity during the morph.

In the case where $\beta$ is a right turn and $\gamma$ a left turn let $\mathcal{V}$ be the set of all vertices and bends that are left of or on the ray originating at $\beta$ and passing through $\alpha$, or strictly right of the ray originating at $\gamma$ and passing through $\delta$. Once again we exclude $\gamma$ from $\mathcal{V}$.

A bend-creating slide is a linear morph that introduces a zigzag in a horizontal or vertical line (see Figure 1(c)). For similar reasoning it also maintains planarity.

![Figure 1: (a) A horizontal zigzag. (b) A zigzag-eliminating slide is a linear morph that straightens a zigzag. (c) A bend-creating slide is a linear morph that introduces a zigzag.](image-url)
Homotopic paths. Our algorithm relies on the concept of wires separating the vertices of the drawings. Wires are linked up between different drawings via their homotopy classes. We consider the vertices of a drawing as the set of obstacles $B$. Let $\pi_1, \pi_2 : [0, 1] \to \mathbb{R}^2 \setminus B$ be two paths in the plane avoiding the vertices. We say that $\pi_1$ and $\pi_2$ are homotopic ($\pi_1 \sim_h \pi_2$) if they have the same endpoints and there exists a continuous function avoiding $B$ that deforms $\pi_1$ into $\pi_2$. That is, there exists a function $H : [0, 1] \times [0, 1] \to \mathbb{R}^2$ such that

- $H(0, t) = \pi_1(t)$ and $H(1, t) = \pi_2(t)$ for all $0 \leq t \leq 1$.
- $H(s, 0) = \pi_1(0) = \pi_2(0)$ and $H(s, 1) = \pi_1(1) = \pi_2(1)$ for all $0 \leq s \leq 1$.
- $H(\lambda, t) \notin B$ for all $0 \leq \lambda \leq 1, 0 \leq t \leq 1$.

Since the homotopic relation is an equivalence relation, every path belongs to a homotopy class. The geometric intersection number of a pair of paths $\pi_1, \pi_2$ is the minimum number of intersections between any pair of paths homotopic to $\pi_1$, respectively $\pi_2$. Freedman, Hass, and Scott proved the following theorem$^2$.

**Theorem 1** (from [6] Theorem 3.3). Let $M^2$ be a closed, Riemannian 2-manifold, and let $\sigma_1 \subset M^2$ and $\sigma_2 \subset M^2$ be two shortest loops of their respective homotopy classes. If $\pi_1 \sim_h \sigma_1$ and $\pi_2 \sim_h \sigma_2$, then the number of crossings between $\sigma_1$ and $\sigma_2$ is at most the number of crossings between $\pi_1$ and $\pi_2$.

In other words, the number of crossings between two loops of fixed homotopy classes are minimized by the shortest respective loops. This theorem can easily be extended to paths instead of loops, if we can consider the endpoints of the paths as obstacles. For orthogonal paths the shortest path is not uniquely defined, however using lowest paths the theorem still holds. An orthogonal path is lowest if it is shortest with respect to its homotopy class and each staircase subpath is as low as possible (the staircase subpath follows the lower envelope of all homotopic staircase paths between the endpoints). Refer to [8, Lemma 6] for details.

Conventions. Two equivalent drawings $\Gamma$ and $\Gamma'$ may not have the same complexity (number of vertices and bends) as the orthogonal polylines in $\Gamma$ and $\Gamma'$ representing the same edge may have a different number of segments. To simplify the discussion of our algorithm, we first ensure that each edge has the same number of segments in $\Gamma$ and $\Gamma'$. This can be achieved by subdividing segments, creating additional virtual bends. Next, we replace all bends by vertices. As a result, all edges of the graph are represented by straight segments in both $\Gamma$ and $\Gamma'$. We call the resulting graph the unification of $\Gamma$ and $\Gamma'$. If the maximum complexity of $\Gamma$ and $\Gamma'$ is $O(n)$ then clearly the complexity of the unification of $\Gamma$ and $\Gamma'$ is $O(n)$.

We say that two planar drawings $\Gamma$ and $\Gamma'$ of a unified graph are similar if the horizontal and vertical order of the vertices is the same in both drawings.

**Observation 1.** A planar orthogonal drawing can be morphed to a similar planar orthogonal drawing using a single linear morph while maintaining planarity.

**Proof.** We can introduce a planarity violation only if two vertices swap in the horizontal or vertical order, which cannot happen during a linear morph between two similar drawings. $\square$

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$^2$Reformulated (and simplified) to suit our notation rather than the more involved notation in [6].
3 Wires

In the following we show how to morph an orthogonal planar drawing $\Gamma_I$ of $G = (V, E)$ to another orthogonal planar drawing $\Gamma_O$ of $G$ while maintaining planarity and orthogonality. We assume that $\Gamma_I$ and $\Gamma_O$ are equivalent, that $G$ is the unification of $\Gamma_I$ and $\Gamma_O$, and that $G$ contains $n$ vertices. To morph $\Gamma_I$ to $\Gamma_O$, our main strategy is to first make $\Gamma_I$ similar to $\Gamma_O$, after which we can morph $\Gamma_I$ to $\Gamma_O$ using a single linear morph (by Observation 1).

To capture the difference between $\Gamma_I$ and $\Gamma_O$ we use two sets of wires. The lr-wires $W_{\rightarrow}$, going from left to right through the drawings, capture the vertical order of the vertices in comparison to the vertical order in $\Gamma_O$. The tb-wires $W_{\downarrow}$, going from top to bottom through the drawings, capture the horizontal order of the vertices with respect to $\Gamma_O$.

3.1 Basic properties

Since we want to match the horizontal and vertical order of vertices in $\Gamma_O$, the wires $W_{\rightarrow}$ and $W_{\downarrow}$ are simply horizontal and vertical lines in $\Gamma_O$, respectively, separating any two consecutive coordinates used by vertices (see Figure 2(a)). Now assume that we have a planar orthogonal morph from $\Gamma_I$ to $\Gamma_O$ (the existence of such a morph follows from [3]). If we were to apply this morph in the reverse direction on the wires in $\Gamma_O$, we end up with another set of wires in $\Gamma_I$ with the following properties (see Figure 2(b)):

P1 Two wires are non-crossing if they both belong to $W_{\rightarrow}$ or $W_{\downarrow}$ and cross exactly once otherwise.

P2 The order of the wires in $W_{\rightarrow}$ ($W_{\downarrow}$) is the same as in $\Gamma_O$ and the same vertices are between consecutive wires.

P3 The wires cross exactly the same sequence of edges and links as in $\Gamma_O$.

These properties follow directly from the fact that a planar morph cannot introduce or remove any crossings, and thus these properties are invariant under planar morphs. We refer to a set of wires in $\Gamma_I$ that has the above properties as an equivalent set of wires (to

[Figure 2: (a) The set $W_{\rightarrow}$ of lr-wires (red) and the set $W_{\downarrow}$ of tb-wires (blue) in the output drawing $\Gamma_O$. (b) Equivalent wires in the input drawing $\Gamma_I$.]
the wires in $\Gamma_O$). Interestingly, any equivalent set of wires can be used to construct a planar morph from $\Gamma_I$ to $\Gamma_O$. We first use a planar morph to straighten the wires. Then, by $P1$ and $P2$, for the resulting drawing $\Gamma$ the wires for a grid where each cell contains at most one vertex. By $P3$ the wires also intersect the same sequence of edges and links as in $\Gamma_O$. As all wires are straight-line we conclude that $\Gamma$ is similar to $\Gamma_O$. Hence, we can eliminate all bends in a single morph by combining individual morphs per cell. For each cell we morph all bends (and the vertex) to the center of the cell. The resulting drawing is similar to $\Gamma_O$ and has no bends, and thus we can finish the planar morph with a single linear morph.

In the following we assume that we are given an equivalent set of wires in $\Gamma_I$ to the set of straight-line wires in $\Gamma_O$. To keep the distinction between wires and edges clear, we refer to the horizontal and vertical segments of wires as links. It directly follows that every set of wires in $\Gamma_I$ equivalent to straight-line wires in $\Gamma_O$ fulfills the following properties:

**Observation 2.**

- Every wire crosses an edge at most once.
- Every edge is crossed by wires from either $W_\downarrow$ or $W_\rightarrow$.
- All wires crossing an edge cross it in the same direction and the order of intersections matches the horizontal (vertical) order of the wires in $\Gamma_O$.

**Proof.** As edges are straight lines in $\Gamma_O$, each edge can only be crossed once by a wire. As edges are straight-line segments and wires straight lines in $\Gamma_O$, each edge is either crossed by wires from $\Gamma_\downarrow$ or wires from $\Gamma_\rightarrow$ and every wire crosses the edge in the same direction. As equivalent wires in $\Gamma_I$ cross the same sequence of edges and separate the vertices identically (and wires from the same set are non-crossing) the intersection of the edges in $\Gamma_I$ and $\Gamma_O$ must be identical. □

Without loss of generality we also make two assumptions to simplify the exposition.

**Assumption 1.** No two links overlap in more than a single point in $\Gamma_I$.

If two links overlap in a segment along both links then by carefully $\varepsilon$-perturbing the links we may prevent the overlap.

**Assumption 2.** A bounding box surrounds the drawing in both $\Gamma_I$ and $\Gamma_O$ and each wire starts (ends) on this bounding box at the same place in the same orientation in $\Gamma_I$ and $\Gamma_O$.

Our goal is to straighten the wires. As even a single wire in $\Gamma_I$ may have $\Omega(n^2)$ links (see Figure 3), it is inefficient to straighten the wires one link at a time. Instead we consider the spirality of the wires. For a wire $w \in W_\downarrow$, let $\ell_1 \ldots \ell_k$ be the links of $w$ in order from left to right. Let $b_i$ be the orientation of the bend between $\ell_i$ and $\ell_{i+1}$, where $b_i = 1$ for a left turn, $b_i = -1$ for a right turn, and $b_i = 0$ otherwise. The **spirality** of a link $\ell_i$ is defined as $s(\ell_i) = \sum_{j=1}^{i-1} b_j$. By definition the spirality of $\ell_1$ is 0, and by construction the spirality of $\ell_k$ is also 0. The **spirality** of a wire is defined as the maximum absolute spirality over all its links. The spirality of wires in $W_\downarrow$ is defined analogously, going from top to bottom.
The complexity of a wire can be $\Omega(n^2)$. To satisfy property $P3$, the wire must spiral through the same polygon a linear number of times. The spirality is still $O(n)$.

Spirality is closely related to the number of convex and reflex corners in a simple closed orthogonal curve. Particularly, if we traverse a wire as part of the boundary of a closed orthogonal curve then for each convex corner (left turn) the spirality of the link traversed increases by one and for each reflex corner (right turn) the spirality decreases by one. Thus, when we trace a wire from a link with spirality $s$ to a link with spirality $t$, then the curve has $t - s$ more convex than reflex corners along the traced segment. Symmetrically, if we traverse a wire backwards starting from a link with spirality $s'$ to a link with spirality $t'$, then the curve has $s' - t'$ more convex than reflex corners. We repeatedly use this fact together with the fact that a simple closed orthogonal curve has four more convex corners than reflex corners, to prove the spirality of links in the wires.

**Lemma 1.** If a wire $w \in W_\to$ and a wire $w' \in W_\downarrow$ cross in links $\ell_i$ and $\ell'_j$, then $s(\ell_i) = s(\ell'_j)$.

**Proof.** By property $P1$ $w$ and $w'$ cross exactly once. Consider the bounding box $B$ that contains the complete drawing and intersects $w$ and $w'$ in $\ell_1$ respectively $\ell'_1$. The wires $w$ and $w'$ subdivide $B$ into four simple faces (see Figure 4). Consider the top-left face. Since the face is simple and orthogonal, it contains four more convex corners than reflex corners. Two convex corners are at the intersection of $w$ and $w'$ with $B$, and one is a corner of $B$.

By definition the spirality of $\ell_1$ and $\ell'_1$ is zero. Thus if the spirality of $\ell_i$ is $x$, then $w$ contains $x$ more convex than reflex corners. Symmetrically if the spirality of $\ell_j$ is $y$, then $w'$ contains $y$ less convex than reflex corners. As the intersection of $\ell_i$ and $\ell'_j$ also forms a convex corner, using a double-counting argument we get that $x - y + 4 = 4$. But then $x = y$. \qed

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**Figure 3:** The complexity of a wire can be $\Omega(n^2)$. To satisfy property $P3$, the wire must spiral through the same polygon a linear number of times. The spirality is still $O(n)$.

**Figure 4:** As $\ell_1$ and $\ell'_1$ have spirality zero and a closed orthogonal curve has four more convex than reflex corners, the crossing links of $w$ and $w'$ have equal spirality.

**Figure 5:** As $\ell_1$ and $\ell'_1$ have spirality zero and $w$ and $w'$ must have equally many more left turns than right turns, the two links crossing $\sigma$ have the same spirality.
Lemma 2. All links intersecting the same segment of an edge have the same spirality.

Proof. Consider a segment $\sigma$ of an edge $e$. Assume without loss of generality that $e$ is horizontal in $\Gamma_O$. Only wires from $W_\downarrow$ intersect $e$ (and thus $\sigma$) and all wires cross $e$ ($\sigma$) in the same direction. Consider two consecutive wires $w, w' \in W_\downarrow$ intersecting $\sigma$. Let $B$ be the bounding box of the drawing intersecting $w$ and $w'$ at $\ell_1$ respectively $\ell'_1$. By definition the spirality of $\ell_1$ and $\ell'_1$ is zero. The area enclosed by $w, w', \sigma$, and $B$ between the intersection with $\ell_1$ and $\ell'_1$ forms a simple closed orthogonal curve (see Figure 5). A counter-clockwise tour of the curve has two convex turns at $\sigma$ and two convex turns at $B$, all remaining turns are caused by $w$ and $w'$. If $x$ and $x'$ are the spiralities of $w$ and $w'$ when intersecting $\sigma$, then using the fact that the polygon has four more convex than reflex corners and a double-counting argument similar to Lemma 1 we get $x + 2 - x' + 2 = 4$, and thus $x = x'$. □

3.2 Spirality bound

In Section 4 we show that we can straighten a set of wires using only $O(k)$ linear morphs, if the spirality of each wire is bounded by $k$. It is therefore pertinent to bound the spirality of an equivalent set of wires. Let $\Gamma_I$ and $\Gamma_O$ be two equivalent planar orthogonal drawings of a graph $G$. We show that we can find an equivalent set of wires in $\Gamma_I$ with spirality $O(n)$ with respect to the straight-line wire-grid in $\Gamma_O$.

Below we show that we can choose homotopy classes for the wires in $\Gamma_I$ incrementally, first for the lr-wires and then for the tb-wires, while maintaining the correct intersection pattern and hence equivalence with $\Gamma_O$. For each of the resulting equivalence classes we add the shortest wire to the set of wires. It remains to argue that the resulting set of wires has spirality $O(n)$, despite the interdependence of the homotopy classes and despite the fact that the arrangement of drawing and wires can have super-linear complexity. We consider only wires in $W_\rightarrow$ and links with positive spirality. Structurally identical symmetric results for wires from $W_\downarrow$ and for links with negative spirality can be setup. For a wire $w$ let $w[\ell_i]$ be the partial wire consisting of links $\ell_1, \ldots, \ell_i$.

Lemma 3. Let $\ell_i$ be a horizontal link of a wire $w \in W_\rightarrow$ with even spirality, and let $L$ be a vertical line crossing $\ell_i$. The lowest link of $w[\ell_i]$ crossing $L$ has spirality 0 or $-2$ and the highest link of $w[\ell_i]$ crossing $L$ has spirality 0 or 2.

Proof. Let $\ell_l$ be the lowest link from $w[\ell_i]$ crossing $L$. We can create a simple closed counter-clockwise orthogonal curve (see Figure 6) by (1) first going down from $\ell_l$ along the bounding box, (2) going right along the bounding box until reaching $L$, (3) going up until reaching $\ell_l$, and (4) following $w[\ell_i]$ backwards until reaching the starting point of the curve. The curve has four more convex than reflex corners, and it contains 3 convex corners by construction. Let $x$ be the contribution of the turn at $\ell_l$ (which can be left (+1) or right (−1)). Then we get that $3 + x - s(\ell_l) = 4$, so $s(\ell_l) = x - 1$, directly implying $s(\ell_l) \in \{0, -2\}$.

Let $\ell_h$ be the highest link from $w[\ell_i]$ crossing $L$. We create a simple closed counter-clockwise orthogonal curve by (1) following the wire from $\ell_l$ to $\ell_h$, (2) going up until the bounding box, (3) going left to the corner of the bounding box, (4) going down to $\ell_l$. Let $x$ be the contribution of the turn at $\ell_h$. We get $s(\ell_h) + x + 3 = 4$, implying $s(\ell_h) \in \{0, 2\}$. □
**Lemma 4.** Let $\ell_i$ be a left-oriented link with spirality $s$ of a wire $w \in W_\leftarrow$ and $L$ be a vertical line crossing $\ell_i$. If $w[\ell_i]$ (the wire upto $\ell_i$) crosses $L$ below $\ell_i$ then the highest link from $w[\ell_i]$ that crosses $L$ below $\ell_i$ has spirality $s - 2$ or $s + 4$. If $w[\ell_i]$ crosses $L$ above $\ell_i$ then the lowest link from $w[\ell_i]$ that crosses $L$ above $\ell_i$ has spirality $s - 4$ or $s + 2$.

**Proof.** Let $\ell_h$ be the highest (horizontal) link from $w[\ell_i]$ that crosses $L$ lower than $\ell_i$. Thus $w[\ell_i]$ (and moreover $w[\ell_h]$) cannot cross through $L$ between $\ell_i$ and $\ell_h$. But then the sub-wire from $\ell_h$ to $\ell_i$ cannot enclose the origin of $\ell_h$ as this would force $w[\ell_h]$ to intersect $L$ between $\ell_i$ and $\ell_h$ (see Figure 7(a)). Thus if $\ell_h$ is right-oriented then the wire from $\ell_h$ to $\ell_i$ followed by the segment along $L$ from the intersection with $\ell_i$ to the intersection with $\ell_h$ must form a simple closed counter-clockwise orthogonal curve (see Figure 7(b)). As this curve has two left turns at $L$, there must be two more left turns than right turns along the wire from $\ell_h$ to $\ell_i$, leading to $s(\ell_h) = s(\ell_i) - 2$. If $\ell_h$ is left-oriented, then the segment along $L$ from the intersection with $\ell_h$ to the intersection with $\ell_i$ followed by the wire backwards from $\ell_i$ to $\ell_h$ forms a simple closed orthogonal curve and we get $s(\ell_h) = s(\ell_i) + 4$ (see Figure 7(c)).

Symmetrically for the lowest link $\ell_i$ from $w[\ell_i]$ crossing $L$ above $\ell_i$ it follows through a case distinction that if $\ell_i$ is left-oriented then $s(\ell_i) = s(\ell_i) - 4$ and if $\ell_i$ is right-oriented then $s(\ell_i) = s(\ell_i) + 2$ (see Figure 7(d)).

**Lemma 5.** Let $\ell_r$ be a right-oriented link with spirality $s$ of a wire $w \in W_\rightarrow$ and $L$ be a vertical line crossing $\ell_r$. If $w[\ell_r]$ crosses $L$ below $\ell_r$ then the highest link from $w[\ell_r]$ that crosses $L$ below $\ell_r$ has spirality $s + 2$ or $s - 4$. If $w[\ell_r]$ crosses $L$ above $\ell_r$ then the lowest link from $w[\ell_r]$ that crosses $L$ above $\ell_r$ has spirality $s + 4$ or $s - 2$.

**Proof.** Using a similar case distinction as Lemma 4 but mirroring the cases.
Lemma 6. For each right-oriented link $\ell \rightarrow$ of a wire $w \in W_{\rightarrow}$ with spirality $s > 0$ and each vertical line $L$ crossing $\ell \rightarrow$, there exists a subsequence $S$ of $\Omega(s)$ links of $w$ crossing $L$, such that the spiralities of the links in sequence are $(0, 2, 4, \ldots, s - 2, s)$, and when ordered top-to-bottom along $L$ form the sequence $(2, 6, 10, \ldots, s - 2, s, s - 4, \ldots, 4, 0)$.

Proof. Consider a vertical line $L$ through $\ell \rightarrow$. We inductively find the desired subsequence $S$ of $w$ by constructing it in reverse order starting from $\ell \rightarrow$. Let $\ell^i \in S$ be the unique link with spirality $0 \leq i \leq s$ in $S$. We maintain the following stronger induction hypothesis for $0 \leq t \leq s$ and $t \pmod{2} = 0$. The first two sub-hypotheses form our claim.

IH1 A subsequence $S_t$ of $w$ exists where the links have spiralities $(t, t + 2, \ldots, s - 2, s)$.

IH2 When the links from $S_t$ are ordered top-to-bottom along $L$ the resulting sequence is a monotone increasing sequence of all left-oriented links (with spirality $s(\ell_j) \pmod{4} = 2$), followed by a monotone decreasing sequence of all right-oriented links (with spirality $s(\ell_j) \pmod{4} = 0$).

(Specifically this gives rise to the sequence $(t + 2, t + 6, \ldots, s - 2, s, s - 4, \ldots, 4)$ when $t \pmod{4} = 0$ or $(t, t + 4, t + 8, \ldots, s - 2, s, s - 2, s, s - 4, \ldots, 4)$ when $t \pmod{4} = 2$)

IH3 For every two links $\ell^u$ and $\ell^{u-2}$ from $S_t$, where $u \pmod{4} = 2$ and $t < u \leq s$, link $\ell^{u-2}$ is the highest link with spirality $u - 2$ from $w[\ell^u]$ that crosses $L$ below $\ell^u$.

IH4 For every two links $\ell^u$ and $\ell^{u-2}$ from $S_t$, where $u \pmod{4} = 0$ and $t < u \leq s$, link $\ell^{u-2}$ is the lowest link with spirality $u - 2$ from $w[\ell^u]$ that crosses $L$ above $\ell^u$.

In the base case we use $S_s = (\ell_{\rightarrow})$ which trivially satisfies all requirements. For the inductive step we may assume a subsequence $S_t = (\ell^i, \ldots, \ell_{\rightarrow})$ of $w$ exists meeting the requirements.

Case 1 $\ell^t$ is left-oriented: $(t \pmod{4} = 2)$

As $\ell^t$ has spirality $t \pmod{4} = 2$, by IH2 link $\ell^t$ is part of the initial monotone increasing subsequence in $S_t$ when ordered top-to-bottom along $L$. Moreover, also by IH2, as $\ell^t$ has the smallest spirality from all selected links in $S_t$, link $\ell^t$ must be the first link in $S_t$ and hence be the highest link that crosses $L$. We show that (1) a link from $w[\ell^t]$ that crosses $L$ below $\ell^t$ and that has spirality $t - 2$ exists and (2) the highest such link can be added to $S_t$ while maintaining all properties of the induction hypothesis.

We first prove that a link with the desired properties exists (1). Starting with $\ell_i = \ell^t$ we repeat the following search. Find the highest (horizontal) link from $w[\ell_i]$ that crosses $L$ lower than $\ell_i$. By Lemma 3 such a link must exist while $s(\ell_i) \geq 2$. Let this link be $\ell_j$. If $s(\ell_j) \neq t - 2$ then repeat the search with $\ell_i = \ell_j$. If $s(\ell_j) = t - 2$ then, as $\ell_j \in w[\ell_i] \subset w[\ell^t]$, link $\ell_j$ has the desired properties. It remains to show that such a link must be found.

Assume for contradiction that no link of spirality $t - 2$ was found. As each next link found occurs strictly lower along $L$ the search must terminate. By Lemma 3 the lowest link
of $w[\ell_t]$ that crosses $L$ has spirality at most 0. Thus the search must terminate with a link of spirality at most 0. Moreover, by Lemma 4 if $s(\ell_t) \equiv 2 \pmod{4}$ then link $\ell_t$ has spirality $s(\ell_t) - 2$ or $s(\ell_t) + 4$. Similarly, by Lemma 5, if $s(\ell_t) \equiv 0 \pmod{4}$ then link $\ell_t$ has spirality $s(\ell_t) + 2$ or $s(\ell_t) - 4$. But then it must be that at some point during the search $s(\ell_t) = t$ and $s(\ell_t) = t - 4$ as initially the spirality is $t \geq 2$ and at termination the spirality is at most 0. Contradiction as $t \equiv 2 \pmod{4}$ and thus by Lemma 4 $s(\ell_t) \geq t - 2$.

As a link from $w[\ell_t]$ exists that crosses $L$ below $\ell^t$ and that has spirality $t - 2$, the highest link $\ell^{t-2}$ with these properties also exists. Adding $\ell^{t-2}$ to $S_t$ trivially maintains IH1, IH3, IH4. We show that it also maintains IH2. Particularly we show that $\ell^{t-2}$ crosses $L$ lower than all links from $S_t$. If $S_t = \{\ell^t\}$ this is trivially true. Otherwise, by IH1 there exists a link $\ell^{t+2}$ in $S_t$ with spirality $t + 2$. Link $\ell^{t+2}$ is right-oriented and, as it has the smallest spirality of all right-oriented links in $S_t$, it is the link from $S_t$ that crosses $L$ in the lowest point. We show $\ell^{t-2}$ crosses $L$ in an even lower point.

Assume for contradiction that $\ell^{t-2}$ crosses $L$ above $\ell^{t+2}$ (see Figure 9(a)). We maintain an interval on $L$ bounded from above by a link $\ell_a$ that crosses $L$ and from below by a link $\ell_b$ that crosses $L$. We show that within this interval a link from the subwire $w[\ell^{t+2}]$ with spirality $t$ exists. Initially $\ell_a = \ell^{t-2}$ and $\ell_b = \ell^{t+2}$. We have $s(\ell_a) = t - 2$ and $s(\ell_b) = t + 2$, and both $\ell_a$ and $\ell_b$ occur on $w$ before (or at) $\ell^{t+2}$. During our search we maintain stronger conditions. Particularly, $s(\ell_a) \leq t - 2$ if $s(\ell_a) \equiv 0 \pmod{4}$ and $s(\ell_a) \leq t - 4$ otherwise. Moreover, $s(\ell_b) \geq t + 2$ if $s(\ell_b) \equiv 0 \pmod{4}$ and $s(\ell_b) \geq t + 4$ otherwise.

We make a case distinction on whether $\ell_a \in w[\ell_b]$ or $\ell_b \in w[\ell_a]$. Initially $\ell_a \in w[\ell_b]$. 

**Case 1a** $\ell_a \in w[\ell_b]$ (see Figure 9(b))

If $s(\ell_t) \equiv 0 \pmod{4}$ ($\ell_b$ is right-oriented) then by Lemma 5 the lowest link from $w[\ell_b]$ crossing $L$ above $\ell_b$ has spirality $s(\ell_b) - 2$ or $s(\ell_b) + 4$. If $s(\ell_b) \equiv 2 \pmod{4}$, and thus $s(\ell_b) \geq t + 4$, then by Lemma 4 this link has spirality $s(\ell_b) - 4$ or $s(\ell_b) + 2$. In each case the spirality is at least $t$. As $s(\ell_a) \leq t - 2$, there must be a different link $\ell_m$ from $w[\ell_b]$ that intersects $L$ between $\ell_a$ and $\ell_b$. We have either $s(\ell_m) = t$, or $s(\ell_m) \geq t + 2$ and $s(\ell_m) \equiv 0 \pmod{4}$, or $s(\ell_m) \geq t + 4$ and $s(\ell_m) \equiv 2 \pmod{4}$. If $s(\ell_m) \geq t + 2$ then let $\ell_b = \ell_m$.

![Figure 9](image-url)
Case 1b \( \ell_b \in w[\ell_a] \): (see Figure 9(c))

If \( s(\ell_a) \mod 4 = 0 \) then by Lemma 5 the highest link from \( w[\ell_a] \) crossing \( L \) below \( \ell_a \) has spirality \( s(\ell_a) + 2 \leq t \) or \( s(\ell_a) - 4 \leq t \). If \( s(\ell_a) \mod 4 = 2 \), and thus \( s(\ell_a) \leq t - 4 \), then by Lemma 4 this link has spirality \( s(\ell_a) + 4 \leq t \) or \( s(\ell_a) - 2 \leq t \). As \( s(\ell_b) \geq t + 2 \), there must be a different link \( \ell_m \) from \( w[\ell_a] \) that intersects \( L \) between \( \ell_a \) and \( \ell_b \), such that either \( s(\ell_m) = t \), or \( s(\ell_m) \leq t - 2 \) and \( s(\ell_m) \mod 4 = 0 \), or \( s(\ell_m) \leq t - 4 \) and \( s(\ell_m) \mod 4 = 2 \). If \( s(\ell_m) \leq t - 2 \) then let \( \ell_a = \ell_m \).

As we can repeat case 1a and 1b as long as \( s(\ell_a) \neq t \) and \( s(\ell_b) \neq t \), and as \( \ell_a \) and \( \ell_b \) have fewer links crossing \( L \) between them after each iteration, at some point we must have that either \( s(\ell_a) = t \) or \( s(\ell_b) = t \). As both \( \ell_a \) and \( \ell_b \) are part of \( w[\ell^t + 2] \), and as \( \ell_a \) and \( \ell_b \) both cross \( L \) lower (or at) \( \ell^{t - 2} \), we reach a contradiction with the fact that by IH4 \( \ell^t \) is the lowest link from \( w[\ell^t + 2] \) with spirality \( t \) that crosses \( L \) above \( \ell^{t + 2} \) (e.g., see Figure 9(d)).

Case 2 \( \ell^t \) is right-oriented: \( (t \mod 4 = 0) \)

This case is symmetric to case 1, we abbreviate the argument for conciseness. As \( \ell^t \) has spirality \( t \mod 4 = 0 \), by IH2 the link \( \ell^t \) is part of the monotone decreasing subsequence in \( S_t \) when ordered top-to-bottom along \( L \). Moreover \( \ell^t \) is the lowest link that crosses \( L \). Let \( \ell_j \) be the lowest (horizontal) link from \( w[\ell^t] \) that crosses \( L \) higher than \( \ell^t \). By Lemma 5 we have \( s(\ell_j) = t - 2 \) or \( s(\ell_j) = t + 4 \). If \( s(\ell_j) = t - 2 \) then the claim is proven. Otherwise, we continue the upwards search from \( \ell_j \). As by Lemma 3 the highest link crossing \( L \) has spirality at least 0, we can repeat the upwards search until a link of spirality \( t - 2 \) is found. As we continuously search earlier along the wire this link must also be part of \( w[\ell^t] \).

Adding the lowest link \( \ell^{t - 2} \) with these properties to \( S_t \) trivially maintains IH1, IH3, IH4. We show that it also maintains that \( \ell^{t - 2} \) crosses \( L \) higher than all other links from \( S_t \) (IH2). Assume for contradiction that \( \ell^{t - 2} \) crosses \( L \) below the previously highest link \( \ell^{t + 2} \). Then a link \( \ell_m \) must exist that crosses \( L \) between \( \ell^{t - 2} \) and \( \ell^{t + 2} \) because as a consequence of Lemma 4 \( \ell^{t - 2} \) cannot be the highest link from \( w[\ell^{t + 2}] \) crossing below \( \ell^{t + 2} \). Using \( \ell_m \) we can repeatedly shrink the search interval along \( L \) until we find a link with spirality \( t \) that is part of \( w[\ell^{t + 2}] \) and that crosses \( L \) higher than \( \ell^t \) contradicting IH3.

Let \( \ell_{\leftarrow} \) be a right-oriented link on a wire \( w \) and w.l.o.g. let \( s > 0 \) be the spirality of \( \ell_{\leftarrow} \). Further, let \( L \) be a vertical line through \( \ell_{\leftarrow} \) and \( S \) a subsequence from \( w \) with the properties guaranteed by Lemma 6. Finally, let \( \ell^i \in S \) be the unique link with spirality \( 0 \leq i \leq s \) in \( S \). We define the \( i \)-core for \( S \) (for \( 4 \leq i \leq s \) and \( i \mod 4 = 0 \)) as the region enclosed by the wire \( w \) from the intersection between \( \ell^{i-4} \) and \( L \) to the intersection between \( \ell^i \) and \( L \) and the line segment along \( L \) connecting them (see Figure 10(a)). We define a layer of \( S \) as the difference between the \( i \)-core and the \( (i + 4) \)-core, for \( 4 \leq i \leq s - 4 \) and \( i \mod 4 = 0 \) (see Figure 10(b)). There are \( \Theta(s) \) layers surrounding a link of spirality \( s \geq 8 \).

Lemma 7. An equivalent set of \( r \)-wires with spirality \( O(n) \) exists in \( \Gamma_1 \) with respect to a straight-line wire-grid in \( \Gamma_O \).

Proof. We prove the statement constructively by induction on the size of the constructed set. Assume a set \( S \) of \( k \) \( r \)-wires with spirality \( O(n) \) exists, that is equivalent to a subset \( T \) of the wires from \( W_{\leftarrow} \) in \( \Gamma_O \). Moreover, assume that each wire in \( S \) is shortest with respect to the previously inserted wires in \( \Gamma_I \). In the base case \( S = T = \emptyset \).
For the inductive step consider an arbitrary wire $w_O \in W_\rightarrow$ from $\Gamma_O$ where $w_O \notin T$. Find the Euclidean shortest lr-wire $w$ in $\Gamma_I$ such that $S \cup w$ is equivalent to $T \cup w_O$. Consider a right-oriented link $\ell_\rightarrow \in w$ with maximum absolute spirality $s$. Based on Lemma 6 we conclude that there are $\Theta(s)$ layers surrounding $\ell_\rightarrow$. We bound the number of layers surrounding $\ell_\rightarrow$ to $O(n)$. Thus, it must also be that $s = O(n)$ and the spirality of $w$ is $O(n)$.

To achieve this we classify the layers of the spiral by their containment of the drawing: (1) a layer contains a vertex, or (2) a layer is crossed by an edge, or (3) a layer contains no part of the drawing (but may contain wires).

There are at most $O(n)$ layers that contain a vertex of the drawing. If a layer contains no vertex but does intersect an edge, then that edge must cross through the layer. As each edge is crossed at most once by $w$ (Observation 2), an edge can only exit the layer on one side by crossing through $w$. Hence each edge crossing a layer must cross $L$ exactly once. Each edge can at most cross through the (two) layers directly adjacent to the crossing with $L$ in this way (see Figure 10(c)). Thus only $O(n)$ layers intersect an edge.

We now show that these are all layers, by showing every layer must contain a part of $G$. Assume for contradiction a layer $R$ exists that does not contain any part of $G$. The boundary of $R$ is formed by $w$ and two straight-line segments along $L$. We refer to the two segments along $L$ as the gates of the layer.

The layer $R$ may still contain (subsections of) lr-wires, including $w$ itself. lr-wires do not cross and hence must enter and leave $R$ through the gates. As $R$ contains no part of the drawing and the lr-wires are each shortest with respect to the previously inserted wires they cannot consecutively enter and leave $R$ through the same gate. Moreover, as lr-wires do not cross or self-intersect, the order of the wires at both gates must be identical.

Disconnect all lr-wires at the gates of $R$. Also disconnect $w$ at the lower link adjacent to each gate. Remove all disconnected components. Reconnect the remaining parts locally along $L$ ensuring no crossings occur and all wires visit the remaining links in the same order (see Figure 10(d)). All wires crossing $R$ are shortened by this. Contradiction, because particularly the rerouted wire that was inserted first was not shortest with respect to the previously inserted wires. We conclude there are at most $O(n)$ layers, and thus the maximum absolute spirality of any link of $w$ and thereby $w$ is $O(n)$.

Figure 10: (a) The $i$-core of a spiral for a link $\ell^i \in S$ (gray). (b) The $i$-layer of the spiral (gray). (Shape simplified for exposition.) (c) Each edge can cross through (at most) the two layers directly adjacent to the crossing with $L$. (d) A layer cannot only contain wires as then all wires can be shortened.
Theorem 2. Let $\Gamma_I$ and $\Gamma_O$ be two unified planar orthogonal drawings of a (potentially disconnected) graph $G$. There exists a set of wires in $\Gamma_I$ that is equivalent to the straight-line wires in $\Gamma_O$ and that has spirality $O(n)$.

Proof. By Lemma 7 we can insert all lr-wires with spirality $O(n)$. By Lemma 1 intersecting links have equal spirality. Thus when a tb-wire intersects a lr-wire it has spirality $O(n)$ and we can consider the regions between two consecutive lr-wires independently. Within the region between two consecutive lr-wires no pair of wires intersect. Using the same proof as Lemma 7 we conclude that within the region between two lr-wires the spirality of each tb-wire may increase (decrease) by at most $O(n)$, but the spirality will be $O(n)$ again when crossing the second lr-wire. Thus the maximum spirality of the tb-wires is also $O(n)$. \qed

4 Basic algorithm using a linear number of morphs

We now describe our algorithm to morph $\Gamma_I$ to $\Gamma_O$ using $O(s) = O(n)$ linear morphs, where $s$ is the spirality of an equivalent set of wires in $\Gamma_I$. The complexity of the intermediate drawing may rise to $O(n^3)$. In Section 5 we refine the algorithm by batching linear morphs to reduce the required number of linear morphs to $s$. Finally, in Section 6 we describe an extension to the algorithm to ensure the complexity of intermediate drawings remains $O(n)$ at the cost of a single extra linear morph. Thus using $s+1$ linear morphs we maintain planarity, orthogonality, and linear complexity of the drawing during the morph.

For our analysis of the initial spirality we required $\Gamma_I$ and $\Gamma_O$ to be straight-line drawings of the unified graph. For the morph itself we let go of this stringent requirement. During the morph we introduce bends in the edges to change their orientation while maintaining orthogonality. We will show that the spirality of the wires only decreases during the morph.

The idea of the algorithm is to reduce the maximum spirality of the wires using only $O(1)$ linear morphs. Then, by Theorem 2 we need only $O(n)$ linear morphs to straighten the wires. The resulting drawing can be made similar to $\Gamma_O$, by adding additional bends in $\Gamma_O$, after which we can morph the resulting drawing to $\Gamma_O$ using a single linear morph (Observation 1). We show how to reduce the spirality of wires in $W_\leftarrow$ without increasing the spirality of wires in $W_\rightarrow$ and vice versa. In the description below, we limit ourselves to straightening the wires in $W_\rightarrow$.

Figure 11: Different slide-types used on the wires. (a) The slide from [3] executed on a wire. (b) A bend-introducing slide splits the link without increasing the overall spirality. (c) A single crossing edge (link) causes the introduction of new bends in the edge (link).
To straighten the wires we use zigzag-eliminating slides (see Section 2). Consider a link $\ell^*$ with maximum absolute spirality. We consider two cases. In the first case $\ell^*$ is not crossed by any link or segment. As $\ell^*$ is a link with maximum absolute spirality, the links before and after $\ell^*$ are on opposite sides of the line through $\ell^*$ and hence $\ell^*$ is the middle link of a zigzag in the wire. Performing a zigzag-eliminating slide on $\ell^*$ eliminates $\ell^*$ and consequently may lower the spirality of the wire containing $\ell^*$.

In the second case, $\ell^*$ is crossed by one or more links or segments. We describe two additional steps to account for this by introducing additional bends in the wires.

**Step 1:** (only if $\ell^*$ intersects multiple segments and/or links)
We split $\ell^*$ into several links each crossing exactly one segment/link by performing a bend-introducing slide on $\ell^*$ inbetween each pair of crossings (see Figure 11(b)). We can pick a direction for the bend-creating slides to ensure only links are created with an absolute spirality smaller than or equal to the maximum absolute spirality.

**Step 2:**
Note that $\ell^*$ intersects exactly one segment (link) $(u,v)$. Before executing the zigzag-eliminating slide we first introduce two bends in $(u,v)$ at the intersection of $(u,v)$ and $\ell^*$ (see Figure 11(c)). Without loss of generality assume that $\ell^*$ is a vertical link. We symbolically offset the first bend to be left of $\ell^*$ and the second to be right of $\ell^*$. Segment $(u,v)$ is now split into three segments, the first and last being completely on one side of the zigzag and the middle being a degenerate segment. Upon performing the linear slide all three segments will stay (degenerately) horizontal or vertical during the slide. A zigzag-eliminating slide eliminates $\ell^*$ and may thereby lower the maximum absolute spirality of the wire containing $\ell^*$. In the next part we show that these slides also maintain the spirality of all other wires.

To reduce the number of linear morphs, we combine all slides of the same type into a single linear morph. For all links with the same spirality, all bend-creating slides are combined into one linear morph, and all zigzag-eliminating slides are combined into another linear morph. For convenience of argument, links with positive spirality and links with negative spirality are combined into separate linear morphs. Thus, using at most 4 linear morphs, we can reduce the maximum absolute spirality of all wires in $W_{\rightarrow}$ by one.

**Analysis.** We first show that performing slides on links in $W_{\rightarrow}$ does not have adverse effects on wires in $W_{\downarrow}$. This is easy to see for bend-creating slides, as wires in $W_{\rightarrow}$ and wires in $W_{\downarrow}$ never overlap in more than a point.

**Lemma 8.** Performing a zigzag-eliminating slide on a link with maximum absolute spirality in $W_{\rightarrow}$ does not increase the spirality of a wire in $W_{\downarrow}$.

**Proof.** Let $\ell^*$ be the middle link of the zigzag causing the zigzag-eliminating slide. The zigzag-eliminating slide can only change a wire $w'$ in $W_{\downarrow}$ if $\ell^*$ crosses a link $\ell'$ in $w'$. By Lemma 1 $s(\ell') = s(\ell^*)$ before the linear slide is performed. The slide does not change the spirality of any existing link in $w'$ as the introduction of a left and right bend in $w'$ has an overall neutral effect on the spirality of $w'$. A new link is created however and this link crosses the link formed by the merging of the link before and after $\ell^*$, which both have absolute spirality $|s(\ell^*)| - 1$. By Lemma 1 the new link in $w'$ must also have absolute spirality $|s(\ell^*)| - 1$, and thus the spirality of $w'$ has not been increased. \qed
Lemma 9. *Multiple bend-creating or zigzag-eliminating slides on links of the same spirality in* $W_{\rightarrow}$ *can be combined into a single linear morph that maintains planarity and orthogonality.*

**Proof.** As bend-creating slides and zigzag-eliminating slides operate similarly, we restrict our argument to the latter. As all zigzag-eliminating slides operate on links of the same spirality, they are either all horizontal or all vertical. Without loss of generality, assume that all zigzags are horizontal. Then all vertices in the drawing are moved only vertically and vertical edges remain vertical. Furthermore, since we introduce bends at edges that intersect the middle segment of zigzags, horizontal edges are either subdivided or remain horizontal during the linear morph. As the horizontal order of vertices and bends is maintained, planarity can only be violated if there exists a vertical line where the vertical order is changed. However, by construction of the slides, on any vertical line points with higher $y$-coordinates are moved up at least as far as points with lower $y$-coordinates, maintaining the vertical order.

Theorem 3. *Let* $\Gamma_I$ and $\Gamma_O$ *be two orthogonal planar drawings of* $G$, *where* $G$ *is the unification of* $\Gamma_I$ and $\Gamma_O$, *and* $\Gamma_I$ and $\Gamma_O$ *are equivalent. Then we can morph* $\Gamma_I$ to $\Gamma_O$ *using* $O(n)$ *linear morphs, where* $n$ *is the number of vertices of* $G$.

**Proof.** Let $W_{\rightarrow}$ and $W_\downarrow$ be an equivalent set of wires for $\Gamma_I$, with respect to the straight-line wire-grid in $\Gamma_O$, having maximum spirality $O(n)$. By Theorem 2 such a set exists. By Lemma 8 the wires in $W_{\rightarrow}$ ($W_\downarrow$) can be straightened without affecting the spirality of the wires in $W_\downarrow$ ($W_{\rightarrow}$). By Lemma 9, we can reduce the maximum spirality of all wires in $W_{\rightarrow}$ ($W_\downarrow$) by one using at most four linear morphs. Hence, all wires can be straightened with at most $O(n)$ linear morphs. Afterwards, the resulting drawing $\Gamma$ is similar to $\Gamma_O$ except for additional bends. After adding matching bends in $\Gamma_O$, we can use a single linear morph to morph $\Gamma$ to $\Gamma_O$ (Observation 1).

5 Batching linear morphs

In the previous section we have shown that a morph that maintains planarity and orthogonality exists that consists of $O(n)$ linear morphs. However, the intermediate complexity of the drawing could increase to $O(n^3)$ as each of the $O(n)$ morphing steps may reduce $O(n)$ links each of which crosses $O(n)$ edges, thus adding an additional $O(n^2)$ bends per step of the morph. Before we show how to also maintain $O(n)$ complexity of the drawing (Section 6), we first further strengthen the result. The proof of Theorem 3 implies that a morph between two unified planar orthogonal drawings $\Gamma_I$ and $\Gamma_O$ exists using $O(s)$ linear morphs, where $s$ is the spirality of $\Gamma_I$. In this section we show how to effectively combine consecutive linear slides, resulting in a planarity and orthogonality preserving morph using only $s$ linear morphs.

In this section we initially consider the previously defined morph without any merging of linear slides. Thus each linear morph is directly defined by a single zigzag-eliminating or bend-introducing slide. This sequence of linear slides can be encoded by a sequence of drawings, starting with $\Gamma_I$ and ending with $\Gamma_O$, such that every consecutive pair of drawings is caused by a linear slide. For notational convenience let $\Gamma_i \rightarrow \Gamma_j$ indicate that $\Gamma_i$ occurs before $\Gamma_j$ during the morph and $\Gamma_i \Rightarrow \Gamma_j$ that $\Gamma_i \rightarrow \Gamma_j$ or $\Gamma_i = \Gamma_j$. 


Let the spirality of a drawing, with respect to a given set of wires, be defined as the maximum spirality over all wires. Note that spirality of a drawing \( \Gamma \) is always relative to another drawing \( \Gamma' \) and dependent on the set of wires selected for \( \Gamma \). There may be multiple potential sets of equivalent wires in \( \Gamma \) for a given drawing \( \Gamma' \). Still, whenever the drawing \( \Gamma' \) and the matching set of wires are clear from the context, then by abuse of notation we will speak of the spirality of \( \Gamma \). Unless stated otherwise, we consider spirality relative to \( \Gamma_0 \).

Let an iteration of the original morph consist of all linear slides that jointly reduce spirality of the drawing by one. Let the first drawing of iteration \( s \) be the first drawing in the original morph with spirality \( s \) and the last drawing of iteration \( s \) be the first drawing with spirality \( s - 1 \). Consecutive iterations overlap in exactly one drawing. These drawings in the overlap of iterations are the intermediate steps of the final morph. Within this section let \( \Gamma_I \Rightarrow \Gamma_a \Rightarrow \Gamma_b \Rightarrow \Gamma_O \), where \( \Gamma_a \) is the first drawing with spirality \( s \) and \( \Gamma_b \) is the first drawing with spirality \( s - 1 \).

We prove that within one iteration for any two vertices the relative order along at most one axis can be changed. We use this to show that the spirality of \( \Gamma_a \) with respect to \( \Gamma_b \) is one. We then show that we can also find a drawing \( \Gamma'_b \), based on \( \Gamma_b \), such that \( \Gamma_a \) has spirality one to \( \Gamma'_b \), \( \Gamma'_b \) has spirality \( s - 1 \), and \( \Gamma'_b \) is a straight-line drawing. Finally we show that any two drawings \( \Gamma, \Gamma' \) where \( \Gamma \) has spirality one relative to \( \Gamma' \) can be linearly interpolated without violating planarity of the drawing during the morph. Together this implies we can reduce spirality by one using only a single linear morph while maintaining a straight-line intermediate drawing (though each individual morph may temporarily increase complexity by \( O(n^2) \)). Consequently using only \( s \) linear morphs we can morph \( \Gamma_I \) into \( \Gamma_O \) while maintaining planarity and orthogonality.

As bend-introducing slides (Section 4, Figure 11(b)) can trivially be offset to not have any vertices along the vertical (horizontal) link that is split, we can prevent them from changing the \( x \) or \( y \)-order of any pair of vertices in the drawing. Therefore, we leave them out of consideration in this section.

### 5.1 Staircases

Consider two distinct vertices \( v \) and \( w \) of the drawing. Define an \( x \)-inversion (\( y \)-inversion) of \( v \) and \( w \) between \( \Gamma_a \) and \( \Gamma_b \) when the sign \((+,−,0)\) of \( v.x − w.x \) \((v.y − w.y)\) differs in \( \Gamma_a \) and \( \Gamma_b \). In that case we say two vertices are \( x \)-inverted (\( y \)-inverted), or simply inverted. Two vertices \( v \) and \( w \) are separated in a drawing by a link \( \ell \) when they are both in the vertical (horizontal) strip spanned by \( \ell \), and \( v \) and \( w \) are on opposite sides of \( \ell \).

**Lemma 10.** Let \( \Gamma_a \) be a drawing and \( \Gamma_b \) be obtained from \( \Gamma_a \) by a zigzag-removing slide along link \( \ell \). If two vertices \( v \) and \( w \) are inverted, then \( v \) and \( w \) were separated by \( \ell \) in \( \Gamma_a \).

**Proof.** W.l.o.g. assume \( \ell \) is vertical and the spirality is positive (see Figure 12). Let \( V \) be the set of vertices moved by a zigzag-removing slide on \( \ell \). If \( v, w \in V \) or \( v, w \not\in V \) then \( v, w \) are moved equally in the same direction and cannot have been inverted. Hence either \( v \in V \) or \( w \in V \); assume \( v \in V \). All vertices in \( V \) move up by the length of \( \ell \). To be inverted we need that initially \( w.y > v.y \), but also that \( w \not\in V \). Then \( v \) and \( w \) must be separated by \( \ell \). \( \square \)
A downward horizontal staircase is a sequence of horizontal links where:
(1) the left-endpoints are $x$-monotone increasing and strictly $y$-monotone decreasing,
(2) the projection on the $x$-axis intersects for a pair if and only if they are consecutive in the sequence, and
(3) all links have positive spirality.
Two vertices $v$ and $w$ are separated by a downward staircase if $v$ is in the vertical strip spanned by the first link of the staircase and above it and $w$ is in the vertical strip spanned by the last link and below it. Similar concepts can be defined for upward horizontal staircases and for vertical staircases.

**Lemma 11.** Two vertices $v$ and $w$ whose order during a morph from $\Gamma_a$ to $\Gamma_b$ is $x$-inverted ($y$-inverted) before it optionally is $y$-inverted ($x$-inverted), are separated by a horizontal (vertical) staircase of maximum spirality links in $\Gamma_a$.

**Proof.** Assume w.l.o.g. that only one inversion occurs and it occurs from $\Gamma_b^{-1}$ to $\Gamma_b$, otherwise consider the submorph from the start until the first slide causing an inversion. Assume that $v.x < w.x$, $v.y > w.y$ in all drawings from $\Gamma_a$ to $\Gamma_b^{-1}$ and $v.x > w.x$, $v.y > w.y$ in $\Gamma_b$. We prove the claim inductively in backwards direction. For convenience of the argument we will treat $v$ as the right endpoint of an initial link of the staircase and $w$ as the left endpoint of a final link.

The base case is $\Gamma_b^{-1}$, where $v$ and $w$ are separated by a single maximum absolute spirality link (Lemma 10). As $v.y > w.y$ and $v.x < w.x$ in $\Gamma_b^{-1}$ and $v.x > w.x$ in $\Gamma_b$, this link has positive spirality and trivially (together with the imaginary initial and final links) forms a downward staircase.

For the step let the sequence $S$ compose a downwards staircase in $\Gamma_i$, where $\Gamma_a \Rightarrow \Gamma_i \Rightarrow \Gamma_{b^{-1}}$. Let $\ell_{i-1}$ be the link in $\Gamma_{i^{-1}}$ of maximum absolute spirality that specified the linear slide from $\Gamma_{i^{-1}}$ to $\Gamma_i$. We define four rectangular regions $A, B, C, D$ surrounding $\ell_{i-1}$ that partition the plane in $\Gamma_{i^{-1}}$. In the case where $\ell_{i-1}$ is horizontal in $\Gamma_{i^{-1}}$ we define region $A$ as the halfspace left of the left endpoint of $\ell_{i-1}$, region $D$ as the halfspace right of the right endpoint of $\ell_{i-1}$, and the remaining strip is split in region $B$ above $\ell_{i-1}$ and region $C$ below (e.g., see Figure 13(a)). In the case when $\ell_{i-1}$ is vertical we define $A$ to be above, $D$ below, $B$ left of, and $C$ right of $\ell_{i-1}$. During the linear slide from $\Gamma_{i^{-1}}$ to $\Gamma_i$ all four regions are maintained. Moreover, two new regions $F$ and $G$ appear in $\Gamma_i$. As regions $A, B, C, D$ are maintained and together contain all vertices, regions $F$ and $G$ do not contain any vertices.
Figure 13: (a) Regions surrounding $\ell_{i-1}$ in $\Gamma_{i-1}$ and the matching regions in $\Gamma_i$. Regions $A, B, C, D$ are maintained between the two drawings. (b) A vertical slide may merge two links from staircase $S$ in $\Gamma_{i-1}$. (c) If $S$ is split in $\Gamma_{i-1}$ then it can be extended to a staircase (white outline) by adding $\ell_{i-1}$. (d) If non-adjacent links from $S$ overlap in $\Gamma_{i-1}$ then we can select a subsequence from $S$ forming a staircase in $\Gamma_{i-1}$ (white outline).

Assume $S$ is not a downwards staircase in $\Gamma_{i-1}$, for otherwise we are done. A staircase is defined solely by the $x$- and $y$-order of the endpoints of the links. Thus, as $S$ is not a downwards staircase, there must be at least two endpoints of links in $S$ whose $x$- or $y$-order is different in $\Gamma_{i-1}$ from $\Gamma_i$. The respective endpoints are separated by $\ell_{i-1}$ in $\Gamma_{i-1}$, for otherwise their order could not change (Lemma 10). Hence at least one link from $S$ intersects region $B$ and one link intersects region $C$. (As $B$ and $C$ are not horizontally or vertically adjacent in $\Gamma_i$ these must be two different links.)

Let $S_1$ be the subsequence of $S$ up to the last link $\ell_1$ that intersects, or is contained in, region $B$. Let $S_2$ consist of the remaining links. The first link $\ell_2$ of $S_2$ must intersect or be contained in region $C$. The endpoints of the links in $S$ are monotone decreasing in $y$ (respectively monotone increasing in $x$) in $\Gamma_i$. As region $C$ is below (respectively right) of $B$, no link in $S_1$ intersects $C$ and no links from $S_2$ intersect $B$. Hence, within $S_1$ and $S_2$ there are no changes in the $x$- and $y$-order of the endpoints. Any staircase properties must be broken by the interaction between $S_1$ and $S_2$. We make a case distinction on the orientation and spirality of $\ell_{i-1}$.

**Case 1:** ($\ell_{i-1}$ is vertical – Figure 13(b))

Then $\ell_{i-1}$ must have positive spirality. Assume for contradiction that $\ell_{i-1}$ has negative spirality. Then in $\Gamma_i$, in mirror to the displayed case, the top of region $B$ is at the same height as the bottom of region $C$. But then $\ell_1$ must have a lower or equivalent $y$-coordinate to $\ell_2$ in $\Gamma_i$. This contradicts the fact that $S$ was a valid staircase in $\Gamma_i$. 

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Thus \( \ell_{i-1} \) has positive spirality. As the \( x \)-projection of \( \ell_1 \) and \( \ell_2 \) touches (overlaps) in \( \Gamma_i \) and regions \( F \) and \( G \) contain no vertices, it must be that \( \ell_1 \) ends at the right border of \( B \) and \( \ell_2 \) starts at the left border of \( C \). As link \( \ell_{i-1} \) can only be crossed by a single edge, the right endpoint of \( \ell_1 \) must be equal with the left endpoint of \( \ell_2 \) in \( \Gamma_{i-1} \) forming a single link. The order of all other pairs of links is maintained as a vertical slide does not affect the \( x \)-order and on each vertical line the order of all points is maintained. We conclude that the sequence \( S \) is also a valid staircase in \( \Gamma_{i-1} \).

**Case 2:** \( (\ell_{i-1} \text{ is horizontal and has positive spirality – Figure 13(c)}) \)

A horizontal slide can falsify the \( x \)-monotonicity or the overlap of links. As \( S \) is a valid staircase in \( \Gamma_i \) the projection on the \( x \)-axis of any two non-adjacent links is non-overlapping. Then \( \ell_1 \) and \( \ell_2 \) cannot be fully contained in \( B \) respectively \( C \) as otherwise either \( \ell_1 \) is contained in \( \ell_2 \) or vice versa, and therefore at least one pair of non-adjacent links must overlap. Moreover, from \( S \) only \( \ell_1 \) enters \( B \) and only \( \ell_2 \) enters \( C \). In \( \Gamma_{i-1} \) only the right endpoint of \( \ell_1 \) and the left endpoint of \( \ell_2 \) may be inverted. If so, then by Lemma 10 \( \ell_1 \) and \( \ell_2 \) are separated by \( \ell_s \). Thus \( (S_1, \ell_s, S_2) \) forms a downwards staircase in \( \Gamma_{i-1} \).

**Case 3:** \( (\ell_{i-1} \text{ is horizontal and has negative spirality – Figure 13(d)}) \)

As \( S \) is a valid staircase in \( \Gamma_i \) link \( \ell_1 \) must intersect both region \( B \) and \( D \). As \( \ell_2 \) intersects region \( C \), and as only for adjacent links in \( S \) the projection on the \( x \)-axis can overlap, link \( \ell_1 \) is the only link intersecting region \( B \) and \( D \). Similarly only \( \ell_2 \) intersects region \( C \). However, in \( \Gamma_{i-1} \) the projection on the \( x \)-axis of several links from \( S_1 \) may overlap with the projection of \( \ell_2 \). Select a subsequence from \( S \) satisfying all constraints by removing all links from \( S_1 \) after the first link from \( S_1 \) that overlaps \( \ell_2 \).

\( \square \)

### 5.2 Inversions

We show that every pair of vertices is inverted along at most one axis during the morph from \( \Gamma_a \) to \( \Gamma_b \). We then prove that \( \Gamma_a \) has spirality one relative to \( \Gamma_b \). This will be used in the next sections to prove a single linear morph is sufficient to morph from \( \Gamma_a \) to \( \Gamma_b \).

**Lemma 12.** Two vertices \( v \) and \( w \) can be inverted along only one axis during any sub-morph of the morph from \( \Gamma_a \) to \( \Gamma_b \).

*Proof.* Suppose for a contradiction, that a pair of vertices \( v, w \) exists that is inverted along both axes during the submorph from \( \Gamma_i \) to \( \Gamma_j \), where \( \Gamma_a \rightarrow \Gamma_i \rightarrow \Gamma_j \rightarrow \Gamma_b \). Assume they are inverted along both axes exactly once, otherwise consider the smaller submorph where this is the case. W.l.o.g. let \( v.x < w.x, v.y < w.y \) in \( \Gamma_i \), \( v.x < w.x, v.y > w.y \) in all drawings from \( \Gamma_{i+1} \) to \( \Gamma_{j-1} \), and \( v.x > w.x, v.y > w.y \) in \( \Gamma_j \).

By Lemma 11, and the relative position of \( v \) and \( w \), there exists a downwards staircase separating \( v \) and \( w \) in \( \Gamma_{i+1} \). By Lemma 10 and the inversion of \( v, w \) from \( \Gamma_i \) to \( \Gamma_{i+1} \), the morph from \( \Gamma_i \) to \( \Gamma_{i+1} \) is defined by a vertical maximum-spirality link separating \( v \) and \( w \) in \( \Gamma_i \). The resulting vertical slide cannot break a downward staircase (see Case 1 in the proof of Lemma 11) and hence if a downwards staircase separating \( v \) and \( w \) was present in \( \Gamma_{i+1} \) then there is also a downwards staircase separating \( v \) and \( w \) in \( \Gamma_i \). But then it must be that \( v.y > w.y \) in \( \Gamma_i \). Contradiction. \( \square \)
Figure 14: (a) Sets $S_L$ and $S_R$ in $\Gamma_a$ and $\Gamma_i$. (b) A vertical edge $e \in S_R$ cannot cross $L_\downarrow$ left of a vertical edge $f \in S_L$ as vertex $u$ must be $x$- and $y$-inverted with one of the endpoints of $f$ during the morph. (c) The $y$-monotone line cannot cross the edges in the wrong order as then vertex $u$ must be $x$- and $y$-inverted with an endpoint of $f$.

**Lemma 13.** For any drawing $\Gamma_i$ where $\Gamma_a \to \Gamma_i \Rightarrow \Gamma_b$, each vertical (horizontal) straight-line wire in $\Gamma_i$ not crossing a vertex has an equivalent $y$- ($x$-) monotone wire in $\Gamma_a$.

**Proof.** Let $L_\downarrow$ be a vertical line in $\Gamma_i$ not crossing any vertex. Line $L_\downarrow$ partitions the set of vertices and vertical edges in $\Gamma_i$ into two subsets $S_L, S_R$ (see Figure 14(a)). Let $L_\downarrow$ be an arbitrary horizontal line in $\Gamma_a$. We first prove that on $L_\downarrow$, every element from $S_L$ comes before every element from $S_R$. It follows that a $y$-monotone wire must exist in $\Gamma_a$ that correctly partitions the vertices and vertical edges.

Assume for contradiction that there exist elements $e \in S_R$ and $f \in S_L$, with $e.x < f.x$ in $\Gamma_a$ and $e.x > f.x$ in $\Gamma_i$, that lie on a horizontal line in $\Gamma_a$. As $e$ and $f$ are $x$-inverted, they (or their endpoints) cannot be $y$-inverted between $\Gamma_a$ and $\Gamma_i$ (Lemma 12). Assume $e, f$ are vertical edges in $\Gamma_a$ and an endpoint $u$ of $e$ is in the horizontal strip defined by $f$ (see Figure 14(b)). The case where $e$ or $f$ are vertices is analogous. As the morph is planar $u$ cannot move through $f$ while morphing to $\Gamma_i$. But then $u$ changed in the $y$-order with one of the endpoints of $f$ during the morph. Contradiction.

We now show that a $y$-monotone polyline separating $S_L$ and $S_R$ in $\Gamma_a$ intersects the horizontal edges in the same order as $L$ in $\Gamma_b$. Consider an arbitrary pair of horizontal edges $e, f$ that is intersected by $L_\downarrow$ in this order in $\Gamma_i$. If $e, f$ have the same vertical order in $\Gamma_a$ then the claim trivially holds. Otherwise the endpoints of $e, f$ are $y$-inverted in $\Gamma_a$ (see Figure 14(c)) and thus by Lemma 12 the $x$-order of the end-points is the same in $\Gamma_a$ and $\Gamma_i$. Using the same argument as in the previous paragraph there must exist an endpoint of $e$ and an endpoint of $f$ that have also changed in the $x$-order during the morph. Contradiction.

**Lemma 14.** Drawing $\Gamma_a$ has spirality one relative to $\Gamma_i$, where $\Gamma_a \to \Gamma_i \Rightarrow \Gamma_b$.

**Proof.** The spirality of $\Gamma_a$ relative to $\Gamma_i$ is one if there exists a wire grid of spirality one in $\Gamma_a$ that is equivalent to a straight-line wire grid in $\Gamma_i$. A wire with spirality one is, by definition, equivalent to a monotone wire. Thus, it is sufficient to show a wire grid exists where each wire is monotone. To this end, consider a new straight-line wire grid in $\Gamma_i$. By Lemma 13 we can find an equivalent $x$- ($y$-) monotone wire in $\Gamma_a$ for each straight wire in $\Gamma_i$. We show that there is also a set of monotone wires in $\Gamma_a$ that form an equivalent set to the wires in $\Gamma_i$. 

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Let $W$ be a set of wires in $\Gamma_a$ with the following properties. Firstly, every wire is monotone. Secondly, each wire is individually equivalent to a distinct straight wire in $\Gamma_i$. Thirdly, $W$ has the minimum number of intersections over all sets satisfying the previous requirements. We show that $W$ is equivalent to the wire-grid in $\Gamma_i$. That is, each pair of tb- (lr-) wires does not intersect and each pair of a tb- and a lr-wire intersect exactly once.

Assume for contradiction that a pair of $y$-monotone tb-wires intersect at least twice in $W$ (see Figure 15(a)). Consider the top-most two intersections. The region enclosed by the wires cannot contain vertices as both wires partition the vertices equivalently to $\Gamma_i$. As the enclosed region is simple and every edge is intersected at most once by a single wire, the order in which edges are intersected along both sides is the same. Locally reroute both wires along the enclosed region to remove both intersections. Contradiction as $W$ has the minimum number of intersections. Thus, each pair of $y$-monotone tb-wires in $W$ does not intersect. A symmetric argument show no pair of $x$-monotone lr-wires can intersect.

Assume for contradiction that an $x$-monotone lr-wire $w\rightarrow$ and a $y$-monotone tb-wire $w\downarrow$ intersect more than once. As they must intersect an odd number of times, they intersect at least three times. Consider a region $R$ enclosed between consecutive intersections $p_1, p_2$. Assume w.l.o.g. that $w\rightarrow$ intersects $w\downarrow$ left to right in $p_1$ (see Figure 15(b)). If $R$ does not contain vertices, then consider the left-most, lowest $x$-monotone increasing path $\pi$ through $R$. As the boundary right of $\pi$ is $y$-monotone decreasing, $\pi$ must also be $y$-monotone decreasing. Reroute both wires along $\pi$ between $p_1$ and $p_2$ to remove the intersections. Contradiction as $W$ has the minimum number of intersections. Each enclosed region must contain a vertex.

Consider the leftmost three consecutive intersections $p_1, p_2, p_3$ along $w\rightarrow$. We have
\[ p_1.x \leq p_2.x \leq p_3.x. \] Assume for contradiction \( w \) crosses through \( p_2 \) first. Then \( w \) up to \( p_2 \), \( w \) up to \( p_2 \), and the bounding box \( B \) surrounding the drawing (and wires) enclose a simple region (see Figure 15(c)). Wire \( w \) enters this region through \( p_1 \), but cannot exit it without intersecting \( w \) somewhere before \( p_2 \). Contradiction, as \( p_1, p_2, p_3 \) are the first three intersections along \( w \). Similarly, if \( w \) crosses through \( p_2 \) last then \( w \) between \( p_1 \) and \( p_3 \), and \( w \) between \( p_1 \) and \( p_3 \) enclose a simple region, which \( w \) enters through \( p_2 \). However, then there must be another intersection between \( w \) and \( w \) occurring along \( w \) before \( p_2 \), which contradicts our assumption. Thus either \( p_1.y \leq p_2.y \leq p_3.y \) or \( p_1.y \geq p_2.y \geq p_3.y \).

Assume w.l.o.g. \( p_1.y \geq p_2.y \geq p_3.y \) (see Figure 15(d)). The wires between these intersections enclose two disjoint regions \( R_1, R_2 \). Each region contains at least one vertex, let \( u \in R_1 \) and \( v \in R_2 \). Subdivide the plane into four axis-aligned quadrants at \( p_2 \). Region \( R_1 \) lies in the top-left quadrant and \( R_2 \) in the bottom-right quadrant. Thus, \( u.x < v.x \) and \( u.y > v.y \) in \( \Gamma \). As the wires are equivalent to \( \Gamma \), by construction \( u.x > v.x \) and \( u.y < v.y \) in \( \Gamma \). Contradiction, as vertices \( u, v \) cannot be inverted along both axes (Lemma 12). Thus \( W \) cannot contain a lr-wire and a tb-wire that intersect each other more than once.

We conclude that \( W \) is an equivalent set of \( x \)- respectively \( y \)-monotone wires that matches the straight-line wire-grid in \( \Gamma \). As all wires are monotone they have spirality one and thus, by definition, \( \Gamma \) has spirality one relative to \( \Gamma \). □

The claim that \( \Gamma \) has spirality one relative to \( \Gamma \) follows directly by choosing \( \Gamma_i = \Gamma_b \).

5.3 Simplification

In the previous subsection we showed that the spirality of \( \Gamma \) relative to \( \Gamma_b \) is one, where \( \Gamma \) is the initial drawing of an iteration and \( \Gamma_b \) the last drawing. Let \( s \) be the spirality of \( \Gamma \) and, therefore, \( s - 1 \) the spirality of \( \Gamma \). As the morph from \( \Gamma \) to \( \Gamma_b \) inserts additional bends in the edges the complexity of \( \Gamma_b \) may be superlinear. In this section we show that an alternative drawing \( \Gamma_b' \) exists that has linear complexity, has the same spirality as \( \Gamma_b \), and such that \( \Gamma \) has spirality one relative to \( \Gamma_b \).

For future use (Section 6) we discuss the results in a more general case for a drawing \( \Gamma_h \), where \( \Gamma_h \) is a drawing resulting from \( \Gamma \) by performing linear slides on a subset of the maximum absolute spirality links in \( \Gamma \). This subset must contain at least one of the crossing links for each segment in \( \Gamma \) crossed by maximum absolute spirality links. Trivially \( \Gamma_b \) satisfies this requirement.

Let \( \varepsilon \) be a suitably small value such that for each vertex \( v \) in \( \Gamma \), a \( 6\varepsilon \)-sized square box centered at \( v \) contains only \( v \) and a \( 3\varepsilon \)-part of each outgoing segment from \( v \). A rotation for edge \( e \) leaving \( v \) rightwards, is a redrawing of \( e \) within the \( 6\varepsilon \)-box centered at \( v \) using the coordinates \( (v, v +(0, -\varepsilon), v + (2\varepsilon, -\varepsilon), v + (2\varepsilon, 0), v + (3\varepsilon, 0)) \) (see Figure 16). Analogous rerouting can be done for edges leaving \( v \) in other directions. For an edge crossed by a negative spirality link invert the left and right turns. An edge rotation is a rotation at both endpoints of the edge. We use the following technical lemma to prove that this redraw step can safely be done without introducing planarity violations.
Figure 16: A 6ε-box surrounding a vertex v (dashed) with four redrawn edges.

Figure 17: (a) Configuration in ΓO with e left-oriented and f up-oriented. (b) A configuration in Γa when e is counter-clockwise adjacent to f in ΓO. (c) A configuration when e is clockwise adjacent to f in ΓO. (d) A configuration when e and f are horizontal in ΓO.

Lemma 15. Let v be a vertex with at least two outgoing edges e, f and let c be the turn made at v going from e to f, where c = −1 for a right turn, c = 1 for a left turn, and c = 0 otherwise. Let ℓe be a link crossing e and ℓf be a link crossing f. We have \( s(ℓe) + c - 1 \leq s(ℓf) \leq s(ℓe) + c + 1 \).

Proof. In ΓO edges e and f are either both horizontal (vertical) or they have different orientations. For the case where the edges have different orientations, w.l.o.g. assume that in ΓO e is a left-edge for v and f a top-edge for v. By construction e and f are intersected by a pair of wires \( w \in W_\rightarrow \) and \( w' \in W'_\rightarrow \), and they cross before crossing e respectively f (see Figure 17(a)). Wires w and w' together with edges e and f enclose a simple region in ΓO. As the wires in Γa form an equivalent set this simple region also exists in Γa, though its shape and the orientation of the outgoing edges at v may be different.

By construction the border of this region contains a left turn at the intersection between the wires and two left turns at the intersection of the wires with the respective edges e and f (Figure 17(b)). The turn at v depends on the configuration of e and f in Γa. Let \( ℓ_\epsilon, ℓ_f \) be the links of \( w, w' \) crossing e and f. Furthermore, let k be the spirality of the links of w and w' at the crossing between w and w'. As the number of left turns is four larger than the number of right turns when traversing a cycle counter-clockwise we have \((k - s(ℓ_\epsilon)) + (s(ℓ_f) - k) + 3 + c = 4\), simplified \( s(ℓ_f) = s(ℓ_\epsilon) + c - 1 \). When, in ΓO, e is clockwise adjacent to f at v then we get \( s(ℓ_f) = s(ℓ_\epsilon) + c + 1 \) (Figure 17(c)).

In the case where both edges are horizontal (vertical) a similar argument holds, but now the cycle is formed by two wires from \( W'_\rightarrow \) and one wire from \( W_\rightarrow \), resulting in one more left turn. We find \( s(ℓ_f) = s(ℓ_\epsilon) + c \) (Figure 17(d)). Combining all bounds gives the result. □

Lemma 16. We can rotate all edges in Γh that were crossed by a maximal spirality link in Γa, without causing planarity violations and without changing the cyclic order.

Proof. Trivially planarity violations occur only inside the 6ε-boxes at the endpoints. There are two possible cases causing a planarity violation when redrawing Γh. First, two perpendicular edges leaving v coincide internally after the redraw step. This occurs if one of the edges is crossed by links of absolute spirality s and the other is not. Second, two edges leaving v in
opposing direction coincide internally after the redraw step. This occurs if one of the edges is crossed by links of spirality $s$ and the other by links of spirality $-s$.

For the first case, w.l.o.g. assume $e$ is a right-edge of $v$ and $f$ a bottom-edge. Let $\ell_e$ be a link crossing $e$ and $\ell_f$ a link crossing $f$ in $\Gamma_a$. Assume w.l.o.g. $s(\ell_e) = s$ and $s(\ell_f) < s$. By Lemma 15, using $c = 1$, in $\Gamma_a$ we have $s(\ell_e) \leq s(\ell_f) \leq s(\ell_e) + 2$. As $s(\ell_e) = s$ and no larger spiralities exists in $\Gamma_a$, we must have $s(\ell_f) = s(\ell_e)$. Contradiction.

For the second case, w.l.o.g. assume $e$ is a right-edge of $v$ and $f$ a left-edge. Let $\ell_e$ be a link crossing $e$ and $\ell_f$ a link crossing $f$, where $s(\ell_e) = s$ and $s(\ell_f) = -s$. Using Lemma 15, with $c = 0$, we get $-s < 0 \leq s(\ell_e) - 1 \leq s(\ell_f)$. Contradiction.

The cyclic order can only change when there are two perpendicular edges $e, f$ in $\Gamma_a$ that are redrawn in opposing directions and hence are crossed by a link of positive respective negative spirality. W.l.o.g. let the turn from $e$ to $f$ at $v$ be a left-turn. To change in the cyclic order $e$ must be crossed by a link of spirality $s$ and $f$ must be crossed by a link of spirality $-s$. However, by Lemma 15 $s(\ell_e) \leq s(\ell_f)$. Contradiction.

Lemma 17. There exists a straight-line drawing $\Gamma_h'$ with spirality equal to $\Gamma_h$.

Proof. Consider an edge $e$ that in $\Gamma_a$ is crossed by links of maximum absolute spirality $s$. Without loss of generality let $s > 0$. For the case where $s < 0$ exchange left and right turns in the following argument. All links crossing $e$ have positive (equivalent) spirality (Lemma 2), therefore a linear slide along any such link $\ell_a$ introduces a right bend followed by a left bend in $e$. Thus $e$ has an odd number of segments in $\Gamma_h$ and the turns along $e$ are alternating right and left turns, starting with a right turn (see Figure 18(b)). Moreover, at least one linear slide was performed on a link crossing $e$ and hence in $\Gamma_h$ at least one link of spirality $s - 1$ crosses $e$ and it crosses $e$ in a segment started by a right bend and followed by a left bend.

Rotate $e$ within the $6\varepsilon$-boxes near the endpoints to create two left turns and a right turn at the start of the edge, and one left turn and two right turns at the end of the edge (see Figure 18(c)). We can do this while maintaining planarity of the drawing (Lemma 16). Thus the bends in $e$ in $\Gamma_h$ can be encoded as $LLR (RL)^+ LRR$, where $L$ encodes a left turn, $R$ a right turn, and $(RL)^+$ is the alternating sequence of turns starting with a right turn and ending with a left turn. Split differently we have $LLRR (LR)^+ LLRR$, where $(LR)^+$ is the possibly empty alternating sequence of left and right turns starting with a left turn and ending with a right turn. Two turns in this sequence enclose a segment of $e$.

We can remove a pair of consecutive bends $LR$ by performing a zigzag-removing slide on the segment $\sigma$ between the bends. As every link with spirality $s - 1$ crosses a segment started by a right bend, any link crossing $\sigma$ must have spirality $s$. If multiple wires intersect $\sigma$, then we can first split $\sigma$ using a bend-introducing slide. Thus, assume $\sigma$ is crossed by at most one link. Removing $\sigma$ with a zigzag-removing slide introduces a single new link in the crossing wire and it crosses $e$. After removing every pair of consecutive $LR$ bends, the edge forms a single straight-line segment (see Figure 18(d)). This segment is crossed by at least one link of spirality $s - 1$, and by all newly introduced links. As all crossing links must have the same spirality (by Lemma 2) all newly introduced links have spirality $s - 1$. Thus this operation does not increase the spirality. The result is a straight-line version $\Gamma_h'$ of $\Gamma_h$ with equal spirality.
Figure 18: (a) Drawing \( \Gamma_a \) (black) and the maximum absolute spirality links (red, blue) crossing the edges of \( \Gamma_a \). (b) Drawing \( \Gamma_h \) that results by performing linear slides along all maximum absolute spirality links except the top-left link. (c) The drawing after locally redrawing incoming edges that are not a straight-line segment. (d) Straightening the drawing removes all bends from the edges. Additional segments are introduced in remaining maximum spirality links, however this does not increase the maximum absolute spirality.

**Lemma 18.** Drawing \( \Gamma_a \) has spirality one relative to \( \Gamma'_h \).

**Proof.** Consider a straight-line wire grid in \( \Gamma'_h \). We revert \( \Gamma'_h \) to \( \Gamma_h \) using bend-introducing slides to re-insert the additional bends in the edges. During this process we move the wires from \( \Gamma'_h \) along. As bend-introducing slides do not create any additional bends in the straight-line wires, there exists a set of equivalent wires in \( \Gamma_h \), to the straight-line wire-grid in \( \Gamma'_h \), that are also straight lines. This set of equivalent wires in \( \Gamma_h \) must be a subset of the complete straight-line wire-grid in \( \Gamma'_h \), and thus by Lemma 14 there must exist an equivalent set of wires in \( \Gamma_a \) that is monotone. But then \( \Gamma_a \) has spirality one relative to \( \Gamma'_h \). \( \square \)

As trivially \( \Gamma_b \) can be created from \( \Gamma_a \) by performing linear slides along a suitable subset of the maximum spirality links (viz. the complete set), we conclude that a straight-line drawing \( \Gamma'_b \) exists such that \( \Gamma_a \) has spirality one relative to \( \Gamma'_b \), and \( \Gamma'_b \) has spirality \( s - 1 \).

### 5.4 Single linear morph

We have shown that given a straight-line drawing \( \Gamma_a \) with spirality \( s \), we can find a straight-line drawing \( \Gamma'_b \) with spirality \( s - 1 \) such that \( \Gamma_a \) has spirality one relative to \( \Gamma'_b \). We now show that any two planar (unified) orthogonal drawings \( \Gamma_i \) and \( \Gamma_j \), where \( \Gamma_i \) has spirality one relative to \( \Gamma_j \), can be morphed using a single linear morph while maintaining planarity.

Two drawings are **shape-equivalent** if for each edge the sequence of left and right turns is identical and the initial segment is horizontal (vertical) in both drawings. We say two drawings are **degenerate shape-equivalent** if edges may contain zero-length segments but an assignment of orientations to the segments exists that is consistent with both drawings. Two (degenerate) shape-equivalent drawings are per definition also unified. We discuss how to make \( \Gamma_i \) degenerate shape-equivalent to \( \Gamma_j \) such that the linear interpolation from \( \Gamma_i \) to \( \Gamma_j \) is planar. To achieve this we use the intersections of the drawing with the wires (with spirality one). Note that we need not concern ourselves with planarity of the wires during the morph as this is not required for planarity of the final morph (of the drawing).
Lemma 19. Let $\Gamma_i$ and $\Gamma_j$ be two unified planar orthogonal drawings, where $\Gamma_i$ has spirality one to $\Gamma_j$. There exists a single linear morph from $\Gamma_i$ to $\Gamma_j$ that maintains planarity and orthogonality.

Proof. We say two points $p$ and $q$ on a drawing are split by a wire when $p$ and $q$ lie on different sides of the wire. The partition of the drawing by all wires defines cells; regions of the plane not split by any wire.

We first make $\Gamma_i$ degenerate shape-equivalent to $\Gamma_j$. Consider all intersections in $\Gamma_i$ between a maximum-spirality link and a segment of the drawing. For each intersection between an edge $e$ and a maximum-spirality link $\ell_s$ we add a zero-length segment in $\Gamma_i$ at the intersection of $e$ and $\ell_s$. We symbolically perturb the endpoints of the zero-length segment such that each endpoint is in a different cell. To ensure that $\Gamma_j$ remains unified we also add an additional segment in $\Gamma_j$. However here we place the endpoints strictly inside the cells of the drawing while ensuring that in $\Gamma_j$: (1) the endpoints of the segment are in the same cells as the (symbolically perturbed) endpoints in $\Gamma_i$, (2) all bends and vertices enclosed by two consecutive horizontal wires have the same $y$-coordinate, (3) all bends and vertices enclosed by two consecutive vertical wires have the same $x$-coordinate (see Figure 19).

For each cell containing at least one bend or vertex, linearly interpolate all vertices and bends in $\Gamma_i$ to the unique vertex or bend location in $\Gamma_j$. This directly defines a linear interpolation for each point (not necessarily a vertex or bend) between $\Gamma_i$ and $\Gamma_j$.

First, we prove that the described linear morph maintains orthogonality. The endpoints of all (zero-length) segments crossing a tb-wire have the same $y$-coordinates in $\Gamma_i$ and $\Gamma_j$, hence they remain horizontal. Symmetrically all segments crossing a lr-wire remain vertical. All other segments morph to a single point and, as the $x$-coordinates ($y$-coordinates) of the endpoints are equivalent at both the start and end of the linear morph, they remain equivalent throughout the interpolation as well.

Second, we prove that during the described linear morph the drawing remains planar. Assume for contradiction there exist two distinct points $p, q$ on an edge or vertex of the drawing that coincide during the linear interpolation (excluding $\Gamma_i, \Gamma_j$). By linear motion the $x$-coordinates and $y$-coordinates of $p$ and $q$ change linearly. To be identical at a time $0 < t < 1$ during the morph we need that either $p$ and $q$ have identical $x$-coordinates in
Figure 20: (a) Two points $p$ and $q$ on the same vertical line in $\Gamma_i$ and $\Gamma_j$ require spirality at least two. (b) Two points $p$ and $q$ on vertical segments of the drawing that are inverted along both axes imply wires in $\Gamma_i$ that are not equivalent to $\Gamma_j$. (c) Points $p$ and $q$ on a horizontal and vertical segment. (d) Points $p$ and $q$ on horizontal segments.

$\Gamma_i$ and $\Gamma_j$ but are $y$-order inverted, $p$ and $q$ have identical $y$-coordinates but are $x$-order inverted, or $p$ and $q$ are inverted along both axes.

**Case 1:** ($p$ and $q$ have identical $x$-coordinates but are $y$-order inverted)

We prove by contradiction that this cannot be the case. Assume w.l.o.g. that $p.y < q.y$ in $\Gamma_i$ and $p.y > q.y$ in $\Gamma_j$. Note that if $p.y = q.y$ in either $\Gamma_i$ or $\Gamma_j$, then by linear motion there can not be an intersection during the linear morph itself as there already is an intersection at one of the endpoints of the morph.

Points $p$ and $q$ cannot be on the same segment as the orientation of each segment is (degenerately) maintained. Thus $p$ and $q$ must be on different segments. Let $s$ be a lowest endpoint of the segment containing $q$ and $r$ a highest endpoint of the segment containing $p$ (see Figure 20(a)). We must have either $r.y < s.y$ in $\Gamma_i$ and $r.y > s.y$ in $\Gamma_j$ or $r.y = s.y$. The assumption $r = s$ results in a contradiction as in $\Gamma_j$ we need that $r.y \geq p.y > q.y \geq s.y$. Thus we must have that $r.y < s.y$ in $\Gamma_i$ and $r.y > s.y$ in $\Gamma_j$. But if $r.y > s.y$ in $\Gamma_j$ then $r$ and $s$ (and thus $p$ and $q$) are split by a horizontal straight-line wire. This wire passes below $p$ and above $q$. However, there exists no $x$-monotone wire in $\Gamma_i$ that passes below $p$ and above $q$. Contradiction as $\Gamma_i$ has spirality one.

**Case 2:** ($p$ and $q$ have identical $y$-coordinates but are $x$-order inverted)

This case is symmetrical to case 1.

**Case 3:** ($p$ and $q$ are inverted along both axes)

Assume w.l.o.g. that $p.x < q.x$ and $p.y < q.y$ in $\Gamma_i$. We distinguish whether $p$ and $q$ are on a horizontal or vertical segment. We will work out the first case in detail and indicate the setup for the other cases, which are analogous.
Case 3a: \( (p \text{ and } q \text{ are both on a vertical segment in } \Gamma_j \text{ – Figure 20(b)}) \)

Let \( r \) be the top endpoint of the segment containing \( p \) and \( s \) the bottom endpoint of the segment containing \( q \). In \( \Gamma_j \) we have \( r.y > s.y \) and \( r.x > s.x \). As \( r \) and \( s \) have distinct \( x \)- and \( y \)-coordinates they are split by at least one tb-wire and one lr-wire in \( \Gamma_j \). As \( \overline{pr} \) and \( \overline{qs} \) are also vertical segments in \( \Gamma_i \) we must have \( r.x < s.x \). The matching (monotone) tb-wire in \( \Gamma_i \) splits \( p \) and \( q \) identically and, similarly to \( \Gamma_j \) does not cross the segments \( \overline{pr} \) and \( \overline{qs} \). Due to the relative position of \( p \) and \( q \), the monotonicity of the tb-wire, and \( r.x < s.x \), it must also be that \( r.y < s.y \) in \( \Gamma_i \). But then the lr-wire splitting \( r \) and \( s \) must cross the tb-wire at least three times in \( \Gamma_i \). Contradiction.

Case 3b: \( (p \text{ is on a horizontal segment and } q \text{ on a vertical segment – Figure 20(c)}) \)

Let \( r \) be the right endpoint of the segment containing \( p \) and \( s \) be the bottom endpoint of the segment containing \( q \). Once again we reach the contradiction that the tb-wire and the lr-wire splitting \( r \) and \( s \) must cross at least three times in \( \Gamma_i \).

Case 3c: \( (p \text{ and } q \text{ are both on a horizontal segment – Figure 20(d)}) \)

Let \( r \) be the right endpoint of the segment containing \( p \) and \( s \) be the left-endpoint of the segment containing \( q \).

We conclude there do not exist two distinct points \( p, q \) on the edges (vertices) of the drawing that coincide during the linear morph.

To reduce the need for a final linear morph from a drawing similar to \( \Gamma_O \) to \( \Gamma_O \) itself we make one more observation.

**Corollary 1.** If a morph from a drawing \( \Gamma_a \) with spirality one to a drawing \( \Gamma_b \) is planar, then the morph from \( \Gamma_a \) to any drawing similar to \( \Gamma_b \) is planar.

**Proof.** Two similar drawings have the same \( x \)- and \( y \)-order. The planarity of a linear morph remains intact if the \( x \)- and \( y \)-order of the vertices and bends does not change.

**Theorem 4.** Let \( \Gamma_I \) and \( \Gamma_O \) be two unified planar orthogonal drawings of a (potentially disconnected) graph \( G \), where \( \Gamma_I \) has spirality \( s \) to \( \Gamma_O \). We can morph \( \Gamma_I \) into \( \Gamma_O \) using exactly \( s \) linear morphs while maintaining planarity and orthogonality, and keeping the intermediate complexity of the drawing reduced to \( O(n^2) \).

**Proof.** By Lemma 14 for every iteration the initial drawing \( \Gamma_s \) has spirality one relative to the first drawing \( \Gamma_{s-1} \) of the next iteration. By Lemma 17 and 18 there also exists a straight-line drawing \( \Gamma'_{s-1} \) such that \( \Gamma_s \) has spirality one to \( \Gamma'_{s-1} \) and \( \Gamma'_{s-1} \) has spirality \( s - 1 \) (to \( \Gamma_O \)). By Lemma 19 we can morph \( \Gamma_s \) to \( \Gamma'_{s-1} \) with a single linear morph, reducing the spirality by one. After repeating this process \( s \) times the spirality is reduced to zero and the resulting drawing must be similar to \( \Gamma_O \). A final single linear morph simplifies the drawing to \( \Gamma_O \). We can slightly improve this as by Corollary 1 we can merge the last two morphs without affecting the planarity of the morph.

As each single linear morph can increase the intermediate complexity by at most \( O(n^2) \) and after each single linear morph we have a straight-line drawing again (of complexity \( O(n) \)), the maximum intermediate complexity during the morph is \( O(n^2) \).
6 Maintaining linear complexity

The approach from Section 5 maintains planarity and orthogonality and reduces complexity of the drawing to $O(n)$ after each linear morph. However, during a single linear morph complexity may increase to $\Theta(n^2)$ as each edge may be crossed by $O(n)$ wires each introducing two additional bends. We refine the approach to ensure that the drawing also maintains $O(n)$ complexity during the morph. For each edge intersected by links of maximum absolute spirality, we perform a linear slide along only one of the intersecting links. Thus at most two bends are introduced in each edge, directly ensuring the linear complexity of the drawing is maintained.

The remaining wires are rerouted to cross the newly introduced segment of the edge. In general rerouting the wires cannot be done without increasing their absolute spirality, which is problematic as we aim to reduce the maximum absolute spirality with each single linear morph. We show that we can perform an initial step locally inserting windmills in the wires adjacent to the crossed edges. Windmills introduce additional initial complexity to the wires to prevent the need to add extra complexity to the drawing or extra spirality to the wires during the morph itself. This initial rerouting of the wires in $\Gamma_I$ increases the maximum absolute spirality by one. Thus, using Theorem 4, $s + 1$ morphs are sufficient to morph two equivalent drawings into each other while maintaining planarity, orthogonality, and linear complexity.

Windmills. The initial rerouting step reroutes all wires in $W_\downarrow$ and $W_\rightarrow$ locally at each crossed edge if the wires satisfy the following criteria: 1) the absolute spirality of the crossing links is greater than zero and 2) at least two links cross the edge. W.l.o.g. consider a horizontal edge $e$ that is crossed by at least two wires in $\Gamma_I$. By Lemma 2 all crossing links have the same spirality. Assume w.l.o.g. that this spirality is positive, otherwise mirror the rotations and replace right by left. Let $\varepsilon$ be a small distance such that the $\varepsilon$-band above and below $e$ is empty except for the links crossing $e$ (see Figure 21(a)). We reroute the wires within the $\varepsilon$-band around $e$. First disconnect all crossing links within the $\varepsilon$-band. Then reroute all wires below $e$ in a parallel bundle to the left, past the left-most wire $w_1$ crossing $e$. We show that the complexity remains linear each time a windmill is inserted.

![Figure 21](image-url)

Figure 21: (a) The empty $\varepsilon$-band. (b) Introducing an $s$-windmill ($s = 1$). The outlined edges form a $s$-windmill. The $s$-windmill is connected to the original wire with links of spirality $s + 1$. (c) The $s$-windmill after reducing all links of spirality $s + 1$. Marked bends are separated by the right-most crossing link. (d) After a linear slide along the right-most crossing link. (e) After rerouting.
We now insert a so-called $s$-windmill which is defined as follows (see Figure 21(b)). Start by spiralling the bundle using right turns until the spirality of the links reaches minus one. Next we unwind the bundle again within the spiral until we reach links of spirality $s - 2$. We then route the wires parallel back along $e$ to ensure each wire crosses $e$ in its original location and with a link of spirality $s$. We repeat this process above the edge, where we reroute all wires in a parallel bundle above $e$ to the right beyond the right-most wire $w_r$ crossing $e$. Again we spiral the bundle using right turns until the links have spirality minus one, then unwind the bundle until spirality $s$. This concludes the $s$-windmill. Finally we reconnect the wires by routing back parallel to $e$ to maintain the original crossing points.

The creation of the initial $s$-windmill locally increases the spirality by one. After the first iteration only the $s$-windmill is left (see Figure 21(c)).

**Morphing with rerouting.** Once again consider the morph as a sequence of individual linear slides. Specifically consider a single iteration of the resulting morph. The linear slides performed within an iteration are all based on links of the same maximum absolute spirality. As we have not enforced any order on these linear slides we can perform them in any order. We now define a partial ordering on the linear slides that, indirectly, prevents excess complexity from being introduced in the drawing during the individual morphs.

Specifically consider an ordering that first performs all slides caused by maximum-spirality links that do not cross a segment of the drawing. Then, of the remaining maximum-spirality links, the linear slides defined by links meeting the following criteria are performed: For each horizontal edge $e$ crossed by $k > 1$ links of maximum absolute spirality we perform the linear slide caused by the right-most link crossing $e$. This slide creates a new vertical segment (see Figure 21(d)). For each vertical edge crossed by $k > 1$ links of maximum-spirality we perform the slide defined by the top-most link crossing the edge.

Let $\Gamma_s$ be the first drawing of a given iteration and let $\Gamma_s$ have spirality $s$. Furthermore consider the intermediate drawing $\Gamma_{s-1}^{r}$ that results from performing all the slides defined above. Using Lemma 14 we directly derive the following lemma.

**Lemma 20.** Drawing $\Gamma_s$ has spirality one relative to $\Gamma_{s-1}^{r}$.

We reroute the wires by shortcutting them as a bundle across the new segment (see Figure 21(e)).

**Lemma 21.** In $\Gamma_{s-1}^{r}$ all remaining maximum-spirality links crossing an edge $e$ can be shortcut across the newly created segment in $e$.

**Proof.** Assume w.l.o.g. that the spirality is positive and that $e$ is horizontal. Let $\ell$ be the (right-most) link that caused the linear slide introducing a new segment in $e$. Let $w$ be a wire crossing $e$ that is not the right-most wire. Let $\ell_w$ be the link from $w$ crossing $e$. Finally let $\ell_{w+2}$ be the first link along $w$ after $\ell_w$ that does not share an endpoint with $\ell_w$. In $\Gamma_s$ the start point $s_w$ of $\ell_w$ is separated by $\ell$ from the endpoint $t_{w+2}$ of $\ell_{w+2}$ (see Figure 21(c)). However, the start point $s_{w+2}$ of $\ell_{w+2}$ is not separated from $s_w$. After performing the linear slide along $\ell$, endpoint $t_{w+2}$ has a smaller $y$-coordinate than $s_w$, whereas $s_w$ still has a larger $y$-coordinate. But then after performing the linear slide defined by $\ell$ a rightwards ray from...
the start point of $\ell_w$ must cross $\ell_{w+2}$. We can horizontally shortcut each wire. As the order of the wires is identical on both sides we can do so without introducing new crossings.

**Corollary 2.** Drawing $\Gamma_{s-1}^r$ has spirality $s - 1$ relative to $\Gamma_O$.

**Proof.** Rerouting the wires removes all links of maximum-spirality. Thus the resulting wires have maximum-spirality $s - 1$, and are equivalent to the straight-line wire grid in $\Gamma_O$. But then $\Gamma_{s-1}^r$ has spirality $s - 1$ relative to $\Gamma_O$.

**Lemma 22.** At the start of iteration $s - 1$ of the final morph including intermediate simplification and rerouting, all wires crossing an edge $e$ with links of spirality $s - 1$ form an $(s - 1)$-windmill in an empty $\varepsilon$-band next to $e$.

**Proof.** As the $\varepsilon$-band is empty except for the windmill, every link that is not part of the windmill cannot separate two bends that are part of the windmill. Such links by Lemma 10 cannot destroy the structure (the $x$- and $y$-order of the bends) of the windmill, so we only concern ourselves with links that are part of the windmill. Performing a linear slide along the right-most link $\ell$ crossing $e$ maintains all links in the windmill except for $\ell$ (see Figure 21(d)). Moreover, after the linear slide every wire, after crossing $e$, will be routed along the newly generated segment past the newly produced intersection of $e$ with the right-most wire. Rerouting the remaining wires (see Figure 21(e)) removes all spirality $s$ links, but maintains all other links. The spirality of the resulting set of wires is at most $s - 1$ and they form an $(s - 1)$-windmill next to $e$.

By Lemma 20 drawing $\Gamma_s$ has spirality one relative to $\Gamma_{s-1}^r$. Thus by Lemma 19 we can reduce the spirality by one using a single linear morph, while increasing the complexity of the drawing by only $O(n)$. By Lemma 22 this process can be repeated without increasing spirality intermittently. All that is left is to reintroduce the intermittent simplification to maintain $O(n)$ complexity.

**Lemma 23.** Drawing $\Gamma_s$ has spirality one relative to drawing $\Gamma_{s-1}^r$ and $\Gamma_{s-1}^r$ has spirality $s - 1$.

**Proof.** By Lemma 20 $\Gamma_s$ has spirality one relative to $\Gamma_{s-1}^r$. By Lemma 16 and 17 we can redraw $\Gamma_{s-1}^r$ to a straight-line drawing $\Gamma_{s-1}^r$ while maintaining that $\Gamma_s$ has spirality one relative to $\Gamma_{s-1}^r$.

We conclude that given a straight-line orthogonal drawing $\Gamma_s$ of complexity $O(n)$ and with spirality $s$ to $\Gamma_O$, a straight-line orthogonal drawing $\Gamma_{s-1}^r$ exists with complexity $O(n)$ and spirality $s - 1$ to $\Gamma_O$. Moreover, $\Gamma_s$ has spirality one to $\Gamma_{s-1}^r$ and the unified complexity of $\Gamma_s$ and $\Gamma_{s-1}^r$ is $O(n)$. By Lemma 19 we can morph $\Gamma_s$ to $\Gamma_{s-1}^r$ with a single linear morph while maintaining planarity, orthogonality, and linear complexity.
7 Final Algorithm

Including all improvements we obtain the following algorithm (Algorithm 1). Let $\Gamma_I$ and $\Gamma_O$ be two equivalent orthogonal drawings of a (potentially disconnected) graph $G$. We first unify $\Gamma_I$ and $\Gamma_O$ to ensure they are equivalent orthogonal straight-line drawings. This can be done without increasing the complexity. Find two sets of wires $W_\rightarrow$ and $W_\downarrow$ that are an equivalent set of wires for $\Gamma_I$, with respect to the straight-line wire-grid in $\Gamma_O$, having maximum spirality $s = O(n)$. By Theorem 2 such a set exists. Introduce windmills locally at each crossed edge if 1) the absolute spirality of the edge is greater than zero, 2) at least two wires cross the edge. This increases the maximum absolute spirality by one to $s + 1$.

Consider the drawing $\Gamma_{s+1} = \Gamma_I$ together with the rerouted wire set.

We now repeat the following for each orthogonal straight-line drawing $\Gamma_t$ with spirality $0 < t \leq s + 1$. Perform linear slides on all maximum-spirality links that do not intersect the drawing. Perform linear slides on all maximum-spirality links that are the right-most link crossing a horizontal edge or the top-most link crossing a vertical edge. Reroute the wires to remove all other maximum-spirality links. Straighten the drawing to the resulting drawing $\Gamma_{t-1}$. Drawing $\Gamma_t$ has spirality one relative to $\Gamma_{t-1}$ and $\Gamma_{t-1}$ has spirality $t - 1$ (Lemma 23) and $O(n)$ complexity. By Lemma 19 we can linearly interpolate $\Gamma_t$ to $\Gamma_{t-1}$ while maintaining planarity, and orthogonality.

Algorithm 1 Final morphing algorithm

Require: Two equivalent planar orthogonal drawings $\Gamma_I$ and $\Gamma_O$ with maximum complexity $O(n)$. Moreover let $s$ be the spirality of $\Gamma_I$ with respect to $\Gamma_O$.

Ensure: A sequence of $s + 1 = O(n)$ linear morphs that morph $\Gamma_I$ to $\Gamma_O$.

- Unify $\Gamma_I$ and $\Gamma_O$. (Section 2)
- Find an equivalent set of wires for $\Gamma_I$ compared to $\Gamma_O$. (Section 3)
- Let $s$ be the maximum absolute spirality of the wires in $\Gamma_I$.
- Add windmills to $\Gamma_I$. (Section 6)
- Let $\Gamma_{s+1}$ be the current drawing and let $t = s + 1$.

while $t > 0$ do

- Based on $\Gamma_t$ compute an orthogonal straight-line drawing $\Gamma_{t-1}$. (Section 6)
- Morph $\Gamma_t$ to $\Gamma_{t-1}$ with a single linear morph. (Section 5.4)
- $t \leftarrow t - 1$

end while

As complexity only increases by $O(n)$ during each linear morph and each drawing is simplified to $O(n)$ complexity again we also maintain linear complexity of the drawing during the morph. Thus using $s + 1$ linear morphs we can morph $\Gamma_I$ to $\Gamma_O$ while maintaining planarity, orthogonality, and linear complexity.

Thus we directly obtain the final Theorem:

Theorem 5. Let $\Gamma_I$ and $\Gamma_O$ be two equivalent drawings of a (potentially disconnected) graph $G$, where $\Gamma_I$ has spirality $s$. We can morph $\Gamma_I$ into $\Gamma_O$ using $s + 1$ linear morphs while maintaining planarity, orthogonality, and linear complexity of the drawing during the morph.
8 Conclusion

We have described a morph of two planar orthogonal drawings of a (potentially disconnected) graph $G$ of complexity $n$, using $O(n)$ linear morphs. To this end we used the spirality $s$ of the drawing $\Gamma$ of $G$, which we have shown is $O(n)$ even for disconnected graphs. We further refined the analysis to show that not only are $O(n)$ linear morphs sufficient, but indeed $s + 1$ linear morphs are sufficient to maintain planarity, orthogonality, and linear complexity during the complete morph. There remain a number of open questions surrounding morphs of orthogonal drawings.

First, it is clear that $s$ morphs is a lowerbound for our approach. It is unclear whether $s + 1$ linear morphs are needed to maintain linear complexity though. Preliminary investigations indicate this may be the case, but as it stands $s + 1$ is just one off from a trivial lowerbound. Second, it is not yet clear what the required time complexity is to compute the described morphs. Our proofs are mostly constructive, but the efficiency of computing the morph was not taken into account. Third, from a visualization perspective the reduced number of linear morphs is satisfying, but the drawing may be arbitrarily scaled during the morph. The final morph does not take stability of the drawing into account, something that would be desirable for practical application.

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