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A unicorn solution to the Finslerian extension of the Einstein field equations

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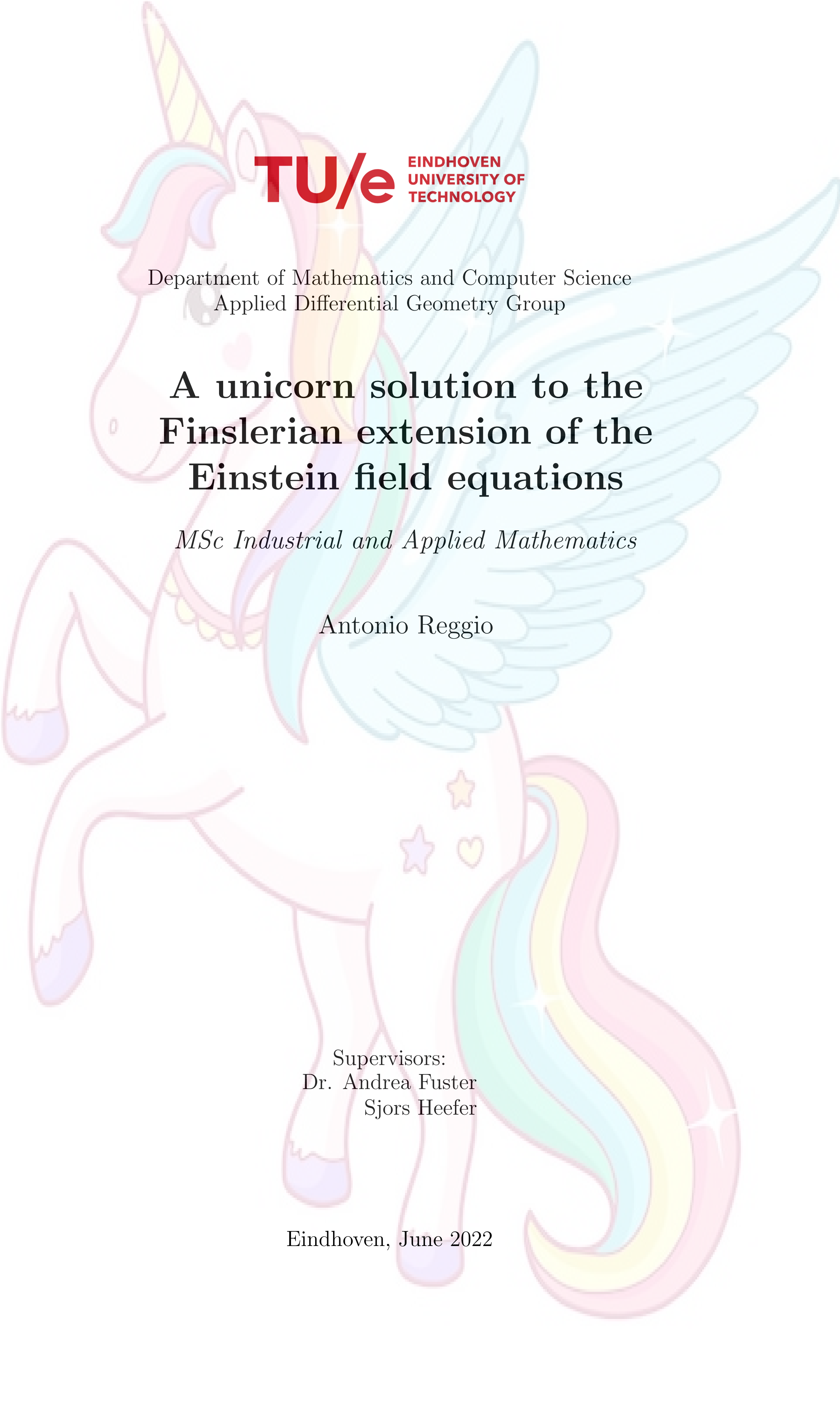

Department of Mathematics and Computer Science
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A unicorn solution to the Finslerian extension of the Einstein field equations

MSc Industrial and Applied Mathematics

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Contents

Introduction

In this project, the presence of non-Berwaldian solutions to the Finslerian extension of the Einstein field equations (FEFE) is investigated. More specifically, after introducing the necessary theory, the first non-Berwaldian exact solution to the FEFE is produced. This is done by building on the work of Elgendi and proving that a subclass of one of the unicorn classes presented by him [10] is a solution to the field equation in Finsler gravity.

Finsler spaces can be thought of as generalizations of pseudo-Riemannian spaces [8]. In pseudo-Riemannian geometry the length of a curve $x = x(\lambda), \lambda \in [a, b]$ is measured by

$$S(x) = \int_a^b |g_{ab}\dot{x}^a \dot{x}^b|^{\frac{1}{2}} d\lambda, \tag{1}$$

where $\dot{x} = (\dot{x}^0, .., \dot{x}^{n-1})$ are the coordinates of the tangent vector to the curve. In Finsler geometry, the concept of length of a curve is broadened. It seems natural that a very general way to measure the length of curves is to use

$$S(x) = \int_a^b F(x, \dot{x}) d\lambda, \tag{2}$$

where $F : A \to \mathbb{R}$ ($A \subset TM \setminus 0$ being conic) is a smooth function, which is called the Finsler function. This definition should not depend on the parametrization of x and this can be done by requiring F to be one-homogeneous in the fibre coordinates when restricted to every tangent space [20]. From the function F, we can derive the so-called fundamental tensor, which at each point v with coordinates (x, y) of A gives a bilinear map $g_v : T_vM \times T_vM \to \mathbb{R}$. In case the map applied to two vectors gives a quadratic expression in the fibre coordinates, we are back in the pseudo-Riemannian case. Several applications of Finsler geometry have been found. As an example, many researchers (e.g. [6], [23] and [31]) have made contributions to the field of Finsler gravity starting from the assumption that at every point the Finsler space should resemble a space with a Minkowski norm. In this context, the Einstein field equations have been generalized by Pfeifer and Wohlfarth [26] to the Finsler case and several analytical solutions have already been found, for example [13] and [11]. All these solutions have in common that they are Berwald solutions, the closest a Finsler space can be to a pseudo-Riemannian space. To the best knowledge of the author, this work presents the first non-Berwaldian exact solution to the FEFE which has ever been found.

This thesis starts with a thorough introduction to Finsler geometry. Initially, the concepts of a Finsler space and a Finsler spacetime are introduced. In this context, two definitions of a Finsler spacetime are presented. The first one is less restrictive and requires the Finsler function to only be defined on A. The second one is specifically designed to define geometric objects on the whole of $TM \setminus 0$ even when the Finsler function might not be smooth on this whole set. The following chapter hosts a discussion on the various kinds of connections and their curvature which can be encountered on the fibre bundles involved. Concerning connections, the standard approach deals with two objects: a non-linear connection on the fibre bundle $A \to M$ and a linear connection on the vertical bundle $\mathcal{V}A \to A$. Motivated by the complexity of this setting, some researchers have introduced the concept of anisotropic connection, an intuitive generalization of the (pseudo)-Riemannian Koszul derivative which incorporates in a natural way the dependence on direction of Finsler geometry [20]. In [16] Javaloyes has developed the anisotropic calculus, which allows to do computations with anisotropic derivations. This framework as well as the relationships between the classical connections in the standard setting and the anisotropic connection are deeply analysed in this chapter. Next, the curvature tensor for the anisotropic connection is defined and it is related to the curvature of the non-linear connection on $A \to M$ by means of an explicit computation. The chapter ends with a presentation of the Finsler-Ricci scalar, a central object in the Finsler formulation of the Einstein field equations. Later, several classical tensors in Finsler geometry are discussed. In this chapter, it is seen that the Chern connection is the natural analog of the Levi-Civita connection in the Finsler setting. In addition, Berwald and Landsberg spaces are introduced and the concept of a non-Berwaldian Landsberg unicorn is formulated. This term was coined by Bao to highlight how rare these geometries are [30].

The last part of the thesis is devoted to studying in more detail one of the unicorn classes proposed by Elgendi [10]:

$$F = \left(a\beta + \sqrt{\alpha^2 - \beta^2}\right) e^{\frac{a\beta}{a\beta + \sqrt{\alpha^2 - \beta^2}}}, \tag{3}$$

where $a \neq 0, \beta = f(x^0)y^0$ and $\alpha = f(x^0)\sqrt{(y^0)^2 + \varphi(\hat{y})}$, with φ being an arbitrary quadratic form in $\hat{y} = y^1, ..., y^{n-1}$ such that α is non-degenerate and $f(x^0)$ a positive smooth function on $\mathbb{R}$. We will show that, by choosing $f(x^0) = e^{bx^0}$, $b \neq 0$, the Finsler-Ricci scalar associated with this Finsler function will vanish. For Landsberg unicorns this implies that this function is a solution to the FEFE, provided that the fundamental tensor of F has Lorentzian signature. We will see that for $\alpha = f(x^0)\sqrt{(y^0)^2 + (y^1)^2 + (y^2)^2 - (y^3)^2}$, this is indeed the case. Furthermore, it will be noted that every choice of φ such that α has Lorentzian signature actually describes the same class of geometries as $\alpha = f(x^0)\sqrt{(y^0)^2 + (y^1)^2 + (y^2)^2 - (y^3)^2}$. To the best knowledge of the author, this solution represents the first non-Berwald and the first unicorn solution which has ever been found. The appendix contains a short discussion of parallel transport, both of an instantaneous observer and with respect to an istantaneous observer.

Chapter 1

Finsler spaces and spacetimes

1.1 Introduction

In Finsler geometry, definitions of Finsler spaces and Finsler spacetimes tend to vary based on the author. For example, [28] defines Finsler spaces as those where the fundamental tensor of the Finsler function is positive definite. In this case, a distinction between Finsler and pseudo-Finsler spaces is made, with the latter only being required to have non-degenerate fundamental tensor. In this chapter, the concepts of a Finsler space and a Finsler spacetime are defined, following [20] and [22]. For Finsler spacetimes, this definition is very well-suited for applications and it is the only one that can be used in the context of non-regular (i.e. not smooth on the whole of $TM \setminus 0$), non-Berwaldian Landsberg spaces. Later, a second, alternative definition of a Finsler spacetime is given, closely following the formalism of [25]. This definition puts particular emphasis on defining geometric objects on the whole of $TM \setminus 0$ even when the null structure of the Finsler function F is non-trivial.

1.2 Finsler spaces

In this thesis, given an n-dimensional differentiable manifold M and a coordinate patch (U, ψ) on it, we will consider the induced coordinates on $\pi : TM \to M$ unless stated otherwise. The local frame $\{\partial_i\}_{i=0,...,n-1}$ on TM is canonically obtained by setting

$$\partial_i(x) = (d\psi_x)^{-1} \left(\frac{\partial}{\partial x_i}_{|\psi(x)} \right), \tag{1.1}$$

where $x \in U$ and $\{\frac{\partial}{\partial x^i}_{|\psi(x)}\}_{i=0,...,n-1}$ is a basis of $T_{\psi(x)}\mathbb{R}^n$. The coordinates corresponding to this local frame will be denoted by $(x, y) = (x^0, ..., x^{n-1}, y^0, ..., y^{n-1})$, whereas the last $n-1$ fibre coordinates will be denoted $\hat{y} = y^1, ..., y^{n-1}$. For local frames on TTM, the local sections induced by the base manifold coordinates will still be denoted by $\{\partial_i\}_{i=0,...,n-1}$, whereas the local sections induced by the fibre coordinates of TM will be denoted by $\{\dot{\partial}_i\}_{i=0,...,n-1}$. Points on TM will be denoted by v or by their coordinates (x, y). Throughout the whole text the Einstein summation convention is used.

In order to guarantee the successful completion of this project, the guidance, direction and supervision of Andrea Fuster and Sjors Heefer have been crucial. For this and for their contagious enthusiasm, they deserve a big thank you.

Definition 1 (Finsler function)**.** *Let M be an n-dimensional manifold, TM its tangent bundle and $A \subset TM \setminus 0$, with A conic. A smooth real function $F : A \to \mathbb{R}$ is called a Finsler function (or Finsler metric) if it satisfies:*

1. *F is homogeneous of degree one with respect to the fibre coordinates of TM:*

$$F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0. \tag{1.2}$$

2. *The Hessian g^F_{ab} of F^2 with respect to the tangent space coordinates is non-degenerate on A.*

$$g^F_{ab} = \frac{1}{2}\bar{\partial}_a\bar{\partial}_b F^2. \tag{1.3}$$

We can now define a Finsler space.

Definition 2 (Finsler space)**.** *An n-dimensional ($n \geq 2$) manifold M equipped with a Finsler function F is called a Finsler space, denoted by (M, F).*

The following theorem will be of crucial importance in calculations and proofs of this thesis.

Theorem 1 (Euler's Theorem)**.** [1] *Let $f : V \to \mathbb{R}$ be a differentiable function which is homogeneous of degree r, i.e. $f(\lambda x) = \lambda^r f(x)$, for $\lambda > 0$, from some vector space V into $\mathbb{R}$. Then the following holds:*

$$x^a \partial_a f(x) = r f(x) \tag{1.4}$$

Definition 3. *The first derivative of F^2 with respect to the tangent space coordinates defines the components p_a of a one-homogeneous one-form*

$$p_a(x, y) = \frac{1}{2}\bar{\partial}_a F^2(x, y). \tag{1.5}$$

Then, by Theorem 1, $y^a p_a = F^2$.

Definition 4 (Fundamental tensor)**.** *The Hessian of F^2 (Equation 1.3) with respect to the fibre coordinates defines the component g^F_{ab} of a zero-homogeneous symmetric $(0, 2)$-tensor, called the fundamental tensor.*

By Theorem 1, we get $y^a y^b g^F_{ab} = F^2$. It is important to notice that the fundamental tensor is not a metric on TM and definitely not a metric on M. The former is because in that case the fundamental tensor should act on vectors in the TTM. The latter would be true if the fundamental tensor did not depend on the fibre coordinates.

1.3 Finsler spacetimes

Definition 5 (Finsler spacetime)**.** *A Finsler spacetime is a four dimensional manifold M endowed with a Finsler function F such that the fundamental tensor g^F_{ab} of the Finsler function has index $n - 1$.*

[1] For a proof see [1].

This requirement is very intuitive. In this way, we make sure that locally the spacetime approaches Minkowski spacetime. In fact, it can be shown that g^F_{ab} is the best scalar product approximation of F^2 at a given point (x, y). In the following chapters, if F will be smooth on $TM \setminus 0$, we will have $A = TM \setminus 0$. Otherwise we will choose A such that F is smooth on it. It is important to notice that, in some cases, it is possible to describe the geometry of the Finsler spacetime on the whole $TM \setminus 0$, even though F is not smooth on it. Consider for example the Finsler function F induced by a symmetric $(0, d)$-tensor G on the manifold M.

$$F(x, y) = |G_{a_1 \cdots a_d}(x) y^{a_1} \cdots y^{a_d}|^{\frac{1}{d}}\,. \tag{1.6}$$

We can see that if the null structure

$$N = \{(x, y) \in TM | F(x, y) = 0)\} \tag{1.7}$$

is non-trivial, then F^2 is only smooth on $TM \setminus N$, and not on $TM \setminus 0$. The next definition of a Finsler spacetime is exactly designed to overcome this problem and to study spaces with a Finsler function with a non trivial null structure on the whole $TM \setminus 0$.

Definition 6 (Finsler spacetime)**.** *A Finsler spacetime, denoted by the tripel (M,L,F) is a four dimensional smooth manifold M equipped with a continuous fundamental geometry function $L : TM \to \mathbb{R}$ which has the following properties:*

1. *L is smooth on $TM \setminus 0$.*
2. *L is positively homogeneous of degree $r \geq 2$ with respect to the fibre coordinates of TM:*

$$L(x, \lambda y) = \lambda^r L(x, y) \quad \forall \lambda > 0. \tag{1.8}$$

3. *L is reversible, meaning:*

$$|L(x, -y)| = |L(x, y)|. \tag{1.9}$$

4. *The Hessian of L is non-degenerate nearly everywhere on TM and especially nearly everywhere along the null structure of the spacetime $N_L = \{v \in TM | L(v) = 0\}$.*

$$g^L_{ab} = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b L \tag{1.10}$$

That is, g^L_{ab} is non-degenerate on $TM \setminus A$ with A for a measure 0 subset A, so that $B = A \cap N_L$ is a lower dimensional set of N_L.

5. *The unit timelike condition holds, i.e., for all $x \in M$ the set*

$$\Omega_x = \left\{ v \in T_x M \big| |L(v)| = 1, g^L(v) \;\; \textit{has} \;\; \textit{signature} \; (\epsilon, -\epsilon, -\epsilon, -\epsilon), \epsilon = \frac{|L(v)|}{L(v)} \right\} \tag{1.11}$$

contains a non-empty closed connected component S_x.

Furthermore, we define the following objects associated with L:

$$F(x, y) = |L(x, y)|^{\frac{1}{r}}, \qquad g^F_{ab} = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b F^2. \tag{1.12}$$

Consider the previous example where F was induced by a symmetric $(0, d)$-tensor G. We can then define:

$$L(x, y) = G_{a_1 \cdots a_d} y^{a_1} \cdots y^{a_d} \to F(x, y) = |G_{a_1 \cdots a_d} y^{a_1} \cdots y^{a_d}|^{\frac{1}{d}} . \tag{1.13}$$

The wonderful feature of this construction is that the geometry of the Finsler spacetime can be formulated in terms of derivatives acting on L. From the previous discussion, we can see that there are some problems on N_L with respect to the differentiability of F. Nevertheless, L is smooth on $TM \setminus 0$, so especially on N_L. In this formalism, geometric objects which are usually defined in the literature in terms of F (e.g. the Cartan tensor) can now be defined in terms of L. Therefore we can now study examples which have differentiability issues along a non-trivial null structure in terms of the function L on the whole of $TM \setminus 0$. For a detailed explanation of how this is done, the reader is referred to [25].

1.3.1 Examples

Example 1 (Lorentzian manifold [25])**.** *Lorentzian manifolds $(M, \tilde{g})$ with metric $\tilde{g}$ of signature $(-, +, +, +)$ are a special case of Finsler spacetimes (M, L, F). They are described by the function*

$$L(x, y) = \tilde{g}_{ab}(x) y^a y^b, \tag{1.14}$$

which is homogeneous of degree $r = 2$. Then, $L(x, y)$ leads to the Finsler function $F(x, y) = \left|\tilde{g}_{ab} y^a y^b\right|^{\frac{1}{2}}$. In addition, we straightforwardly notice that L is smooth on TM and obeys the reversibility property. The metric $g^L_{ab} = \tilde{g}_{ab}$. Therefore, the signature of g^L is globally $(-, +, +, +)$. The Finsler function F is non-differentiable on N_L and we just say that the fundamental tensor g^F is not defined on this set. We obtain $g^F_{ab}(x, y) = -\tilde{g}_{ab}(x)$ on the $\tilde{g}$-timelike vectors and $g^F_{ab}(x, y) = \tilde{g}_{ab}(x)$ on the $\tilde{g}$-spacelike vectors.

Example 2 (Simple bimetric Finsler structures)**.** *We can define an example of a Finsler spacetime (M, L, F) through two Lorentzian metrics h and k of signature $(-, +, +, +)$ for which the cone of h-timelike vectors is contained and centred in the cone of k-timelike vectors. Consider the function*

$$L(x, y) = h_{ab}(x) y^a y^b k_{cd}(x) y^c y^d. \tag{1.15}$$

Clearly, L is homogeneous of degree 4, smooth on TM and obeys the reversibility condition. The corresponding Finsler function is $F(x, y) = |h_{ab}(x) y^a y^b k_{cd}(x) y^c y^d|^{\frac{1}{4}}$. In [25] Pfeifer argues that the null structure N_L is the union of the null structure of the metrics h and k, and that the metric g^L_{ab} is degenerate on a measure 0 subset A that forms an additional structure between the null surfaces without intersecting them; hence in this example $B = \emptyset$. Across A, the metric g^L changes its signature from $(+, -, -, -)$ to $(-, +, +, +)$. Just like in the previous example, the function F of this Finsler spacetime is not differentiable where $L = 0$. There, the fundamental tensor g^F is not defined. But since on Finsler spacetimes of this kind all geometric objects (e.g. geodesics, the Cartan tensor) are defined through the function L, they are well defined on $TM \setminus A$, in particular on N_L.

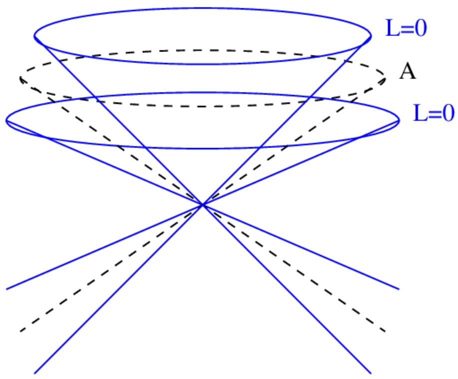

Figure 1.1: The null structure of bimetric Finsler spacetimes [25]

When, given a Finsler function, it is possible to define a function L such that they describe the same Finsler spacetime, then Definition 6 is to be preferred because of the advantages outlined above. For example, this is the case of [12]. Nevertheless, in the context of non-regular, non-Berwaldian Landsberg unicorns, it is not possible to define such a function. As a result, from now on, Definition 5 will be the default one when referring to Finsler spacetimes. Even though Definition 5 will be used, the previous discussion is nevertheless important as it shows that a more holistic, rigorous approach can be taken in some cases. The superscript will be dropped from the fundamental tensor of the Finsler function, as there is no chance for confusion anymore.

1.4 Alpha-beta metrics

(α, β)-metrics are a special case of Finsler metrics and deserve to be thoroughly analyzed in this thesis. This is because all the examples of Landsberg unicorns which will be given are (α, β)-metrics. In addition, there exists an explicit expression to calculate the determinant of the fundamental tensor originating from such a metric. This fact will turn out useful in the search for solutions to the Finslerian extension of the Einstein field equations which are also Landsberg unicorns.

Let M be an n-dimensional manifold, α a pseudo-Riemannian metric on M and β a 1-form on M. Using the standard coordinates, we can write $\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(x,y) = b(x)_i y^i$. Also, we define the α-norm of a 1-form β as $||\beta||_\alpha = sup_{y \in T_x U} \frac{\beta(x,y)}{\alpha(x,y)}$, where $U \subset M$ is our coordinate patch. In addition, let $\phi : (-b_0, b_0) \to \mathbb{R}$ with $b_0 \in \mathbb{R}$ be a smooth positive function. If the function

$$F(x,y) = \alpha(x,y)\phi(s) \qquad s = \frac{\beta(x,y)}{\alpha(x,y)} \tag{1.16}$$

satisfies the axioms of Definition 1, it is called an (α, β)-metric.

Proposition 1 ([28]). *The determinant of the fundamental tensor of an (α, β)-metric is given by*

$$det(g_{ij}) = \left[\phi^{n+1}(\phi - s\phi')^{n-2}(\phi - s\phi')(||\beta||^2_\alpha - s^2)\phi''\right] det(a_{ij}) \tag{1.17}$$

From this Proposition, the following Lemma follows.

Lemma 1 ([19])**.** *An (α, β) metric has positive definite fundamental tensor for any α and β with $||\beta||_\alpha < b_0$ if and only if the function $\phi(s)$ satisfies:*

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \tag{1.18}$$

with $s, b \in \mathbb{R}$ and $|s| \leq b < b_0$

Example 3 (Randers metric [28])**.** *One straightforward example of an (α, β)-metric is a Randers metric. In this case $\phi(s) = 1 + s$. It is straightforward then to see that for the fundamental tensor to be positive definite we need to have a ϕ such that $b_0 = 1$, since the function needs to be positive according to Lemma 1. The Finsler metric will then have the form*

$$F := \alpha + \beta. \tag{1.19}$$

This type of (α, β)-metrics is particularly relevant in the context of a Berwald space (see Section 3.3). In fact, Heefer et al. [13] have proved the following theorem

Theorem 2. *Let (M, F) be a Finsler spacetime of Berwald type, where F is a Randers function. Then F is a solution for the vacuum FEFE if and only if α is a solution to the Einstein field equations in vacuum.*

Example 4. *We now present the class of $(\alpha, \beta)-$metrics to which the unicorn solution to the FEFE which will be presented in Chapter 4 belongs. Consider the following Finsler function*

$$F = \alpha\phi\left(\frac{\beta}{\alpha}\right), \tag{1.20}$$

where

$$\phi(s) = \left(as + \sqrt{1 - s^2}\right) exp\left(\frac{as}{as + \sqrt{1 - s^2}}\right). \tag{1.21}$$

With regards to positive definiteness of the fundamental tensor, we see that this function is positive when $as + \sqrt{1 - s^2}$ is. We now distinguish two cases, namely $a > 0$ and $a < 0$. When $a > 0$, the function is positive on the interval $-\sqrt{\frac{1}{1+a^2}} < s \leq 1$. On the other hand, when $a < 0$, the function is positive on $-1 \leq s < \sqrt{\frac{1}{1+a^2}}$. We therefore see that a good candidate for b_0 is $b_0 = \sqrt{\frac{1}{1+a^2}}$. It is easy to check that choosing this b_0 the other condition for the metric to have a positive definite fundamental tensor is indeed satisfied. Therefore, according to Lemma 1, fixing α choosing a β such that $||\beta||_\alpha < \sqrt{\frac{1}{1+a^2}}$ will guarantee that the fundamental tensor is positive definite.
In Chapter 4, we will see that a subclass of this class solves the field equation in Finsler gravity, provided that the fundamental tensor is Lorentzian. There, we will choose $\beta = f(x^0)y^0$ and $\alpha = f(x^0)\sqrt{(y^0)^2 + \varphi(\hat{y})}$, where φ is a quadratic form such that α is non-degenerate and $f(x^0)$ is a positive smooth function. In a $4-$dimensional manifold, for $\varphi = (y^1)^2 + (y^2)^2 - (y^3)^2$, we obtain:

$$det(g_{ij}) = -a^2 e^{\frac{8ay^0}{ay^0 + \sqrt{\varphi}}} f(x^0)^8. \tag{1.22}$$

As in 4 dimensions a negative determinant means that three eigenvalues have the same sign and one has the opposite sign, a manifold together with this function F describes a Finsler spacetime according to Definition 5. It is actually a well know result that every choice of φ such that α has Lorentzian signature can be traced back to $\varphi = (y^1)^2 + (y^2)^2 - (y^3)^2$ by means of a coordinate transformation (see for example [14]).

Chapter 2

Anisotropic calculus

2.1 Introduction

This chapter is largely based on the report written by the author for his Master internship. Following the formalism outlined in [16], [20] and [18], we examine the theory needed to understand the concepts of connections and curvature in Finsler geometry. This formalism is different from the standard approach in the literaure (e.g. [4]), but it bears many advantages; the main one being the use of anisotropic connections. As an example, with this approach it is possible to show that the Berwald connection is the most natural linear connection on $\mathcal{V}A \to A$ associated with the Finsler function.

2.2 Anisotropic tensors

Let M be an n-dimensional manifold and $\pi : TM \to M$ its tangent bundle. Furthermore, let $A \subset TM \setminus 0$ be such that $\pi(A) = M$. We now pull-back the tangent bundle along $\pi_{|A}$ and consider the pull-back bundle with A as base manifold. In summary:

$$\begin{array}{ccc} \pi^*_{|A}TM & & TM \\ \big\downarrow \tilde{\pi} & & \big\downarrow \pi \\ A & \xrightarrow{\pi_{|A}} & M \end{array}$$

Notice that, set theoretically, $\pi^*_{|A}TM = \cup_{v \in A} T_{\pi(v)}M$ and so, as a manifold, its dimension is $3n$. We can then analogously define the pull-back bundle of the cotangent bundle $\tilde{\pi} : TM^* \to M$.

Before diving into formal definitions, we make an important observation which will turn out very useful later on in the context of connections. Consider the tangent bundle $TA \to A$. We then define its vertical space $\mathcal{V}A = ker(d\pi_{|A})$, where $d\pi_{|A} : TA \to TM$ is the differential map. In a natural system of coordinates (U, ϕ) we have that a vector $v \in TA$ has the representation $(x, y, \dot{x}, \dot{y})$. The vector $d\pi_{|A}(v) \in TM$ will then have coordinates $(x, \dot{x})$. We thus conclude that the kernel of $d\pi_{|A}$ is made of those vectors for which $\dot{x} = 0$. Therefore, we see that we have a natural set of coordinates on $\mathcal{V}A$, namely $(x, y, \dot{y})$. As a consequence, by carefully choosing local trivializations on $\mathcal{V}A \to A$ and $\pi^*_{|A}TM \to A$, it can be shown that these two bundles are isomorphic.

Definition 7. *Given non-negative integer numbers* $r, s \in \mathbb{N}^*$ *such that* $r + s > 0$*, we define an A-anisotropic* (r, s) *tensor in* M *as a smooth section of* $\pi^* TM^{\otimes r} \otimes \tilde{\pi}^* TM^{*\otimes s}$*, namely a smooth map* $T : A \to \pi^* TM^{\otimes r} \otimes \tilde{\pi}^* TM^{*\otimes s}$ *such that its composition with the projection on the base manifold is the identity.*

We will denote the space of (r, s) A-anisotropic tensors as $\mathcal{T}^r_s(M, A)$. Then, $\mathcal{T}^0_0(M, A)$ will be the space of C^∞ real functions on A, which will be denoted by $\mathcal{F}(A)$.

Example 5. *Consider the basic case of* $\mathcal{T}^1_0(M, A)$*, which is just a section of* $\pi^*_{|A} TM$*. By choosing a local frame in* TM*, we also get a local frame on* $\pi^*_{|A} TM$*, simply by pulling back these sections. We then have a local frame in the pull-back bundle, which means that any* $h \in \mathcal{T}^1_0(M, A)$ *can be written as follows:*

$$h(x, y) = \sum_{i=1}^{n} f^i(x, y) \pi^*_{|A} \partial_i, \tag{2.1}$$

where $f^i(v) \in \mathcal{F}(A)$*. This reasoning can then be extended to tensor products and we see that, in a local coordinate system , we can express* $T \in \mathcal{T}^r_s(M, A)$ *as a* $\mathcal{F}(A)$ *multi-linear map:*

$$T(x, y) = T^{i_1, i_2, .., i_r}_{j_1, j_2, ... j_s}(x, y) \pi^* \partial_{i_1} \otimes ... \otimes \tilde{\pi}^* dx^{j_s} \tag{2.2}$$

where $T^{i_1, i_2, .., i_r}_{j_1, j_2, ..., j_s} \in \mathcal{F}(A)$ *and* $v \in A$*.*

Remark 1. *We can identify an element of* $\mathfrak{X}(M)$ *with its pull-back along* π*. In that way, we can regard an element* $X \in \mathfrak{X}(M)$ *as an element* $X^* \in \mathcal{T}^1_0(M, A)$*, defined as* $X^*(v) = X(\pi(v))$*. In this way, we may rewrite Equation 2.2 as*

$$T(x, y) = T^{i_1, i_2, .., i_r}_{j_1, j_2, ... j_s}(x, y) \partial_{i_1} \otimes ... \otimes dx^{j_s} \tag{2.3}$$

With this interpretation in mind, we can then see that to specify an A-anisotropic tensor, it is enough to specify its value on vector fields and differential 1-forms on M*. Interestingly, to specify an anisotropic tensor it is actually enough to define that such a map is* $\mathcal{F}(M)$ *multi-linear. The tensor can then be extended to* $\mathcal{T}^0_1(M, A)^r \times \mathcal{T}^1_0(M, A)^s$ *by using a local frame and* $\mathcal{F}(A)$ *multi-linearity. It is then straightforward that this extension does not depend on the frame.*

From now on simplify notation, we will omit the subset A and we will just talk about anisotropic tensors.

Definition 8. *We say that a vector field* V *defined on* $U \subset M$ *is A-admissible if* $V(x) \in A$ *for every* $x \in U$*. The set of A-admissible vector fields on* $U \subset M$ *will be denoted by* $\mathfrak{X}^A(U)$*.*

We now proceed to define the tools which are needed to construct tensor derivations. Once again, these definitions will turn out very useful later on in the context of connections.

Definition 9. *Given an anisotropic tensor* $T \in (T)^r_s(M, A)$*, we define its vertical derivative as the tensor* $(\partial^{\mathcal{V}} T) \in \mathcal{T}^r_{s+1}(M, A)$ *given by*

$$\partial^{\mathcal{V}} T_v(\theta^1, .., \theta^r, X_1, .., X_s, Z) = \frac{\partial}{\partial t} T_{v + tZ(\pi(v))}(\theta^1, .., \theta^r, X_1, .., X_s)_{|t=0} \tag{2.4}$$

with $(\theta^1, .., \theta^r, X_1, .., X_s, Z) \in \Omega(M) \times ... \times \mathfrak{X}(M)$ *and* $v \in A$*.*

Notice that, according to Remark 1, this is enough to fully specify the tensor. Also, notice that in the above definition, the subscript T_v just means $T(v)$. We are now ready to turn to tensor derivations.

2.2.1 Tensor derivations

Definition 10. *A tensor derivations $\mathcal{D}$ is an $\mathbb{R}$-linear map*

$$\mathcal{D} : \cup_{r,s\geq 0}\mathcal{T}^r_s(M,A) \to \cup_{r,s\geq 0}\mathcal{T}^r_s(M,A) \tag{2.5}$$

such that the restriction $\mathcal{D}^r_s = D_{|\mathcal{T}^r_s(M,A)}$ has image included in $\mathcal{T}^r_s(M,A)$ for any $r,s \geq 0$, and it satisfies the following two properties:

1. *For any tensors T_1 and T_2: $\mathcal{D}(T_1 \otimes T_2) = \mathcal{D}T_1 \otimes T_2 + T_1 \otimes \mathcal{D}T_2$*
2. *for any tensor T and any contraction C, $\mathcal{D}(C(T)) = C(\mathcal{D}(T))$*

By using property 1 above, we can derive the following product rule for tensor derivations:

$$\mathcal{D}(T(\theta^1,..,\theta^r,X_1,..,X_s))(v) = \mathcal{D}T(\theta^1,..,\theta^r,X_1,..,X_s)(v) \tag{2.6}$$

$$+\sum_{i=1}^{r} T(\theta^1,..,\mathcal{D}(\theta^i),...,\theta^r,X_1,..,X_s)(v) \tag{2.7}$$

$$+\sum_{i=1}^{r} T(\theta^1,...,\theta^r,X_1,..,\mathcal{D}(X_i),...,X_s)(v) \tag{2.8}$$

with $(\theta^1,..,\theta^r,X_1,..,X_s) \in \Omega(M) \times ... \times \mathfrak{X}(M)$ and $v \in A$.

Theorem 3. *A tensor derivation is fully determined by its action on $\mathcal{F}(A)$ and $\mathfrak{X}(M)$.*

Proof. By the product rule 2.6, it is clear that it depends only on $\mathfrak{X}(M), \Omega(M)$ and $\mathcal{F}(A)$. Nevertheless, by the product rule we also know that

$$\mathcal{D}(\theta)(X) = \mathcal{D}(\theta(X)) - \theta(\mathcal{D}(X)) \tag{2.9}$$

for $\theta \in \Omega(M), X \in \mathfrak{X}(M)$. Hence the result. □

Definition 11. *An anisotropic derivation δ with associated vector field $Z \in \mathfrak{X}(M)$ is a map*

$$\delta : A \times \mathfrak{X}(M) \to TM, \quad (v,X) \to \delta^v X \in T_{\pi(v)}M. \tag{2.10}$$

where, fixing $X \in \mathfrak{X}(M)$, $v \to \delta^v X$ is an element of $\mathcal{T}^1_0(M,A)$. This map also satisfies:

1. *$\delta^v(X+Y) = \delta^v X + \delta^v Y$, $X,Y \in \mathfrak{X}(M)$.*
2. *$\delta^v(fX) = Z(f)X + f\delta^v X$ for any $f \in \mathcal{F}(M), X \in \mathfrak{X}(M)$.*

In addition, $\delta^V X(x) := \delta^{V(x)}X$, with $x \in M$ is smooth for any A-admissible vector field V and for any $X \in \mathfrak{X}(M)$.

We will now proceed to use the tools we have defined so far to construct a tensor derivation corresponding to an anisotropic derivation. Recall from Theorem 3 that it is enough to define our derivation on $\mathcal{T}^0_0(M,A) := \mathcal{F}(A)$ and on $\mathfrak{X}(M) \subset \mathcal{T}^1_0(M,A)$ (Remark 1). We now proceed with the first of the two.

Definition 12. *Given an anisotropic derivation δ with associated vector field $Z \in \mathfrak{X}(M)$, we can define the derivation of a function $h \in \mathcal{F}(A)$ as follows:*

$$\mathcal{D}h(v) := Z(h(V))(\pi(v)) - (\partial^{\mathcal{V}} h)_v(\delta^v V) \tag{2.11}$$

for any A-admissible vector field V such that $V(\pi(v)) = v$.

We now define the tensor derivation on $\mathfrak{X}(M)$ by simply requiring that $\mathcal{D}X(v) = \delta^v X$, where we have considered $\mathfrak{X}(M) \subset \mathcal{T}^1_0(M, A)$ as in Remark 1. By using property 1 of Definition 10, we can easily see how the tensor derivation acts on $\mathcal{T}^1_0(M, A)$, namely:

$$\mathcal{D}(\xi)(v) = \mathcal{D}(a^i)X_i(\pi(v)) + a^i \delta^v X_i \tag{2.12}$$

where $\xi(v) = a^i X_i$ with $a^i \in \mathcal{F}(A), X_i \in \mathfrak{X}(M)$. Notice once again the identification of a section with its pull-back under $\pi_{|A}$. We now have all the ingredients required to define a tensor derivation once we are given an anisotropic derivation. We summarize these results in the following theorem.

Theorem 4. *Let δ be an anisotropic derivation with associated vector field $Z \in \mathfrak{X}(M)$. Then there exists a unique tensor derivation $\mathcal{D}$ such that $\mathcal{D}(X)(v) = \delta^v X$ for every $X \in \mathfrak{X}(M)$, and $\mathcal{D}(h)$ is defined as in Definition 12 for $h \in \mathcal{F}(A)$.*

2.3 Connections

2.3.1 Anisotropic linear connections

Definition 13. *An anisotropic linear connection is a map*

$$\nabla : A \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to TM \tag{2.13}$$

$$\nabla(v, X, Y) \to \nabla^v_X Y \in T_{\pi(v)}M \tag{2.14}$$

that satisfies the following:

1. $\nabla^v_X(Y + Z) = \nabla^v_X Y + \nabla^v_X Z$ *for any* $X, Y, Z \in \mathfrak{X}(M)$.
2. $\nabla_{fX+hY} Z = f\nabla_X Z + h\nabla_Y Z$ *for any* $X, Y, Z \in \mathfrak{X}(M),\ f, h \in \mathcal{F}(M)$.
3. $\nabla^v_X(fY) = X(f)Y(\pi(v)) + f(\pi(v))\nabla^v_X Y$ *for any* $X, Y \in \mathfrak{X}(M),\ f \in \mathcal{F}(M)$.
4. $v \to \nabla^v_X Y$ *belongs to* $\mathcal{T}^1_0(M, A)$ *for fixed* $X, Y \in \mathfrak{X}(M)$.

Definition 14. *We call an anisotropic connection homogeneous if for all $\lambda > 0$*

$$\nabla^{\lambda v}_X Y = \nabla^v_X Y \quad X, Y \in \mathfrak{X}(M) \tag{2.15}$$

Just like in the isotropic case, given a coordinate chart (U, ψ), we can define the anisotropic Christoffel symbols in the following way:

$$\Gamma^i_{jk} : A \cap TU \to \mathbb{R} \tag{2.16}$$

$$\nabla^v_{\partial_i} \partial_k = \Gamma^j_{ik}(v)\partial_j \tag{2.17}$$

Remark 2. *Because of point 3 of Definition 13, an anisotropic linear connection together with fixed $X \in \mathfrak{X}(M)$ specifies an anisotropic derivation ∇_X with associated vector field X. Therefore, we can use anisotropic derivations to take derivatives of arbitrary anisotropic tensors $T \in \mathcal{T}^r_s(M,A)$. Hence, we see that*

$$\nabla^v_X T(\theta^1,..,\theta^r,X_1,..,X_s) = \nabla^v_X(T(\theta^1,..,\theta^r,X_1,..,X_s)) \tag{2.18}$$

$$- \sum_{i=1}^{r} T(\theta^1,..,\nabla^v_X(\theta^i),...,\theta^r,X_1,..,X_s)(v) \tag{2.19}$$

$$- \sum_{i=1}^{r} T(\theta^1,...,\theta^r,X_1,..,\nabla^v_X(X_i),...,X_s)(v) \tag{2.20}$$

with $(\theta^1,..,\theta^r,X_1,..,X_s) \in \Omega(M) \times ... \times \mathfrak{X}(M)$ and $v \in A$.

2.3.2 Non-linear connections

Recall from Section 2.2 that the vertical bundle $\mathcal{V}A \to A$ is identifiable with the pull-back bundle $\pi^*_{|A}TM \to A$. Hence, we also have a bijection between anisotropic vector fields $\mathcal{T}^1_0(M,A)$ and sections of $\mathcal{V}A \to A$. Let's now consider the following short exact sequence:

$$0 \longrightarrow \mathcal{V}_vA \xrightarrow{\iota} T_vA \xrightarrow{d_v\pi} T_{\pi(v)}M \longrightarrow 0$$

where ι is the inclusion map. This short sequence splits by definition and, from a very well-known theorem in linear algebra, we know that choosing a homomorphism $\omega_v : T_vA \to \mathcal{V}_vA$ which is the identity when restricted to $\mathcal{V}_vA$ is the same as choosing a horizontal space $\mathcal{H}_v$ on T_vA, such that $\mathcal{H}_v$ is isomorphic to $T_{\pi(v)}M$ via $d_v\pi$. The two are related by $\mathcal{H}_v = ker(\omega_v)$. By varying v, we obtain a short exact sequence of fiber bundles:

$$0 \longrightarrow \mathcal{V}A \xrightarrow{\iota} TA \xrightarrow{d\pi} TM \longrightarrow 0$$

We then define a non-linear connection on $A \to M$ as a morphism of fibre bundles $\omega : TA \to \mathcal{V}A$ such that it is the identity when restricted to $\mathcal{V}A$. By the arguments above, this is then equivalent to choosing a vector subbundle $\mathcal{H}$ of TA, with $\mathcal{H} = \cup_{v\in A}\mathcal{H}_v = \cup_{v\in A}ker(\omega_v)$.
Recall from Section 2.2 that, choosing a coordinate system (U,ψ) with coordinates $(x,y,\dot{x},\dot{y})$ on TA, the vertical space $\mathcal{V}A$ is made by vectors for which $\dot{x} = 0$. Using these natural coordinates, we define:

$$\omega\left(\dot{x^i}\partial_i\big|_{(x,y)} + \dot{y^i}\dot{\partial}_i\big|_{(x,y)}\right) = \left(\dot{y^a} + \dot{x^i}N^a_i(x,y)\right)\dot{\partial}_a\big|_{(x,y)} \tag{2.21}$$

where $N^a_i : A \cap TU \to \mathbb{R}$. We will call the N^a_i the non-linear coefficients of the connection. Furthermore, we see that $\mathcal{H}_{(x,y)}$ will be:

$$\mathcal{H}_{(x,y)} = Span_{\{i=0,..,n-1\}}\left\{\frac{\delta}{\delta x^i}\big|_{(x,y)} := \partial_i\big|_{(x,y)} - N^a_i(x,y)\dot{\partial}_a\big|_{(x,y)}\right\} \tag{2.22}$$

Furthermore, we will require the non-linear connection on $A \to M$ to be homogeneous, meaning:

$$N^a_i(x,\lambda y) = \lambda N^a_i(x,y) \quad \lambda > 0 \tag{2.23}$$

In our context a non-linear connection means non necessarily linear. We will now proceed to introduce linear connections as a subset of the previously defined non-linear ones.

Definition 15. *We say that a non-linear connection on $A \to M$ is linear when in every coordinate chart (U, ψ) there exist smooth functions $\Gamma^k_{ij} : U \to \mathbb{R}$ such that*

$$N^a_i(x, y) = \Gamma^a_{ij}(x) y^j \tag{2.24}$$

2.3.3 Anisotropic vs non-linear connections

Given an anisotropic connection, we can determine a non-linear connection on $A \to M$. Calling ∇ the anisotropic connection, let us define the following map:

$$D : \mathfrak{X}(M) \times \mathfrak{X}^A(U) \to TM \tag{2.25}$$

$$(X, Y) \to D_X Y := \nabla^Y_X Y \tag{2.26}$$

Clearly, this map is linear in the first, but not in the second argument. In coordinates:

$$D_X Y = \Big(X(Y^k) + X^i Y^j \Gamma^k_{ij}(Y) \Big) \partial_k \tag{2.27}$$

where $X = X^i \partial_i, Y = Y^i \partial_i$. Notice that, by relabeling $Y^j \Gamma^k_{ij}(Y) = N^k_i(Y)$, we get something very similar to Equation 2.21. In fact, given vector fields $(X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}^A(M)$, we can determine a vector W in $T_{Y(p)}A$. In the usual natural coordinates:

$$W(X, Y) = (x^0, .., x^{n-1}, Y^0, .., Y^{n-1}, X^0, .., X^{n-1}, X(Y^0), .., X(Y^{n-1})) \tag{2.28}$$

We can then define the non-linear connection associated with the anisotropic connections as the vectors W in TA such that $D_X Y = 0$. In fact, it is easy to see that for these vectors $\omega(W) = 0$. To derive the corresponding $\mathcal{H}$, let $W = W^i_1 \partial_i + W^i_2 \dot{\partial}_i$. Then W is in the horizontal space if and only if

$$W^k_2 + W^i_1 N^k_i(x, y) = 0 \quad k = 0, .., n-1 \tag{2.29}$$

But then a basis for the horizontal space is given by

$$\left\{ \frac{\delta}{\delta x^i}\Big|_{(x,y)} := \partial_i\big|_{(x,y)} - N^a_i(x, y) \dot{\partial}_a\big|_{(x,y)} \right\}_{i=0,..,n-1} \tag{2.30}$$

We now state all the correspondences between anisotropic connections and non-linear connections in the following theorem.

Theorem 5.
1. *Any anisotropic connection ∇ defines canonically a non-linear connection ω^∇ whose components N^a_i in any chart are given as*

$$N^a_i(x, y) = \Gamma^a_{ij}(x, y) y^j \tag{2.31}$$

   In addition, if ∇ is homogeneous then ω^∇ is too.

2. *Any non-linear connection ω defines canonically an anisotropic connection ∇^ω whose symbols Γ^a_{ij} in any chart are given by*

$$\Gamma^a_{ij}(x, y) = \dot{\partial}_j N^a_i(x, y). \tag{2.32}$$

   Moreover, if ω is homogeneous then ∇^ω is homogeneous and $\omega^{\nabla^\omega} = \omega$.

3. *For any homogeneous ω, the set $\{\nabla$ anisotropic connection, $\omega^{\nabla} = \omega\}$ is:*

$$\left\{\nabla^{\omega} + Q : \quad Q \in \mathcal{T}^1_2(M, A), Q^k_{ij} y^j = 0\right\}. \tag{2.33}$$

4. *When restricted to affine linear connections, the map $\nabla \to \omega^{\nabla}$ is well defined and bijective.*

Proof. (Sketch)

1. The non-linear connection is defined exactly like in the discussion above. With regards to the homogeneity of the N^a_i, we see that for homogeneous ∇

$$N^a_i(x, \lambda y) = \Gamma^a_{ij}(x, \lambda y)\lambda y^j = \lambda \Gamma^a_{ij}(x, y) y^j = \lambda N^a_i(x, y) \quad \lambda > 0 \tag{2.34}$$

2. Using cocycle transformations, it can be shown that the Γ^a_{ij} defined in this way transform in the correct fashion. Then, since ω is homogeneous, notice that applying Euler's Theorem we obtain

$$N^a_i(x, y) = \partial_{y^j} N^a_i(x, y) y^j. \tag{2.35}$$

We then see that ∇^{ω} is homogeneous and that $\omega^{\nabla^{\omega}} = \omega$.

3. We see this from the previous part and from Equation 2.31. Let $\tilde{\nabla}$ be such that $\omega^{\tilde{\nabla}} = \omega$. Consider $\tilde{\nabla} - \nabla^{\omega}$, where ∇^{ω} is the canonical anisotropic connection from the previous point. The tensorial nature of $\tilde{\nabla} - \nabla^{\omega}$ stems from verifying that it transforms like a tensor. We then calculate:

$$Q^k_{ij} y^j = \left[\tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}\right] y^j = N^k_i - N^k_i = 0. \tag{2.36}$$

4. For this ∇, $\Gamma^k_{ij}(x, y) = \Gamma^k_{ij}(x)$ (recall Definition 15). Given a second $\tilde{\nabla}$ with $\omega^{\nabla} = \omega^{\tilde{\nabla}}$, the difference $Q = \tilde{\nabla} - \nabla$ is also independent of y and $Q^k_{ij}(x) y^j = 0$. This in turn implies $Q = 0$ by taking the derivative with respect to each fibre coordinate.

□

Remark 3. *Given $X \in \mathfrak{X}(M)$ such that $X(x) = X^i(x)\partial_i|_x \in T_x M$, we can define an horizontal lift $X^{\mathcal{H}}_v = X^i(\pi(v))\frac{\delta}{\delta x^i}|_{(x,y)}$, where the $\frac{\delta}{\delta x^i}|_{(x,y)}$ are the basis vectors of the horizontal space (recall Equation 2.30). Now, let $h \in \mathcal{F}(A)$. We then see that*

$$X^{\mathcal{H}}_v(h) = X^i(\pi(v))\left(\partial_i h - \Gamma^k_{ij} y^j \dot{\partial}_k h\right)(v) = (\nabla_X h)(v) \tag{2.37}$$

always interpreting ∇_X as a tensor derivation (see Remark 2).

2.3.4 Anisotropic vs linear connections

Anisotropic connections as vertically trivial linear connections

In this section, we will identify anisotropic connections with a class of linear connections on the fibre bundle $\mathcal{V}A \to A$. Recall from Section 2.2 that on $\mathcal{V}A$ we have natural coordinates, which we will now relabel (x, y, z). Then we can write any element of $\mathcal{V}_{(x,y)}A$ as $z^a(x, y)\dot{\partial}_a|_{(x,y)}$

(recall that $\dot{x} = 0$). Notice that this fibre bundle has a tangent space of its own, namely $T\mathcal{V}A$. This space will on its own have a vertical space $\mathcal{V}\mathcal{V}A \to \mathcal{V}A$ composed by vectors of the type $\dot{z}^a(x, y, z)\partial_{z^a}|_{(x,y,z)}$.
In order to define a non-linear connection on $\mathcal{V}A \to A$, we consider the following short exact sequence of bundles:

$$0 \longrightarrow \mathcal{V}\mathcal{V}A \xrightarrow{\iota} T\mathcal{V}A \xrightarrow{d\pi} TA \longrightarrow 0$$

Recall that choosing a (non)-linear connection on $\mathcal{V}A \to A$ amounts to choosing a horizontal space to $T\mathcal{V}A$ which is isomorphic to TA via $d\pi$ (π being the projection map $\pi : \mathcal{V}A \to A$). Choosing a linear connection amounts to choosing a connection for which the $N_i^a(x, y, z) = \Gamma_{ij}^a(x, y)z^j$ for some functions Γ_{ij}^a in each coordinate system.
It is a standard result of the theory of linear connections that a linear connection ω^* is characterized by its Koszul derivative ∇^*. Recall from Section 2.2 that we identify the vertical bundle $\mathcal{V}A \to A$ with the pull-back bundle $\pi^*_{|A}TM \to A$. Hence we can think of the sections of $\mathcal{V}A \to A$ as anisotropic vector fields $\mathcal{T}_0^1(M, A)$. Therefore ∇^* is a map

$$\nabla^* : \mathfrak{X}(A) \times \mathcal{T}_0^1(M, A) \to \mathcal{T}_0^1(M, A) \tag{2.38}$$

$$(W, Z) \to \nabla^*_W Z. \tag{2.39}$$

Notice that, although this maps into $\mathcal{T}_0^1(M, A)$, it is not to be confused with the anisotopic connection.
We now wish to express ∇^* in terms of its Christoffel symbols. We then have to choose a basis for $\mathfrak{X}(A)$ and one for $\mathcal{T}_0^1(M, A)$. Suppose we have a prescribed non-linear connection $\dot{\omega}$ on $A \to M$. Then a convenient choice of basis might be $\{\delta_i, \dot{\partial}_i\}$:

$$\delta_i|_{(x,y)} = \frac{\delta}{\delta x^i}|_{(x,y)} = \partial_{x^i}|_{(x,y)} - \dot{N}_i^a(x, y)\partial_{y^a}|_{(x,y)}, \quad \dot{\partial}_i|_{(x,y)} \tag{2.40}$$

Now, notice that $\dot{\partial}_i|_{(x,y)}$ is a basis for $\mathcal{V}A$ (recall the discussion on the identification of $\mathcal{V}A$ with the pull-back bundle at the beginning of this subsection). On the other hand, the δ_i are a basis of the horizontal space of TA. Hence, we will use the $\dot{\partial}_i$ to construct elements of $\mathcal{T}_0^1(M, A)$ and the $\dot{\partial}_i$ (which we can see as the "vertical components") as well as the δ_i (which can be seen as the "horizontal" components) to construct elements of $\mathfrak{X}(A)$.

Definition 16. *The horizontal and vertical Christoffel symbols of ∇^* with respect to a prescribed non-linear connection $\dot{\omega}$ on $A \to M$ are, respectively, given by the functions H_{ij}^a, V_{ij}^a determined by*

$$H_{ij}^a \dot{\partial}_a = \nabla^*_{\delta_i} \dot{\partial}_j \tag{2.41}$$

$$V_{ij}^a \dot{\partial}_a = \nabla^*_{\dot{\partial}_i} \dot{\partial}_j \tag{2.42}$$

in a coordinate system (U, ψ). Easily, we can see that the horizontal Christoffel symbols are obtained using the horizontal components of $\mathfrak{X}(A)$, with the opposite happening for the vertical symbols.

Proposition 2. *1. The cocycle for H^a_{ij} (resp. V^a_{ij}) under a change of coordinates coincides with the one for the Christoffel symbols Γ^a_{ij} of an anisotropic connection (resp. the components of a $(1,2)$-anisotropic tensor). In addition, if all the V^a_{ij} vanish for some coordinates in U, then they vanish for any coordinate choice.*

2. Once a homogeneous non-linear connection $\dot{\omega}$ on $A \to M$ is prescribed, any local choice of functions H^a_{ij}, V^a_{ij} satisfying (1) for a coordinate atlas determine a unique linear connection ∇^*, whose Christoffel symbols with respect to $\dot{\omega}$ in that atlas coincide with the original H^a_{ij}, V^a_{ij}.

Remark 4. *Even though it might seem arbitrary to assume that we have a canonical homogeneous non-linear connection on $A \to M$, we will see that this indeed happens in the Finsler case where $\dot{\omega}$ is provided by the geodesic spray and is homogeneous.*

Definition 17. *Let ∇^* be a linear connection on $\mathcal{V}A \to A$. We say that ∇^* is vertically trivial if $V^a_{ij} = 0$ everywhere for some coordinates (and then for all coordinates because of Proposition 2).*

Remark 5. *It is clear from Proposition 2 that any homogeneous non-linear connection $\dot{\omega}$ on $A \to M$ induces a projection of the set of all ∇^*'s onto the vertically trivial ones, namely $(H^a_{ij}, V^a_{ij}) \to (H^a_{ij}, 0)$*

We proceed by stating the following theorem, which gives us the identification of vertically trivial linear connections on $\mathcal{V}A \to A$ with anisotropic connections.

Theorem 6. *Let $\dot{\omega}$ be a homogeneous non-linear connection on $A \to M$. We then have a map between the sets of vertically trivial and anisotropic connections, defined in natural coordinates as*

$$\{\textit{vertically trivial conn. } \nabla^* \textit{ on } \mathcal{V}A \to A\} \longleftrightarrow \{\textit{anisotropic conn. } \nabla\} \tag{2.43}$$

$$(H^a_{ij}, 0) \longleftrightarrow \Gamma^a_{ij} = H^a_{ij}. \tag{2.44}$$

Moreover, it can be shown that this map does not depend on $\dot{\omega}$, so that there exists a natural identification between vertically trivial linear connections and anisotropic connections.

Explicit construction of anisotropic connections

In this subsection, we look at the relationship between linear connections on $\mathcal{V}A \to A$ and anisotropic connections from a different angle. Suppose we are given a non-linear connection $\mathcal{H}$ on $A \to M$.
Recalling Remark 3, given $X = X^i \partial_i \in \mathfrak{X}(M)$, we can define $X^{\mathcal{H}} \in \mathfrak{X}(A)$ as

$$X^{\mathcal{H}} = X^i \left(\partial_i - N^j_i \dot{\partial}_j \right). \tag{2.45}$$

Once again, we identify the vertical bundle $\mathcal{V}A \to A$ with the pull-back bundle $\pi^*_{|A} TM \to A$. Therefore, just like in the previous section, we see that a linear connection on $\mathcal{V}A \to A$ can be also interpreted as a map

$$\tilde{\nabla} : \mathfrak{X}(A) \times \mathcal{T}^1_0(M, A) \to \mathcal{T}^1_0(M, A). \tag{2.46}$$

Definition 18. *Given a non-linear connection* $\mathcal{H}$ *on* $A \to M$*, the anisotropic connection associated with the classical connection* $\tilde{\nabla}$ *is defined as*

$$\nabla^v_X Y = (\tilde{\nabla}_{X^{\mathcal{H}}} Y)(v) \tag{2.47}$$

for any $v \in A$ *and* $X, Y \in \mathfrak{X}(M)$*. Once again, we have identified* Y*, an element of* $\mathfrak{X}(M)$*, with its pull-back under* $\pi|_A$*.*

Furthermore, given a vector field $X \in \mathfrak{X}(M)$, we can also define a derivative of a function $f \in \mathcal{F}(A)$ using the non-linear connection, namely $D(f) = X^{\mathcal{H}}(f)$. Moreover, we can define an anisotropic tensor

$$\mathcal{C}_v(X, Y) = (\tilde{\nabla}_{X^{\mathcal{V}}} Y)(v) \tag{2.48}$$

for any $v \in A$ and $(X, Y) \in \mathfrak{X}(M)$. Here, $X^{\mathcal{V}} = X^i \dot{\partial}_i$. Observe that $\mathcal{C}$ is a tensor because if $f \in \mathcal{F}(M)$, then $X^{\mathcal{V}}(f \circ \pi) = 0$ by definition, as the vertical space is the kernel of the differential. We then have that the term in the Leibniz rule for a connection vanishes and that the map is thus $\mathcal{F}(M)$ linear in both arguments. Recalling Remark 1, if the tensor is $\mathcal{F}(\mathcal{M})$ linear, it can then be extended using a local frame and $\mathcal{F}(A)$ multi-linearity, with the extension being independent of the frame. We now summarize our results in the following proposition.

Proposition 3. *A classical connection* $\tilde{\nabla}$ *on the vertical bundle and a horizontal connection on* $A \to M$ *determine the anisotropic connection* ∇ *and the anisotropic tensor* $\mathcal{C} \in \mathcal{T}^0_2(M, A)$*.*

2.3.5 Anisotropic vs Finsler connections

In the classical approach, when we are given a Finsler metric F (see Chapter 1) on A, there are two geometric objects which we can focus on. The first one is its geodesic spray on A and its associated canonical homogeneous non-linear connection on $A \to M$. The second one is a homogeneous linear connection on $\mathcal{V}A \to A$. Nevertheless, we do not have a canonical way of constructing the latter, as it involves some arbitrary choices.

Geodesic spray and associated non-linear connection

Definition 19. *A spray* G *on* A *is a vector field (section* $TA \to A$*) satisfying:*

1. *G can be written as*

$$G_{(x,y)} = y^i \partial_i\big|_{(x,y)} - 2G^a(x, y)\dot{\partial}_a\big|_{(x,y)} \tag{2.49}$$

2. *$G^a(x, \lambda y) = \lambda^2 G^a(x, y)$ for $\lambda > 0$.*

Given a Finsler function F, we can define its geodesic spray as having components $G^a = \gamma^a_{ij} y^i y^j$, where

$$\gamma^a_{ij} = \frac{1}{2} g^{ak} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{2.50}$$

The γ^a_{ij} are called the formal Christoffel symbols.
The following theorem gives the relationship between sprays and non-linear connections on $A \to M$.

Theorem 7. *1. A homogeneous non-linear connection ω defines a natural spray G. In coordinates:*

$$G^a(x,y) = \frac{1}{2} N_i^a(x,y) y^i \tag{2.51}$$

2. A spray G gives a natural homogeneous non-linear connection ω on $A \to M$ with components

$$N_i^a(x,y) = \dot{\partial}_i G^a(x,y) \tag{2.52}$$

Proof. (Sketch)

1. The 2-homogeneity of the G^a comes from the 1-homogeneity of the N_i^a.

2. It can be verified that the N_i^a satisfy the required cocycle transformation and the 1-homogeneity comes from the 2-homogeneity of the G^a's.

□

We now recall Theorem 5 (items 2 and 3) and we see that, given a spray G we have a canonical anisotropic connection ∇^G. This connection is called the Berwald connection. On the other hand, the anisotropic connections which canonically yield the non-linear connection corresponding to G are controlled by an anisotropic tensor Q satisfying $Q_{ij}^k y^j = 0$.

Finslerian linear connections

Definition 20. *Given a manifold M endowed with a Finsler metric on $A \subset TM$, a linear connection on $\mathcal{V}A \to A$ is called a Finslerian linear connection.*

As we have seen in the previous section, given a Finsler metric, we have a geodesic spray G with a canonical associated non-linear connection on $A \to M$. We will use this connection as the prescribed linear connection discussed in Subsection 2.3.4. Then, we can apply Theorem 6 and we obtain that the vertically trivial linear connections on $\mathcal{V}A$ correspond to the anisotropic linear connections. Furthermore, using the prescribed non-linear connection, we can project the linear connections on the the vertically trivial ones. Then choosing a linear connection, we automatically get an anisotropic connection by first projecting and then applying Theorem 6.

Example 6. *In Finsler geometry, two very frequent choices of vertically trivial linear connection ∇^* on $\mathcal{V}A \to A$ are:*

1. *Berwald:* $H_{ij}^k = \dot{\partial}_j N_i^k$.

2. *Chern-Rund:* $H_{ij}^k = \frac{1}{2} g^{ak} \left(\frac{\delta g_{ai}}{\delta x^j} + \frac{\delta g_{aj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^a} \right)$.

We conclude this section by summarizing all the different objects that we have seen in the following graph:

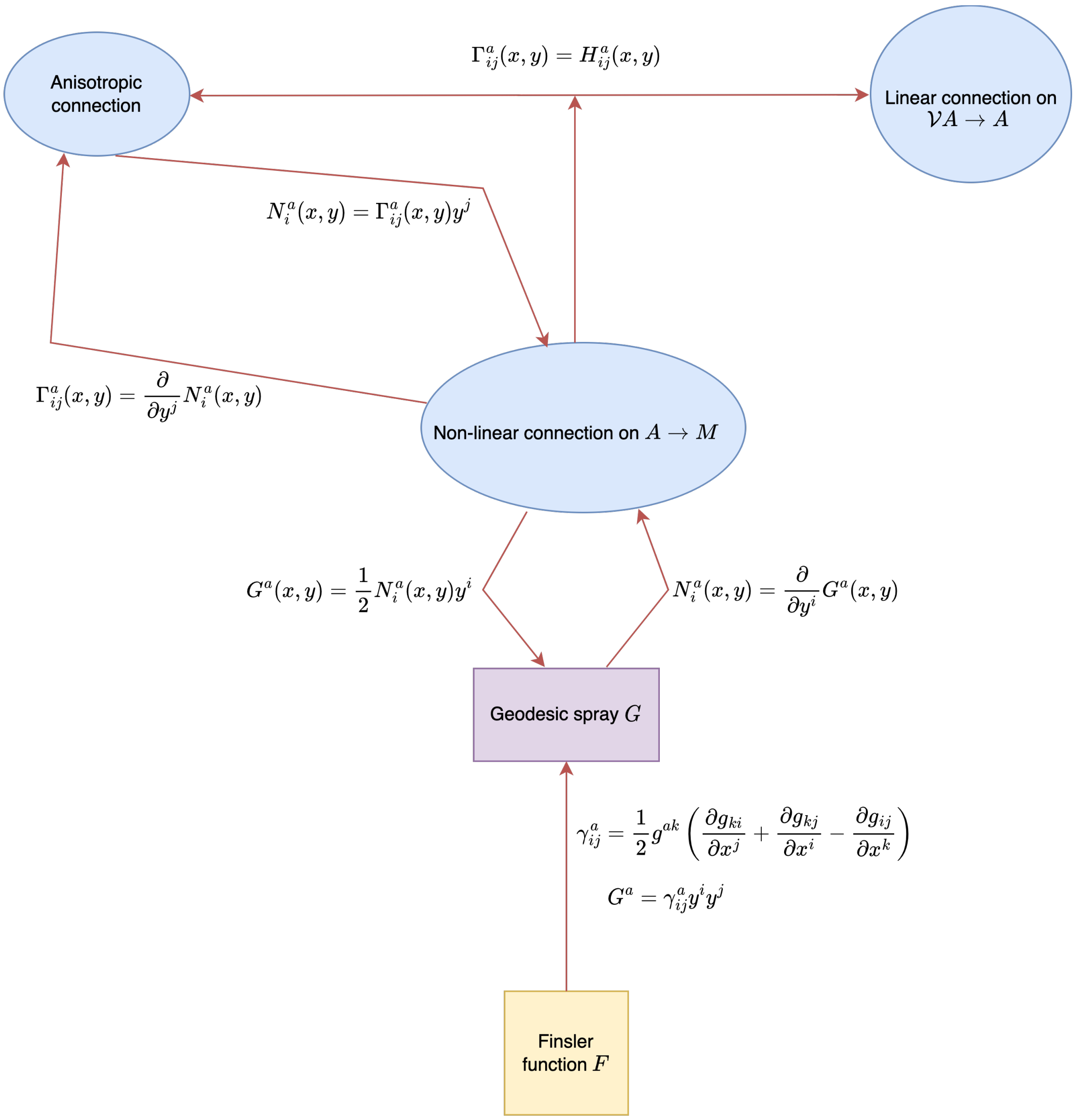

Figure 2.1: Relationships between cardinal geometric objects in Finsler geometry.

2.4 Curvature

In this section, the idea of curvature in the Finsler setting is introduced. We will see that it is possible to define an anisotropic tensor analogous to the Riemann curvature tensor in Riemannian geometry. Furthermore, the notion of curvature of the non-linear connection will be introduced and the relationship between the two will be shown. To conclude, the Finsler version of the Ricci tensor and the Ricci scalar will be introduced.

2.4.1 Curvature tensor of the anisotropic connection

Given an anisotropic connection ∇, we can define the associated curvature tensor $\mathcal{R}_v$ as follows:

$$\mathcal{R}_v : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to T_{\pi(v)}M \tag{2.53}$$

$$\mathcal{R}_v(X,Y)Z = \nabla^v_X(\nabla^v_Y Z) - \nabla^v_Y(\nabla^v_X Z) - \nabla^v_{[X,Y]}Z \tag{2.54}$$

where $[X,Y] = X \circ Y - Y \circ X$ and $v \in A; X, Y, Z \in \mathfrak{X}(M)$. It can be easily proved that R_v is $\mathcal{F}(M)$ multi-linear and that it is therefore a tensor. In coordinates, if $R_v(\partial_k, \partial_l)\partial_j = R^i_{jkl}(v)\partial_i$, we get:

$$R^i_{jkl} = \frac{\delta\Gamma^i_{lj}}{\delta x^k} - \frac{\delta\Gamma^i_{kj}}{\delta x^l} + \Gamma^i_{kh}\Gamma^h_{lj} - \Gamma^i_{lh}\Gamma^h_{kj} \tag{2.55}$$

Recalling Proposition 3, we know that a classical linear connection on the vertical bundle together with an horizontal connection on $A \to M$ determine an anistotropic connection and the anisotropic tensor $\mathcal{C} \in \mathcal{T}^0_2(M,A)$. The following proposition links the curvature tensor of the classical linear connection on the vertical bundle with the curvature tensor of the corresponding anisotropic connection and $\mathcal{C}$.

Proposition 4. *Let $\tilde{\nabla}$ (with curvature tensor $\tilde{R}$) be a linear connection on $\mathcal{V}A \to A$ and let ∇ (with curvature tensor R) and $\mathcal{C}$ be the associated anisotropic connection and tensor as in Proposition 3. Then:*

$$\tilde{R}_v(X^{\mathcal{H}}, Y^{\mathcal{H}})Z = R_v(X,Y)Z + \mathcal{C}_v(\iota_v^{-1}(\mathcal{V}[X^{\mathcal{H}}, Y^{\mathcal{H}}]), Z), \tag{2.56}$$

where $i_v : T_{\pi(v)}M \to \mathcal{V}_vTM$ is the canonical vertical lift map, mapping $X = X^i\partial_i$ to $X^{\mathcal{V}} = X^i\dot{\partial}_i$.

2.4.2 Curvature of a non-linear connection

Given a non-linear connection ω on $A \to M$, we can define its curvature as

$$F(\omega) = -\frac{1}{2}[\omega,\omega], \tag{2.57}$$

where $[\omega,\omega]$ denotes the Frölicher–Nijenhuis bracket [1] of differential forms. To see what this tensor looks like in coordinates, we choose a basis for TA, namely $(\delta_i, \dot{\partial}_j)$. Then:

$$F(\omega) = \left(\frac{\delta}{\delta x^k}N^i_l - \frac{\delta}{\delta x^l}N^i_k\right) dx^k \otimes dx^l \otimes \dot{\partial}_i. \tag{2.58}$$

This happens because if we feed it two basis vectors $\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}$:

$$-\frac{1}{2}[\omega,\omega]\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}\right) = \left[\frac{\delta}{\delta x^l}, \frac{\delta}{\delta x^k}\right] = \left(\frac{\delta}{\delta x^k}N^i_l - \frac{\delta}{\delta x^l}N^i_k\right)\dot{\partial}_i. \tag{2.59}$$

[1] See [5].

We will now show the connection between this coordinate representation and the coordinate representation of the curvature tensor of the anisotropic connection. Given the position vector $v = y^i\partial_i$, we calculate:

$$R_v(v, \partial_k)\partial_l = y^j R^m_{jkl}\partial_m \tag{2.60}$$

$$= y^j \left(\frac{\delta \Gamma^i_{lj}}{\delta x^k} - \frac{\delta \Gamma^i_{kj}}{\delta x^l} + \Gamma^i_{kh}\Gamma^h_{lj} - \Gamma^i_{lh}\Gamma^h_{kj} \right) \partial_i \tag{2.61}$$

$$= \Bigg[\left(\frac{\partial}{\partial x_k} N^i_l - N^a_k \frac{\partial}{\partial y^a} N^i_l + N^a_k \Gamma^i_{la} \right) \tag{2.62}$$

$$- \left(\frac{\partial}{\partial x_l} N^i_k - N^a_l \frac{\partial}{\partial y^a} N^i_k + N^a_l \Gamma^i_{ka} \right) \tag{2.63}$$

$$+ \Gamma^i_{kh} N^h_l - \Gamma^i_{lh} N^h_k \Bigg] \partial_i \tag{2.64}$$

$$= \left(\frac{\delta}{\delta x^k} N^i_l - \frac{\delta}{\delta x^l} N^i_k \right) \partial_i. \tag{2.65}$$

Here, we have used that given an anisotropic connection, the components of the corresponding non-linear connection are obtained as $N^a_i(x, y) = y^j \Gamma^a_{ij}(x, y)$ (see subsection 2.3.3). Therefore, we see that the components of $R_v(v, \cdot, \cdot)$ and $F(\omega)$ are the same.

2.4.3 Finsler-Ricci scalar and Finsler-Ricci tensor

As anticipated earlier, it is possible to define an equivalent of the Ricci tensor and the Ricci scalar in the Finsler setting. We will do so along the structure outlined in [21]. From Subsection 2.4.2, we see that the curvature of a nonlinear connection has components

$$R^i_{lk} = \frac{\delta}{\delta x^k} N^i_l - \frac{\delta}{\delta x^l} N^i_k. \tag{2.66}$$

We then define the Finsler-Ricci scalar as

$$Ric = R^i_{ik} y^k \tag{2.67}$$

and the Finsler-Ricci tensor as having components

$$R_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j Ric. \tag{2.68}$$

Chapter 3

Classical tensors

3.1 Introduction

In this chapter, the most common tensors encountered in Finsler geometry will be introduced. Definitions will be given both in terms of the anisotropic calculus formalism and in terms of the classical formalism, according to [16], [20] and [22]. The anisotropic calculus formalism is especially congenial because it gives a deeper meaning to the Chern connection. In the classical approach, the Chern connection is the Finsler equivalent of the Levi-Civita connection only for Berwald spaces [29], whereas in this approach this is always the case. At the end of the chapter, the concepts of Berwald and Landsberg tensors and spaces will finally be introduced. Equivalent characterizations of these spaces will be discussed.

3.2 Cartan tensor

Definition 21 (Torsion)**.** *Given an anisotropic connection* ∇ *we define the torsion of* ∇ *as*

$$T_v(X,Y) = \nabla^v_X Y - \nabla^v Y X - [X,Y] \tag{3.1}$$

for $X, Y \in \mathfrak{X}(M), v \in A$.

It should be noted that the anisotropic tensor T cannot be evaluated in the elements of $\mathcal{T}^1_0(M,A)$ directly. This is because we do not have a Lie bracket for anisotropic tensors. Nevertheless, we can extend the definition by linearity to anisotropic tensors making use of Remark 1 and once again identifying vector fields with their pullback.

Definition 22 (Cartan tensor)**.** *Given a Finsler space* (M,F)*, we define the Cartan tensor associated with* F *as having components*

$$C_v(\partial_i, \partial_j, \partial_k) = \frac{1}{4}\frac{\partial^3}{\partial s_3 \partial s_2 \partial s_1} F^2(v + s_1\partial_i + s_2\partial_j + s_3\partial_k) \tag{3.2}$$

or alternatively

$$C_v(\partial_i, \partial_j, \partial_k) = \left(\frac{1}{4}\bar{\partial}_i\bar{\partial}_j\bar{\partial}_k F^2\right)(v) \tag{3.3}$$

$$= \frac{1}{2}\bar{\partial}_i g_{jk}(v) \tag{3.4}$$

As a consequence, we see that the Cartan tensor kind of measures how far the fundamental tensor is from being pseudo-Riemannian. This is because it measures how much the fundamental tensor changes in the fibre coordinates directions.
Observe that, according to Definition 12, if we have an anisotropic derivation, we can take a tensor derivation of the function F^2. Furthermore, recall that according to Remark 2, an anisotropic connection together with $X \in \mathfrak{X}(M)$ specifies an anisotropic derivation ∇_X with associated vector field X. As a consequence we can calculate:

$$(\nabla_X F^2)(v) = X(F^2(V))(\pi(v)) - (\partial^\nu F^2)_v(\nabla^V_X V) \tag{3.5}$$

where V is any A-admissible extension of v and X is a vector field on M. Furthermore, we know that the vertical derivative of F^2 satisfies $(\partial^\nu F^2)_v(w) = 2g_v(v, w)$ and so

$$(\nabla_X F^2)(v) = X(F^2(V))(\pi(v)) - 2g_v(V, \nabla^V_X V). \tag{3.6}$$

In addition, according to Remark 2, the anisotropic derivation of the fundamental tensor of F^2 can be computed as follows:

$$(\nabla_X g)_v(Y, Z) = X(g_V(Y, Z))(\pi(v)) - g_v(\nabla^V_X Y, Z) - g_v(Y, \nabla^V_X Z) - \tag{3.7}$$

$$(\partial^\nu g)_v)(Y, Z, \nabla^V_X V) \tag{3.8}$$

for any $X, Y, Z \in \mathfrak{X}(M)$ and V an A-admissible vector field such that $V(\pi(v)) = v$. Since $(\partial^\nu g)_v(X, Y, Z) = 2C_v(X, Y, Z)$ (Equation 3.4), we obtain

$$(\nabla_X g)_v(Y, Z) = X(g_V(Y, Z))(\pi(v)) - g_v(\nabla^V_X Y, Z) - g_v(Y, \nabla^V_X Z) - 2C_v(Y, Z, \nabla^V_X V) \tag{3.9}$$

We now consider the task of finding an anisotropic connection which is compatible with the fundamental tensor of the Finsler space in the sense that $\nabla g = 0$. It follows from using the expression for the Chern connection (Example 6) and plugging it into Equation 3.9 that the Chern connection satisfies this requirement (here we have identified an anisotropic connection with a vertically trivial linear connection on $\mathcal{V}A \to A$). Since this connection also turns out to be the only one with this property which is also torsion free, it can be considered as the analog of the Levi-Civita connection in the Finsler setting.

In the previous chapter, we have seen how, given a manifold M endowed with a Finsler function F, we can canonically obtain an anisotropic connection. This anisotropic connection is then called the Berwald connection. From Theorem 5 and Theorem 7, we obtain that the relationship of the anisotropic Christoffel symbol with the spray of the Finsler function is

$$\Gamma^k_{ij}(x, y) = \frac{\partial^2 G^k}{\partial y^i \partial y^j}(x, y). \tag{3.10}$$

According to Theorem 6, there is a bijective correspondence between anisotropic connections and vertically trivial linear connections on $\mathcal{V}A \to A$. As a consequence, we will use the term Berwald connection for both. It will be clear from the context which one is being considered at the moment.

3.3 Berwald tensor

Definition 23 (Berwald tensor)**.** *Let S be a spray in $A \in TM$ and $\tilde{\nabla}$ the Berwald connection associated with it. The Berwald tensor is defined as the vertical derivative (Definition 9) of the Berwald connection evaluated with vector fields that are extensions of the arguments:*

$$B_v(u, w, z) = \frac{\partial}{\partial t} \tilde{\nabla}_X^{V+tZ} Y_{|t=0} \tag{3.11}$$

where V, X, Y, Z are extensions of respectively $v, u, w, z \in T_{\pi(v)}M$ and V is A-admissible.

From this definition, it is possible to get an expression for the components of the Berwald tensor which does not make use of the anisotropic connection. This expression is the most well-known and popular one in the literature.

$$B_v(\partial_i, \partial_j, \partial_k) = \frac{\partial}{\partial t} \tilde{\nabla}_{\partial_i}^{V+t\partial_k} \partial_{j_{|t=0}} \tag{3.12}$$

$$= \frac{\partial}{\partial t} \left(\frac{\partial^2 G^l}{\partial y^i \partial y^j} (v + t\partial_k) \partial_l \right)_{|t=0} \tag{3.13}$$

$$= \frac{\partial^3 G^l}{\partial y^i \partial y^j \partial y^k} (v) \partial_l = B^l_{ijk}(v) \partial_l \tag{3.14}$$

It follows that the Berwald tensor is symmetric, since the order in which the derivatives are taken does not matter.

Remark 6 (Chern tensor)**.** *Just like the Berwald tensor is obtained by taking the vertical derivative of the anisotropic Berwald connection, the same can be done for the anisotropic Chern connection (the Levi Civita equivalent for anisotropic connections).*
Let S be a spray defined on $A \subset TM$ and ∇ the Chern connection. The Chern tensor is defined as the vertical derivative of the Chern connection:

$$P_v(u, w, z) = \frac{\partial}{\partial t} \nabla_X^{V+tZ} Y_{|t=0} \tag{3.15}$$

where V, X, Y, Z are extensions of respectively $v, u, w, z \in T_{\pi(v)}M$ and V is A-admissible.

We now state the important definition of a Berwald space

Definition 24 (Berwald space)**.** *Given a Finsler space (M, F), we call it a Berwald space if the Berwald tensor associated with it vanishes everywhere.*

The most straightforward example of a Berwald space is a manifold equipped with a Lorentzian metric (recall Example 1). There, we saw that the fundamental tensor coincides with the metric tensor, and it therefore does not depend on the fibre coordinates. It follows directly that it is a Berwald space. The following Proposition gives equivalent characterizations of Berwald spaces.

Proposition 5 ([29])**.** *Let (M, F) be a Finsler space. The following are equivalent:*

1. *The Christoffel symbols of the Berwald connection depend only on the base manifold coordinates in all coordinate systems.*

2. *The Berwald tensor associated with (M, F) vanishes identically.*

3. *The canonical spray of the space (M, F) is an affine spray, meaning that the G^i are quadratic in the fibre coordinates.*

4. *The canonical connection on the bundle $A \to M$ associated with (M, F) is a linear connection.*

Example 7. *Recall the discussion on Randers metrics in Example 3. The following Theorem gives a sufficient and necessary condition for a Finsler space equipped with a Randers metric to be a Berwald space.*

Theorem 8 ([32])**.** *A Randers metric $F = \alpha + \beta$ is a Berwald metric if and only if β is parallel with respect to α.*

3.4 Landsberg tensor

We are now ready to give the definition of the Landsberg curvature tensor, which is crucial to understanding unicorns in Finsler geometry.

Definition 25 (Landsberg curvature tensor)**.** *Given a Finsler space (M, F), we define the Landsberg curvature tensor associated with it as:*

$$L_v(u, w, z) = -\frac{1}{2} g_v(B_v(u, w, z), v) \tag{3.16}$$

where $v, u, w, z \in T_{\pi(v)}M$.

The components of the tensor can be easily computed to be $S_{jkl} = -\frac{1}{2} y_i B^i_{jkl}$, using the fact that $y_i = g_{im}(x, y)y^m$.

Lemma 2. [1] *The components of the Landsberg curvature can be expressed in the following equivalent ways:*

$$\begin{aligned} S_{jkl}(v) &= \left(-\frac{1}{2} y_i B^i_{jkl}\right) && (3.17)\\ &= \left(-\frac{1}{4} y_i \bar{\partial}_j \bar{\partial}_k \bar{\partial}_l G^i\right) && (3.18)\\ &= \left(g_{ij}(\tilde{\Gamma}^i_{kl} - \Gamma^i_{kl})\right) && (3.19)\\ &= \left(-\frac{1}{2} \nabla_{\delta_j} g_{kl}\right) && (3.20) \end{aligned}$$

where ∇ is the Berwald connection and $v \in A$ with coordinates (x, y).

In addition, we can define the mean Landsberg curvature as the tensor obtained by raising the last two indexes of the Landsberg curvature using the inverse of the fundamental tensor [7]:

$$S_j(v) = (g^{kl} S_{jkl})(v), \qquad v \in A. \tag{3.21}$$

Given these definitions, we are now ready to define a Landsberg space.

[1] For proof see [4] for example.

Definition 26. *We say that a Finsler space (M, F) is a Landsberg space if the Landsberg tensor associated with it vanishes identically.*

From Lemma 2, we easily see that every Berwald space is also a Landsberg space. As a consequence, the earlier trivial example of a Lorentzian manifold is also valid as an example of a Landsberg space.

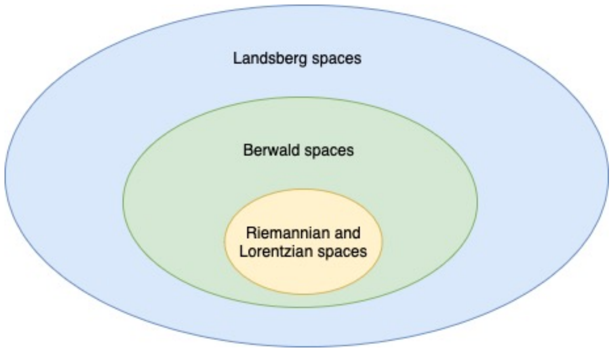

Figure 3.1: Every Lorentzian space is also a Berwald space and then automatically a Landsberg space.

In the following chapter the question whether there are Landsberg spaces which are not Berwald will be explored. In other words: is the blue shaded area in Figure 3.1 empty or does it contain some Finsler spaces?

Chapter 4

Landsberg unicorns

4.1 Introduction

At the end of Chapter 3 we have seen that every Berwald space is also a Landsberg space. Now, we explore the question whether the opposite is true; namely: is every Landsberg space a Berwald space? The answer turns out to be no, at least as far as non regular (i.e. not smooth on the whole of $TM \setminus 0$) Finsler metrics are concerned. In fact, as of today, it is still not yet known whether regular Landsberg and Berwald spaces actually coincide. In 2003, Matsmutomo described this as the most important problem in Finsler geometry [30]. As a matter of fact, even non-regular, non-Berwaldian Landsberg metrics are extremely rare and hard to find. For this reason, Bao has named them unicorns [30], like the supposedely rare mythological creatures. This chapter will start with a well-know example of a unicorn, taken from [10] and based on the class discovered by Asanov [2]. Subsequently, three general classes of unicorns discovered by Elgendi [10] will be presented. Thereafter, we will analyze the first class of unicorns proposed in [10] more closely. In particular, we prove that for a specific choice of $f(x^0)$, the Finsler-Ricci scalar associated with this class vanishes. As the considerd class of metrics is made by unicorns, the vanishing of the Finsler-Ricci scalar is a sufficient condition for the Finsler function to solve the field equation in Finsler gravity, provided that the fundamental tensor has Lorentzian signature.

Example 8. *Consider the Finsler space* $(\mathbb{R}^4, F)$ *with*

$$F = 2^{x^0}\sqrt{(y^0)^2 + y^1y^2 + (y^3)^2 + y^0\sqrt{y^1y^2 + (y^3)^2}}\,e^{\frac{1}{\sqrt{3}}arctan\left(\frac{2y^0+\sqrt{y^1y^2+(y^3)^2}}{\sqrt{3(y^1y^2+(y^3)^2)}}\right)} \tag{4.1}$$

Right away, we see that this function is only defined when $(y^0)^2+y^1y^2+(y^3)^2+y^0\sqrt{y^1y^2+(y^3)^2} \geq 0$ *and* $y^1y^2+(y^3)^2 > 0$*, hence we see that it is not defined for all values of the fibre coordinates and it is thus not regular. The geodesic spray is then computed with the help of Wolfram Mathematica. It is recommended to use the function FullSimplify together with Expand on the expressions when trying to replicate this example and the following ones. For the spray,* *we obtain:*

$$G^0 = \frac{ln(2)}{2}\left((y^0)^2 - y^1y^2 - (y^3)^2\right) \tag{4.2}$$

$$G^1 = \frac{ln(2)}{2}y^1\left(2y^0 + \sqrt{y^1y^2 + (y^3)^2}\right) \tag{4.3}$$

$$G^2 = \frac{ln(2)}{2}y^2\left(2y^0 + \sqrt{y^1y^2 + (y^3)^2}\right) \tag{4.4}$$

$$G^3 = \frac{ln(2)}{2}y^3\left(2y^0 + \sqrt{y^1y^2 + (y^3)^2}\right) \tag{4.5}$$

$$\tag{4.6}$$

According to Theorem 7, given the coefficients of the geodesic spray, we can calculate the coefficients of the non-linear connection associated to the Finsler function by using the relation $N^a_b = \dot{\partial}_b G^a$*. So for example:*

$$N^0_0 = y^0 ln(2), \quad N^1_2 = \frac{(y^2)^2}{\sqrt{y^1y^3 + (y^3)^2}}\frac{ln(2)}{2}, \quad N^3_1 = \frac{y^3y^1}{\sqrt{y^1y^3 + (y^3)^2}}\ln(2) \tag{4.7}$$

At a first glance, it seems indeed that only the first component of the spray is quadratic in the fibre coordinates. If this were true, then according to Proposition 5 we would not indeed be dealing with a Berwald space. One may check that

$$B^1_{111} = \frac{-3}{16}\frac{ln(2)[(y^2)^2(y^1y^2 + 2(y^3)^2)]}{[y^1y^2 + (y^3)^2]^{\frac{5}{2}}}. \tag{4.8}$$

By contrast we see that for the Landsberg tensor it holds that $L_{ijk} = 0$ *for* $i, j, k = 0, 1, 2, 3$*. This shows that this example is indeed a non-regular Landsberg unicorn.*

4.2 Three classes of unicorns

In this section, three classes of Landsberg unicorns proposed by Elgendi [30] will be discussed. These classes were obtained by taking the spray of the unicorn class proposed by Asanov and making a small deformation to it. An inverse problem was then solved in order to find the Finsler function that generated such deformed sprays. It is important to highlight that not every deformation to a spray yields a unicorn.
Before stating the first theorem, we consider an example.

Example 9. *Let* $(\mathbb{R}^3, F)$ *be a Finsler space with* F *given by*

$$F = f(x^0)(y^0 + \sqrt{y^1y^2})e^{\frac{y^0}{y^0+\sqrt{(y^1)(y^2)}}}, \tag{4.9}$$

with $f(x_0)$ *a smooth positive function. We see right away that this Finsler function is not defined for all fibre coordinates. In fact, we need to restrict ourselves to the cone where*

$y^1y^2 > 0$ and $y^0 + \sqrt{y^1y^2} \neq 0$. Calculating the spray coefficients we obtain:

$$G^0 = [(y^0)^2 - y^1y^2]\frac{f'(x^0)}{2f(x)} \tag{4.10}$$

$$G^1 = y^1\left(y^0 + \sqrt{y^1y^2}\right)\frac{f'(x^0)}{f(x^0)} \tag{4.11}$$

$$G^2 = y^2\left(y^0 + \sqrt{y^1y^2}\right)\frac{f'(x^0)}{f(x^0)} \tag{4.12}$$

Computing the Landsberg curvature and Berwald tensors we see that:

$$L_{ijk} = 0 \quad i,j,k = 0,1,2 \tag{4.13}$$

$$B^1_{111} = \frac{-3}{8}\frac{y^2}{y^1\sqrt{y^1y^2}}\frac{f'(x^0)}{f(x^0)}. \tag{4.14}$$

We have therefore verified that, as long as $f(x^0) \neq C$ (with $C > 0$ an arbitrary constant) this space is Landsberg but not Berwald and we are therefore in the presence of a unicorn.

From now on, we will deal with abstract classes of (α, β)-metrics. We will use $\beta = f(x^0)y^0$ and $\alpha = f(x^0)\sqrt{(y^0)^2 + \varphi(\hat{y})}$, where $f(x^0)$ is a positive non-constant function on $\mathbb{R}$ and φ is an arbitrary quadratic form in $\hat{y} = y^1, ..., y^{n-1}$ such that α is non-degenerate. We can thus write φ as $\varphi = c_{\lambda\nu}y^\lambda y^\nu$, where $c_{\lambda\nu}$ are the components of a symmetric invertible $n-1 \times n-1$ matrix. With the following theorem, we see that Example 9 is actually part of a class of (α, β)-metrics, all of which are unicorns.

Theorem 9 (First unicorn class)**.** *For $n \geq 3$, the class*

$$F = \left(a\beta + \sqrt{\alpha^2 - \beta^2}\right)e^{\frac{a\beta}{a\beta + \sqrt{\alpha^2-\beta^2}}} \tag{4.15}$$

on an n-dimensional manifold M is a class of Landsberg functions which are not Berwaldian. In addition, the geodesic spray of F is given by

$$G^0 = \left(\frac{2f(x^0)^2(y^0)^2 - \alpha^2}{2f(x^0)^2} + \frac{a^2-1}{2a^2}\frac{\alpha^2-\beta^2}{f(x^0)^2}\right)\frac{f'(x^0)}{f(x^0)} \tag{4.16}$$

$$G^i = Py^i \tag{4.17}$$

where $a \neq 0$, $i = 1, ..., n-1$ and

$$P = \left(y^0 + \frac{1}{af(x^0)}\sqrt{\alpha^2 - \beta^2}\right)\frac{f'(x^0)}{f(x^0)}. \tag{4.18}$$

The following two theorems give another two classes of (α, β)-metrics which are Landsberg unicorns.

Theorem 10 (Second unicorn class)**.** *For $n \geq 3$, the class*

$$F = f(x^0)\left(a\beta + \frac{\alpha^2 - \beta^2}{a\beta + 2\sqrt{\alpha^2 - \beta^2}}\right) \tag{4.19}$$

on an n-dimensional manifold M is a class of Landsberg functions which are not Berwaldian. In addition, the geodesic spray of F is given by

$$G^0 = \left(\frac{2f(x^0)^2(y^0)^2-\alpha^2}{2f(x^0)^2} + \frac{a^2-2}{2a^2}\frac{\alpha^2-\beta^2}{f(x^0)^2}\right)\frac{f'(x^0)}{f(x^0)} \tag{4.20}$$

$$G^i = Py^i \tag{4.21}$$

where $a \neq 0$, $i = 1,...,n-1$ and

$$P = \left(y^0 + \frac{3}{2a}\sqrt{\alpha^2-\beta^2}\right)\frac{f'(x^0)}{f(x^0)}. \tag{4.22}$$

Theorem 11 (Third unicorn class)**.** *For $n \geq 3$, the class*

$$F = \left((a+1)\beta + \sqrt{\alpha^2-\beta^2}\right)^{\frac{1+a}{2}}\left((a-1)\beta + \sqrt{\alpha^2-\beta^2}\right)^{\frac{1-a}{2}} \tag{4.23}$$

on an n-dimensional manifold M is a class of Landsberg functions which are not Berwaldian. In addition, the geodesic spray of F is given by

$$G^0 = \left(\frac{2f(x^0)^2(y^0)^2-\alpha^2}{2f(x^0)^2} + \frac{a^2-2}{2(a^2-1)}\frac{\alpha^2-\beta^2}{f(x^0)^2}\right)\frac{f'(x^0)}{f(x^0)} \tag{4.24}$$

$$G^i = Py_i \tag{4.25}$$

where $a \neq 0, 1, -1$, $i = 1,...,n-1$ and

$$P = \left(y^0 + \frac{a}{a^2-1}\sqrt{\alpha^2-\beta^2}\right)\frac{f'(x^0)}{f(x^0)}. \tag{4.26}$$

4.3 Vanishing of the Finsler-Ricci scalar

In the next theorem, it will be shown that for $f(x^0) = e^{bx^0}(b \neq 0)$ the Finsler-Ricci scalar associated with the class of Theorem 9 vanishes identically. It also turns out that this choice is the only one for which the Finsler-Ricci scalar vanishes and the class is still made by unicorns.

Theorem 12. *Consider the unicorn class of Theorem 9. For $f(x^0) = e^{bx^0}, b \neq 0$, the Finsler-Ricci scalar associated with this class of Finsler functions vanishes identically.*

Proof. From Theorem 9, substituting $f(x^0) = e^{bx^0}$ we know that the geodesic spray of this class has the form

$$G^0 = b\left(\frac{2e^{2bx^0}(y^0)^2-\alpha^2}{2e^{2bx^0}} + \frac{a^2-1}{2a^2}\frac{\alpha^2-\beta^2}{e^{2bx^0}}\right) \tag{4.27}$$

$$G^i = Py^i$$

where $i = 1,...,n-1$ and

$$P = b\left(y^0 + \frac{1}{ae^{bx^0}}\sqrt{\alpha^2-\beta^2}\right). \tag{4.28}$$

From Subsection 2.4.3, we know that the curvature of the non-linear conenction and the Finsler-Ricci scalar are

$$R^a_{bc} = \delta_c N^a_b - \delta_b N^a_c \qquad (4.29)$$
$$Ric = R^a_{ac} y^c.$$

Furthermore, from Theorem 7, we know that the non-linear connection coefficients associated with F have the form

$$N^a_b = \dot{\partial}_b G^a. \qquad (4.30)$$

We are now ready to substitute the expressions for the geodesic spray in the expression for the Ricci scalar.

$$\begin{aligned} Ric &= y^c(\delta_c N^a_a - \delta_a N^a_c) \qquad (4.31)\\ &= y^c\left(\delta_c \dot{\partial}_a G^a - \delta_a \dot{\partial}_c G^a\right)\\ &= y^c\left[\left(\partial_c - N^d_c \dot{\partial}_d\right)\dot{\partial}_a G^a - \left(\partial_a - N^e_a \dot{\partial}_e\right)\dot{\partial}_c G^a\right]\\ &= \left[y^c\left(\partial_c \dot{\partial}_a G^a - \partial_a \dot{\partial}_c G^a\right) - y^c\left(N^d_c \dot{\partial}_d \dot{\partial}_a G^a - N^e_a \dot{\partial}_e \dot{\partial}_c G^a\right)\right]\\ &= \left[y^c\left(\partial_c \dot{\partial}_a G^a - \partial_a \dot{\partial}_c G^a\right) - y^c\left(\dot{\partial}_c G^d \dot{\partial}_d \dot{\partial}_a G^a - \dot{\partial}_a G^e \dot{\partial}_e \dot{\partial}_c G^a\right)\right]\\ &= \underbrace{y^c\left(\partial_c \dot{\partial}_a G^a - \partial_a \dot{\partial}_c G^a\right)}_{\text{I}} - \underbrace{y^c\left(\dot{\partial}_c G^d \dot{\partial}_d \dot{\partial}_a G^a - \dot{\partial}_a G^e \dot{\partial}_e \dot{\partial}_c G^a\right)}_{\text{II}}. \qquad (4.32)\end{aligned}$$

We start by noticing that the geodesic spray of F does not depend on the base manifold coordinates. Therefore term I will vanish. Notice that we could have achieved this result also by choosing $f(x^0) = C$, with $C > 0$. Nevertheless, in this case the class would not be made by unicorns anymore, as it is easy to see that the Berwald tensor associated with it vanishes identically. We now turn our attention to term II and we further decompose it into two terms:

$$y^c\left(\dot{\partial}_c G^d \dot{\partial}_d \dot{\partial}_a G^a - \partial_a G^e \dot{\partial}_e \dot{\partial}_c G^a\right) = \underbrace{y^c \dot{\partial}_c G^d \dot{\partial}_d \dot{\partial}_a G^a}_{\text{II'}} - \underbrace{y^c \dot{\partial}_a G^e \dot{\partial}_e \dot{\partial}_c G^a}_{\text{II''}} \qquad (4.33)$$

In the following, latin indices will denote sums from 0 to $n-1$, whereas greek indices will denote sums from 1 to $n-1$. Furthermore, for short we will use the notation $P_i = \dot{\partial}_i P$ and $P_{ij} = \dot{\partial}_i \dot{\partial}_j P$. We start by computing II$'$:

$$\begin{aligned} y^c \dot{\partial}_c G^d \dot{\partial}_d \dot{\partial}_a G^a &= y^c \dot{\partial}_c \left[G^0 \dot{\partial}_0 + (P y^\alpha)\dot{\partial}_\alpha\right]\left[\dot{\partial}_0 G^0 + \dot{\partial}_\beta (P y^\beta)\right] \qquad (4.34)\\ &= y^c \dot{\partial}_c G^0 \dot{\partial}^2_0 G^0 + y^c \dot{\partial}_c G^0 \dot{\partial}_0 \dot{\partial}_\beta (P y^\beta) + y^c \dot{\partial}_c (P y^\alpha) \dot{\partial}_\alpha \dot{\partial}_0 G^0\\ &\quad + y^c \dot{\partial}_c (P y^\alpha) \dot{\partial}_\alpha \dot{\partial}_\beta (P y^\beta)\\ &= y^c \dot{\partial}_c G^0 \dot{\partial}^2_0 G^0 + y^c \dot{\partial}_c G^0 \dot{\partial}_0 (P_\beta y^\beta) + y^c \dot{\partial}_c G^0 P_0 (n-1)\\ &\quad + y^c (P_c y^\alpha) \dot{\partial}_\alpha \dot{\partial}_0 G^0 + y^c (P \delta^\alpha_c) \dot{\partial}_\alpha \dot{\partial}_0 G^0 + y^c (P_c y^\alpha) \dot{\partial}_\alpha (P_\beta y^\beta)\\ &\quad + y^c (P_c y^\alpha) P_\alpha (n-1) + y^c (P \delta^\alpha_c) \dot{\partial}_\alpha (P_\beta y^\beta) + y^c (P \delta^\alpha_c)(n-1) P_\alpha.\end{aligned}$$

Notice the following:

- The second term $y^c\dot\partial_c G^0 \dot\partial_0(P_\beta y^\beta)$ vanishes. This is because $\dot\partial_0(P_\beta y^\beta) = P_{1\beta}y^\beta + P_\beta\delta_1^\beta$. Then, since $P_{0c} = 0$ and β cannot be equal to 1, this term equals 0.
- $y^c\dot\partial_c G^0 = 2G^0$, applying Theorem 1 because the spray is quadratic in the fibre coordinates.
- $P_0 = 1$ and $\partial_0^2 G^0 = 1$.
- The fourth and fifth term vanish. This is because they contain $\dot\partial_\alpha\dot\partial_0 G^0$ and $\dot\partial_0 G^0 = y^1$.
- $P_c y^c = P$ because P is homogeneous of degree 1.

Given these observations, we can rewrite the last term of our expression for II$'$ as

$$\begin{aligned}
&2nG^0 + Py^\alpha\dot\partial_\alpha(P_\beta y^\beta) + Py^\alpha P_\alpha(n-1) + y^c(P\delta_c^\alpha)\dot\partial_\alpha(P_\beta y^\beta) + y^c(P\delta_c^\alpha)(n-1)P_\alpha \qquad (4.35)\\
&= 2nG^0 + Py^\alpha P_{\alpha\beta}y^\beta + Py^\alpha P_\beta\delta_\alpha^\beta + Py^\alpha P_\alpha(n-1) + y^c(P\delta_c^\alpha)P_{\alpha\beta}y^\beta\\
&\quad + y^c(P\delta_c^\alpha)P_\beta\delta_\alpha^\beta + y^c(P\delta_c^\alpha)(n-1)P_\alpha\\
&= 2nG^0 + 2Py^\alpha P_{\alpha\beta}y^\beta + 2ny^\alpha P_\alpha P.
\end{aligned}$$

We also notice that the term $Py^\alpha P_{\alpha\beta}y^\beta = 0$. This is because P_α is zero-homogeneous and $P_{0c} = 0$ so the sum from 1 to $n-1$ actually includes all non-zero terms. We now compute term II". We start by noticing that $\dot\partial_e G^a$ is first order homogeneous. As a consequence, applying Euler's theorem, $y^c\dot\partial_c\dot\partial_e G^a = \dot\partial_e G^a$. We can then expand the simplified expression

$$\begin{aligned}
\dot\partial_a G^e\dot\partial_e G^a &= \dot\partial_0 G^e\dot\partial_e G^0 + \dot\partial_\alpha G^e\dot\partial_e G^\alpha \qquad (4.36)\\
&= \dot\partial_0\left[G^0\dot\partial_0 + \left(Py^\beta\right)\dot\partial_\beta\right]G^0 + \dot\partial_\alpha\left[G^0\dot\partial_0 + \left(Py^\beta\right)\dot\partial_\beta\right](Py^\alpha)\\
&= (\dot\partial_0 G^0)^2 + y^\beta\dot\partial_\beta G^0 + +\dot\partial_\alpha G^0 y^\alpha + \left(P_\alpha y^\beta + P\delta_\beta^\alpha\right)\left(P_\beta y^\alpha + P\delta_\alpha^\beta\right)\\
&= 2G^0 + \dot\partial_\alpha G^0 y^\alpha + P_\alpha y^\alpha P_\beta y^\beta + Py^\alpha P_\alpha + Py^\alpha P_\alpha + P^2(n-1)\\
&= 2G^0 + \dot\partial_\alpha G^0 y^\alpha + (P_\alpha y^\alpha)^2 + 2Py^\alpha P_\alpha + P^2(n-1). \qquad (4.37)
\end{aligned}$$

Using the new, simplified expressions, we subtract II" from II' and we find:

$$\begin{aligned}
\mathrm{II}' - \mathrm{II}'' &= 2nG^0 + 2ny^\alpha P_\alpha P - 2G^0 - \dot\partial_\alpha G^0 y^\alpha - (P_\alpha y^\alpha)^2 - 2Py^\alpha P_\alpha - P^2(n-1) \qquad (4.38)\\
&= 2(n-1)G^0 + 2(n-1)PP_\alpha y^\alpha - (P_\alpha y^\alpha)^2 - \dot\partial_\alpha G^0 y^\alpha - P^2(n-1).\\
&= 2(n-1)G^0 + 2(n-1)P\frac{\sqrt{\varphi}}{a} - \frac{\varphi}{a^2} - P^2(n-1) + \varphi - \frac{a^2-1}{a^2}\varphi \qquad (4.39)\\
&= 2(n-1)G^0 + 2(n-1)P\frac{\sqrt{\varphi}}{a} - P^2(n-1) = 0.
\end{aligned}$$

□

We now turn to the question whether we can choose $f(x^0)$ such that the Finsler-Ricci scalar associated with the classes presented in Theorem 10 and Theorem 11 vanishes. We start by recalling the expression for the Finsler-Ricci scalar

$$Ric = y^c\left(\partial_c\dot\partial_a G^a - \partial_a\dot\partial_c G^a\right) - y^c\left(\dot\partial_c G^d\dot\partial_d\dot\partial_a G^a - \dot\partial_a G^e\dot\partial_e\dot\partial_c G^a\right) = \mathrm{I} - \mathrm{II}. \qquad (4.40)$$

For both classes, without specifying $f(x^0)$ in advance term I becomes

$$\mathrm{I} = y^0 \partial_0 \dot{\partial}_a G^a - y^c \partial_0 \dot{\partial}_c G^0 = \mathrm{I}' - \mathrm{I}'' \tag{4.41}$$

As derivatives of the spray components in the fibre coordinates still contain the term $\frac{f'(x^0)}{f(x^0)}$, we obtain that both I′ and I″ will contain a non-zero expression in the fibre coordinates multiplied by the factor $\frac{(f'(x^0))^2 - f(x^0) f''(x^0)}{f^2(x^0)}$. With regards to term II, if we plug in the expression for the spray we obtain that it does not vanish. As term II only contains derivatives in the fibre coordinates, we know that that term will be made of a non-vanishing expression in the fibre coordinates multiplied by $\frac{f'(x^0)}{f(x^0)}$. It follows that choosing $f(x^0) = e^{bx^0}$ with $b \neq 0$ sets term I to 0, whereas term II does not vanish. Now, $f(x^0)$ needs to be such that $\mathrm{I} = \mathrm{II} \neq 0$. Unfortunately, it is not possible to find an expression for $f(x^0)$ such that the Finsler-Ricci scalar vanishes for all choices of φ for any of these classes. In fact, terms I and II both have the same form: a term in the base manifold coordinates multiplied by a term in the fibre coordinates. We will now denote the part of i^{th} component of the spray in the fibre coordinates as $\tilde{G^i}$, so that $G^i = \frac{f'(x^0)}{f(x^0)} \tilde{G^i}$. As a consequence, in order for the Finsler-Ricci scalar to vanishing, we need to have the following:

$$1 = \frac{\mathrm{I}}{\mathrm{II}} = \frac{(f'(x^0))^2 - f(x^0) f''(x^0)}{f(x^0) f'(x^0)} \frac{y^0 \dot{\partial}_a \tilde{G^a} - y^c \dot{\partial}_c \tilde{G^0}}{y^c \left(\dot{\partial}_c \tilde{G^d} \dot{\partial}_d \dot{\partial}_a \tilde{G^a} - \dot{\partial}_a \tilde{G^e} \dot{\partial}_e \dot{\partial}_c \tilde{G^a} \right)} \tag{4.42}$$

In order for this to happen, we need to have that $\frac{y^0 \dot{\partial}_a \tilde{G^a} - y^c \dot{\partial}_c \tilde{G^0}}{y^c \left(\dot{\partial}_c \tilde{G^d} \dot{\partial}_d \dot{\partial}_a \tilde{G^a} - \dot{\partial}_a \tilde{G^e} \dot{\partial}_e \dot{\partial}_c \tilde{G^a} \right)} = D$, with $D \in \mathbb{R}$. We then can set $\frac{(f'(x^0))^2 - f(x^0) f''(x^0)}{f(x^0) f'(x^0)} = \frac{1}{D}$ and solve for $f(x^0)$. Nevertheless, trying different expressions of φ and using the software Wolfram Mathematica, it becomes apparent that the fraction in the fibre coordinates is not a constant. As a consequence, it looks like this ratio will always depend on the fibre coordinates, even though we cannot be completely sure.

4.4 A unicorn solution to the FEFE

In the literature, several equations have been proposed as a Finsler generalization of the Einstein field equations when the fundamental tensor has a Lorentzian signature. Here, we consider two of them. The first one has been proposed by Rutz and, in vacuum, it reads [27]

$$Ric = 0. \tag{4.43}$$

This is analogous to the isotropic case of the Einstein field equations, which in vacuum amount to the vanishing of the Ricci scalar. The second equation has been obtained by Pfeifer and Wohlfart, using a variational argument. This equation is the most commonly used in the literature since, as opposed to Rutz's equation, it can be obtained as an Euler-Lagrange equation [15]:

$$2Ric - \frac{2F^2}{3} g^{ij} R_{ij} + \frac{2F^2}{3} g^{ij} \left(\dot{\partial}_i (\nabla S_j) - S_i S_j + \nabla_{\delta_i} S_j \right) = 0, \tag{4.44}$$

where ∇ is the Chern connection and S_j is the mean Landsberg curvature tensor. In case (M, F) is a Landsberg space, the mean Landsberg curvature tensor vanishes and the equation reduces to:

$$2Ric - \frac{2F^2}{3} g^{ij} R_{ij} = 0. \tag{4.45}$$

Hence, we see that if the Finsler-Ricci scalar associated with the Finsler function F vanishes, then we have a solution to both Equation 4.43 and Equation 4.44. As a consequence, once we have a Landsberg function with vanishing Finsler-Ricci scalar, we only need to make sure that it describes a Finsler spacetime (i.e. the signature of the fundamental tensor is Lorentzian) in order for it to solve the Finsler version of the Einstein field equations. In Theorem 12, we have proven that the Finsler-Ricci scalar vanishes for a class of unicorn metrics. As a consequence, if we manage to find a Finsler function belonging to this class which has a fundamental tensor with signature $(n-1)$, we then obtain a unicorn solution to the FEFE. Convienently, from Example 1.4, we know that in 4 coordinates, by choosing φ such that α has index $n-1$, we obtain a fundamental tensor with Lorentzian signature. We summarize the findings of this Chapter in the following theorem:

Theorem 13. *Let (M, F) belong to the class of Theorem 9, with $f(x^0) = e^{bx^0}$, $b \neq 0$ and φ such that α has Lorentzian signature in its matrix representation. Then the Finsler function F solves both versions of the field equation in Finsler gravity 4.43 and 4.44 .*

Chapter 5

Discussion and conclusion

In this thesis, a detailed introduction to Finsler geometry has been given and the concepts of a Landsberg space and a Landsberg unicorn have been introduced. Thereafter, a unicorn solution to the Finsler version of the Einstein field equations has been produced. To the best knowledge of the author, this is the first non-Berwald exact solution to the FEFE which has ever been found. Nevertheless, this work only touches the tip of the iceberg. Several, exciting possibilities are open ahead. To begin with, Elgendi has introduced the concept of a conformal transformation of a Finsler function [9]. This amounts to using a Finsler function to define a new one by setting

$$\bar{F} = e^{\theta(x)} F, \tag{5.1}$$

where $\theta(x)$ is a smooth function on the base manifold. In that paper, it is proven that a Berwald space can be transformed into a Landsberg unicorn under a conformal transformation, provided that some technical conditions are satisfied. It is very interesting to notice that the solution to the FEFE proposed in this thesis has something in common with this construction. In particular, by taking

$$F = \left(ay^0 + \sqrt{\varphi(\hat{y})}\right) e^{\frac{ay^0}{ay^0+\sqrt{\varphi(\hat{y})}}} \tag{5.2}$$

$$\theta(x) = bx^0 \tag{5.3}$$

we see that, if F is a Berwald Finsler function which satisfies the technical conditions, this case falls under the unicorns descibed by Elgendi in [9].

As the field equation in Finsler gravity simplifies considerably for Landsberg spaces, future research should start by focusing on producing new classes of unicorns and checking whether they solve this equation. There are two ways in which, intuitively, one could maximise the possibility that the unicorn found is a solution to Equation 4.44. Elgendi [10] has found a solid strategy to produce new examples of unicorns starting from existing ones via deformations of the spray followed by solving an inverse problem. In doing so, one could start from the spray of the solution presented in Theorem 13. When applying the deformation, one should make sure that terms II' and II" still simplify, as in Theorem 12. We have seen that this is equivalent to imposing

$$2(n-1)G^1 + 2(n-1)PP_\alpha y^\alpha - (P_\alpha y^\alpha)^2 - \dot{\partial}_\alpha G^1 y^\alpha - P^2(n-1) = 0 \tag{5.4}$$

when solving the inverse problem to find the Finsler function generated by the spray. The second way is to find a Berwald solution of the FEFE which satisfies the technical conditions described in [9] and applying a conformal transformation. Another possible expansion of this research is to consider the positive definite version of the Finsler gravity field equation, developed in [3]. In that context, a Landsberg unicorn with a vanishing Finsler-Ricci tensor also produces a solution. Unfortunately, according to Example 1.4, the fundamental tensor of the $(\alpha, \beta)-$metric of Theorem 9 is positive definite only when $||\beta||_\alpha < \sqrt{\frac{1}{1+a^2}}$. Nevertheless, the norm of the chosen β is always 1 for every choice of φ. Future research might try to find a positive definite version of the solution presented in Theorem 13.
Finally, another direction future research could go into is the physical interpretation of the solution presented in Theorem 13. A potential explanation could be that this function describes the expansion of the universe. In fact, considering the full expression for the Finsler function

$$F = \left(ay^0 + \sqrt{\varphi(\hat{y})}\right) e^{bt + \frac{ay^0}{ay^0 + \sqrt{\varphi(\hat{y})}}} \tag{5.5}$$

we see that as $t \to -\infty$, we have $F \to 0$. This seems to suggest that, going far back in time, the potential physical model described by this function allows for the possibility of the Big Bang. In addition, it is possible to compute the proper time experienced by a stationary observer from the Big Bang until now. At coordinate time t the stationary observer has coordinates $x = (t, 0, 0, 0)$ and therefore $\dot{x} = (1, 0, 0, 0)$. We can now compute the "age" of the universe as experienced by an observer who has been around since the Big Bang. Denoting the present coordinate time as T and the proper time elapsed as τ:

$$\begin{aligned} \tau &= \int_{-\infty}^{T} F(x, \dot{x}) dt \\ &= \int_{-\infty}^{T} a e^{bt+1} dt \\ &= \frac{a}{b} e^{bT+1}. \end{aligned} \tag{5.6}$$

We therefore see that the stationary observer sitting on the point of the Big Bang measures a finite time on her watch. This is indeed in line with the most widely accepted cosmology model in the literature, the ΛCDM model [24].

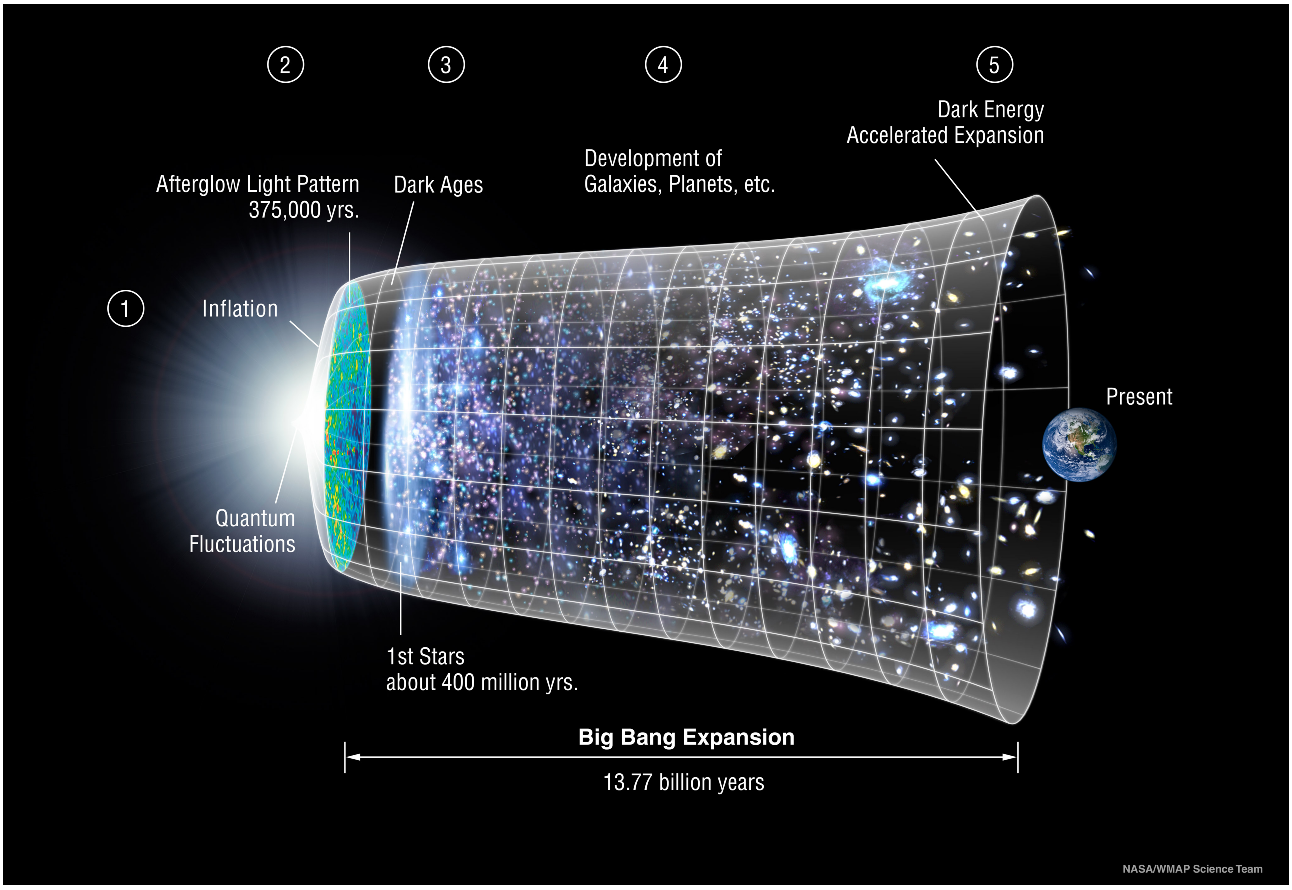

Figure 5.1: ΛCDM model of cosmology [24]

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Appendix A

Parallel Transport

A.1 Introduction

This section features a short overview of parallel transport, both of an instantaneous observer and with respect to an instantaneous observer and is based on the report written by the author for his Master internship. The discussion is follows the structure of [17], [20] and [19].

A.2 Covariant derivatives along curves

In the following, given a smooth curve $\gamma : [a, b] \to M$, $\mathfrak{X}(\gamma)$ will denote the space of smooth vector fields along γ and $\mathcal{F}(I)$ the smooth real functions on $I = [a, b]$.

Definition 27. *An anisotropic covariant derivation D_γ in A along a curve $\gamma : [a, b] \to M$ is a map*

$$D_\gamma : \gamma^*(A) \times \mathfrak{X}(\gamma) \to TM \tag{A.1}$$
$$(v, X) \to D^v_\gamma X \in T_\pi(v)M \tag{A.2}$$

with a smooth dependence on v, such that if $\pi(v) = \gamma(t_0)$ with $t_0 \in [a, b]$:

1. $D^v_\gamma(X + Y) = D^v_\gamma X + D^v_\gamma Y$ *with* $X, Y \in \mathfrak{X}(\gamma)$
2. $D^v_\gamma(fX) = \frac{df}{dt}(t_0)X(t_0) + f(t_0)D^v_\gamma X$ *for* $f \in \mathcal{F}(I), X \in \mathfrak{X}(\gamma)$

The notion of anisotropic connection summarizes the information of the covariant derivative along all possible curves. In fact, given an anisotropic connection, there is a canonical unique covariant derivative along curves

Proposition 6. *Given a smooth curve $\gamma : [a, b] \to M$, an anisotropic connection ∇ determines an induced covariant derivative along γ with the following property:*

$$D^v_\gamma(X_\gamma) = \nabla^v_{\dot\gamma} X \quad X \in \mathfrak{X}(M) \tag{A.3}$$

where X_γ is defined as $X_\gamma(t) = X(\gamma(t))$ for all $t \in [a, b]$ and $\dot\gamma$ is the tangent vector to the curve.

In local coordinates (U, ϕ) it is possible to specify the covariant derivative in terms of the Christoffel symbols of the anisotropic connection. In fact, if $X = X^k \partial_k$, then

$$D_\gamma^W X = (\dot{X}^i + \Gamma^i_{jk}(W)\dot{\gamma}^j X^k)\partial_i \tag{A.4}$$

where $W \in \mathfrak{X}(\gamma)$ and by definition the covariant derivative at t_0 depends only on the value of this vector field at t_0.

A.3 Parallel transport

Given a curve $\gamma : [a, b] \to M$ and a fixed reference vector field $Z \in \mathfrak{X}(\gamma)$ and $t_1, t_2 \in [a, b]$, then it is possible to define a parallel transport

$$P^Z_{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M \tag{A.5}$$

such that $P^Z_{t_1,t_2}(w) = W(t_2)$, for $W \in \mathfrak{X}(\gamma)$ such that $D^Z_\gamma W = 0$ and $W(t_1) = w$.
There is also another type of parallel transport, which is not always well defined. The goal of this type of parallel transport is to find a vector field such that $D^V_\gamma V = 0$ along γ.

Definition 28. *A smooth curve $\gamma : [a, b] \to$ is said to be parallel transport complete if for every $v \in A \cap T_{\gamma(a)}M$, there exists a unique A-admissible vector field $V \in \mathfrak{X}(\gamma)$ such that $D^V_\gamma V = 0$ and $V(a) = v$.*

In short, for these kinds of curves the second kind of parallel transport is always possible.

Remark 7. *From ODE theory, for every $v \in A \cap T_{\gamma(a)}M$, there exists always some $\epsilon > 0$ such that a parallel V as above is well-defined in $[a, a + \epsilon]$. Of course, usually $\epsilon < b - a$.*

From now on, we assume that our curves are all parallel transport complete. We are now ready to give a formal definition for the parallel transport of an instantaneous observer.

Definition 29. *Let ∇ be an anisotropic connection and $\gamma : [a, b] \to M$ a parallel transport complete curve. For each $t_1, t_2 \in [a, b]$, the instantaneous observer's parallel transport is the map:*

$$P_{t_1,t_2} : A \cap T_{\gamma(t_1)}M \to A \cap T_{\gamma(t_2)}M \tag{A.6}$$

given by $P_{t_1,t_2}(v) = V(t_2)$ for $V \in \mathfrak{X}(\gamma)$ satisfying $V(t_1) = v$ and $D^V_\gamma V = 0$.

Given this definition, we are now ready to define parallel transport with respect to an instantaneous observer.

Definition 30. *Let ∇ be an anisotropic connection and $\gamma : [a, b] \to M$ a parallel transport complete curve. For each $t_1, t_2 \in [a, b]$ and vector $v \in T_\gamma(t_1)M \cap A$, the parallel transport with respect to v (the instantaneous observer) is the map*

$$P^v_{t_1,t_2} : T_\gamma(t_1)M \to T_{\gamma(t_2)}M \tag{A.7}$$

obtained as $P^v_{t_1,t_2}(w) = W(t_2)$, where $W \in \mathfrak{X}(\gamma)$ satisfies $W(t_1) = w$ and $D^V_\gamma W = 0$ with V satisfying $D^V_\gamma V = 0$ and $V(t_1) = v$.

Recall the first kind of parallel transport briefly discussed at the beginning of this section. When we have a parallel transport complete curve, it is possible to choose the vector field V such that $D^V_\gamma V = 0$ as the reference vector field of parallel transport.

A.3.1 Parallel transport of anisotropic tensors

Firstly, notice that we can define the parallel transport of dual vectors with respect to an a vector $v \in T_{\gamma(a)}M \cap A$ (which we will call the instantaneous observer).

$$P^v_{a,b} : T_{\gamma(a)}M^* \to T_{\gamma(b)}M^* \tag{A.8}$$

$$P^v_{a,b}(\theta)(w) = \theta(P^v b, a(w)) \tag{A.9}$$

for $\theta \in T_{\gamma(a)}M^*$ and $w \in T_{\gamma(b)}M$. We have now defined the parallel transport on both vectors and covectors with respect to an instantaneous observer. If we require that the parallel transport agrees (i.e. commutes) with the tensor product, then we can parallel transport arbitrary anisotropic tensors.

Definition 31. *Given an anisotropic tensor $T \in \mathcal{T}^r_s(M, A)$ and a parallel transport complete curve $\gamma : [a, b] \to M$, we can define the parallel transport $P_{t_1,t_2} \in [a, b]$ as the map:*

$$P_{t_1,t_2}(T)_v : T_{\pi(v)}M^* \times ... \times T_{\pi(v)}M \to \mathbb{R} \tag{A.10}$$

$$P_{t_1,t_2}(T)_v(\theta^1, .., v_s) = T_{P_{t_1,t_2}(v)}(P^v_{t_2,t_1}(\theta^1), ..., P^v_{t_2,t_1}(v_s)) \tag{A.11}$$

with $\theta^1, ...\theta^r \in T_{\gamma(t_2)}M^*, v_1, ..., v_s \in T_{\gamma(t_2)}M$.