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Left Invariant Evolution Equations on Gabor Transforms

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Abstract By means of the unitary Gabor transform one can relate operators on signals to operators on the space of Gabor transforms. In order to obtain a translation and modulation invariant operator on the space of signals, the corresponding operator on the reproducing kernel space of Gabor transforms must be left invariant, i.e. it should commute with the left regular action of the reduced Heisenberg group H_r. By using the left invariant vector fields on H_r and the corresponding left-invariant vector fields on on phase space in the generators of our transport and diffusion equations on Gabor transforms we naturally employ the essential group structure on the domain of a Gabor transform. Here we mainly restrict ourselves to non-linear adaptive left-invariant convection (reassignment), while maintaining the original signal.

1 Introduction

The Gabor transform of a signal $f \in \mathbb{L}_2(\mathbb{R}^d)$ is a function $\mathscr{G}_\psi[f] : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ that can be roughly understood as a musical score of f, with $\mathscr{G}_\psi[f](p,q)$ describing the contribution of frequency q to the behaviour of f near p [19, 22]. This interpretation is necessarily of limited precision, due to the various uncertainty principles, but it has nonetheless turned out to be a very rich source of mathematical theory as well as practical signal processing algorithms.

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The use of a window function for the Gabor transform results in a smooth, and to some extent blurred, time-frequency representation; though keep in mind that by the uncertainty principle, there is no such thing as a "true time-frequency representation". For purposes of signal analysis, say for the extraction of instantaneous frequencies, various authors tried to improve the resolution of the Gabor transform, literally in order to sharpen the time-frequency picture of the signal; this type of procedure is often called "reassignment" in the literature. For instance, Kodera et al. [26] studied techniques for the enhancement of the spectrogram, i.e. the squared modulus of the short-time Fourier transform. Since the phase of the Gabor transform is neglected, the original signal is not easily recovered from the reassigned spectrogram. Since then, various authors developed reassignment methods that were intended to allow (approximate) signal recovery [2, 5, 7]. We claim that a proper treatment of phase may be understood as *phase-covariance*, rather than *phase-invariance*, as advocated previously. An illustration of this claim is contained in Figure 1.

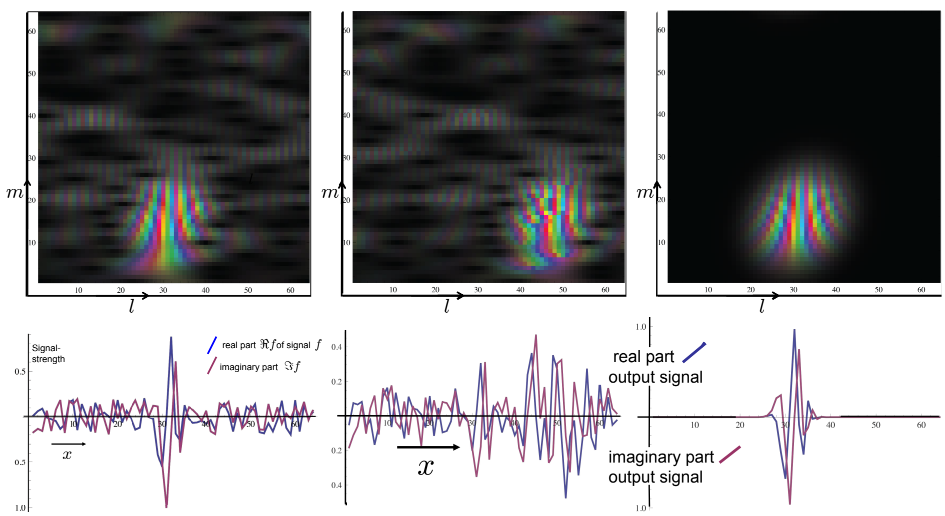

Fig. 1 Top row from left to right, (1) the Gabor transform of original signal f, (2) processed Gabor transform $\Phi_t(\mathscr{W}_\psi f)$ where Φ_t denotes a phase invariant shift (for more elaborate adaptive convection/reassignment operators see Section 6 where we operationalize the theory in [7]) using a discrete Heisenberg group, where l represents discrete spatial shift and m denotes discrete local frequency, (3) processed Gabor transform $\Phi_t(\mathscr{W}_\psi f)$ where Φ_t denotes a *phase covariant* diffusion operator on Gabor transforms with stopping time $t > 0$. For details on phase covariant diffusions on Gabor transforms, see [14, ch:7] and [25, ch:6]. Note that phase-covariance is preferable over phase invariance. For example restoration of the old phase in the phase invariant shift (the same holds for the adaptive phase-invariant convection) creates noisy artificial patterns (middle image) in the phase of the transported strong responses in the Gabor domain. Bottom row, from left to right: (1) Original complex-valued signal f, (2) output signal $\Upsilon_\psi f = \mathscr{W}_\psi^* \Phi_t \mathscr{W}_\psi f$ where Φ_t denotes a *phase-invariant* spatial shift (due to phase invariance the output signal looks bad and clearly phase invariant spatial shifts in the Gabor domain do not correspond to spatial shifts in the signal domain), (3) Output signal $\Upsilon_\psi f = \mathscr{W}_\psi^* \Phi_t \mathscr{W}_\psi f$ where Φ_t denotes *phase-covariant* adaptive diffusion in the Gabor domain with stopping time $t > 0$.

We adapt the group theoretical approach developed for the Euclidean motion groups in the recent works [9, 18, 12, 13, 15, 11], thus illustrating the scope of the methods devised for general Lie groups in [10] in signal and image processing. Reassignment will be seen to be a special case of left-invariant convection. A useful source of ideas specific to Gabor analysis and reassignment was the paper [7].

The chapter is structured as follows: Section 2 collects basic facts concerning the Gabor transform and its relation to the Heisenberg group. Section 3 contains the formulation of the convection-diffusion schemes. We explain the rationale behind these schemes, and comment on their interpretation in differential-geometric terms. Section 4 is concerned with a transfer of the schemes from the full Heisenberg group to phase space, resulting in a dimension reduction that is beneficial for implementation. The resulting scheme on phase space is described in Section 5. For a suitable choice of Gaussian window, it is possible to exploit Cauchy-Riemann equations for the analysis of the algorithms, and the design of more efficient alternatives. Section 6 describes a discrete implementation, and presents some experiments.

2 Gabor transforms and the reduced Heisenberg group

Throughout the paper, we fix integers $d \in \mathbb{N}$ and $n \in \mathbb{Z} \setminus \{0\}$. The continuous Gabor-transform $\mathscr{G}_\psi[f] : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ of a square integrable signal $f : \mathbb{R}^d \to \mathbb{C}$ is commonly defined as

$$\mathscr{G}_\psi[f](p,q) = \int_{\mathbb{R}^d} f(\xi)\overline{\psi(\xi - p)} e^{-2\pi n i (\xi - p)\cdot q} \, \mathrm{d}\xi \;, \tag{1}$$

where $\psi \in \mathbb{L}_2(\mathbb{R}^d)$ is a suitable window function. For window functions centered around zero both in space and frequency, the Gabor coefficient $\mathscr{G}_\psi[f](p,q)$ expresses the contribution of the frequency nq to the behaviour of f near p.

This interpretation is suggested by the Parseval formula associated to the Gabor transform, which reads

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathscr{G}_\psi[f](p,q)|^2 \, \mathrm{d}p \, \mathrm{d}q = C_\psi \int_{\mathbb{R}^d} |f(p)|^2 \, \mathrm{d}p, \;\; \text{where } C_\psi = \frac{1}{n} \|\psi\|^2_{\mathbb{L}_2(\mathbb{R}^d)} \tag{2}$$

for all $f, \psi \in \mathbb{L}_2(\mathbb{R}^d)$. This property can be rephrased as an inversion formula:

$$f(\xi) = \frac{1}{C_\psi} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{G}_\psi[f](p,q) \, e^{i2\pi n(\xi - p)\cdot q} \, \psi(\xi - p) \, \mathrm{d}p \mathrm{d}q \;, \tag{3}$$

to be read in the weak sense. The inversion formula is commonly understood as the decomposition of f into building blocks, indexed by a time and a frequency parameter; most applications of Gabor analysis are based on this heuristic interpretation. For many such applications, the phase of the Gabor transform is of secondary importance (see, e.g., the characterization of function spaces via Gabor coefficient decay [20]). However, since the Gabor transform uses highly oscillatory complex-valued functions, its phase information is often crucial, a fact that has been specifically acknowledged in the context of reassignment for Gabor transforms [7].

For this aspect of Gabor transform, as for many others, the group-theoretic viewpoint becomes particularly beneficial. The underlying group is the *reduced Heisenberg group* H_r. As a set, $H_r = \mathbb{R}^{2d} \times \mathbb{R}/\mathbb{Z}$, with the group product

$$(p,q,s+\mathbb{Z})(p',q',s'+\mathbb{Z}) = (p+p',q+q',s+s'+\frac{1}{2}(q\cdot p'-p\cdot q')+\mathbb{Z}) \ .$$

This makes H_r a connected (nonabelian) nilpotent Lie group. The Lie algebra is spanned by vectors $A_1,\ldots,A_{2d+1}$ with Lie brackets $[A_i,A_{i+d}] = -A_{2d+1}$, and all other brackets vanishing.

H_r acts on $\mathbb{L}_2(\mathbb{R}^d)$ via the *Schrödinger representations* $\mathscr{U}^n : H_r \to \mathscr{B}(\mathbb{L}_2(\mathbb{R}))$,

$$\mathscr{U}^n_{g=(p,q,s+\mathbb{Z})}\psi(\xi) = e^{2\pi i n(s+q\xi-\frac{pq}{2})}\psi(\xi-p), \qquad \psi\in\mathbb{L}_2(\mathbb{R}). \tag{4}$$

The associated matrix coefficients are defined as

$$\mathscr{W}^n_\psi f(p,q,s+\mathbb{Z}) = (\mathscr{U}^n_{(p,q,s+\mathbb{Z})}\psi,f)_{\mathbb{L}_2(\mathbb{R}^d)}. \tag{5}$$

In the following, we will often omit the superscript n from U and $\mathscr{W}_\psi$, implicitly assuming that we use the same choice of n as in the definition of $\mathbb{G}_\psi$. Then a simple comparison of (5) with (1) reveals that

$$\mathscr{G}_\psi[f](p,q) = \mathscr{W}_\psi f(p,q,s=-\frac{pq}{2}). \tag{6}$$

Since $\mathscr{W}_\psi f(p,q,s+\mathbb{Z}) = e^{2\pi i n s}\mathscr{W}_\psi f(p,q,0+\mathbb{Z})$, the phase variable s does not affect the modulus, and (2) can be rephrased as

$$\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\mathscr{W}_\psi[f](p,q,s+\mathbb{Z})|^2\,\mathrm{d}p\,\mathrm{d}q\mathrm{d}s = C_\psi\int_{\mathbb{R}^d}|f(p)|^2\,\mathrm{d}p. \tag{7}$$

Just as before, this induces a weak-sense inversion formula, which reads

$$f = \frac{1}{C_\psi}\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathscr{W}_\psi[f](p,q,s+\mathbb{Z})\mathscr{U}^n_{(p,q,s+\mathbb{Z})}\psi\,\mathrm{d}p\,\mathrm{d}q\mathrm{d}s\ .$$

As a byproduct of (7), we note that the Schrödinger representation is irreducible. Furthermore, the orthogonal projection $\mathbb{P}_\psi$ of $\mathbb{L}_2(H_r)$ onto the range $\mathscr{R}(\mathscr{W}_\psi)$ turns out to be right convolution with a suitable (reproducing) kernel function,

$$(\mathbb{P}_\psi U)(h) = U*K(h) = \int_{H_r}U(g)K(g^{-1}h)dg\ ,$$

with dg denoting the left Haar measure (which is just the Lebesgue measure on $\mathbb{R}^{2d} \times \mathbb{R}/\mathbb{Z}$) and $K(p,q,s) = \frac{1}{C_\psi}\mathscr{W}_\psi \psi(p,q,s) = \frac{1}{C_\psi}(U_{(p,q,s)}\psi, \psi)$.

The chief reason for choosing the somewhat more redundant function $\mathscr{W}_\psi f$ over $\mathscr{G}_\psi[f]$ is that $\mathscr{W}_\psi$ translates time-frequency shifts acting on the signal f to shifts in the argument. If $\mathscr{L}$ and $\mathscr{R}$ denote the left and right regular representation, i.e., for all $g,h \in H_r$ and $F \in \mathbb{L}_2(H_r)$,

$$(\mathscr{L}_g F)(h) = F(g^{-1}h) \ , \ (\mathscr{R}_g F)(h) = F(hg) \ ,$$

then $\mathscr{W}_\psi$ intertwines $\mathscr{U}$ and $\mathscr{L}$,

$$\mathscr{W}_\psi \circ \mathscr{U}^n_g = \mathscr{L}_g \circ \mathscr{W}_\psi \ . \tag{8}$$

Thus the additional group parameter s in H_r keeps track of the phase shifts induced by the noncommutativity of time-frequency shifts. By contrast, *right shifts* on the Gabor transform corresponds to changing the window:

$$\mathscr{R}_g(\mathrm{W}^n_\psi(h)) = (\mathscr{U}_{hg}\psi, f) = \mathscr{W}_{\mathscr{U}_g\psi} f(h) \ . \tag{9}$$

3 Left Invariant Evolutions on Gabor Transforms

We relate operators $\Phi : \mathscr{R}(\mathscr{W}_\psi) \to \mathbb{L}_2(H_r)$ on Gabor transforms, which actually use and change the relevant phase information of a Gabor transform, in a well-posed manner to operators $\Upsilon_\psi : \mathbb{L}_2(\mathbb{R}^d) \to \mathbb{L}_2(\mathbb{R}^d)$ on signals via

$$\begin{aligned}(\Upsilon_\psi f)(\xi) &= (\mathscr{W}^*_\psi \circ \Phi \circ \mathscr{W}_\psi f)(\xi) \\ &= \frac{1}{C_\psi} \int\limits_{[0,1]} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (\Phi(\mathscr{W}_\psi f))(p,q,s)\, e^{i2\pi n[(\xi,q)+(s)-(1/2)(p,q)]}\, \psi(\xi - p)\, \mathrm{d}p\, \mathrm{d}q\, \mathrm{d}s.\end{aligned} \tag{10}$$

Our aim is to design operators Υ_ψ that address signal processing problems such as denoising or detection.

3.1 Design principles

We now formulate a few desirable properties of Υ_ψ, and sufficient conditions for Φ to guarantee that Υ_ψ meets these requirements.

1. *Covariance with respect to time-frequency-shifts:* The operator Υ_ψ should commute with time-frequency shifts. This requires a proper treatment of the phase. One easy way of guaranteeing covariance of Υ_ψ is to ensure left invariance of Φ: If Φ commutes with $\mathscr{L}_g$, for all $g \in H_r$, it follows from (8) that

$$\Upsilon_\psi \circ \mathscr{U}^n_g = \mathscr{W}^*_\psi \circ \Phi \circ \mathscr{W}_\psi \circ \mathscr{U}^n_g = \mathscr{W}^*_\psi \circ \Phi \circ \mathscr{L}_g \circ \mathscr{W}_\psi = \mathscr{U}^n_g \circ \Upsilon_\psi \ .$$

Generally speaking, left invariance of Φ is not a *necessary* condition for invariance of Υ_ψ: Note that $\mathscr{W}_\psi^* = \mathscr{W}_\psi^* \circ \mathbb{P}_\psi$. Thus if Φ is left-invariant, and $A : \mathbb{L}_2(H_r) \to \mathscr{R}(\mathscr{W}_\psi^n)^\perp$ an arbitrary operator, then $\Phi + A$ cannot be expected to be left-invariant, but the resulting operator on the signal side will be the same as for Φ, thus covariant with respect to time-frequency shifts.
The authors [7] studied reassignment procedures that leave the phase invariant, whereas we shall put emphasis on phase-covariance. Note however that the two properties are not mutually exclusive; convection along equiphase lines fulfills both. (See also the discussion in Subsection 3.4.)

2. *Nonlinearity:* The requirement that Υ_ψ commute with $\mathscr{U}^n$ immediately rules out linear operators Φ. Recall that $\mathscr{U}^n$ is irreducible, and by Schur's lemma [8], any linear intertwining operator is a scalar multiple of the identity operator.
3. By contrast to left invariance, right invariance of Φ is undesirable. By a similar argument as for left-invariance, it would provide that $\Upsilon_\psi = \Upsilon_{\mathscr{U}_g^n \psi}$.

We stress that one cannot expect that the processed Gabor transform $\Phi(\mathscr{W}_\psi f)$ is again the Gabor transform of some function constructed by the same kernel ψ, i.e. we do not expect that $\Phi(\mathscr{R}(\mathscr{W}_\psi^n)) \subset \mathscr{R}(\mathscr{W}_\psi^n)$.

3.2 Invariant differential operators on H_r

The basic building blocks for the evolution equations are the left-invariant differential operators on H_r of degree one. These operators are conveniently obtained by differentiating the *right* regular representation, restricted to one-parameter subgroups through the generators $\{A_1, \ldots, A_{2d+1}\} = \{\partial_{p_1}, \ldots, \partial_{p_d}, \partial_{q_1}, \ldots, \partial_{q_d}, \partial_s\} \subset T_e(H_r)$,

$$\mathrm{d}\mathscr{R}(A_i)U(g) = \lim_{\varepsilon \to 0} \frac{U(g e^{\varepsilon A_i}) - U(g)}{\varepsilon} \text{ for all } g \in H_r \text{ and smooth } U \in \mathscr{C}^\infty(H_r), \quad (11)$$

The resulting differential operators $\{\mathrm{d}\mathscr{R}(A_1), \ldots, \mathrm{d}\mathscr{R}(A_{2d+1})\} =: \{\mathscr{A}_1, \ldots, \mathscr{A}_{2d+1}\}$ denote the left-invariant vector fields on H_r, and brief computation of (11) yields:

$$\mathscr{A}_i = \partial_{p_i} + \frac{q_i}{2}\partial_s, \quad \mathscr{A}_{d+i} = \partial_{q_i} - \frac{p_i}{2}\partial_s, \quad \mathscr{A}_{2d+1} = \partial_s, \qquad \text{for } i = 1, \ldots, d. \,,$$

The differential operators obey the same commutation relations as their Lie algebra counterparts $A_1, \ldots, A_{2d+1}$

$$[\mathscr{A}_i, \mathscr{A}_{d+i}] := \mathscr{A}_i \mathscr{A}_{d+i} - \mathscr{A}_{d+i} \mathscr{A}_i = -\mathscr{A}_{2d+1}, \quad (12)$$

and all other commutators are zero. I.e. $\mathrm{d}\mathscr{R}$ is a Lie algebra isomorphism.

3.3 Setting up the equations

For the effective operator Φ, we will choose left-invariant evolution operators with stopping time $t > 0$. To stress the dependence on the stopping time we shall write Φ_t rather than Φ. Typically, such operators are defined by $W(p,q,s,t) = \Phi_t(\mathscr{W}_\psi f)(p,q,s)$ where W is the solution of

$$\boxed{\begin{cases} \partial_t W(p,q,s,t) = Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d}) W(p,q,s,t), \\ W(p,q,s,0) \quad = \mathscr{W}_\psi f(p,q,s). \end{cases}} \tag{13}$$

where we note that the left-invariant vector fields $\{\mathscr{A}_i\}_{i=1}^{2d+1}$ on H_r are given by

$$\mathscr{A}_i = \partial_{p_i} + \tfrac{q_i}{2}\partial_s, \mathscr{A}_{d+i} = \partial_{q_i} - \tfrac{p_i}{2}\partial_s, \mathscr{A}_{2d+1} = \partial_s, \qquad \text{for } i = 1, \ldots, d, \ ,$$

with left-invariant quadratic differential form

$$Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d}) = -\sum_{i=1}^{2d} a_i(|\mathscr{W}_\psi f|)(p,q)\mathscr{A}_i + \sum_{i=1}^{2d}\sum_{j=1}^{2d} \mathscr{A}_i \, D_{ij}(|\mathscr{W}_\psi f|)(p,q) \, \mathscr{A}_j. \tag{14}$$

Here $a_i(|\mathscr{W}_\psi f|)$ and $D_{ij}(|\mathscr{W}_\psi f|)$ are functions such that $(p,q) \mapsto a_i(|\mathscr{W}_\psi f|)(p,q) \in \mathbb{R}$ and $(p,q) \mapsto a_i(|\mathscr{W}_\psi f|)(p,q) \in \mathbb{R}$ are smooth and either $D = 0$ (pure convection) or $D^T = D > 0$ holds pointwise (with $D = [D_{ij}]$) for all $i = 1, \ldots, 2d, \ j = 1, \ldots 2d$. Moreover, in order to guarantee left-invariance, the mappings $a_i : \mathscr{W}_\psi f \mapsto a_i(|\mathscr{W}_\psi f|)$ need to fulfill the covariance relation

$$a_i(|\mathscr{L}_h \mathscr{W}_\psi f|)(g) = a_i(|\mathscr{W}_\psi f|)(p-p', q-q'), \tag{15}$$

for all $f \in \mathbb{L}_2(\mathbb{R})$, and all $g = (p,q,s+\mathbb{Z}), h = (p',q',s'+\mathbb{Z}) \in H_r$.

For $a_1 = \ldots = a_{2d+1} = 0$, the equation is a diffusion equation, whereas if $D = 0$, the equation describes a convection. We note that existence, uniqueness and square-integrability of the solutions (and thus well-definedness of Υ) are issues that will have to be decided separately for each particular choice of a_i and D. In general existence and uniqueness are guaranteed, see Section 7.

This definition of Φ_t satisfies the criteria we set up above:

1. Since the evolution equation is left-invariant (and provided uniqueness of the solutions), it follows that Φ_t is left-invariant. Thus the associated Υ_ψ is invariant under time-frequency shifts.
2. In order to ensure non-linearity, not all of the functions a_i, D_{ij} should be constant, i.e. the schemes should be *adaptive convection* and/or *adaptive diffusion*, via *adaptive* choices of convection vectors $(a_1, \ldots, a_{2d})^T$ and/or conductivity matrix D. We will use ideas similar to our previous work on adaptive diffusions on invertible orientation scores [17], [12], [11], [13] (where we employed evolution equations for the 2D-Euclidean motion group). We use the absolute value to adapt the diffusion and convection to avoid oscillations.

3. The two-sided invariant differential operators of degree one correspond to the center of the Lie algebra, which is precisely the span of A_{2d+1}. Both in the cases of diffusion and convection, we consistently removed the $\mathscr{A}_{2d+1} = \partial_s$-direction, and we removed the s-dependence in the coefficients $a_i(|\mathscr{W}_\psi f|)(p,q)$, $D_{ij}(|\mathscr{W}_\psi f|)(p,q)$ of the generator $Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d})$ by taking the absolute value $|\mathscr{W}_\psi f|$, which is independent of s. A more complete discussion of the role of the s-variable is contained in the following subsection.

3.4 Convection and Diffusion along Horizontal Curves

So far our motivation for (13) has been group theoretical. There is one issue we did not address yet, namely the omission of $\partial_s = \mathscr{A}_{2d+1}$ in (13). Here we first motivate this omission and then consider the differential geometrical consequence that (adaptive) convection and diffusion takes place along so-called horizontal curves.

The reason for the removal of the $\mathscr{A}_{2d+1}$ direction in our diffusions and convections is simply that this direction leads to a scalar multiplication operator mapping the space of Gabor transform to itself, since $\partial_s \mathscr{W}_\psi f = -2\pi i n \mathscr{W}_\psi f$. Moreover, we adaptively steer the convections and diffusions by the modulus of a Gabor transform $|\mathscr{W}_\psi f(p,q,s)| = |\mathscr{G}_\psi f(p,q)|$, which is independent of s, and clearly a vector field $(p,q,s) \mapsto F(p,q)\partial_s$ is left-invariant iff F is constant. Consequently it does *not* make sense to include the separate ∂_s in our convection-diffusion equations, as it can only yield a scalar multiplication, as for all constant $\alpha > 0, \beta \in \mathbb{R}$ we have

$$[\partial_s, Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d})] = 0 \text{ and } \partial_s \mathscr{W}_\psi f = -2\pi i n \mathscr{W}_\psi f \Rightarrow$$
$$e^{t\left((\alpha\partial_s^2+\beta\partial_s)+Q(|\mathscr{W}_\psi f|,\mathscr{A}_1,\ldots,\mathscr{A}_{2d})\right)} = e^{-t\alpha(2\pi n)^2 - t\beta 2\pi i n}\, e^{tQ(|\mathscr{W}_\psi f|,\mathscr{A}_1,\ldots,\mathscr{A}_{2d})}.$$

In other words ∂_s is a redundant direction in each tangent space $T_g(H_r)$, $g \in H_r$. This however does *not* imply that it is a redundant direction in the group manifold H_r itself, since clearly the s-axis represents the relevant phase and stores the non-commutative nature between position and frequency, [14, ch:1].

The omission of the redundant direction ∂_s in $T(H_r)$ has an important geometrical consequence. Akin to our framework of linear evolutions on orientation scores, cf. [13, 17], this means that we enforce horizontal diffusion and convection, i.e. transport and diffusion only takes place along so-called *horizontal* curves in H_r which are curves $t \mapsto (p(t), q(t), s(t)) \in H_r$, with $s(t) \in (0,1)$, along which

$$s(t) = \frac{1}{2} \int_0^t \sum_{i=1}^d q_i(\tau) p_i'(\tau) - p_i(\tau) q_i'(\tau) \mathrm{d}\tau \ ,$$

see Theorem 1. This gives a nice geometric interpretation to the phase variable $s(t)$, since by the Stokes theorem it represents the net surface area between a straight line connection between $(p(0),q(0),s(0))$ and $(p(t),q(t),s(t))$ and the actual horizontal curve connection $[0,t] \ni \tau \mapsto (p(\tau),q(\tau),s(\tau))$. For details, see [14].

In order to explain why the omission of the redundant direction ∂_s from the tangent bundle $T(H_r)$ implies a restriction to horizontal curves, we consider the dual frame associated to our frame of reference $\{\mathscr{A}_1,\ldots,\mathscr{A}_{2d+1}\}$. We will denote this dual frame by $\{\mathrm{d}\mathscr{A}^1,\ldots,\mathrm{d}\mathscr{A}^{2d+1}\}$ and it is uniquely determined by $\langle \mathrm{d}\mathscr{A}^i, \mathscr{A}_j\rangle = \delta^i_j$, $i,j = 1,2,3$ where δ^i_j denotes the Kronecker delta. A brief computation yields

$$\begin{array}{l} \mathrm{d}\mathscr{A}^i\big|_{g=(p,q,s)} = \mathrm{d}p^i \ , \ \mathrm{d}\mathscr{A}^{d+i}\big|_{g=(p,q,s)} = \mathrm{d}q^i \ , \ \ i=1,\ldots,d \\ \mathrm{d}\mathscr{A}^{2d+1}\big|_{g=(p,q,s)} = \mathrm{d}s + \frac{1}{2}(p\cdot \mathrm{d}q - q\cdot \mathrm{d}p), \end{array} \tag{16}$$

Consequently a smooth curve $t \mapsto \gamma(t) = (p(t),q(t),s(t))$ is horizontal iff

$$\langle \mathrm{d}\mathscr{A}^{2d+1}\big|_{\gamma(s)}, \gamma'(s)\rangle = 0 \Leftrightarrow s'(t) = \frac{1}{2}(q(t)\cdot p'(t) - p(t)\cdot q'(t)).$$

Theorem 1. *Let $f \in \mathbb{L}_2(\mathbb{R})$ be a signal and $\mathscr{W}_\psi f$ be its Gabor transform associated to the Schwartz function ψ. If we just consider convection and no diffusion (i.e. $D=0$) then the solution of (13) is given by*

$$W(g,t) = \mathscr{W}_\psi f(\gamma_f^g(t)) \ , \qquad g=(p,q,s)\in H_r,$$

where the characteristic horizontal curve $t\mapsto \gamma_f^{g_0}(t) = (p(t),q(t),s(t))$ for each $g_0 = (p_0,q_0,s_0)\in H_r$ is given by the unique solution of the following ODE:

$$\begin{cases} \dot{p}(t) = -a^1(|\mathscr{W}_\psi f|)(p(t),q(t)), \ p(0)=p_0, \\ \dot{q}(t) = -a^2(|\mathscr{W}_\psi f|)(p(t),q(t)), \ q(0)=q_0, \\ \dot{s}(t) = \frac{q(t)}{2}\dot{p}(t) - \frac{p(t)}{2}\dot{q}(t), \qquad s(0)=s_0, \end{cases}$$

Consequently, the operator $\mathscr{W}_\psi f \mapsto W(\cdot,t)$ is phase covariant (the phase moves along with the characteristic curves of transport):

$$arg\{W(g,t)\} = arg\{\mathscr{W}_\psi f\}(\gamma_f^g) \ for\ all\ t>0.$$

Proof. For proof see [14, p.30, p.31].

Also for the (degenerate) diffusion case with $D = D^T = [D_{ij}]_{i,j=1,\ldots,d} > 0$, the omission of the $2d+1$-th direction $\partial_s = \mathscr{A}_{2d+1}$ implies that diffusion takes place along horizontal curves. Moreover, the omission does not affect the smoothness and uniqueness of the solutions of (13), since the initial condition is infinitely differentiable (if ψ is a Schwarz function) and the Hörmander condition [23], [10] is by (12) still satisfied.

The removal of the ∂_s direction from the tangent space does *not* imply that one can entirely ignore the ∂_s-axis in the domain of a (processed) Gabor transform.

The domain of a (processed) Gabor transform $\Phi_t(\mathscr{W}_\psi f)$ should *not*[1] be considered as $\mathbb{R}^{2d} \equiv H_r/\Theta$. Simply, because $[\partial_p, \partial_q] = 0$ whereas we should have (12). For further differential geometrical details see the appendices of [14], analogous to the differential geometry on orientation scores, [13], [14, App. D , App. C.1].

4 Towards Phase Space and Back

As pointed out in the introduction it is very important to keep track of the phase variable $s > 0$. The first concern that arises here is whether this results in slower algorithms. In this section we will show that this is not the case. As we will explain next, one can use an *invertible* mapping $\mathscr{S}$ from the space $\mathscr{H}_n$ of Gabor transforms to phase space (the space of Gabor transforms restricted to the plane $s = \frac{pq}{2}$). As a result by means of conjugation with $\mathscr{S}$ we can map our diffusions on $\mathscr{H}_n \subset \mathbb{L}_2(\mathbb{R}^2 \times [0,1])$ uniquely to diffusions on $\mathbb{L}_2(\mathbb{R}^2)$ simply by conjugation with $\mathscr{S}$. From a geometrical point of view it is better/easier to consider the diffusions on $\mathscr{H}_n \subset \mathbb{L}_2(\mathbb{R}^{2d} \times [0,1])$ than on $\mathbb{L}_2(\mathbb{R}^{2d})$, even though all our numerical PDE-Algorithms take place in phase space in order to gain speed.

Definition 1. Let $\mathscr{H}_n$ denote the space of all complex-valued functions F on H_r such that $F(p,q,s+\mathbb{Z}) = e^{-2\pi i n s} F(p,q,1)$ and $F(\cdot,\cdot,s+\mathbb{Z}) \in \mathbb{L}_2(\mathbb{R}^{2d})$ for all $s \in \mathbb{R}$, then clearly $\mathscr{W}_\psi f \in \mathscr{H}_n$ for all $f, \psi \in \mathscr{H}_n$.

In fact $\mathscr{H}_n$ is the closure of the space $\{\mathscr{W}^n_\psi f \mid \psi, f \in \mathbb{L}_2(\mathbb{R})\}$ in $\mathbb{L}_2(H_r)$. The space $\mathscr{H}_n$ is bi-invariant, since:

$$\mathscr{W}^n_\psi \circ \mathscr{U}^n_g = \mathscr{L}_g \circ \mathscr{W}^n_\psi \text{ and } \mathscr{W}^n_{\mathscr{U}^n_g \psi} = \mathscr{R}_g \circ \mathscr{W}^n_\psi, \tag{17}$$

where again $\mathscr{R}$ denotes the right regular representation on $\mathbb{L}_2(H_r)$ and $\mathscr{L}$ denotes the left regular representation of H_r on $\mathbb{L}_2(H_r)$. We can identify $\mathscr{H}_n$ with $\mathbb{L}_2(\mathbb{R}^{2d})$ by means of the following operator $\mathscr{S} : \mathscr{H}_n \to \mathbb{L}_2(\mathbb{R}^{2d})$ given by

$$(\mathscr{S}F)(p,q) = F(p,q,\frac{pq}{2}+\mathbb{Z}) = e^{i\pi n p q} F(p,q,0+\mathbb{Z}).$$

Clearly, this operator is invertible and its inverse is given by

$$(\mathscr{S}^{-1}F)(p,q,s+\mathbb{Z}) = e^{-2\pi i s n} e^{-i\pi n p q} F(p,q)$$

The operator $\mathscr{S}$ simply corresponds to taking the section $s(p,q) = -\frac{pq}{2}$ in the left cosets H_r/Θ where $\Theta = \{(0,0,s+\mathbb{Z}) \mid s \in \mathbb{R}\}$ of H_r. Furthermore we recall the common Gabor transform $\mathscr{G}^n_\psi$ given by (1) and its relation (6) to the full Gabor transform. This relation is simply $\mathscr{G}^n_\psi = \mathscr{S} \circ \mathscr{W}^n_\psi$.

---

[1] As we explain in [14, App. B and App. C] the Gabor domain is a principal fiber bundle $P_T = (H_r, \mathbb{T}, \pi, \mathscr{R})$ equipped with the Cartan connection form $\omega_g(X_g) = \langle \mathrm{d}s + \frac{1}{2}(p\mathrm{d}q - q\mathrm{d}p), X_g \rangle$, or equivalently, it is a contact manifold, cf. [3, p.6], [14, App. B, def. B.14] , $(H_r, \mathrm{d}\mathscr{A}^{2d+1})$).

Theorem 2. *Let the operator Φ map the closure $\mathscr{H}_n$, $n \in \mathbb{Z}$, of the space of Gabor transforms into itself, i.e. $\Phi : \mathscr{H}_n \to \mathscr{H}_n$. Define the left and right-regular rep's of H_r on $\mathscr{H}_n$ by restriction*

$$\mathscr{R}_g^{(n)} = \mathscr{R}_g\big|_{\mathscr{H}_n} \text{ and } \mathscr{L}_g^{(n)} = \mathscr{L}_g\big|_{\mathscr{H}_n} \quad \text{for all } g \in H_r. \tag{18}$$

Define the corresponding left and right-regular rep's of H_r on phase space by

$$\tilde{\mathscr{R}}_g^{(n)} := \mathscr{S} \circ \mathscr{R}_g^{(n)} \circ \mathscr{S}^{-1}, \; \tilde{\mathscr{L}}_g^{(n)} := \mathscr{S} \circ \mathscr{L}_g^{(n)} \circ \mathscr{S}^{-1}.$$

For explicit formulas see [14, p.9]. Let $\tilde{\Phi} := \mathscr{S} \circ \Phi \circ \mathscr{S}^{-1}$ be the corresponding operator on $\mathbb{L}_2(\mathbb{R}^{2d})$ and

$$\Upsilon_\psi = (\mathscr{W}_\psi^n)^* \circ \Phi \circ \mathscr{W}_\psi^n = (\mathscr{S}\mathscr{W}_\psi^n)^{-1} \circ \tilde{\Phi} \circ \mathscr{S}\mathscr{W}_\psi^n = (\mathscr{G}_\psi^n)^* \circ \tilde{\Phi} \circ \mathscr{G}_\psi^n.$$

Then one has the following correspondence:

$$\Upsilon_\psi \circ \mathscr{U}^n = \mathscr{U}^n \circ \Upsilon_\psi \Leftarrow \Phi \circ \mathscr{L}^n = \mathscr{L}^n \circ \Phi \Leftrightarrow \tilde{\Phi} \circ \tilde{\mathscr{L}}^n = \tilde{\mathscr{L}}^n \circ \tilde{\Phi}. \tag{19}$$

If moreover $\Phi(\mathscr{R}(\mathscr{W}_\psi)) \subset \mathscr{R}(\mathscr{W}_\psi)$ then the left implication may be replaced by an equivalence. If Φ does not satisfy this property then one may replace $\Phi \to \mathscr{W}_\psi \mathscr{W}_\psi^* \Phi$ in (19) to obtain full equivalence. Note that $\Upsilon_\psi = \mathscr{W}_\psi^* \Phi \mathscr{W}_\psi = \mathscr{W}_\psi^*(\mathscr{W}_\psi \mathscr{W}_\psi^* \Phi)\mathscr{W}_\psi$.

Proof. For details see our technical report [14, Thm 2.2].

5 Left-invariant Evolutions on Phase Space

For the remainder of the paper, for the sake of simplicity, we fix $d = 1$.

Now we would like to apply Theorem 2 to our left invariant evolutions (13) to obtain the left-invariant diffusions on phase space (where we reduce 1 dimension in the domain). To this end we first compute the left-invariant vector fields $\{\tilde{\mathscr{A}_i}\} := \{\mathscr{S}\mathscr{A}_i\mathscr{S}^{-1}\}_{i=1}^3$ on phase space. The left-invariant vector fields on phase space are

$$\begin{array}{l} \tilde{\mathscr{A}_1}U(p',q') = \mathscr{S}\mathscr{A}_1\mathscr{S}^{-1}U(p',q') = ((\partial_{p'} - 2n\pi i q')U)(p',q'), \\ \tilde{\mathscr{A}_2}U(p',q') = \mathscr{S}\mathscr{A}_2\mathscr{S}^{-1}U(p',q') = (\partial_{q'}U)(p',q'), \\ \tilde{\mathscr{A}_3}U(p',q') = \mathscr{S}\mathscr{A}_3\mathscr{S}^{-1}U(p',q') = -2in\pi U(p',q') \, , \end{array} \tag{20}$$

for all $(p,q) \in \mathbb{R}$ and all locally defined smooth functions $U : \Omega_{(p,q)} \subset \mathbb{R}^2 \to \mathbb{C}$.

Now that we have computed the left-invariant vector fields on phase space, we can express our left-invariant evolution equations (13) on phase space

$$\boxed{\begin{cases} \partial_t \tilde{W}(p,q,t) = \tilde{Q}(|\mathscr{G}_\psi f|, \tilde{\mathscr{A}_1}, \tilde{\mathscr{A}_2})\tilde{W}(p,q,t), \\ \tilde{W}(p,q,0) \quad = \mathscr{G}_\psi f(p,q). \end{cases}} \tag{21}$$

with left-invariant quadratic differential form

$$\tilde{Q}(|\mathscr{G}_\psi f|, \tilde{\mathscr{A}_1}, \tilde{\mathscr{A}_2}) = -\sum_{i=1}^{2} a_i(|\mathscr{G}_\psi f|)(p,q)\tilde{\mathscr{A}_i} + \sum_{i=1}^{2}\sum_{j=1}^{2} \tilde{\mathscr{A}_i}\, D_{ij}(|\mathscr{G}_\psi f|)(p,q)\, \tilde{\mathscr{A}_j}. \tag{22}$$

Similar to the group case, the a_i and D_{ij} are functions such that $(p,q) \mapsto a_i(|\mathscr{G}_\psi f|)(p,q) \in \mathbb{R}$ and $(p,q) \mapsto a_i(|\mathscr{G}_\psi f|)(p,q) \in \mathbb{R}$ are smooth and either $D = 0$ (pure convection) or $D^T = D > 0$ (with $D = [D_{ij}]$ $i,j = 1,\ldots,2d$), so Hörmander's condition [23] (which guarantees smooth solutions $\tilde{W}$, provided the initial condition $\tilde{W}(\cdot,\cdot,0)$ is smooth) is satisfied because of (12).

Theorem 3. *The unique solution $\tilde{W}$ of (21) is obtained from the unique solution W of (13) by means of*

$$\tilde{W}(p,q,t) = (\mathscr{S}\, W(\cdot,\cdot,\cdot,t))(p,q)\ ,\ \textit{for all } t \geq 0 \textit{ and for all } (p,q) \in \mathbb{R}^2,$$

with in particular $\tilde{W}(p,q,0) = \mathscr{G}_\psi(p,q) = (\mathscr{S}\mathscr{W}_\psi)(p,q) = (\mathscr{S}W(\cdot,\cdot,\cdot,0))(p,q)$.

Proof. This follows by the fact that the evolutions (13) leave the function space $\mathscr{H}_n$ invariant and the fact that the evolutions (21) leave the space invariant $\mathbb{L}_2(\mathbb{R}^2)$ invariant, so that we can apply direct conjugation with the invertible operator $\mathscr{S}$ to relate the unique solutions, where we have

$$\begin{aligned}
\tilde{W}(p,q,t) &= (e^{t\tilde{Q}(|\mathscr{G}_\psi f|, \tilde{\mathscr{A}_1}, \tilde{\mathscr{A}_2})}\mathscr{G}_\psi f)(p,q) \\
&= (e^{t\tilde{Q}(|\mathscr{G}_\psi f|, \mathscr{S}\mathscr{A}_1\mathscr{S}^{-1}, \mathscr{S}\mathscr{A}_2\mathscr{S}^{-1})}\mathscr{S}\mathscr{W}_\psi f)(p,q) \\
&= (e^{\mathscr{S}\circ t\, Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \mathscr{A}_2)\circ \mathscr{S}^{-1}}\mathscr{S}\mathscr{W}_\psi f)(p,q) \\
&= (\mathscr{S} \circ e^{t\, Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \mathscr{A}_2)} \circ \mathscr{S}^{-1}\mathscr{S} \circ \mathscr{W}_\psi f)(p,q) \\
&= (\mathscr{S}\, W(\cdot,\cdot,\cdot,t))(p,q)
\end{aligned} \tag{23}$$

for all $t > 0$ on densely defined domains. For every $\psi \in \mathbb{L}_2(\mathbb{R}) \cap S(\mathbb{R})$, the space of Gabor transforms is a reproducing kernel space with a bounded and smooth reproducing kernel, so that $\mathscr{W}_\psi f$ (and thereby $|\mathscr{W}_\psi f| = |\mathscr{G}_\psi f| = \sqrt{(\Re \mathscr{G}_\psi f)^2 + (\Im \mathscr{G}_\psi f)^2}$) is uniformly bounded and continuous and equality (23) holds for all $p,q \in \mathbb{R}^2$.

5.1 The Cauchy Riemann Equations on Gabor Transforms

As previously observed in [7], the Gabor transforms associated to Gaussian windows obey Cauchy-Riemann equations which are particularly useful for the analysis of convection schemes, as well as for the design of more efficient algorithms.

More precisely, if $\psi(\xi) = \psi_a(\xi) := e^{-\pi n \frac{(\xi - c)^2}{a^2}}$ and f is some arbitrary signal in $\mathbb{L}_2(\mathbb{R})$ then we have

$$\begin{aligned}
&(a^{-1}\mathscr{A}_2 + ia\mathscr{A}_1)\mathscr{W}_\psi(f) = 0 \Leftrightarrow (a^{-1}\tilde{\mathscr{A}_2} + ia\tilde{\mathscr{A}_1})\mathscr{G}_\psi(f) = 0 \\
&(a^{-1}\mathscr{A}_2 + ia\mathscr{A}_1)\log \mathscr{W}_\psi(f) = 0 \Leftrightarrow (a^{-1}\tilde{\mathscr{A}_2} + ia\tilde{\mathscr{A}_1})\log \mathscr{G}_\psi(f) = 0
\end{aligned} \tag{24}$$

where we recall that $\mathscr{G}_\psi(f) = \mathscr{S}\mathscr{W}_\psi(f)$ and $\mathscr{A}_i = \mathscr{S}^{-1}\tilde{\mathscr{A}}_i\mathscr{S}$ for $i = 1,2,3$. For details see [14], [25, ch:5], where the essential observation is that we can write

$$\mathscr{G}_{\psi_a} f(p,q) = \sqrt{a}\mathscr{G}_{\mathscr{D}_{\frac{1}{a}}\psi}(f)(p,q) = \sqrt{a}\mathscr{G}_\psi \mathscr{D}_{\frac{1}{a}} f(\frac{p}{a},aq)$$

with $\psi = \psi_{a=1}$ and where the unitary dilation operator $\mathscr{D}_a : \mathbb{L}_2(\mathbb{R}) \to \mathbb{L}_2(\mathbb{R})$ is given by $\mathscr{D}_a(\psi)(x) = a^{-\frac{1}{2}} f(x/a)$, $a > 0$. For the case $a = 1$, equation (24) was noted in [7]. As a direct consequence of (24) we have

$$\begin{array}{lll} |\tilde{U}^a|\partial_q\tilde{\Omega}^a = -a^2\partial_p|\tilde{U}^a| & \text{and} & |\tilde{U}^a|\partial_p\tilde{\Omega}^a = a^{-2}\partial_q|\tilde{U}^a| + 2\pi q. \\ \mathscr{A}_2\Omega^a = a^2\mathscr{A}_1|U^a| & \text{and} & \mathscr{A}_1\Omega^a = a^{-2}\mathscr{A}_1|U^a| \ . \end{array} \tag{25}$$

where $\tilde{U}^a$ resp. U^a is short notation for $\tilde{U}^a = \mathscr{G}_{\psi_a}(f)$, $U^a = \mathscr{W}_{\psi_a}(f)$, $\tilde{\Omega}^a = \arg\{\mathscr{G}_{\psi_a}(f)\}$ and $\Omega^a = \arg\{\mathscr{W}_{\psi_a}(f)\}$.

If one equips the *contact-manifold* (for general definition see cf. [3, p.6] or [14, App. B, def. B.14]), given by the pair $(H_r, \mathrm{d}\mathscr{A}^3)$, recall (16) with the following *non-degenerate* [2] left-invariant metric tensor

$$\mathscr{G}_\beta = g_{ij}\mathrm{d}\mathscr{A}^i \otimes \mathrm{d}\mathscr{A}^j = \beta^4\mathrm{d}\mathscr{A}^1 \otimes \mathrm{d}\mathscr{A}^1 + \mathrm{d}\mathscr{A}^2 \otimes \mathrm{d}\mathscr{A}^2, \tag{26}$$

which is bijectively related to the linear operator $G : \mathfrak{H} \to \mathfrak{H}'$, where $\mathfrak{H} = \mathrm{span}\{\mathscr{A}_1, \mathscr{A}_2\}$ denotes the horizontal part of the tangent space, that maps $\mathscr{A}_1$ to $\beta^4\mathrm{d}\mathscr{A}^1$ and $\mathscr{A}_2$ to $\mathrm{d}\mathscr{A}^2$. The inverse operator of G is bijectively related to

$$\mathscr{G}_\beta^{-1} = g^{ij}\mathscr{A}_i \otimes \mathscr{A}_j = \beta^{-4}\mathscr{A}_1 \otimes \mathscr{A}_1 + \mathscr{A}_2 \otimes \mathscr{A}_2 \ .$$

Here the fundamental positive parameter β^{-1} has physical dimension length, so that this first fundamental form is consistent with respect to physical dimensions. Intuitively, the parameter β sets a global balance between changes in frequency space and changes in position space. The Cauchy-Riemann relations (25) that hold between local phase and local amplitude can be written in geometrical form:

$$\boxed{\mathscr{G}^{-1}_{\beta=\frac{1}{a}}(\mathrm{d}\log|U|, \mathbb{P}_{\mathfrak{H}^*}\mathrm{d}\Omega) = 0,} \tag{27}$$

where $U = \mathscr{W}_{\psi_a} f = |U|e^{i\Omega}$ and where the left-invariant gradient equals $\mathrm{d}\Omega = \sum_{i=1}^{3} \mathscr{A}_i\Omega \ \mathrm{d}\mathscr{A}^i$ whose horizontal part equals $\mathbb{P}_{\mathfrak{H}^*}\mathrm{d}\Omega = \sum_{i=1}^{2} \mathscr{A}_i\Omega \ \mathrm{d}\mathscr{A}^i$. This gives us a geometric understanding. The horizontal part $\mathbb{P}_{\mathfrak{H}^*} \ \mathrm{d}\Omega|_{g_0}$ of the normal co-vector $\mathrm{d}\Omega|_{g_0}$ to the surface $\{(p,q,s) \in H_r \mid \Omega(p,q,s) = \Omega(g_0)\}$ is $\mathscr{G}_\beta$-orthogonal to the normal co-vector $\mathrm{d}|U||_{g_0}$ to the surfaces $\{(p,q,s) \in H_r \mid |U|(p,q,s) = |U|(g_0)\}$.

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[2] The metric tensor is degenerate on H_r, but we consider a contact manifold $(H_3, \mathrm{d}\mathscr{A}^3)$ where tangent vectors along horizontal curves do not have an $\mathscr{A}_3$-component.

6 Phase Invariant Convection on Gabor Transforms

First we derive left-invariant and phase-invariant differential operators on Gabor transforms $U := \mathscr{W}_\psi(f)$, which will serve as generators of left-invariant phase-invariant convection (i.e. set $D = 0$ in (13) and (21)) equations on Gabor transforms. This type of convection is also known as differential reassignment, cf. [7, 5], where the practical goal is to sharpen Gabor distributions towards lines (close to minimal energy curves [14, App.D]) in H_r, while maintaining the signal as much as possible.

On the group H_r it directly follows by the product rule for differentiation that the following differential operators $\mathscr{C} : \mathscr{H}_n \to \mathscr{H}_n$ given by

$$\mathscr{C}(U) = \mathscr{M}(|U|)(-\mathscr{A}_2\Omega\mathscr{A}_1 U + \mathscr{A}_1\Omega\mathscr{A}_2 U), \qquad \text{where } \Omega = \arg\{U\},$$

are phase invariant, where $\mathscr{M}(|U|)$ denotes a multiplication operator on $\mathscr{H}_n$ with the modulus of U naturally associated to a bounded monotonically increasing differentiable function $\mu : [0, \max(U)] \to [0, \mu(\max(U))] \subset \mathbb{R}$ with $\mu(0) = 0$, i.e. $(\mathscr{M}(|U|)V)(p,q) = \mu(|U|(p,q))\, V(p,q)$ for all $V \in \mathscr{H}_n, (p,q) \in \mathbb{R}^2$.

The absolute value of Gabor transform is almost everywhere smooth (if ψ is a Schwarz function) bounded and $\mathscr{C}$ can be considered as an unbounded operator from $\mathscr{H}_n$ *into* $\mathscr{H}_n$, as the bi-invariant space $\mathscr{H}_n$ is invariant under bounded multiplication operators which do not depend on $z = e^{2\pi i s}$. Concerning phase invariance, direct computation yields: $\mathscr{C}(e^{i\Omega}|U|) = \mathscr{M}(|U|)\, e^{i\Omega}(-\mathscr{A}_2\Omega\mathscr{A}_1|U| + \mathscr{A}_1\Omega\mathscr{A}_2|U|)$. For Gaussian kernels $\psi_a(\xi) = e^{-a^{-2}\xi^2 n\pi}$ we may apply the Cauchy Riemann relations (24) which simplifies for the special case $\mathscr{M}(|U|) = |U|$ to

$$\mathscr{C}(e^{i\Omega}|U|) = (a^2(\partial_p|U|)^2 + a^{-2}(\partial_q|U|)^2)\, e^{i\Omega}. \tag{28}$$

Now consider the following phase-invariant adaptive convection equation on H_r,

$$\boxed{\begin{cases} \partial_t W(g,t) = -\mathscr{C}(W(\cdot,t))(g), \\ W(g,0) = U(g) \end{cases}} \tag{29}$$

with either

$$\boxed{\begin{array}{l} 1.\ \mathscr{C}(W(\cdot,t)) = \mathscr{M}(|U|)\,(-\mathscr{A}_2\Omega, \mathscr{A}_1\Omega)\cdot(\mathscr{A}_1 W(\cdot,t), \mathscr{A}_2 W(\cdot,t)) \text{ or} \\ 2.\ \mathscr{C}(W(\cdot,t)) = e^{i\Omega}\left(a^2 \frac{(\partial_p|W(\cdot,t)|)^2}{|W(\cdot,t)|} + a^{-2}\frac{(\partial_q|W(\cdot,t)|)^2}{|W(\cdot,t)|}\right). \end{array}} \tag{30}$$

In the first choice we stress that $\arg(W(\cdot,t)) = \arg(W(\cdot,0)) = \Omega$, since transport only takes place along iso-phase surfaces. Initially, in case $\mathscr{M}(|U|) = 1$ the two approaches are the same since at $t = 0$ the Cauchy Riemann relations (25) hold, but as time increases the Cauchy-Riemann equations are violated (this directly follows by the preservation of phase and non-preservation of amplitude), which has been more or less overlooked in the single step convection schemes in [7, 5].

The second choice in (30) in (29) is just a phase-invariant inverse Hamilton Jakobi equation on H_r, with a Gabor transform as initial solution. Rather than computing the viscosity solution of this non-linear PDE, we may as well store the phase and apply an inverse Hamilton Jakobi system on $\mathbb{R}^2$ with the amplitude $|U|$ as initial condition and multiply with the stored phase factor afterwards.

With respect to the first choice in (30) in (29), which is much more cumbersome to implement, the authors in [7] considered the equivalent equation on phase space:

$$\begin{cases} \partial_t \tilde{W}(p,q,t) = -\tilde{\mathscr{C}}(\tilde{W}(\cdot,t))(p,q), \\ \tilde{W}(p,q,0) = \mathscr{G}_\psi f(p,q) =: \tilde{U}(p,q) = e^{i\tilde{\Omega}(p,q)}|\tilde{U}(p,q)| = e^{i\tilde{\Omega}(p,q)}|U|(p,q) \end{cases} \tag{31}$$

with $\tilde{C}(\tilde{W}(\cdot,t)) = \mathscr{M}(|U|)\left(-\tilde{\mathscr{A}_2}\tilde{\Omega}\tilde{\mathscr{A}_1}\tilde{W}(\cdot,t) + (\partial_q\tilde{\Omega} - 2\pi q)\tilde{\mathscr{A}_2}\tilde{W}(\cdot,t)\right)$, where we recall $\mathscr{G}_\psi = \mathscr{S}\mathscr{W}_\psi$ and $\mathscr{A}_i = \mathscr{S}^{-1}\tilde{\mathscr{A}_i}\mathscr{S}$ for $i = 1,2,3$. Note that the authors in [7] consider the case $\mathscr{M} = 1$. However the case $\mathscr{M} = 1$ and the earlier mentioned case $\mathscr{M}(|U|) = |U|$ are equivalent :

$$\frac{\partial}{\partial t}|U| = a^2\frac{(\partial_p|U|)^2}{|U|} + a^{-2}\frac{(\partial_q|U|)^2}{|U|} \Leftrightarrow \frac{\partial}{\partial t}\log|U| = a^2(\partial_p\log|U|)^2 + a^{-2}(\partial_q\log|U|)^2.$$

Although the approach in [7] is highly plausible, the authors did not provide an explicit computational scheme like we provide in the next section.

On the other hand with the second approach in (30) one does not need the technicalities of the previous section, since here the viscosity solution of the system (31) is given by a basic inverse convolution over the $(\max,+)$ algebra, [4], (also known as erosion operator in image analysis)

$$\tilde{W}(p,q,t) = (K_t \ominus |U|)(p,q)e^{i\Omega(p,q,t)}\ , \tag{32}$$

with the kernel

$$K_t(p,q) = -\frac{a^{-2}p^2 + a^2q^2}{4t} \tag{33}$$

where $(f \ominus g)(p,q) = \inf_{(p',q')\in\mathbb{R}^2}[g(p',q') - f(p-p',q-q')]$. Here the homomorphism between dilation/erosion and diffusion/inverse diffusion is given by the Cramer transform $C = \mathfrak{F}\circ\log\circ\mathscr{L}$, [4], [1], which is a concatenation of the multivariate Laplace transform, logarithm and Fenchel transform. The Fenchel transform maps a convex function $c : \mathbb{R}^2 \to \overline{\mathbb{R}}$ onto $\mathbf{x} \mapsto [\mathfrak{F}c](\mathbf{x}) = \sup\{\mathbf{y}\cdot\mathbf{x} - c(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^2\}$. The isomorphic property of the Cramer transform is

$$\mathscr{C}(f*g) = \mathfrak{F}\log\mathscr{L}(f*g) = \mathfrak{F}(\log\mathscr{L}f + \log\mathscr{L}g) = \mathscr{C}f \oplus \mathscr{C}g,$$

with convolution on the $(\max,+)$-algebra given by $f \oplus g(\mathbf{x}) = \sup\limits_{\mathbf{y}\in\mathbb{R}^d}[f(\mathbf{x}-\mathbf{y}) + g(\mathbf{y})]$.

6.1 Algorithm for the PDE-approach to Differential Reassignment

Here we provide an explicit algorithm on the discrete Gabor transform $G^D_{\psi}\mathbf{f}$ of the discrete signal $\mathbf{f}$, that consistently corresponds to the theoretical PDE's on the continuous case as proposed in [7], i.e. convection equation (29) where we apply the first choice (30). Although that the PDE by [7] is not as simple as the second approach in (30) (which corresponds to a standard erosion step on the absolute value $|\mathscr{G}_{\psi} f|$ followed by a restoration of the phase afterwards) we do provide an explicit numerical scheme of this PDE, where we stay entirely in the *discrete phase space*.

It should be stressed that taking straightforward central differences of the continuous differential operators of section 6 does not work. For details and non-trivial motivation of left-invariant differences on discrete Heisenberg groups see [14].

Explicit upwind scheme with left-invariant finite differences in pseudo-code for $\mathscr{M} = 1$

For $l = 1, \ldots, K-1$, $m = 1, \ldots M-1$ set $\tilde{W}[l,m,0] := G^D_{\psi}\mathbf{f}[l,m]$.
For $t = 1, \ldots, T$
For $l = 0, \ldots, K-1$, for $m = 1, \ldots, M-1$ set
$\tilde{v}^1[l,m] := -\frac{aK}{2}(\log|\tilde{W}|[l+1,m,t=0] - \log|\tilde{W}|[l-1,m,t=0])$
$\tilde{v}^2[l,m] := -\frac{aM}{2}(\log|\tilde{W}|[l,m+1,t=0]| - \log|\tilde{W}|[l,m-1,t=0])$
$\tilde{W}[l,m,t] := \tilde{W}[l,m,t-1] + K\,\Delta t \left(z^+(\tilde{v}^1)[l,m]\; [\tilde{\mathscr{A}}_1^{D^-}\tilde{W}][l,m,t] + z^-(\tilde{v}^1)[l,m]\; [\tilde{\mathscr{A}}_1^{D^+}\tilde{W}][l,m,t] \right) +$
$M\Delta t \left(z^+(\tilde{v}^2)[l,m]\; [\tilde{\mathscr{A}}_2^{D^-}\tilde{W}][l,m,t] + z^-(\tilde{v}^2)[l,m]\; [\tilde{\mathscr{A}}_2^{D^+}\tilde{W}][l,m,t] \right)$.

Explanation of all involved variables:

l	discrete position variable $l = 0, \ldots, K-1$.
m	discrete frequency variable $m = 1, \ldots, M-1$.
t	discrete time $t = 1, \ldots T$, where T is the stopping time.
ψ	discrete kernel $\psi = \psi^C_a = \{\psi_a(nN^{-1})\}_{n=-(N-1)}^{N-1}$ or $\psi = \{\psi^D_a[n]\}_{n=-(N-1)}^{N-1}$see below.
$G^D_{\psi}\mathbf{f}[l,m]$	discrete Gabor transform computed by diagonalization via Zak transform [24].
$\tilde{W}[l,m,t]$	discrete evolving Gabor transform evaluated at position l, frequency m and time t.
$\tilde{\mathscr{A}}_i^{D^\pm}$	forward (+), backward (-) left-invariant position ($i = 1$) and frequency ($i = 2$) shifts.
$z^\pm$	$z^+(\phi)[l,m,t] = \max\{\phi(l,m,t),0\}, z^-(\phi)[l,m,t] = \min\{\phi(l,m,t),0\}$ for upwind.

The discrete left-invariant shifts on discrete phase space are given by

$$
\begin{aligned}
(\tilde{\mathscr{A}}_1^{D^+}\tilde{\Phi})[l,m] &= K(e^{-\frac{2\pi i L m}{M}}\tilde{\Phi}[l+1,m] - \tilde{\Phi}[l,m]), \quad & (\tilde{\mathscr{A}}_1^{D^-}\tilde{\Phi})[l,m] &= K(\tilde{\Phi}[l,m] - e^{\frac{2\pi i L m}{M}}\tilde{\Phi}[l,m]),\\
(\tilde{\mathscr{A}}_2^{D^+}\tilde{\Phi})[l,m] &= MN^{-1}(\tilde{\Phi}[l,m+1] - \tilde{\Phi}[l,m]), \quad & (\tilde{\mathscr{A}}_2^{D^-}\tilde{\Phi})[l,m] &= MN^{-1}(\tilde{\Phi}[l,m] - \tilde{\Phi}[l,m-1]),
\end{aligned}
\tag{34}
$$

The discrete Gabor transform equals $G^D_{\psi}\mathbf{f}[l,m] = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\psi[n-lL]} f[n] e^{-\frac{2\pi i n(n-lL)}{M}}$, where M/L denotes the (integer) oversampling factor and $N = KL$. The discrete Cauchy Riemann kernel ψ^D_a is derived in [14] and satisfies the system

$$
\forall_{l=0,\ldots,K-1}\forall_{m=0,\ldots,M-1}\forall_{\mathbf{f}\in\ell_2(I)} \;:\; \tfrac{1}{a}(\tilde{\mathscr{A}}_2^{D^+} + \tilde{\mathscr{A}}_2^{D^-}) + i\,a(\tilde{\mathscr{A}}_1^{D^+} + \tilde{\mathscr{A}}_1^{D^-})(G^D_{\psi^D_a}\mathbf{f})[l,m] = 0\;, \tag{35}
$$

which has a unique solution in case of extreme oversampling $K = M = N$, $L = 1$.

6.2 Evaluation of Reassignment

We distinguished between two approaches to apply left-invariant adaptive convection on discrete Gabor-transforms.[3] Either we apply the numerical upwind PDE-scheme described in subsection 6.1 using the discrete left-invariant vector fields (34), or we apply erosion (32) on the modulus and restore the phase afterwards. Within each of the two approaches, we can use the discrete Cauchy-Riemann kernel ψ_a^D or the sampled continuous Cauchy-Riemann kernel ψ_a^C.

To evaluate these 4 methods we apply the reassignment scheme to the reassignment of a linear chirp that is multiplied by a modulated Gaussian and is sampled using $N = 128$ samples. The input signal is an analytic signal so it suffices to show its Gabor transform from 0 to π. A visualization of this complex valued signal can be found Fig. 3 (top). The other signals in this figure are the reconstructions from the reassigned Gabor transforms that are given in Fig. 5. Here the topmost image shows the Gabor transform of the original signal. One can also find the reconstructions and reassigned Gabor transforms respectively using the four methods of reassignment. The parameters involved in generating these figures are $N = 128$, $K = 128$, $M = 128$, $L = 1$. Furthermore $a = 1/6$ and the time step for the PDE based method is set to $\Delta t = 10^{-3}$. All images show a snapshot of the reassignment method stopped at $t = 0.1$. The signals are scaled such that their energy equals the energy of the input signal. This is needed to correct for the numerical diffusion the discretization scheme suffers from. Clearly the reassigned signals resemble the input signal quite well. The PDE scheme that uses the sampled continuous window shows some defects. In contrast, the PDE scheme that uses ψ_a^D resembles the modulus of the original signal the most. Table 6.2 shows the relative ℓ_2-errors for all 4 experiments. Advantages of the erosion scheme (32) over the PDE-scheme of Section 6.1 are :

1. The erosion scheme does not produce numerical approximation-errors in the phase, which is evident since the phase is not used in the computations.

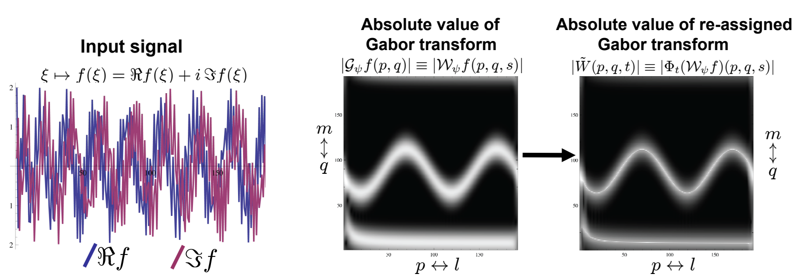

Fig. 2 Illustration of reassignment by adaptive phase-invariant convection explained in Section 6, using the upwind scheme of Subsection 6.1 applied on a Gabor transform.

[3] The induced frame operator can be efficiently diagonalized by Zak-transform, [24], boiling down to diagonalization of inverse Fourier transform on H_r, [14, ch:2.3]. We used this in our algorithms.

	ε_1	ε_2	t
Erosion continuous window	$2.41 10^{-2}$	$8.38 10^{-3}$	0.1
Erosion discrete window	$8.25 10^{-2}$	$7.89 10^{-2}$	0.1
PDE continuous window	$2.16 10^{-2}$	$2.21 10^{-3}$	0.1
PDE discrete window	$1.47 10^{-2}$	$3.32 10^{-4}$	0.1
PDE discrete window	$2.43 10^{-2}$	$6.43 10^{-3}$	0.16

Table 1 The first column shows $\varepsilon_1 = (\|\mathbf{f}-\tilde{\mathbf{f}}\|_{\ell_2(I)})\|\mathbf{f}\|_{\ell_2}^{-1}$, the relative error of the complex valued reconstructed signal compared to the input signal. In the second column $\varepsilon_2 = (\|\,|\mathbf{f}|-|\tilde{\mathbf{f}}|\,\|_{\ell_2(I)})\|\mathbf{f}\|_{\ell_2}^{-1}$ can be found which represents the relative error of the modulus of the signals. Parameters involved are $K=M=N=128$, window scale $a=\frac{1}{8}$ and convection time $t=0.1$, with times step $\Delta t = 10^{-3}$ if applicable. PDE stand for the upwind scheme presented in Subsection 6.1 and erosion means the morphological erosion method given by eq. (32).

2. The erosion scheme does not involve numerical diffusion as it does not suffer from finite step-sizes.
3. The separable erosion scheme is much faster from a computational point of view.

The convection time in the erosion scheme is different than the convection time in the upwind-scheme, due to violation of the Cauchy-Riemann equations. Typically, to get similar visual sharpening of the re-assigned Gabor transforms, the convection time of the PDE-scheme should be taken larger than the convection time of the erosion scheme (due to numerical blur in the PDE-scheme). For example $t = 1.6$ for the PDE-scheme roughly corresponds to $t = 1$ in the sense that the ℓ_2-errors nearly coincide, see Table 6.2. The method that uses a sampled version of the continuous window shows large errors. in Fig. 5 the defects are clearly visible. This shows the importance of the window selection, i.e. *in the PDE-schemes* it is better to use window ψ_a^D rather than window ψ_a^C. However, Fig. 4 and Table 6.2 clearly indicate that *in the erosion schemes* it is better to choose window ψ_a^C than ψ_a^D.

7 Existence and Uniqueness of the Evolution Solutions

The convection diffusion systems (13) have unique solutions, since the coefficients a_i and D_{ij} depend smoothly on the modulus of the initial condition $|\mathscr{W}_\psi f| = |\mathscr{G}_\psi f|$. So for a *given* initial condition $\mathscr{W}_\psi f$ the left-invariant convection diffusion generator $Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d})$ is of the type

$$Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d}) = \sum_{i=1}^{d} \alpha_i \mathscr{A}_i + \sum_{i,j=1}^{d} \mathscr{A}_i \beta_{ij} \mathscr{A}_j.$$

Such hypo-elliptic operators with almost everywhere smooth coefficients given by $\alpha_i(p,q) = a_i(|\mathscr{G}_\psi f|)(p,q)$ and $\beta_{ij}(p,q) = D_{ij}(|\mathscr{G}_\psi f|)(p,q)$ generate strongly continuous, semigroups on $\mathbb{L}_2(\mathbb{R}^2)$, as long as we keep the functions α_i and β_{ij} fixed, [21], yielding unique solutions where at least formally we may write

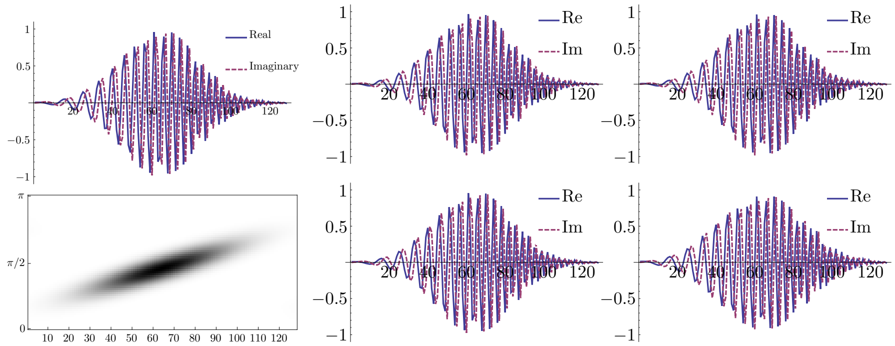

Fig. 3 Reconstructions of the reassigned Gabor transforms of the original signal that is depicted on the top left whose absolute value of the Gabor transform is depicted on bottom left. In the right: 1st row corresponds to reassignment by the upwind scheme ($\mathscr{M} = 1$) of Subsection 6.1, where again left we used ψ_a^C and right we used ψ_a^D. Parameters involved are grid constants $K = M = N = 128$, window scale $a = 1/6$, time step $\Delta t = 10^{-3}$ and time $t = 0.1$. 2nd row to reassignment by morphological erosion where in the left we used kernel ψ_a^C and in the right we used ψ_a^D. The goal of reassignment is achieved; all reconstructed signals are close to the original signal, whereas their corresponding Gabor transforms depicted in Fig. 5 are much sharper than the absolute value of the Gabor transform of the original depicted on the bottom left of this figure.

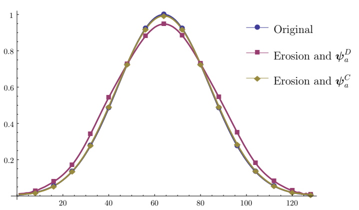

Fig. 4 The modulus of the signals in the bottom row of Fig. 3. For erosion (32) ψ_a^C performs better than erosion applied on a Gabor transform constructed by ψ_a^D.

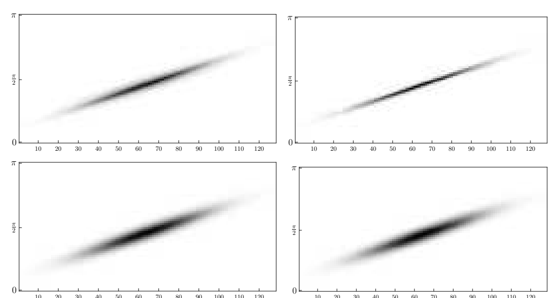

Fig. 5 Absolute value of reassigned Gabor transforms of the signals depicted in the right of Fig. 3.

$$W(p,q,s,t) = \Phi_t(\mathscr{W}_\psi f)(p,q,s) = e^{t\left(\sum_{i=1}^{d} \alpha_i \mathscr{A}_i + \sum_{i,j=1}^{d} \mathscr{A}_i \beta_{ij} \mathscr{A}_j\right)} \mathscr{W}_\psi f(p,q,s)$$

with $\lim_{t\downarrow 0} W(\cdot,t) = \mathscr{W}_\psi f$ in $\mathbb{L}_2$-sense. Note that if ψ is a Gaussian kernel and $f \neq 0$ the Gabor transform $\mathscr{G}_\psi f \neq 0$ is real analytic on $\mathbb{R}^{2d}$, so it can not vanish on a set with positive measure, so that $\alpha_i : \mathbb{R}^2 \to \mathbb{R}$ are almost everywhere smooth.

This applies in particular to the first reassignment approach in (30) (mapping everything consistently into phase space using Theorem 3), where we have set $D = 0$, $a_1(|\mathscr{G}_\psi f|) = \mathscr{M}(|\mathscr{G}_\psi f|)|\mathscr{G}_\psi f|^{-1}\partial_p|\mathscr{G}_\psi f|$ and $a_2(|\mathscr{G}_\psi f|) = \mathscr{M}(|\mathscr{G}_\psi f|)|\mathscr{G}_\psi f|^{-1}\partial_q|\mathscr{G}_\psi f|$.

Now we have to be careful with the second approach in (30), as here the operator $\tilde{U} \mapsto \tilde{\mathscr{C}}(\tilde{U})$ is *non-linear* and we are not allowed to apply the general theory. Nevertheless the operator $\tilde{U} \mapsto \tilde{\mathscr{C}}(\tilde{U})$ is left-invariant and maps the space $\mathbb{L}_2^+(\mathbb{R}^2) = \{f \in \mathbb{L}_2(\mathbb{R}^2) \mid f \geq 0\}$ into itself again. In these cases the erosion solutions (32) are the *unique viscosity solutions*, of (29), see [6].

Remark: For the diffusion case, [14, ch:7], [25, ch:6], we have $D = [D_{ij}]_{i,j=1,\ldots,2d} > 0$, in which case the (horizontal) diffusion generator $Q(|\mathscr{W}_\psi f|, \mathscr{A}_1, \ldots, \mathscr{A}_{2d})$ on the group is *hypo-elliptic*, whereas the corresponding generator $\tilde{Q}(|\mathscr{G}_\psi f|, \tilde{\mathscr{A}_1}, \ldots, \tilde{\mathscr{A}_{2d}})$ on phase space is *elliptic*. By the results [16, ch:7.1.1] and [27] we conclude that there exists a unique weak solution $\tilde{W} = \mathscr{S}W \in \mathbb{L}_2(\mathbb{R}^+, \mathbb{H}_1(\mathbb{R}^2)) \cap \mathbb{H}_1(\mathbb{R}^+, \mathbb{L}_2(\mathbb{R}^2))$ and thereby we can apply continuous point evaluation in time and operator
$\mathbb{L}_2(\mathbb{R}^2) \ni \mathscr{G}_\psi f \mapsto \tilde{\Phi}_t(\mathscr{G}_\psi f) := \tilde{W}(\cdot,\cdot,t) \in \mathbb{L}_2(\mathbb{R}^2)$ is well-defined, for all $t > 0$.

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