

# On infinity norms as Lyapunov functions for piecewise affine systems

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# On Infinity Norms as Lyapunov Functions for Piecewise Affine Systems

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## ABSTRACT

This paper considers off-line synthesis of stabilizing static feedback control laws for discrete-time piecewise affine (PWA) systems. Two of the problems of interest within this framework are: (i) incorporation of the  $\mathcal{S}$ -procedure in synthesis of a stabilizing state feedback control law and (ii) synthesis of a stabilizing output feedback control law. Tackling these problems via (piecewise) quadratic Lyapunov function candidates yields a bilinear matrix inequality at best. A new solution to these problems is proposed in this work, which uses infinity norms as Lyapunov function candidates and, under certain conditions, requires solving a single linear program. This solution also facilitates the computation of piecewise polyhedral positively invariant (or contractive) sets for discrete-time PWA systems.

## Categories and Subject Descriptors

G.1.0 [Numerical analysis]: General—Stability (and instability)

## General Terms

Theory

## Keywords

Stability, Lyapunov methods, Piecewise affine systems, Output feedback, Infinity norms

## 1. INTRODUCTION

The problems encountered in stability analysis and synthesis of stabilizing control laws for hybrid systems led to many interesting developments and relaxations of Lyapunov theory. Perhaps the most important breakthrough was the concept of multiple Lyapunov functions, which was introduced in the seminal paper [1]. Ever since, the focus has been on designing multiple Lyapunov functions for specific relevant classes of hybrid systems. One of the most successful approaches, which was initiated in the PhD thesis [2] (later published in the book [3]), considers piecewise affine (PWA) systems [4] and piecewise quadratic (PWQ) Lyapunov functions. The relaxation proposed therein requires each

quadratic function, which is part of a PWQ global function, to be positive definite and/or satisfy decreasing conditions only in a subset of the state-space, relaxation often referred to as the  $\mathcal{S}$ -procedure [5]. The stability analysis with the  $\mathcal{S}$ -procedure relaxation can be carried out efficiently, both for continuous-time and discrete-time PWA systems, as it requires solving a semidefinite programming problem. However, when it comes to *synthesis*, which consists of simultaneously searching for a PWQ Lyapunov function and a static state feedback control law, the  $\mathcal{S}$ -procedure leads to a nonlinear matrix inequality that has not been solved systematically so far, although several works considered this problem [3, 6–10]. Another relevant, non-trivial problem for PWA systems is the synthesis of a stabilizing output feedback control law. When tackled via quadratic Lyapunov functions this problem is known to be challenging even for linear systems, as it leads to a nonlinear matrix inequality. For PWA systems in particular the output feedback problem is of great interest, as for this class of systems, the observer design is a difficult problem, see, e.g., [11–13] and the references therein.

In this paper we consider discrete-time PWA systems and infinity norm based candidate Lyapunov functions. Notice that trading quadratic forms for infinity norms as Lyapunov functions does not necessarily make any of the above-mentioned problems easier, on the contrary. The application of most Lyapunov criteria expressed using infinity norms, see, for example, the seminal papers [14–18], is limited to stability analysis for linear systems or linear polytopic inclusions [19]. An extension of stability analysis via piecewise linear Lyapunov functions to certain classes of smooth nonlinear systems was proposed in [20]. Recent results on stability analysis of discrete-time linear systems via polyhedral Lyapunov functions can be found in [21]. As far as synthesis is concerned, it is worth to mention the set-based approach for constructing polyhedral control Lyapunov functions presented in [22]. Unfortunately, neither of the above procedures translates to PWA systems straightforwardly. To the best of the authors' knowledge, the existing results for hybrid systems consist of: stability analysis of continuous-time PWA systems via piecewise linear Lyapunov functions [2], switching stabilizability for continuous-time switched linear systems via polyhedral-like Lyapunov functions [23] and, synthesis of stabilizing state-feedback control laws for discrete-time PWA systems via nonlinear programming [24, 25]. In this context it is worth to mention also the *on-line* synthesis method based on linear programming and trajectory-dependent Lyapunov functions, proposed recently in [26], which can be used to stabilize the closed-loop trajectory of a PWA system for a given initial condition.

Perhaps the main reason for the limited (in terms of synthesis in particular) applicability of infinity norms as Lyapunov functions lies in the corresponding necessary stabilization conditions

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[14, 15, 18] that require the solution of a bilinear matrix equation subject to a full-column rank constraint. Starting from the standard Lyapunov sufficient conditions, in this work we propose a novel, geometric approach to infinity norms as Lyapunov functions that leads to a new set of sufficient conditions that can be expressed via a finite number of linear inequalities. For discrete-time PWA systems and static output feedback PWA control laws, we provide a solution for implementing the corresponding Lyapunov conditions that requires solving a single linear program. Moreover, in the case of a polyhedral state space partition, we demonstrate that the developed geometric approach provides a natural and simple way to implement the  $\mathcal{S}$ -procedure relaxation into synthesis, while still requiring solving a single linear program. This method also allows the direct specification of polytopic state and/or input constraints, as additional linear inequalities.

## 2. PRELIMINARIES

In this section we recall preliminary notions and fundamental stability results.

Let  $\mathbb{R}$ ,  $\mathbb{R}_+$ ,  $\mathbb{Z}$  and  $\mathbb{Z}_+$  denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every  $c \in \mathbb{R}$  and  $\Pi \subseteq \mathbb{R}$  we define  $\Pi_{\geq c(\leq c)} := \{k \in \Pi \mid k \geq c(k \leq c)\}$  and  $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$ . For a set  $\mathcal{S} \subseteq \mathbb{R}^n$ , we denote by  $\text{int}(\mathcal{S})$  the interior and by  $\text{cl}(\mathcal{S})$  the closure of  $\mathcal{S}$ . A polyhedron (or a polyhedral set) in  $\mathbb{R}^n$  is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A piecewise polyhedral (PWP) set is a set that consists of a finite union of polyhedra. For a vector  $x \in \mathbb{R}^n$ ,  $[x]_i$  denotes the  $i$ -th element of  $x$ . A vector  $x \in \mathbb{R}^n$  is said to be non-negative (nonpositive) if  $[x]_i \geq 0$  ( $[x]_i \leq 0$ ) for all  $i \in \mathbb{Z}_{[1,m]}$ , and in that case we write  $x \geq 0$  ( $x \leq 0$ ). For a vector  $x \in \mathbb{R}^n$  let  $\|\cdot\|$  denote an arbitrary  $p$ -norm. Let  $\|x\|_\infty := \max_{i=1,\dots,n} |[x]_i|$ , where  $|\cdot|$  denotes the absolute value. In the Euclidean space  $\mathbb{R}^n$  the standard inner product is denoted by  $\langle \cdot, \cdot \rangle$  and the associated norm is denoted by  $\|\cdot\|_2$ , i.e. for  $x \in \mathbb{R}^n$ ,  $\|x\|_2 = \langle x, x \rangle^{\frac{1}{2}} = (x^\top x)^{\frac{1}{2}}$ . For a matrix  $Z \in \mathbb{R}^{m \times n}$ ,  $[Z]_{ij}$  denotes the element in the  $i$ -th row and  $j$ -th column of  $Z$ . Given  $Z \in \mathbb{R}^{m \times n}$  and  $l \in \mathbb{Z}_{[1,m]}$ , we write  $[Z]_{l\bullet}$  to denote the  $l$ -th row of  $Z$ . For a matrix  $Z \in \mathbb{R}^{m \times n}$  let  $\|Z\| := \sup_{x \neq 0} \frac{\|Zx\|}{\|x\|}$  denote its corresponding induced matrix norm. It is well known that  $\|Z\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |[Z]_{ij}|$ .  $I_n \in \mathbb{R}^{n \times n}$  denotes the  $n$ -dimensional identity matrix.

Let  $\{x_1, \dots, x_m\}$  with  $x_i \in \mathbb{R}^n$  be an arbitrary set of points. A point of the form  $\sum_{i=1}^m \alpha_i x_i$  with  $\alpha_i \in \mathbb{R}_+$  is called a *conic combination* of the points  $\{x_1, \dots, x_m\}$ , while a point of the form  $\sum_{i=1}^m \alpha_i x_i$  with  $\alpha_i \in \mathbb{R}_+$  and  $\sum_{i=1}^m \alpha_i = 1$  is called a *convex combination* of  $\{x_1, \dots, x_m\}$ . A set  $\Omega$  is called a cone if for every  $x \in \Omega$  and  $\alpha \in \mathbb{R}_+$  we have  $\alpha x \in \Omega$ . A set  $\Omega$  is a convex cone if it is convex and a cone, which means that for any  $x_1, x_2 \in \Omega$  and  $\alpha_1, \alpha_2 \in \mathbb{R}_+$ , we have  $\alpha_1 x_1 + \alpha_2 x_2 \in \Omega$ . A convex hull of a set  $\Omega$ , denoted  $\text{Co}(\Omega)$ , is the set of all convex combinations of points in  $\Omega$ .

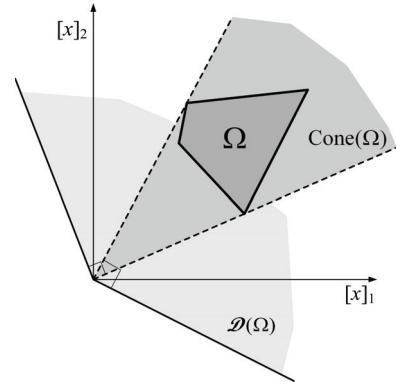
Let  $\Omega \subset \mathbb{R}^n$  be an arbitrary set. Then the set

$$\mathcal{D}(\Omega) := \{v \in \mathbb{R}^n \mid \langle v, x \rangle \geq 0, \forall x \in \Omega\}$$

is called the dual cone to the set  $\Omega$ .  $\text{Cone}(\Omega) := \{\mathcal{D}(\mathcal{D}(\Omega))\}$  denotes the closure of the minimal<sup>1</sup> convex cone that contains the set  $\Omega$ . An illustration of the cones  $\mathcal{D}(\Omega)$  and  $\text{Cone}(\Omega)$  for some set  $\Omega$  is presented in Figure 1.

A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\varphi(0) = 0$ . A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

<sup>1</sup>By this we mean the convex cone that contains  $\Omega$  and it is contained by all the other convex cones that contain  $\Omega$ .



**Figure 1: Illustration of the cones  $\mathcal{D}(\Omega)$  and  $\text{Cone}(\Omega)$  for some set  $\Omega \subset \mathbb{R}^2$ .**

belongs to class  $\mathcal{K}_\infty$  if  $\varphi \in \mathcal{K}$  and it is radially unbounded (i.e.  $\lim_{s \rightarrow \infty} \varphi(s) = \infty$ ). A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if for each fixed  $k \in \mathbb{R}_+$ ,  $\beta(\cdot, k) \in \mathcal{K}$  and for each fixed  $s \in \mathbb{R}_+$ ,  $\beta(s, \cdot)$  is decreasing and  $\lim_{k \rightarrow \infty} \beta(s, k) = 0$ .

Next, consider the discrete-time autonomous nonlinear system

$$x(k+1) = \Phi(x(k)), \quad k \in \mathbb{Z}_+, \quad (1)$$

where  $x(k) \in \mathbb{R}^n$  is the state at the discrete-time instant  $k$  and the mapping  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an arbitrary nonlinear, possibly discontinuous, function. For simplicity, we assume that the origin is an equilibrium of (1), i.e.  $\Phi(0) = 0$ .

**Definition 2.1** Let  $\lambda \in \mathbb{R}_{[0,1]}$ . We call a set  $\mathcal{P} \subseteq \mathbb{R}^n$   $\lambda$ -contractive (or shortly, contractive) for system (1) if for all  $x \in \mathcal{P}$  it holds that  $\Phi(x) \in \lambda \mathcal{P}$ . When this property holds with  $\lambda = 1$  we call  $\mathcal{P}$  a *positively invariant (PI)* set.

**Definition 2.2** Let  $\mathbb{X}$  with  $0 \in \text{int}(\mathbb{X})$  be a subset of  $\mathbb{R}^n$ . We call system (1) AS( $\mathbb{X}$ ) if there exists a  $\mathcal{KL}$ -function  $\beta(\cdot, \cdot)$  such that, for each  $x(0) \in \mathbb{X}$  it holds that the corresponding state trajectory of (1) satisfies  $\|x(k)\| \leq \beta(\|x(0)\|, k)$ ,  $\forall k \in \mathbb{Z}_+$ . We call system (1) globally asymptotically stable (GAS) if it is AS( $\mathbb{R}^n$ ).

**Theorem 2.3** Let  $\mathbb{X}$  be a PI set for (1) with  $0 \in \text{int}(\mathbb{X})$ . Furthermore, let  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\rho \in \mathbb{R}_{[0,1]}$  and let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a function such that:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}, \quad (2a)$$

$$V(\Phi(x)) \leq \rho V(x), \quad \forall x \in \mathbb{X}. \quad (2b)$$

Then system (1) is AS( $\mathbb{X}$ ).

A proof of the above theorem can be found in [27, 28]. We call a function  $V$  that satisfies (2) a *Lyapunov function*.

## 3. PROBLEM FORMULATION

In the remainder of this article we focus on discrete-time, possibly discontinuous, PWA systems of the form

$$x(k+1) = \phi(x(k), u(k)) := A_i x(k) + B_i u(k) + a_i \quad \text{if } x(k) \in \Omega_i, \quad (3)$$

where  $x(k) \in \mathbb{R}^n$  is the state vector at time  $k \in \mathbb{Z}_+$ ,  $u(k) \in \mathbb{R}^m$  is the input vector at time  $k \in \mathbb{Z}_+$ ,  $A_i \in \mathbb{R}^{n \times n}$ ,  $B_i \in \mathbb{R}^{n \times m}$ ,  $a_i \in \mathbb{R}^n$  for all  $i \in \mathcal{I}$  and  $\mathcal{I} \subset \mathbb{Z}_{\geq 1}$  is a finite set of indices. The collection of sets  $\{\Omega_i \mid i \in \mathcal{I}\}$  defines a partition of  $\mathbb{R}^n$ , meaning that  $\cup_{i \in \mathcal{I}} \Omega_i = \mathbb{R}^n$ ,

$\Omega_i \cap \Omega_j = \emptyset$  for  $(i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j$  and  $\text{int}(\Omega_i) \neq \emptyset$  for all  $i \in \mathcal{I}$ . Each  $\Omega_i$  is assumed to be a polyhedron (not necessarily closed). Let  $\mathcal{S}_0 := \{i \in \mathcal{I} \mid 0 \in \text{cl}(\Omega_i)\}$ ,  $\mathcal{S}_1 := \{i \in \mathcal{I} \mid 0 \notin \text{cl}(\Omega_i)\}$  and let  $\mathcal{S}_{\text{aff}} := \{i \in \mathcal{I} \mid a_i \neq 0\}$ ,  $\mathcal{S}_{\text{lin}} := \{i \in \mathcal{I} \mid a_i = 0\}$ , so that

$$\mathcal{S}_0 \cup \mathcal{S}_1 = \mathcal{S}_{\text{aff}} \cup \mathcal{S}_{\text{lin}} = \mathcal{I}.$$

Associated with system (3) we define the output

$$y(k) = C_i x(k) \quad \text{if } x(k) \in \Omega_i, \quad (4)$$

where  $y(k) \in \mathbb{R}^l$  is the output vector at time  $k \in \mathbb{Z}_+$  and  $C_i \in \mathbb{R}^{l \times n}$  for all  $i \in \mathcal{I}$ . We parameterize the control input as  $u(k) = h(y(k))$ ,  $k \in \mathbb{Z}_+$ , with  $h: \mathbb{R}^l \rightarrow \mathbb{R}^m$  defined as:

$$\begin{aligned} h(y) &= F_i y + f_i \quad \text{if } x \in \Omega_i \\ &= F_i C_i x + f_i \quad \text{if } x \in \Omega_i, \end{aligned} \quad (5)$$

and where  $F_i \in \mathbb{R}^{m \times l}$ ,  $f_i \in \mathbb{R}^m$  for all  $i \in \mathcal{I}$ . The implementation of the above output feedback control law requires the following standing assumption.

**Assumption 3.1** At every time instant  $k \in \mathbb{Z}_+$ , the region  $\Omega_i, i \in \mathcal{I}$  where the state  $x(k)$  lies is known.  $\square$

The above assumption requires estimating the system's mode, i.e., the region  $\Omega_i$  where the state lies, from the output, which is a non-trivial problem, see, for example, [11–13] and the references therein. An alternative would be to consider output based switching and formulate partial stability conditions in terms of the output only. Obviously, for  $C_i = I_n$  we recover the state-feedback case.

Next, we define the candidate Lyapunov function:

$$V(x) = V_i(x) \quad \text{if } x \in \Omega_i \quad (6)$$

with  $V_i(x) := \|P_i x\|_\infty$  for all  $i \in \mathcal{I}$ , where  $P_i \in \mathbb{R}^{p_i \times n}$  is a matrix that satisfies  $\|P_i\|_\infty \neq 0$  for all  $i \in \mathcal{I}$  and  $p_i \in \mathbb{Z}_{\geq n}$ . Observe that this property ensures the upper bound in (2a) for all  $x \in \mathbb{R}^n$ , with  $\alpha_2(s) := \max_{i \in \mathcal{I}} \|P_i\|_\infty s$ . Usually, it is also assumed that each  $P_i$  has full-column rank, which ensures the lower bound in (2a) for all  $x \in \mathbb{R}^n$ , with  $\alpha_1(s) := \min_{i \in \mathcal{I}} \frac{\sigma_i}{\sqrt{p_i}} s$ , where  $\sigma_i > 0$  is the smallest singular value of  $P_i$ , respectively. However, this assumption introduces a certain conservatism as, according to the  $\mathcal{S}$ -procedure relaxation,  $V_i(x) \geq 0$  should only hold for  $x \in \Omega_i$  and not for all  $x \in \mathbb{R}^n$ .

**Remark 3.2** For clarity of exposition we used the system's state-space partition  $\{\Omega_i\}_{i \in \mathcal{I}}$  as the control input and Lyapunov function partition and we assumed that the state, input and output have the same dimension for all regions  $\{\Omega_i\}_{i \in \mathcal{I}}$ . In general one can have different partitions and different dimensions, case in which the developed results still apply directly. Also, it is well known [3] that further partitioning the state-space regions  $\Omega_i$ , which leads to more different input feedback and Lyapunov weight matrices decreases conservativeness. This will be illustrated in the example presented in Section 5.  $\square$

Next, we state the standard stability result for system (3)-(4) in closed-loop with (5) with the  $\mathcal{S}$ -procedure relaxation. Let

$$\Phi(x) := (A_i + B_i F_i C_i)x + (B_i f_i + a_i) \quad \text{if } x \in \Omega_i$$

denote the closed-loop dynamics that correspond to (3)-(5). Notice that for the origin to be an equilibrium in the Lyapunov sense for  $\Phi$  it is necessary that  $f_i = 0$  for all  $i \in \mathcal{S}_0 \cap \mathcal{S}_{\text{lin}}$  and  $B_i f_i + a_i = 0$  for all  $i \in \mathcal{S}_0 \cap \mathcal{S}_{\text{aff}}$ . This is usually circumvented [3] by assuming  $a_i = 0$  and setting  $f_i = 0$  for all  $i \in \mathcal{S}_0$ .

**Theorem 3.3** Let  $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{H}_\infty$  and  $\rho \in \mathbb{R}_{(0,1)}$ . Suppose that there exists a set of functions  $V_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $i \in \mathcal{I}$ , with  $V_i(0) = 0$  for all  $i \in \mathcal{S}_0$ , that satisfy:

$$V_i(x) \leq \alpha_{2,i}(\|x\|) \quad \text{if } x \in \Omega_i, \quad (7a)$$

$$V_i(x) \geq \alpha_{1,i}(\|x\|) \quad \text{if } x \in \Omega_i, \quad (7b)$$

$$V_j(\Phi(x)) \leq \rho V_i(x) \quad \text{if } x \in \Omega_i, \Phi(x) \in \Omega_j, \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \quad (7c)$$

Then,  $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ ,  $V(x) := V_i(x)$  if  $x \in \Omega_i$  is a Lyapunov function in  $\mathbb{R}^n$  for the closed-loop system (3)-(5) and consequently, its origin is GAS.

The proof of the above theorem can be found in [1, 3, 28]. Further generic relaxations for which the results developed in this paper still apply directly include a different  $\rho_i \in \mathbb{R}_{(0,1)}$  for each  $i \in \mathcal{I}$  and reducing the set of pairs of indexes in (7c) via a reachability analysis. However, the most important relaxations with respect to Theorem 2.3 are in conditions (7b) and (7c) that impose the lower bound and the one-step decrease region-wise for each function  $V_i$ . For example, in the case of a PWQ Lyapunov function, these conditions relax the positive definiteness requirement on certain matrices.

In what follows we focus on piecewise polyhedral (PWP) Lyapunov functions of the form (6). Notice that if for such a function (7b) holds, it necessarily holds that  $\|P_i\|_\infty \neq 0$ , which further implies (7a) with  $\alpha_{2,i}(s) := \|P_i\|_\infty s$ . Also, it holds that  $V_i(0) = 0$  for all  $i \in \mathcal{I}$ . As such, it is sufficient to focus on finding a solution to the conditions (7b) and (7c), which is formally stated in the next problem.

**Problem 3.4** Find a set of matrices  $\{P_i, F_i\}_{i \in \mathcal{I}}$ , a set of vectors  $\{f_i\}_{i \in \mathcal{I}}$  with  $f_i = 0$  for all  $i \in \mathcal{S}_0$  and a set of constants  $\{c_i\}_{i \in \mathcal{I}}$  with  $c_i \in \mathbb{R}_{>0}$  for all  $i \in \mathcal{I}$  such that:

$$c_i \|x\|_\infty - \|P_i x\|_\infty \leq 0 \quad \text{if } x \in \Omega_i, \quad (8a)$$

$$\begin{aligned} \|P_j((A_i + B_i F_i C_i)x + B_i f_i + a_i)\|_\infty - \rho \|P_i x\|_\infty &\leq 0 \\ \text{if } x \in \Omega_i, \Phi(x) \in \Omega_j, \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \end{aligned} \quad (8b)$$

The difficulty of Problem 3.4 comes mainly from the following issues: (i) the left-hand expression in both (8a) and (8b) is a non-convex function of  $x$  and (ii) the left-hand expression in (8b) is bilinear in  $P_j, F_j$  and  $P_j, f_j$ , respectively. To the best of the authors' knowledge, the only solution to the above problem was presented in [24, 25] for the case when  $f_i = 0$  and  $a_i = 0$  for all  $i \in \mathcal{I}$ . Therein, (8a) is attained by requiring each  $P_i$  to have full-column rank<sup>2</sup> and (8b) is achieved by asking that

$$\|P_j(A_i + B_i F_i C_i)P_i^{-L}\|_\infty - \rho \leq 1, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I},$$

where  $P_i^{-L} := (P_i^\top P_i)^{-1} P_i^\top$ . This solution, which does not include the  $\mathcal{S}$ -procedure relaxation, requires solving a finite dimensional nonlinear program subject to nonlinear constraints, which can be tackled using standard nonlinear solvers, e.g., *fmincon* of Matlab.

In the next section we will provide a novel solution to Problem 3.4 that can be implemented by solving a single linear program.

<sup>2</sup>Recently, a technique for attaining full-column rank of a unknown matrix via linear inequalities in the elements of the matrix was presented in [26].



## 4. MAIN RESULTS

Before continuing with the solution to Problem 3.4 and the complete presentation of the controller synthesis procedure, we need to introduce an appropriately defined set of vertices that corresponds to each region  $\Omega_i$ , i.e.,

$$\mathcal{V}(\Omega_i) := \{x_i^1, x_i^2, \dots, x_i^{M_i}\}, \quad M_i \in \mathbb{Z}_{\geq 1}, i \in \mathcal{I}. \quad (9)$$

This set of vertices will differ depending on the type of the set  $\Omega_i$ , i.e., bounded or unbounded, and will be instrumental in the formulation of the synthesis algorithm.

For the remainder of the article we assume the following property concerning the state-space partition  $\{\Omega_i \mid i \in \mathcal{I}\}$ .

**Assumption 4.1** For each  $\Omega_i$ ,  $i \in \mathcal{I}$ , there exists a closed half space  $\mathcal{H}_i$  of  $\mathbb{R}^n$  defined by a hyperplane through the origin, i.e.  $\mathcal{H}_i := \{x \in \mathbb{R}^n \mid r_i^\top x \geq 0\}$  for some  $r_i \in \mathbb{R}^n$ , such that  $\text{cl}(\Omega_i) \subset \mathcal{H}_i$  holds.  $\square$

Note that the above assumption eliminates any state-space region  $\Omega_i$  that contains the origin in its interior, as well as the case when  $\text{cl}(\Omega_i)$  is itself a closed half space defined by a hyperplane through the origin. However, in the case that the original partition  $\{\Omega_i \mid i \in \mathcal{I}\}$  of the state space for system (3) is such that for some  $i \in \mathcal{I}$  Assumption 4.1 is not satisfied, the corresponding sets  $\Omega_i$  can easily be further partitioned into new polyhedral sets with the same dynamics, so that the above assumption holds for the new partition.

At this point we will make a distinction between bounded and unbounded sets  $\Omega_i$ . If the set  $\Omega_i$  is bounded, since it is assumed to be a polyhedron, there necessarily exists a finite set of vertices  $\{x_i^1, x_i^2, \dots, x_i^{M_i}\}$ , with  $x_i^j \in \mathbb{R}^n$  for all  $j \in \mathbb{Z}_{[1, M_i]}$ , such that

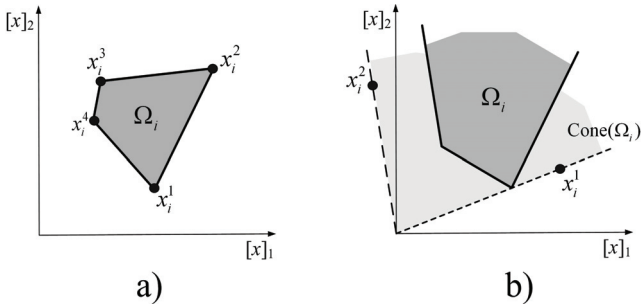
$$\text{cl}(\Omega_i) = \text{Co}(\{x_i^1, x_i^2, \dots, x_i^{M_i}\}).$$

In this case the elements of the set  $\mathcal{V}(\Omega_i)$  are simply the vertices of the polyhedron  $\Omega_i$ . In the case that the set  $\Omega_i$  is unbounded, let  $\{x_i^1, x_i^2, \dots, x_i^{M_i}\}$ , with  $x_i^j \in \mathbb{R}^n$  for all  $j \in \mathbb{Z}_{[1, M_i]}$ , be such that

$$\text{Cone}(\Omega_i) = \text{Cone}(\{x_i^1, x_i^2, \dots, x_i^{M_i}\}),$$

i.e. the set of vertices  $\{x_i^1, x_i^2, \dots, x_i^{M_i}\}$  define a set of points on the rays of the minimal convex cone containing the set  $\Omega_i$ . In this case the elements of the set  $\mathcal{V}(\Omega_i)$  can be chosen arbitrarily as non-zero points on the rays of  $\text{Cone}(\Omega_i)$ . Any element of the set  $\Omega_i$  can be written as a conic combination of the elements of the set  $\mathcal{V}(\Omega_i)$ .

An illustration of the set of vertices  $\mathcal{V}(\Omega_i)$  for a bounded and an unbounded state-space region  $\Omega_i$  is shown in Figure 2.



**Figure 2:** a) Set of vertices for a bounded set  $\Omega_i$ ; b) Set of vertices for an unbounded set  $\Omega_i$ .

Let  $\mathcal{I}_\infty \subseteq \mathcal{I}$  denote the set of indices that correspond to the unbounded regions  $\Omega_i$  and consider the following set of inequalities:

$$[P_i]_{e^\bullet}^\top \in \mathcal{D}(\text{cl}(\Omega_i)), \quad \forall e \in \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \quad (10a)$$

$$c_i \|x_i^y\|_\infty - [P_i x_i^y]_e \leq 0, \quad \forall x_i^y \in \mathcal{V}(\Omega_i), \forall e \in \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \quad (10b)$$

$$\begin{aligned} & \pm [P_j(A_i + B_i F_i C_i)x_i^y + B_i f_i + a_i]_{e_1} - \rho [P_i x_i^y]_{e_2} \leq 0, \\ & \forall x_i^y \in \mathcal{V}(\Omega_i), \forall (e_1, e_2) \in \mathbb{Z}_{[1, p_i]} \times \mathbb{Z}_{[1, p_i]}, \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \end{aligned} \quad (10c)$$

**Lemma 4.2** Suppose that the set of inequalities (10) is feasible and let  $\{P_i, F_i, f_i, c_i\}_{i \in \mathcal{I}}$  denote a solution for which it holds that  $B_i f_i + a_i = 0$  for all  $i \in \mathcal{I}_\infty$ . Then  $\{P_i, F_i, f_i, c_i\}_{i \in \mathcal{I}}$  satisfies the inequalities (8).

**PROOF.** For an arbitrary  $i \in \mathcal{I}$  the constraint (10a) implies that  $[P_i]_{e^\bullet} x \geq 0$  holds for all  $e \in \mathbb{Z}_{[1, p_i]}$  and for all  $x \in \Omega_i$ . In other words  $P_i x \geq 0$  for all  $x \in \Omega_i$  and  $i \in \mathcal{I}$ .

Consider an arbitrary  $x \in \Omega_i$  for an arbitrary  $i \in \mathcal{I}$ . Then  $x$  can be obtained either by a convex or by a conic combination of the elements of  $\mathcal{V}(\Omega_i)$ , i.e., there exists a set of nonnegative scalars  $\{\lambda_v\}_{v \in \mathbb{Z}_{[1, M_i]}}$  such that  $x_i = \sum_{v=1}^{M_i} \lambda_v x_i^v$ . Then (10b), by appropriate multiplication with  $\lambda_v$  and summation, implies  $\sum_{v=1}^{M_i} c_i \lambda_v \|x_i^v\|_\infty - \sum_{v=1}^{M_i} \lambda_v [P_i x_i^v]_e \leq 0$  for all  $e \in \mathbb{Z}_{[1, p_i]}$ . Using the triangular inequality for norms this further implies that  $c_i \|x\|_\infty - [P_i x]_e \leq 0$  for all  $e \in \mathbb{Z}_{[1, p_i]}$ , or equivalently

$$c_i \|x\|_\infty - \max_{e \in \mathbb{Z}_{[1, p_i]}} [P_i x]_e \leq 0. \quad (11)$$

Since  $[P_i x]_e \geq 0$  for all  $e \in \mathbb{Z}_{[1, p_i]}$  it follows that

$$\|P_i x\|_\infty = \max_{e \in \mathbb{Z}_{[1, p_i]}} [P_i x]_e = \max_{e \in \mathbb{Z}_{[1, p_i]}} [P_i x]_e,$$

i.e., (11) implies (8a).

Next we show that (10c) implies (8b). Again, suppose that  $x$  is a point obtained either by a convex or by a conic combination of the elements of  $\mathcal{V}(\Omega_i)$ , i.e.,  $x = \sum_{v=1}^{M_i} \lambda_v x_i^v$  for some set  $\{\lambda_v\}_{v \in \mathbb{Z}_{[1, M_i]}}$ ,  $\lambda_v \in \mathbb{R}_+$ . Appropriate multiplication with  $\lambda_v$  and summation over the indices  $v$  in the inequalities (10c) yield

$$\begin{aligned} & \pm [P_j(A_i + B_i F_i C_i)x + \sum_{v=1}^{M_i} \lambda_v (B_i f_i + a_i)]_{e_1} - \rho [P_i x]_{e_2} \leq 0, \\ & \forall (e_1, e_2) \in \mathbb{Z}_{[1, p_i]} \times \mathbb{Z}_{[1, p_i]}, \forall (i, j) \in \mathcal{I} \times \mathcal{I}, \end{aligned} \quad (12)$$

which further implies that

$$\begin{aligned} & \max_{e_1 \in \mathbb{Z}_{[1, p_i]}} \left( \pm [P_j(A_i + B_i F_i C_i)x + \sum_{v=1}^{M_i} \lambda_v (B_i f_i + a_i)]_{e_1} \right) \\ & - \rho \max_{e_2 \in \mathbb{Z}_{[1, p_i]}} [P_i x]_{e_2} \leq 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \end{aligned} \quad (13)$$

As for an arbitrary vector  $z \in \mathbb{R}^n$ ,  $\max_{e \in \mathbb{Z}_{[1, n]}} (\pm [z]_e) = \|z\|_\infty$  by definition and using the non-negativity of the vector  $P_i x$ , the above inequality yields

$$\begin{aligned} & \left\| P_j(A_i + B_i F_i C_i)x + \sum_{v=1}^{M_i} \lambda_v (B_i f_i + a_i) \right\|_\infty - \rho \|P_i x\|_\infty \leq 0, \\ & \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \end{aligned} \quad (14)$$

To conclude the proof, from this point we make a distinction between unbounded and bounded regions  $\Omega_i$ . Firstly, suppose that  $i \in \mathcal{I} \setminus \mathcal{I}_\infty$ , i.e., that  $\Omega_i$  is a bounded set. In that case any  $x \in \Omega_i$  can

be represented as a *convex* combination of the corresponding set of vertices  $\mathcal{V}(\Omega_i)$ , i.e., the non-negative scalars  $\lambda_v$  from the expression  $x = \sum_{v=1}^{M_i} \lambda_v x_i^v$  have the additional property that  $\sum_{v=1}^{M_i} \lambda_v = 1$ . Using this equality it is obvious that (14) corresponds to (8b). Secondly, suppose that  $i \in \mathcal{I}_\infty$ , i.e., any  $x \in \Omega_i$  can be represented as a *conic* (rather than convex) combination of elements of the corresponding set of vertices  $\mathcal{V}(\Omega_i)$ . In this case (14) corresponds to (8b) since  $B_i f_i + a_i = 0$  for all  $i \in \mathcal{I}_\infty$ .  $\square$

**Remark 4.3** The essential step in overcoming the first difficulty of Problem 3.4, which comes from non-convexity of both (8a) and (8b) in  $x$ , is provided in Lemma 4.2 and consists of constraining the rows of each matrix  $P_i$ , via (10a), so that  $P_i x$  is non-negative for all  $x \in \Omega_i$ .  $\square$

From the inequalities (10) one can deduce that it necessarily holds that  $P_i x_i^v \geq 0$  for all  $x_i^v \in \mathcal{V}(\Omega_i)$ , even in the case when  $x_i^v \notin \Omega_i$ , as it can happen when  $\Omega_i$  is an unbounded set (see Figure 2). This is shown as follows. Since for a closed and convex cone  $C$  it holds that  $\mathcal{D}(\mathcal{D}(C)) = C$ , see e.g., [29], and since the dual cone  $\mathcal{D}(\text{cl}(\Omega_i))$  is always closed and convex [29], we have that  $\mathcal{D}(\text{Cone}(\text{cl}(\Omega_i))) = \mathcal{D}(\mathcal{D}(\mathcal{D}(\text{cl}(\Omega_i)))) = \mathcal{D}(\text{cl}(\Omega_i))$ . Therefore (10a) implies  $P_i x_i^v \geq 0$  for all  $v \in \mathbb{Z}_{[1, M_i]}$ , as  $x_i^v \in \text{Cone}(\text{cl}(\Omega_i))$ .

Similarly, it can be shown that in the case of a bounded set  $\Omega_i$ , the condition (10a) implies  $P_i x \geq 0$  not only for  $x \in \Omega_i$ , but also for all  $x \in \text{Cone}(\mathcal{V}(\Omega_i))$ .

As such, we can conclude that (10a) does not necessarily impose  $P_i x \geq 0$  for any  $x \notin \text{Cone}(\mathcal{V}(\Omega_i))$ , i.e. the  $\mathcal{S}$ -procedure relaxation on the positive definiteness and the lower bound on each function  $V_i$  is attained with respect to each region  $\text{Cone}(\mathcal{V}(\Omega_i))$  that contains  $\Omega_i$ , respectively. This can be more or less conservative, depending on the polyhedron  $\Omega_i$ ,  $i \in \mathcal{I}$ , but it is, however, relaxing the stabilization conditions in comparison with not having an  $\mathcal{S}$ -procedure relaxation at all. It is worth to mention that an exact implementation of the  $\mathcal{S}$ -procedure relaxation is obtained when the regions  $\Omega_i$  are cones, i.e. for conewise linear systems.

Another interesting aspect regarding Lemma 4.2 is the condition  $B_i f_i + a_i = 0$  for  $i \in \mathcal{I}_\infty$ . This condition needs not hold for the indexes  $i \in \mathcal{I}_\infty \cap \mathcal{I}_1$  to allow for a Lyapunov stable equilibrium. As such, if for a certain  $i \in \mathcal{I}_\infty \cap \mathcal{I}_1$ ,  $a_i \neq 0$ , it may be possible that for a certain  $B_i$  there does not exist an  $f_i$  such that  $B_i f_i + a_i = 0$ . This problem is avoided if the dynamics and the control laws are linear, instead of affine, for all  $i \in \mathcal{I}_\infty \cap \mathcal{I}_1$ .

A possible solution to deal with both problems mentioned above, i.e. to further relax (10a) and to discard the condition that  $B_i f_i + a_i = 0$  for  $i \in \mathcal{I}_\infty$ , is to use functions of the form  $V_i(x) := \|P_i x + z_i\|_\infty$  and exploit the freedom from the newly added parameters in  $z_i$ . This makes the object of future work.

**Remark 4.4** The issues discussed above are inherent problems related to affine dynamics and polyhedral regions  $\Omega_i$  with  $0 \notin \text{cl}(\Omega_i)$ . For example, when using PWQ functions either for analysis or synthesis one has to deal with the same problems. The solution proposed in [3] relies on augmenting the state vector with the affine term, which transforms the original PWA system into a lifted, piecewise linear (PWL) form. The techniques developed in this paper would then apply directly to the lifted PWL system. However, it is known [3, 6, 10] that the lifted PWL system also introduces a certain conservatism.  $\square$

Next, we continue with the solution to the second difficulty of Problem 3.4. Let  $\{R_i, r_i\}_{i \in \mathcal{I}}$  with  $R_i \in \mathbb{R}^{m \times l}$  and  $r_i \in \mathbb{R}^m$  for all  $i \in \mathcal{I}$  and  $\xi \in \mathbb{R}_{>0}$  denote unknown variables. Consider the following

inequalities in  $\{P_i, R_i, r_i\}_{i \in \mathcal{I}}$  and  $\xi$ :

$$\begin{aligned} \pm [(\xi A_i + B_i R_i C_i) x_i^v + B_i r_i + \xi a_i]_{e_1} - \rho [P_i x_i^v]_{e_2} \leq 0, \\ \forall x_i^v \in \mathcal{V}(\Omega_i), \forall (e_1, e_2) \in \mathbb{Z}_{[1, m]} \times \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \end{aligned} \quad (15a)$$

$$\|P_i\|_\infty - \xi \leq 0, \quad \forall i \in \mathcal{I}. \quad (15b)$$

**Lemma 4.5** Suppose that the set of inequalities (15) is feasible and let  $\{P_i, R_i, r_i\}_{i \in \mathcal{I}}$  and  $\xi \in \mathbb{R}_{>0}$  denote a solution for which it holds that  $B_i r_i + \xi a_i = 0$  for all  $i \in \mathcal{I}_\infty$ . Then  $\{P_i, F_i, f_i\}_{i \in \mathcal{I}}$  with  $F_i := \frac{1}{\xi} R_i$  and  $f_i := \frac{1}{\xi} r_i$  for all  $i \in \mathcal{I}$  satisfy the inequality (10c) and  $B_i f_i + a_i = 0$  for all  $i \in \mathcal{I}_\infty$ .

**PROOF.** The proof is based on two arguments, i.e. the fact that for an arbitrary vector  $z \in \mathbb{R}^n$  and  $c \in \mathbb{Z}_+$  it holds that  $\|z\| \leq c \Leftrightarrow \pm [z]_j \leq c$  for all  $j \in \mathbb{Z}_{[1, n]}$  and the Cauchy-Schwarz inequality. Starting from (15a) we obtain

$$\begin{aligned} \|(\xi A_i + B_i R_i C_i) x_i^v + B_i r_i + \xi a_i\|_\infty - \rho [P_i x_i^v]_{e_2} \leq 0, \\ \forall x_i^v \in \mathcal{V}(\Omega_i), \forall e_2 \in \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}. \end{aligned}$$

Using the fact that  $\|P_i x\|_\infty \leq \|P_i\|_\infty \|x\|_\infty$  for any  $x \in \mathbb{R}^n$ ,  $i \in \mathcal{I}$  and (15b) in the above inequality we obtain:

$$\begin{aligned} \|P_j (A_i + B_i \frac{1}{\xi} R_i C_i) x_i^v + B_i \frac{1}{\xi} r_i + a_i\|_\infty - \rho [P_i x_i^v]_{e_2} \leq 0, \\ \forall x_i^v \in \mathcal{V}(\Omega_i), \forall e_2 \in \mathbb{Z}_{[1, p_i]}, \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \end{aligned}$$

Letting  $F_i = \frac{1}{\xi} R_i$  and  $f_i = \frac{1}{\xi} r_i$  in the above inequality and using the first argument mentioned above yields (10c). Furthermore, from  $B_i r_i + \xi a_i = 0$  for all  $i \in \mathcal{I}_\infty$  we obtain  $B_i f_i + a_i = 0$  for all  $i \in \mathcal{I}_\infty$ , which completes the proof.  $\square$

Observing that for any matrix  $P_i \in \mathbb{R}^{p_i \times n}$  the condition  $\|P_i\|_\infty \leq \xi$  for some  $\xi \in \mathbb{R}_{>0}$  is equivalent to  $\pm [P_i]_{j1} \pm [P_i]_{j2} \dots \pm [P_i]_{jn} \leq c$  for  $j \in \{1, \dots, p_i\}$ , it follows that both constraints in (15) can be written as linear inequalities in the elements of  $\{P_i, R_i, r_i\}_{i \in \mathcal{I}}$  and  $\xi$ . As such, we are now ready to define a single set of linear inequalities whose solutions solve Problem 3.4. As the set  $\mathcal{D}(\text{cl}(\Omega_i))$  is defined by hyperplanes, there exists a set of matrices and vectors  $\{G_i, g_i\}_{i \in \mathcal{I}}$  such that  $[P_i]_{e \bullet}^\top \in \mathcal{D}(\text{cl}(\Omega_i))$  is equivalent with  $G_i [P_i]_{e \bullet}^\top \leq g_i$ .

Next, consider the following *linear inequality and equality conditions* in the unknowns  $\{P_i, R_i, r_i, c_i\}_{i \in \mathcal{I}}$  and  $\xi$ :

$$G_i [P_i]_{e \bullet}^\top \leq g_i, \quad \forall e \in \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \quad (16a)$$

$$c_i \|x_i^v\|_\infty - [P_i x_i^v]_e \leq 0, \quad \forall x_i^v \in \mathcal{V}(\Omega_i), \forall e \in \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \quad (16b)$$

$$\begin{aligned} \pm [(\xi A_i + B_i R_i C_i) x_i^v + B_i r_i + \xi a_i]_{e_1} - \rho [P_i x_i^v]_{e_2} \leq 0, \\ \forall x_i^v \in \mathcal{V}(\Omega_i), \forall (e_1, e_2) \in \mathbb{Z}_{[1, m]} \times \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \end{aligned} \quad (16c)$$

$$\pm [P_i]_{e1} \pm [P_i]_{e2} \pm \dots \pm [P_i]_{en} \leq \xi, \quad \forall e \in \mathbb{Z}_{[1, p_i]}, \forall i \in \mathcal{I}, \quad (16d)$$

$$B_i r_i + \xi a_i = 0, \quad \forall i \in \mathcal{I}_\infty. \quad (16e)$$

**Theorem 4.6 (i)-Output feedback synthesis:**

Suppose that the set of inequality and equality conditions (16) is feasible and let  $\{P_i, R_i, r_i, c_i\}_{i \in \mathcal{I}}$  with  $c_i \in \mathbb{R}_{>0}$  for all  $i \in \mathcal{I}$  and  $\xi$  denote a solution<sup>3</sup>. Let  $V_i(x) := \|P_i x\|_\infty$  and let  $F_i := \frac{1}{\xi} R_i$  and  $f_i := \frac{1}{\xi} r_i$  for all  $i \in \mathcal{I}$ . Then  $V(x) = V_i(x)$  if  $x \in \Omega_i$  is a Lyapunov function in  $\mathbb{R}^n$  for the closed-loop system (3)-(5) and consequently, its origin is GAS.

<sup>3</sup>Notice that (16b), (16d) and  $c_i \in \mathbb{R}_{>0}$  imply  $\xi \in \mathbb{R}_{>0}$ .

**(ii)-Analysis:**

Suppose that  $a_i = 0$  for all  $i \in \mathcal{I}_0 \cup \mathcal{I}_\infty$  and  $F_i = 0$ ,  $f_i = 0$  for all  $i \in \mathcal{I}$ . Moreover, suppose that the inequalities (16a), (16b) and (10c) are feasible. Let  $\{P_i, c_i\}_{i \in \mathcal{I}}$  with  $c_i \in \mathbb{R}_{>0}$  for all  $i \in \mathcal{I}$  denote a solution of (16a)-(16b)-(10c) and let  $V_i(x) := \|P_i x\|_\infty$ . Then  $V(x) = V_i(x)$  if  $x \in \Omega_i$  is a Lyapunov function in  $\mathbb{R}^n$  for the dynamics  $\Phi(x) = A_i x + a_i$  if  $x \in \Omega_i$  and consequently, its origin is GAS.

PROOF. Both statements follow from their corresponding hypothesis and by applying Lemma 4.5, Lemma 4.2 and Theorem 3.3 for the first statement and Lemma 4.2 and Theorem 3.3 for the second one.  $\square$

**Remark 4.7** The essential step in overcoming the second difficulty of Problem 3.4, which comes from bilinearity in certain variables, is provided in Lemma 4.5 and consists of substituting the matrices  $\{P_j\}_{j \in \mathcal{I}}$  in (10c) with the scalar variable  $\xi$ . This makes it possible to define new variables corresponding to the feedback gain matrices  $\{F_i\}_{i \in \mathcal{I}}$  and vectors  $\{f_i\}_{i \in \mathcal{I}}$  from which these initial variables can be reconstructed. Some conservatism is introduced by assigning a common variable  $\xi$ . Unfortunately, assigning different variables  $\xi_i$  for each matrix  $P_i$  does not allow reconstruction of  $\{F_i\}_{i \in \mathcal{I}}$  and  $\{f_i\}_{i \in \mathcal{I}}$  from the new variables. Future work deals with finding another substitution of variables that is less conservative.  $\square$

**Remark 4.8** The results of Theorem 4.6 could be recovered, *mutatis mutandis*, for continuous-time PWA systems using the results of, for example, [22] to obtain an expression of the derivative of the PWP Lyapunov function (6) and imposing continuity of  $V$  at the boundaries of the regions  $\Omega_i$ ,  $i \in \mathcal{I}$ . Further working out the continuous-time case will be addressed in future work.  $\square$

Next, we propose two alternative sets of inequalities to the set of inequalities (16c), which have a significantly smaller number of inequalities.

**Corollary 4.9** Suppose that the hypotheses of Theorem 4.6 hold for (16c) replaced by

$$\begin{aligned} \pm [(\xi A_i + B_i R_i C_i)x_i^y + B_i r_i + \xi a_i]e_1 - \rho c_i \|x_i^y\|_\infty \leq 0, \\ \forall x_i^y \in \mathcal{V}(\Omega_i), \forall e_1 \in \mathbb{Z}_{[1,m]}, \forall i \in \mathcal{I}. \end{aligned} \quad (17)$$

Then, both statements of Theorem 4.6 hold.

The result of Corollary 4.9 is obtained by substituting  $[P_i x_i^y]_{e_2}$  in (16c) with the lower bound, which is possible due to (16b). In this way, checking (16c) for all possible combinations  $(e_1, e_2) \in \mathbb{Z}_{[1,m]} \times \mathbb{Z}_{[1,p_i]}$  is reduced to checking (17) for  $e_1 \in \mathbb{Z}_{[1,m]}$ .

**Corollary 4.10** Suppose that the hypotheses of Theorem 4.6 hold for (16c) replaced by

$$[P_i x_i^y]_1 \geq [P_i x_i^y]_{e_2}, \forall e_2 \in \mathbb{Z}_{[2,p_i]}, \forall x_i^y \in \mathcal{V}(\Omega_i), \forall i \in \mathcal{I}, \quad (18a)$$

$$\begin{aligned} \pm [(\xi A_i + B_i R_i C_i)x_i^y + B_i r_i + \xi a_i]e_1 - \rho [P_i x_i^y]_1 \leq 0, \\ \forall x_i^y \in \mathcal{V}(\Omega_i), \forall e_1 \in \mathbb{Z}_{[1,m]}, \forall i \in \mathcal{I}. \end{aligned} \quad (18b)$$

Then, both statements of Theorem 4.6 hold.

The result of Corollary 4.10 is obtained by substituting  $[P_i x_i^y]_{e_2}$  in (16c) with  $[P_i x_i^y]_1$ , which is the dominant row due to (18a). In this way, checking (16c) for all possible combinations  $(e_1, e_2) \in \mathbb{Z}_{[1,m]} \times \mathbb{Z}_{[1,p_i]}$  is reduced to checking (18b) for  $e_1 \in \mathbb{Z}_{[1,m]}$  and adding

(18a). It should be mentioned also that the approach of Corollary 4.10 is less conservative than (16c), while the approach of Corollary 4.9 is more conservative.

It is worth to note that the smaller the region  $\Omega_i$  is, the larger the feasible dual cone  $\mathcal{D}(\text{cl}(\Omega_i))$  is. As such, further partitioning each region  $\Omega_i$  does not only decrease conservativeness, but it also increases the feasible cone corresponding to each  $P_i$ . An extreme realization is obtained when each region  $\Omega_i$  consists of a single ray in the state-space, i.e., a possibly different matrix  $P_i$  and hence, a possibly different Lyapunov function  $V_i$ , is assigned to each state vector. Recently, in [26] it was shown that this ‘‘least conservative’’ approach, when the feasible  $\mathcal{D}(\text{cl}(\Omega_i))$  covers the largest area, leads to a trajectory-dependent Lyapunov function that can be searched for via on-line optimization (linear programming) for a given initial condition. In contrast, the results presented in this paper amount to solving a linear program off-line and provide GAS.

An advantage of considering PWP functions over PWQ ones comes from the fact that polytopic state and input constraints, which are often encountered in practice, can easily be specified via linear inequalities. For example, suppose that the constraints are defined by

$$\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathcal{M}x + \mathcal{G}u \leq \mathcal{H}\},$$

for some matrices  $\mathcal{M}, \mathcal{G}$  and vector  $\mathcal{H}$  of appropriate dimensions. Then, these constraints can be imposed by adding the following linear inequalities in  $\{R_i, r_i\}_{i \in \mathcal{I}}$  and  $\xi$  to (16):

$$\begin{aligned} \mathcal{M}((A_i + B_i R_i C_i)x_i^y + B_i r_i + a_i \xi) + \mathcal{G}(R_i x_i^y + r_i) \leq \xi \mathcal{H}, \\ \forall x_i^y \in \mathcal{V}(\Omega_i), \forall i \in \mathcal{I}. \end{aligned}$$

As such, the techniques developed in this paper can be used to synthesize stabilizing PWA control laws for constrained PWA systems, which can constitute an alternative to predictive control laws. Obviously, they can also be used to analyze stability of closed-loop MPC systems that are equivalent to an explicit PWA form.

Another attractive feature of PWP Lyapunov functions is that they generate a family of positively invariant (for  $\rho = 1$ ) or contractive (for  $\rho < 1$ ) sets that are piecewise polyhedral. Given a set of functions  $V_i(x) = \|P_i x\|_\infty$  that are computed via (16), the corresponding family of contractive sets is formally defined as

$$\{\mathcal{V}_c \subseteq \mathbb{R}^n \mid c \in \mathbb{R}_+\}, \quad \mathcal{V}_c := \bigcup_{i \in \mathcal{I}} (\Omega_i \cap \mathcal{V}_c^i),$$

where  $\mathcal{V}_c^i := \{x \in \mathbb{R}^n \mid V_i(x) \leq c\}$ . As each region  $\Omega_i$  is a polyhedron, the above family consists of PWP sets, with each of them further consisting of the union of a finite number of polyhedra equal to the number of regions  $\Omega_i$ . As such, we have obtained a solution for computing PWP contractive or invariant sets for PWA systems that requires solving a single linear program. Notice that the maximal invariant set for PWA systems is a PWP set and its computation suffers from a computational explosion in the number of constituent polyhedra, see, e.g., [25] and the references therein. The solution developed in this paper allows the number of constituent polyhedra of the resulting invariant or contractive PWP set to be fixed a priori, via the partition  $\{\Omega_i\}_{i \in \mathcal{I}}$ . This will be illustrated in the next section.

## 5. ILLUSTRATIVE EXAMPLE

Consider the following open-loop unstable discrete-time PWA

system of the form (3) with  $\mathcal{S} = \mathcal{S}_0 = \mathcal{S}_{\text{lin}} = \{1, \dots, 4\}$ ,

$$A_1 = A_3 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix},$$

$$B_i = I_2, \forall i \in \mathcal{S}, \quad C_1 = C_3 = [0.4 \quad 1], \quad C_2 = C_4 = [0 \quad 0.4].$$

The state-space regions are defined as  $\Omega_i = \{x \in \mathbb{R}^n \mid E_i x \geq 0\}$  for  $i \in \{1, 3\}$  and  $\Omega_i = \{x \in \mathbb{R}^n \mid E_i x > 0\}$  for  $i \in \{2, 4\}$ , with

$$E_1 = -E_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The state-space partition is illustrated in Figure 3.

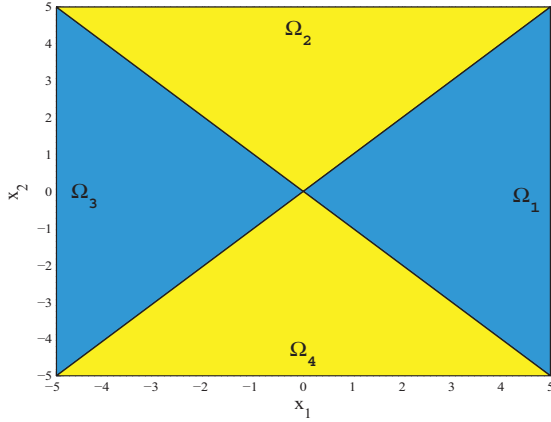


Figure 3: Initial state-space partition.

Following the procedure of Section 4 we have defined a PWP Lyapunov function of the form (6) with  $P_i \in \mathbb{R}^{2 \times 2}$ ,  $P_1 = P_3$  and  $P_2 = P_4$ . Similarly, we have defined the feedback matrices  $F_i \in \mathbb{R}^{2 \times 1}$ ,  $c_i \in \mathbb{R}_{>0}$ ,  $\xi \in \mathbb{R}_{>0}$ ,  $\rho = 0.94$  and the new variables  $R_i \in \mathbb{R}^{2 \times 1}$  for all  $i \in \mathcal{S}$ . As each  $\Omega_i$  is a cone, each set of vertices  $\mathcal{V}(\Omega_i) \subset \mathbb{R}^2$  consists of two non-zero points, one on each ray of  $\Omega_i$ , which can be chosen arbitrarily. For example, we chose  $\mathcal{V}(\Omega_1) = \{x_1^1, x_1^2\}$  with  $x_1^1 = [50 \ 50]^\top$  and  $x_1^2 = [50 \ -50]^\top$ . Unfortunately, the resulting set of inequalities (16) did not yield a feasible solution.

Next, to decrease conservativeness, we further partitioned each region  $\Omega_i$  into two conic sub-regions  $\Omega_{i1}$  and  $\Omega_{i2}$  for all  $i \in \mathcal{S}$ . This is graphically illustrated in Figure 4, for regions  $\Omega_1$  and  $\Omega_2$ . Accordingly, we defined a more complex PWP Lyapunov function of the form (6) with  $P_{ij} \in \mathbb{R}^{2 \times 2}$ ,  $P_{1j} = P_{3j}$  and  $P_{2j} = P_{4j}$ , for all  $(i, j) \in \mathcal{S} \times \{1, 2\}$ . Similarly, we have defined the feedback matrices  $F_{ij} \in \mathbb{R}^{2 \times 1}$ ,  $c_{ij} \in \mathbb{R}_+$ ,  $\xi \in \mathbb{R}_{>0}$ , the new variables  $R_{ij} \in \mathbb{R}^{2 \times 1}$  and the sets of vertices  $\mathcal{V}(\Omega_{ij})$  for all  $(i, j) \in \mathcal{S} \times \{1, 2\}$ . For example, we chose  $\mathcal{V}(\Omega_{11}) = \{x_{11}^1, x_{11}^2\}$  with  $x_{11}^1 = [50 \ 50]^\top$  and  $x_{11}^2 = [50 \ 0]^\top$ .

As it was explained in Section 5, having a smaller region  $\Omega_i$  increases the feasible set where the elements of each  $P_i$  live. The resulting feasible dual cones  $\mathcal{D}(\Omega_{ij})$  are illustrated for the refined regions  $\Omega_{ij}$ ,  $(i, j) \in \{1, 2\} \times \{1, 2\}$  and the corresponding  $P_{ij}$  matrices in Figure 5. By constraining the transpose of each row of  $P_{ij}$  to lie in  $\mathcal{D}(\Omega_{ij})$  we guarantee that  $P_{ij}x \geq 0$  for all  $x \in \Omega_{ij}$ . Solving the resulting set of linear inequalities (16) yielded the following

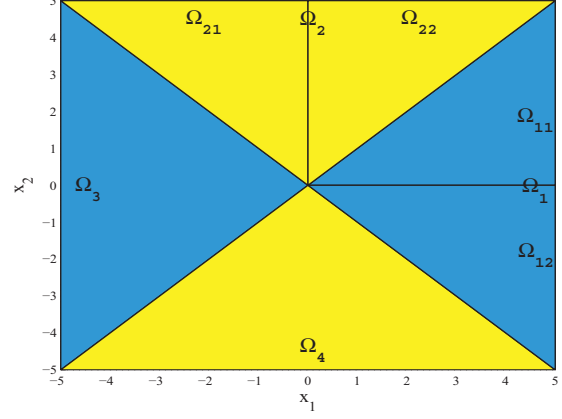


Figure 4: Refined partition versus initial partition.

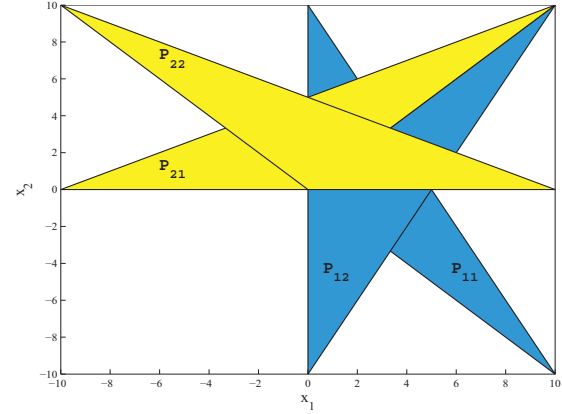


Figure 5: Dual cones  $\mathcal{D}(\Omega_{ij})$ .

feasible solution:

$$P_{11} = \begin{bmatrix} 0.2997 & 4.8651 \\ 4.8651 & -0.2997 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 4.7802 & 0.3846 \\ 0.3846 & -4.0110 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} -5.0549 & 0.1099 \\ 0.1099 & 5.0549 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} -0.1099 & 5.0549 \\ 5.0549 & 0.1099 \end{bmatrix},$$

$$P_{31} = P_{11}, \quad P_{32} = P_{12}, \quad P_{41} = P_{21}, \quad P_{42} = P_{22},$$

$$F_{11} = F_{31} = \begin{bmatrix} -1.3864 \\ -2.1971 \end{bmatrix}, \quad F_{12} = F_{32} = \begin{bmatrix} -1.4250 \\ -2.0750 \end{bmatrix},$$

$$F_{21} = F_{41} = \begin{bmatrix} -1.6600 \\ 1.7250 \end{bmatrix}, \quad F_{22} = F_{42} = \begin{bmatrix} -1.5600 \\ 1.7250 \end{bmatrix},$$

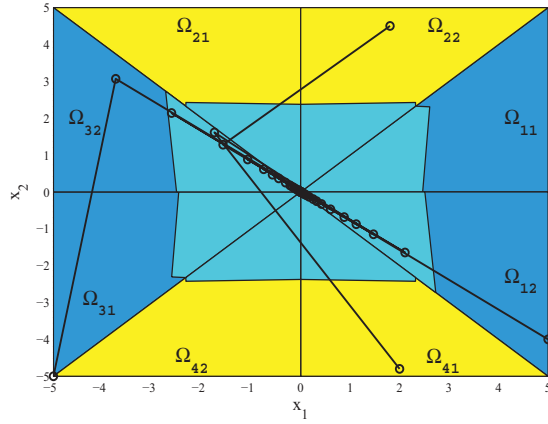
$$\xi = 5.1648, \quad c_{11} = c_{31} = 0.2997,$$

$$c_{ij} = 0.1, \quad \forall (i, j) \in \mathcal{S} \times \{1, 2\} \setminus \{(1, 1), (1, 3)\}.$$

In total, (16) consisted of 125 linear inequalities. To verify the reduction in the number of linear inequalities we solved (16) with (16c) replaced by (18a) and (18b), which also led to a feasible solution, as expected, and consisted of 93 linear inequalities.

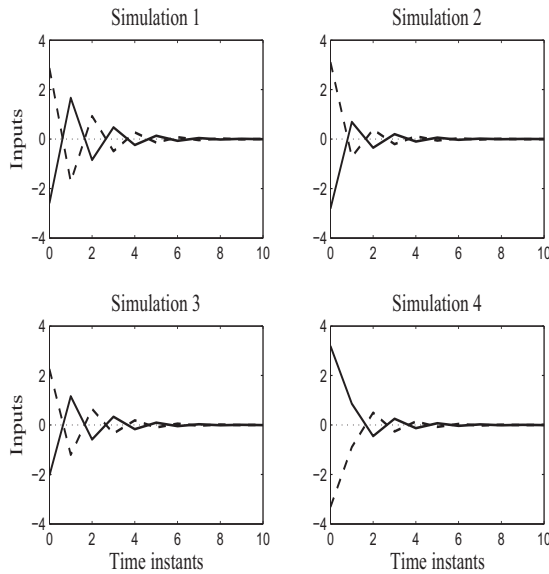
The closed-loop state-trajectories and corresponding input histories obtained in 4 simulations for the initial conditions  $x(0) = [-5 \ -5]^\top$ ,  $x(0) = [1.8 \ 4.5]^\top$ ,  $x(0) = [5 \ -4]^\top$ ,  $x(0) = [2 \ -4.8]^\top$  are plotted in Figure 6 and Figure 7, respectively. The static out-





**Figure 6: Closed-loop state trajectories versus the refined partition and a PWP sublevel set of  $V(x)$ .**

put feedback PWL control law successfully steers the state to the origin for all 4 initial conditions. In Figure 6 we have also plotted the PWP sublevel set  $\mathcal{V}_{12}$  of the corresponding Lyapunov function, which consists of the union of 8 polyhedra, defined by  $\Omega_{ij} \cap \{x \in \mathbb{R}^n \mid \|P_{ij}x\| \leq 12\}$ . As mentioned at the end of Section 5,  $\mathcal{V}_{12}$  is a contractive and positively invariant set for the closed-loop system. This can also be observed in Figure 6, as the trajectory, once it has entered  $\mathcal{V}_{12}$ , it has never left  $\mathcal{V}_{12}$  while converging to the origin. The input histories shown in Figure 7 reveal that convergence to



**Figure 7: Input histories.**

the origin is attained.

## 6. CONCLUSIONS

This paper considered off-line synthesis of stabilizing static feedback control laws for discrete-time PWA systems. The focus was on the implementation of the  $\mathcal{S}$ -procedure in the output feedback synthesis problem. This problem is known to be challenging when

tackled via PWQ Lyapunov function candidates. A new solution was proposed in this work, which uses infinity norms as Lyapunov function candidates and, under certain conditions, requires solving a single linear program. It was demonstrated that this solution also facilitates the computation of piecewise polyhedral positively invariant (or contractive) sets for discrete-time PWA systems and it allows the incorporation of polytopic state and/or input constraints.

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