Output-based event-triggered control with guaranteed L∞-gain and improved event-triggering

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Output-Based Event-Triggered Control with Guaranteed $\mathcal{L}_\infty$-gain and Improved Event-Triggering

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Abstract— Most event-triggered controllers available nowadays are based on static state-feedback controllers. As in many control applications the full state is not available for feedback, it is the objective of this paper to propose event-triggered dynamical output-based controllers. The fact that the controller is based on output feedback instead of state feedback does not allow for straightforward extensions of existing event-triggering mechanisms if a minimum time between two subsequent events, the so-called 'minimum inter-event time', has to be guaranteed. Therefore, we will propose an event-triggering mechanism that invokes execution of the control task when the difference between the measured output or the control input of the plant or controller, respectively, and its previously sampled value becomes 'large' compared to its current value and an additional threshold. For such event-triggering mechanisms, we will study *closed-loop stability and $\mathcal{L}_\infty$-performance* and provide bounds on the minimum inter-event time. In addition, we will model the event-triggered control system using impulsive systems, which truly describe the behaviour of the event-triggered control system. As a result, we can guarantee stability and performance for improved event-triggered controllers with larger minimum inter-event times than existing results in literature.

I. INTRODUCTION

In many control applications nowadays, the controller is implemented on a digital platform. In such an implementation, the control task consists of sampling the outputs of the plant and computing and implementing new actuator signals. Typically, the control task is executed periodically, since this allows the closed-loop system to be analysed and the controller to be designed using the well-developed theory on sampled-data systems, see, e.g., [1], [2]. Although periodic sampling is preferred from an analysis and design point of view, it is sometimes less preferable from an implementation point of view. Namely, executing the control task at times that no disturbances are acting on the system and the system is close to its desired equilibrium might be a waste of computational resources. Moreover, in case the measured outputs and/or the actuator signals have to be transmitted over a shared network, this can lead to unnecessary utilisation of the network, or power consumption of the wireless radios, in case of a wireless network. For these reasons, an alternative to sampled-data control, namely, event-triggered control has been proposed, [3], [4]. Event-triggered control is a control strategy in which the control task is executed after the occurrence of an external event, generated by some

well-designed event-triggering mechanism, rather than the elapse of a certain period of time as in done in conventional sampled-data control.

Although the advantages of event-triggered control are well-motivated and even practical applications show its potential, only few theoretical results exist that study event-triggered control systems, see, e.g., [5]–[14]. In these references, several different event-triggering mechanisms and control strategies are proposed. However, most of the work on event-triggered control considers state-feedback controllers, which assumes that all the plant states can be measured. To the best of the authors' knowledge, the only result on event-triggered control using dynamical output-based controllers is presented in [12]. However, a thorough analysis of the minimum time between two subsequent events, the so-called inter-event time, is not available for [12]. Furthermore, extending the event-triggering mechanisms in [10], [11] to output-based controllers is not straightforward, since for these event-triggering mechanisms, no minimum inter-event times can be shown to exist, even though they have a guaranteed minimum inter-event time for state-feedback controllers.

As in many control applications the full state is not available for feedback, we study in this paper *stability and $\mathcal{L}_\infty$-performance* of event-triggered control systems for *dynamical output-based controllers*. Inspired by [11], we propose a modified event-triggering mechanism that invokes execution of the control task when the difference between the current output and the previously sampled output of either the plant or the controller becomes 'large' compared to the current value of the output of the plant or controller, respectively, plus some additional threshold. This additional threshold ensures the existence of a nonzero minimum inter-event time. The event-triggering mechanism presented in this paper can be seen as a unification of the event-triggering mechanisms proposed in [10], [11] and [12]–[14].

As a second contribution of this paper, we propose to model the event-triggered control system as an impulsive system, see, e.g., [15], [16], which truly describes the behaviour of the event-triggered control system. Furthermore, we extend the framework presented in [11] towards output-feedback controllers and we formally show that the impulsive systems framework allows stability to be guaranteed for event-triggering mechanisms that result in larger minimum inter-event times than the extended results of [11]. We derive conditions for stability in terms of linear matrix inequalities (LMIs) and using two numerical examples, we illustrate that the guaranteed minimum inter-event times is improved with

respect to existing results in literature.

A. Nomenclature

The following notational conventions will be used. $\mathrm{diag}(A_1, \ldots, A_N)$ denotes a block-diagonal matrix with the entries $A_1, \ldots, A_N$ on the diagonal, $A^\top \in \mathbb{R}^{m \times n}$ denotes the transposed of matrix $A \in \mathbb{R}^{n \times m}$, and $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalue of a (symmetric) matrix $A \in \mathbb{R}^{n \times n}$, respectively. For a vector $x \in \mathbb{R}^n$, we denote by $\|x\| := \sqrt{x^\top x}$ its 2-norm, for a signal $x : \mathbb{R}^+ \to \mathbb{R}^n$, we denote by $\|x\|_{\mathcal{L}_\infty} = \mathrm{ess\,sup}_{t \in \mathbb{R}^+} \|x(t)\|$, its $\mathcal{L}_\infty$-norm, and for a matrix $A \in \mathbb{R}^{n \times m}$, we denote by $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$, its induced 2-norm. Finally, we denote $x^+(t) = \lim_{s \downarrow t} x(s)$, i.e., taking the limit from above, and for brevity, we write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ as $\begin{bmatrix} A & \star \\ B^\top & C \end{bmatrix}$.

II. EVENT-TRIGGERED CONTROL

In this section, we present the event-triggered control problem and formulate it as an impulsive system.

A. Problem Formulation

Let us consider a linear time-invariant (LTI) plant given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_p = A_p x_p + B_p \hat{u} + B_w w, \\ y = C_p x_p, \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ the input applied to the plant, $w \in \mathbb{R}^{n_w}$ an unknown disturbance and $y \in \mathbb{R}^{n_y}$ the output of the plant. The plant is controlled using a continuous-time controller given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_c = A_c x_c + B_c \hat{y}, \\ u = C_c x_c + D_c \hat{y}, \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ denotes the state of the controller, $\hat{u} \in \mathbb{R}^{n_y}$ the input of the controller, and $u \in \mathbb{R}^{n_u}$ the output of the controller. We assume that the controller is designed to render (1) and (2) with $y(t) = \hat{y}(t)$ and $u(t) = \hat{u}(t)$, for all $t \in \mathbb{R}^+$, asymptotically stable.

In an event-triggered implementation, the outputs of the plant and controller are not sent continuously, but only at transmission times t_k, $k \in \mathbb{N}$. Therefore, we let the inputs of the plant and controller be described by a zero-order hold, i.e.,

$$\hat{y}(t) = y(t_k) \quad \text{and} \quad \hat{u}(t) = u(t_k), \tag{3}$$

for all $t \in (t_k, t_{k+1}]$, where $t_{k+1} > t_k$, $k \in \mathbb{N}$. In a conventional sampled-data implementation, the transmission times are distributed equidistantly in time, meaning that $t_{k+1} = t_k + h$, for all $k \in \mathbb{N}$ and for some constant transmission interval $h > 0$. In event-triggered control, however, we let these transmissions to be orchestrated by an event-triggering mechanism, as is shown in Fig. 1.

In this paper, we consider an event-triggering mechanism that invokes transmissions of both the outputs of the plant and the controller when either $\hat{y}(t) - y(t)$ or $\hat{u}(t) - u(t)$
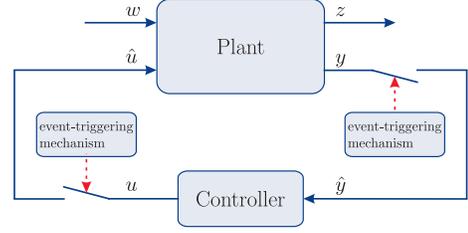

Fig. 1: Event-triggered Control Schematic

becomes too large. In particular, the event-triggering mechanism considered in this paper, results in transmitting the outputs of the plant and the controller at times t_k, $k \in \mathbb{N}$, satisfying

$$t_{k+1} = \inf \big\{ t > t_k \,|\, \|e_y(t)\|^2 = \sigma_y \|y(t)\|^2 + \varepsilon_y \text{ or }$$
$$\|e_u(t)\|^2 = \sigma_u \|u(t)\|^2 + \varepsilon_u \big\}, \tag{4}$$

and $t_0 = 0$, for some $\sigma_y, \sigma_u, \varepsilon_y, \varepsilon_u \geqslant 0$. In these expressions,

$$e_y(t) := \hat{y}(t) - y(t) \quad \text{and} \quad e_u(t) := \hat{u}(t) - u(t) \tag{5}$$

denote the errors induced by the event-triggered implementation of the controller. The event-triggering mechanism is such that when either one of the conditions in (4) is satisfied, both y and u are sent in a synchronised way, in which it is assumed that $y(t_k)$ is sent first. As a result, we have that both $e_y^+(t_k) = 0$ and $e_u^+(t_k) = 0$. Note that this is equivalent to the approach taken in the case the system is controlled by a state-feedback controller, see, e.g., [11]. Namely, in this case the outputs of the plant are transmitted first, after which the controller outputs are updated and transmitted. As a result of (4), the event-triggered-induced errors satisfy, for all $t \in \mathbb{R}^+$

$$\begin{cases} \|e_y(t)\|^2 \leqslant \sigma_y \|y(t)\|^2 + \varepsilon_y, \\ \|e_u(t)\|^2 \leqslant \sigma_u \|u(t)\|^2 + \varepsilon_u. \end{cases} \tag{6}$$

The main objective of the paper is to determine σ_y, σ_u, ε_y and ε_u, such that the closed-loop system is stable in an appropriate sense and a certain level of disturbance attenuation is guaranteed, while the number of transmissions of the outputs of the plant and the controller is minimised. Note that for $\varepsilon_y = \varepsilon_u = 0$, the event-triggering conditions in (4) can be thought of to be an extension of [11] for output-based controllers, and for $\sigma_y = \sigma_u = 0$, it is equivalent to [12]–[14].

B. Discussion and Possible Extensions

The event-triggering mechanism as discussed above requires that it is possible to synchronise the transmissions of both y and u in a particular way in which y is sent just before u, thereby allowing the transmitted u to be based on the newly received $\hat{y}$, (see (2)). The reason for making this assumption is to allow that $D_c \neq 0$ in (2), which also allows us to compare the results presented in this paper with existing results on state-feedback controllers, see below. To explain in more detail what issues might arise in case $D_c \neq 0$, let us consider the case in which the transmissions are not

synchronised and that the outputs of the plant are transmitted at times t_k^y, satisfying

$$t_{k+1}^y = \inf \left\{ t > t_k^y \mid \|e_y(t)\|^2 \geqslant \sigma_y \|y(t)\|^2 + \varepsilon_y \right\}, \quad (7)$$

and the outputs of the controller are transmitted at times t_k^u, satisfying

$$t_{k+1}^u = \inf \left\{ t > t_k^u \mid \|e_u(t)\|^2 \geqslant \sigma_u \|u(t)\|^2 + \varepsilon_u \right\}, \quad (8)$$

and $t_0^y = t_k^u = 0$, for some $\sigma_y, \sigma_u, \varepsilon_y, \varepsilon_u \geqslant 0$. In this case, the transmission times can coincide, i.e., $t_k = t_k^y = t_k^u$, and therefore both $\hat{y}^+(t_k) = y(t_k)$ and $\hat{u}^+(t_k) = u(t_k)$ are implemented simultaneously. Since $\hat{y}$ directly affects u, according to (2), we could have that the condition in (8) is again satisfied immediately, resulting in another transmission of u. Hence, we would have that u is transmitted twice at one time instant, which might not be desirable from a practical point of view and prohibits us from proving the existence of a guaranteed lower bound on the inter-event time.

In the case that synchronisation of the transmissions of the outputs of the plant and the controller is not possible, and it is acceptable that u is sometimes transmitted twice in an infinitesimal amount of time, the modelling steps and the stability results presented in this paper can be applied *mutatis mutandis*. In case that $D_c = 0$, in which the situation discussed above cannot occur, a lower-bound on the inter-event time can be proven to exist.

C. An Impulsive System Formulation

In this section, we reformulate the event-triggered control problem into an impulsive system formulation, see, e.g., [15], [16], of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\bar{x} = \bar{A}\bar{x} + \bar{B}w, & \text{when } \bar{x} \in \mathcal{C} \\ \bar{x}^+ = \bar{G}\bar{x}, & \text{when } \bar{x} \in \mathcal{D}, \end{cases} \quad (9)$$

where $\bar{x} \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ denotes the state of the system and $w \in \mathbb{R}^{n_w}$ an external disturbance. The flow and the jump sets are denoted by $\mathcal{C} \subseteq \mathbb{R}^{n_x}$ and $\mathcal{D} \subseteq \mathbb{R}^{n_x}$, respectively, and satisfy $\mathcal{X} = \mathcal{C} \cup \mathcal{D}$.

To arrive at a system description of the event-triggered control system (1), (2), with (4) of the form (9), we combine (1), (2), (5), and define $\bar{x} := [x_p^\top \ x_c^\top \ e_y^\top \ e_u^\top]^\top \in \mathbb{R}^{n_x}$, where $n_x := n_p + n_c + n_y + n_u$, yielding the flow dynamics of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{x} = \underbrace{\begin{bmatrix} A + BDC & BD \\ -C(A + BDC) & -CBD \end{bmatrix}}_{=: \bar{A}}\bar{x} + \underbrace{\begin{bmatrix} E \\ -CE \end{bmatrix}}_{=: \bar{B}}w, \quad (10)$$

in which

$$A = \mathrm{diag}(A_p, A_c), \quad B = \begin{bmatrix} 0 & B_p \\ B_c & 0 \end{bmatrix}, \quad (11a)$$

$$C = \mathrm{diag}(C_p, C_c), \quad D = \begin{bmatrix} I & 0 \\ D_c & I \end{bmatrix}, \quad E = \begin{bmatrix} B_w \\ 0 \end{bmatrix}. \quad (11b)$$

The system 'flows' as long as the event-triggering conditions are not met, i.e., as long as (6) holds, which is equivalent to $\bar{x} \in \mathcal{C}$, with

$$\mathcal{C} = \{\bar{x} \in \mathbb{R}^{n_x} \mid \bar{x}^\top Q_y \bar{x} \leqslant \varepsilon_y \text{ and } \bar{x}^\top Q_u \bar{x} \leqslant \varepsilon_u\}, \quad (12)$$

and

$$Q_y = \begin{bmatrix} 0 & 0 \\ 0 & \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \end{bmatrix} - \begin{bmatrix} (DC)^\top \\ D^\top - I \end{bmatrix} \begin{bmatrix} \sigma_y I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} DC & D - I \end{bmatrix}, \quad (13a)$$

$$Q_u = \begin{bmatrix} 0 & 0 \\ 0 & \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] \end{bmatrix} - \begin{bmatrix} (DC)^\top \\ D^\top - I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_u I \end{bmatrix} \begin{bmatrix} DC & D - I \end{bmatrix}. \quad (13b)$$

As mentioned before, when the event-triggering mechanism invokes a transmission of the plant and controller outputs, a reset according to $e_y^+(t_k) = 0$ and $e_u^+(t_k) = 0$ occurs. This can be expressed by

$$\bar{x}^+ = \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{=: \bar{G}}\bar{x}, \quad (14)$$

for all $\bar{x} \in \mathcal{D}$, in which

$$\mathcal{D} = \{\bar{x} \in \mathbb{R}^{n_x} \mid \bar{x}^\top Q_y \bar{x} = \varepsilon_y \text{ or } \bar{x}^\top Q_u \bar{x} = \varepsilon_u\}, \quad (15)$$

based on (4). Combining (10), (12), (14) and (15), yields an impulsive system of the form (9).

D. Special Case: State Feedback

In the existing literature, the event-triggered control problem has mostly been applied to state-feedback controlled systems, see, e.g., [10], [11]. In this case, the controller is given by

$$u(t) = Kx(t_k), \quad (16)$$

for all $t \in (t_k, t_{k+1}]$. We can regard this as a special case of the model presented above, and can also formulate it as an impulsive system. In this case (1) has $C_p = I$, i.e., all states are measurable, and (2) is replaced by (16). Furthermore, the states of the plant and the controller outputs are transmitted at transmission times t_k, $k \in \mathbb{N}$, satisfying

$$t_{k+1} = \inf \left\{ t > t_k \mid \|e(t)\|^2 \geqslant \sigma \|x(t)\|^2 + \varepsilon \right\}, \quad (17)$$

in which $e(t) := x(t_k) - x(t)$ and $t_0 = 0$. Based on these simplifications, the resulting impulsive system is given by (10) and $\bar{x} = [x_p^\top \ e^\top]^\top$, in which now

$$A = A_p, \quad B = B_p, \quad C = I, \quad D = K, \quad E = B_w \quad (18)$$

and (10). The flow and jump sets are now given by

$$\mathcal{C} = \{\bar{x} \in \mathbb{R}^{n_x} \mid \bar{x}^\top Q\bar{x} \leqslant \varepsilon\}, \quad \mathcal{D} = \{\bar{x} \in \mathbb{R}^{n_x} \mid \bar{x}^\top Q\bar{x} = \varepsilon\}, \quad (19)$$

where $Q = \mathrm{diag}(-\sigma I, I)$, for some appropriately chosen σ and ε. Taking $\varepsilon = 0$ gives the event-triggered mechanism as studied in [11].

III. STABILITY AND $\mathcal{L}_\infty$-GAIN

In this section, we study stability of the event-triggered control system in the sense of Lyapunov and its $\mathcal{L}_\infty$-gain. We will first review some basic stability and $\mathcal{L}_\infty$-gain results.

Definition III.1 [15] Consider the impulsive system, given by (9) with $w = 0$ and a compact set $\mathcal{A} \subset \mathcal{X}$.

- The set $\mathcal{A}$ is said to be globally stable for the impulsive system (9) with $w = 0$, if for each $\varepsilon > 0$ there exists $\delta > 0$, such that $\min_{x^\star \in \mathcal{A}} \|\bar{x}(0) - x^\star\| \leqslant \delta$ implies

$\min_{x^\star \in \mathcal{A}} \|\bar{x}(t) - x^\star\| \leqslant \varepsilon$, for all solutions $\bar{x}$ to the impulsive system (9) with $w = 0$ and all $t \in \mathbb{R}^+$.

- The set $\mathcal{A}$ is said to be globally attractive if each solution $\bar{x}$ to the impulsive system (9) with $w = 0$ satisfies $\min_{x^\star \in \mathcal{A}} \|\bar{x}(t) - x^\star\| \to 0$ as $t \to \infty$.
- The set $\mathcal{A}$ is *globally asymptotically stable* for (9), with $w = 0$, if it is globally stable and globally attractive.

Let us now define the notion of $\mathcal{L}_\infty$-gain of a system, which was studied for LTI systems in [17], and for which we introduce a performance variable $z \in \mathbb{R}^{n_z}$ given by

$$z = \bar{C}\bar{x} + \bar{D}w. \tag{20}$$

Definition III.2 The $\mathcal{L}_\infty$-gain of the system (9), with (20) is defined as

$$\gamma = \inf\{\bar{\gamma} \in \mathbb{R}^+ \,|\, \exists \omega : \mathbb{R}^n \to \mathbb{R}^+ \text{ such that}$$
$$\|z\|_{\mathcal{L}_\infty} \leqslant \bar{\gamma}\|w\|_{\mathcal{L}_\infty} + \omega(\bar{x}_0), \text{for all } \bar{x}(0) \in \mathcal{X}$$
$$\text{and all } w \text{ with } \|w\|_{\mathcal{L}_\infty} < \infty\}. \tag{21}$$

where z is given by (20), in which $\bar{x}$ is a solution to (9), with initial condition $\bar{x}(0) \in \mathcal{X}$ and input w.

Theorem III.3 *Consider the system given by (9), with (10), (12), (14) and (15), and (20) and assume its solutions exist for all $t \in \mathbb{R}^+$ and all $w \in \mathbb{R}^{n_w}$, satisfying $\|w\|_{\mathcal{L}_\infty} < \infty$. Now suppose there exist a positive definite matrix $P \in \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$, a matrix $U \in \mathbb{R}^{n_x \times n_x}$, such that $\bar{P} := \mathrm{diag}(P,0) + U$, scalars $\alpha, \beta, \gamma > 0$ and $\mu_1, \mu_2, \mu_3 \geqslant 0$, satisfying*

$$\begin{bmatrix} \bar{A}^\top \bar{P} + \bar{P}\bar{A} + \alpha\bar{P} - \mu_1 Q_y - \mu_2 Q_u & \star \\ \bar{B}^\top \bar{P} & -\beta I \end{bmatrix} \preceq 0, \quad (22a)$$

$$\bar{G}^\top \bar{P}\bar{G} - \bar{P} + \mu_3 Q_y \preceq 0, \quad (22b)$$

$$\bar{G}^\top \bar{P}\bar{G} - \bar{P} + \mu_3 Q_u \preceq 0, \quad (22c)$$

$$\begin{bmatrix} \alpha\bar{P} & \star & \star \\ 0 & (\gamma^2 - \beta)I & \star \\ \bar{C} & \bar{D} & I \end{bmatrix} \succeq 0. \quad (22d)$$

Then,

$$\mathcal{A} = \{\bar{x} \in \mathcal{C} \cup \mathcal{D} \,|\, \bar{x}^\top \bar{P}\bar{x} \leqslant \tfrac{\mu_1 \varepsilon_y + \mu_2 \varepsilon_u}{\alpha}\} \tag{23}$$

is an globally asymptotically stable equilibrium set for (9), with $w = 0$, and the $\mathcal{L}_\infty$-gain of (9) is smaller than or equal to γ.

In case disturbances are absent ($w = 0$), we can arrive at simpler conditions to guarantee that $\mathcal{A}$ is an globally asymptotically stable equilibrium set for system (9).

Corollary III.4 *Consider the system given by (9), with (10), (12), (14) and (15), and (20) and assume its solutions exist for all $t \in \mathbb{R}^+$. Now suppose there exist a positive definite matrix $P \in \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$, a matrix $U \in \mathbb{R}^{n_x \times n_x}$, such that $\bar{P} := \mathrm{diag}(P,0) + U \succeq 0$, scalars $\alpha > 0$, $\mu_1, \mu_2, \mu_3 \geqslant 0$, satisfying*

$$\bar{A}^\top \bar{P} + \bar{P}\bar{A} + \alpha\bar{P} - \mu_1 Q_y - \mu_2 Q_u \preceq 0, \quad (24)$$

and (22b) and (22c). Then, (23) is an globally asymptotically stable equilibrium set for (9), with $w = 0$.

Let us now comment on the results presented in Theorem III.3. Feasibility of (22) is determined by the choice of suitable σ_y and σ_u, which determine Q_y and Q_u, and by α and γ, and is not affected by the choice of ε_y and ε_u. Hence, once (22) is feasible, practical stability and the upper bound γ on the $\mathcal{L}_\infty$ gain are guaranteed. The 'size' of the equilibrium set $\mathcal{A}$ is affected by α, γ, σ_y and σ_u, through the resulting P, as well as ε_y and ε_u. However, after choosing σ_y and σ_u that render the event-triggered control system globally asymptotically stable to the equilibrium set $\mathcal{A}$ and has the desired upper bound γ on the $\mathcal{L}_\infty$-gain, the parameters ε_y and ε_u can be freely chosen to determine the size of the equilibrium set $\mathcal{A}$. As we can see from (6), this will affect the number of events, enabling us to make trade-offs between the size of the set $\mathcal{A}$, and the number of transmissions of y and u. Although, the naive choice to take $\varepsilon_y = \varepsilon_u = 0$ guaranteeing the system to converge to the origin, the inter-event times become zero, i.e., infinitely many events occur around times t when either $y(t) = 0$ or $u(t) = 0$ and $\bar{x}(t) \neq 0$.

IV. A Lower Bound on the Inter-Event Times

In this section, we show that the inter-event times $t_{k+1} - t_k$, $k \in \mathbb{N}$, of the event-triggered control system are bounded from below. We will show that although the stability and $\mathcal{L}_\infty$-gain condition of the system hold globally, the guaranteed lower bound on the inter-event time is a local property of the event-triggered control system.

Theorem IV.1 *Consider the event-triggered control system given by (9), with (10), (12), (14) and (15). For every initial condition $\bar{x}(0)$ satisfying $\|\bar{x}(0)\| \leqslant \delta_x$ and every disturbance $w \in \mathbb{R}^{n_w}$ satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \delta_w$, there exists a nonzero minimum inter-event time $h_{\min}$, i.e., $t_{k+1} - t_k \geqslant h_{\min} > 0$, for all $k \in \mathbb{N}$. An explicit expression for a lower bound on $h_{\min}$ is given by*

$$\min\{h > 0 \,|\, \begin{bmatrix} I \\ 0 \end{bmatrix}^\top e^{\bar{A}^\top h} Q_y e^{\bar{A}h} \begin{bmatrix} I \\ 0 \end{bmatrix} - \zeta_y(h)I \not\prec 0$$
$$\text{or } \begin{bmatrix} I \\ 0 \end{bmatrix}^\top e^{\bar{A}^\top h} Q_u e^{\bar{A}h} \begin{bmatrix} I \\ 0 \end{bmatrix} - \zeta_u(h)I \not\prec 0\}, \quad (25)$$

in which

$$\zeta_y(h) = \tfrac{\varepsilon_y - \lambda_{\max}(Q_y)\left(\rho(h) + 2\sqrt{c\rho(h)}\|e^{\bar{A}h}\bar{G}\|\right)}{c}, \quad (26a)$$

$$\zeta_u(h) = \tfrac{\varepsilon_u - \lambda_{\max}(Q_u)\left(\rho(h) + 2\sqrt{c\rho(h)}\|e^{\bar{A}h}\bar{G}\|\right)}{c}, \quad (26b)$$

with $c = \tfrac{1}{\lambda_{\min}(P)}\left(\lambda_{\max}(\bar{P})\delta_x^2 + \tfrac{\beta}{\alpha}\delta_w^2 + \tfrac{\mu_1\varepsilon_y + \mu_2\varepsilon_u}{\alpha}\right)$ and $\rho(h) = \int_0^h e^{\lambda_{\max}(\bar{A}^\top + \bar{A})s}ds\|\bar{B}\|^2\delta_w^2$.

Eqn. (25) in Theorem IV.1 can be solved by computing the eigenvalues of the h-dependent matrix in the left-hand side of the conditions in (25) for $h > 0$ and check when they cross zero for the first time. This determines the lower bound on $h_{\min}$ as in (25). Besides the fact that Theorem IV.1 formally shows that solutions of (9), with (10), (12), (14) and (15), exist for all times $t \in \mathbb{R}^+$ as was assumed in Theorem

III.3 and Corollary III.4, it also provides a lower bound on the inter-event times. This lower bound decreases as $\|\bar{x}(0)\|$ or $\|w\|_{\mathcal{L}_\infty}$ increases, meaning that the control task has to be executed more often if the system's initial state is further away from its equilibrium set or in case the norm of the disturbance is larger. In the special case that C_p and C_c are full rank (implying that Q_y and Q_u are full rank), and there are no disturbances, (i.e., $\delta_w = 0$), taking $\varepsilon_y = \varepsilon_u = 0$ still yields a lower bound larger than zero for all $\bar{x}(0)$. In fact, the resulting condition recovers then the one presented in Theorem 5.1 in [18].

V. IMPROVED EVENT-TRIGGERING CONDITIONS

In the previous sections, we modelled the event-triggered control system as an impulsive system and presented conditions to guarantee its stability. The reason to take an impulsive system approach is that it truly describes the behaviour of the event-triggered control system. In this section, we extend the work of [11] towards output-based controllers, and show how the event-triggering mechanism (4) obtained using the method presented in this section also yields a feasible solution to the conditions of Corollary III.4, (i.e., using the impulsive system description of the event-triggered control system).

Let us consider the following auxiliary system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x = (A + BDC)x + BD\begin{bmatrix} e_y \\ e_u \end{bmatrix}, \\ \begin{bmatrix} y \\ u \end{bmatrix} = DCx + (I - D)\begin{bmatrix} e_y \\ e_u \end{bmatrix}, \end{cases} \quad (27)$$

which is obtained from (10) by setting $E = 0$ and considering the errors e_y and e_y as external inputs. In (27), we let $x = [x_p^\top \quad x_c^\top]^\top$. Since (27) is an LTI system, asymptotic stability also implies that the system has a finite $\mathcal{L}_2$-gain, i.e., there exist a positive definite storage function of the form $V(x) = x^\top P x$ (see [19]), and positive scalars α, σ_y, σ_u, γ, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leqslant -\alpha V(x(t)) - \|y(t)\|^2 - \|u(t)\|^2 \\ + \tfrac{1}{\sigma_y}\|e_y(t)\|^2 + \tfrac{1}{\sigma_u}\|e_u(t)\|^2. \quad (28)$$

The matrix P, and scalars α, σ_y, σ_u can be obtained by solving LMIs. Since (6) holds, we have that

$$\tfrac{1}{\sigma_y}\|e_y(t)\|^2 + \tfrac{1}{\sigma_u}\|e_u(t)\|^2 \leqslant \|y(t)\|^2 + \|u(t)\|^2 + \tfrac{\varepsilon_y}{\sigma_y} + \tfrac{\varepsilon_u}{\sigma_u}, \quad (29)$$

for all $t \in \mathbb{R}^+$. Combining (28) and (29) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leqslant -\alpha V(x(t)) + \tfrac{\varepsilon_y}{\sigma_y} + \tfrac{\varepsilon_u}{\sigma_u}. \quad (30)$$

This expression shows that for $\alpha V(x(t)) \geqslant \tfrac{\varepsilon_y}{\sigma_y} + \tfrac{\varepsilon_u}{\sigma_u}$, the state x of (27), with (6), converges asymptotically to the set $\mathcal{A} = \{x \in \mathbb{R}^{n_p + n_c} \,|\, \alpha V(x) \leqslant \tfrac{\varepsilon_y}{\sigma_y} + \tfrac{\varepsilon_u}{\sigma_u}\}$.

Let us now present a theorem that formally states that any event-triggering condition obtained using (27), with (6), and (28), also renders the set $\mathcal{A}$ and the impulsive system given by (10), (12), (14) and (15) globally asymptotically stable.

Theorem V.1 *Consider the event-triggered control system, given by* (10)*,* (12)*,* (14) *and* (15) *and the auxiliary system*

(27)*. The positive definite matrix P and the scalars α, σ_y, and σ_u satisfying $V(x) = x^\top P x > 0$, which satisfies* (28)*, also satisfy the conditions of Corollary III.4 with $\bar{P} := $ diag$(P, 0)$, $\mu_1 = \tfrac{1}{\sigma_y}$, $\mu_2 = \tfrac{1}{\sigma_u}$ and $\mu_3 = 0$.*

This theorem states that the event-triggering condition resulting from the methodology presented in this section, (i.e., based on (28)), also renders the LMIs in Corollary III.4 (i.e., based on the impulsive system) feasible. As a result, the conditions based on impulsive systems are less conservative than the ones based on system (27), since we can allow $\bar{P}$ in Theorem III.3 and Corollary III.4 to have any structure. Hence, this creates the opportunity to guarantee stability for event-triggering conditions that yield a larger inter-event time.

VI. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the presented theory using two numerical examples. The first example is taken from [11], in which an unstable plant is stabilised using an event-triggered implementation of a state-feedback controller. We will show that by formulating the event-triggered control system as an impulsive system, we can guarantee stability for event-triggered control systems with larger minimum inter-event times. In the second example, we stabilise an unstable plant using a dynamic output-based controller. For both systems, we design an event-triggering condition and reflect on the resulting minimum inter-event time and the size of the globally asymptotically stable set $\mathcal{A}$.

Example 1: Let us consider the numerical example from [11]. The plant (1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad (31)$$

and the state-feedback controller (16), with $K = \begin{bmatrix} 1 & -4 \end{bmatrix}$. In [11], asymptotic stability of the origin is guaranteed for $\bar{\sigma} \leqslant 0.055$ and the event-triggering condition $\|e\| = \bar{\sigma}\|x\|$, obtained using an alternative approach. This yields $\sigma = 0.055^2 = 0.0030$ and $\varepsilon = 0$ for the event-triggering mechanism, as in (4), adapted for state-feedback, see Section II-D. For this event-triggering mechanism, Theorem IV.1, or its counterpart Theorem 5.1 in [18], yields a lower bound on the inter-event times of 0.0318. We now compare this result with the event-triggering mechanism obtained using the results from Section V, i.e., obtained by maximising σ in the LMIs related to (28). Taking this approach allows us to guarantee stability up to $\sigma = 0.0273$, resulting in a lower bound on the inter-event times of 0.0840. Therefore, we can conclude that taking the approach as in Section V already increases the allowable minimum inter-event time. However, if we analyse stability using the result of Corollary III.4, we can guarantee stability of this event-triggered control system up to $\sigma = 0.0588$ and $\varepsilon = 0$, which yields a lower bound on the inter-event times of 0.1136. The increase of inter-event times is expected due to the results of Theorem V.1.

We therefore conclude that by modelling the event-triggering control system using impulsive models, which
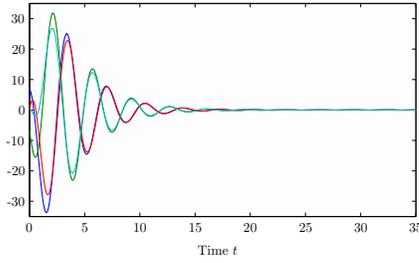
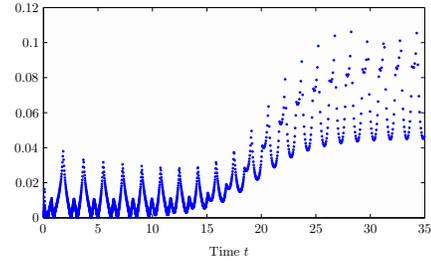Fig. 2: Convergence of the states of the plant and controller.

Fig. 3: The inter-event times as function of time.

truly describes the behaviour of the event-triggered control system, stability can be guaranteed for event-triggering mechanisms that yield larger minimum inter-event times.

Example 2: Let us now consider the plant (1) given by

$$\begin{cases} \frac{d}{dt}x_p = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \hat{u} \\ y = \begin{bmatrix} 2 & 0 \end{bmatrix} x_p, \end{cases} \tag{32}$$

and the controller (2) is given by

$$\begin{cases} \frac{d}{dt}x_c = \begin{bmatrix} -2 & 2 \\ -3 & -2 \end{bmatrix} x_c + \begin{bmatrix} \frac{3}{2} \\ \frac{1}{4} \end{bmatrix} \hat{y} \\ u = \begin{bmatrix} -\frac{1}{4} & -\frac{3}{2} \end{bmatrix} x_c. \end{cases} \tag{33}$$

We assume that no disturbances act on the plant, i.e., $B_w = 0$, and therefore, we simply take $\bar{C} = 0$ and $\bar{D} = 0$. Practical stability of the event-triggered control system (1), (2), with event-triggering mechanism (4), with $\sigma_y = \sigma_u = 0.0011$, can be guaranteed using the impulsive system formulation (9) and the results of Corollary III.4. If we take $\varepsilon_y = \varepsilon_u = 10^{-4}$, we obtain that the event-triggered control system converges asymptotically to $\|[x_p^\top(t) \ x_c^\top(t)]^\top\| \leqslant 3.9$. Using the result of Theorem IV.1, we obtain that if the initial conditions satisfy, e.g., $\|\bar{x}(0)\| \leqslant 10$, a lower bound on the inter-event times of $1.16 \cdot 10^{-5}$ is guaranteed.

When we compare these results with a simulation of the response of the system to the initial condition $\bar{x}(0) = [5\sqrt{2}, -5\sqrt{2}, 0, 0, 0, 0]^\top$, see Fig. 2, we observe that the system indeed converges asymptotically to a vicinity of the origin. However, as $t \to \infty$, the states of the plant and controller approach $\|[x_p^\top(t) \ x_c^\top(t)]^\top\| \approx 0.1$, which is smaller than the predicted value of approximately 3.9, and the observed minimum inter-event time is $h_{\min} \approx 10^{-4}$, which is larger than the predicted value of $1.16 \cdot 10^{-5}$. This seems to hold for many initial conditions satisfying $\|\bar{x}(0)\| \leqslant 10$. This shows that, although we can prove the existence of an globally asymptotically stable compact set and a nonzero lower bound on the inter-event times, the obtained bounds seem to be conservative to some extent, which leaves room for improvement.

VII. CONCLUSIONS

In this paper, we studied stability and $\mathcal{L}_\infty$-performance of event-triggered control systems for dynamical output-based controllers. We modelled the event-triggered control system as an impulsive system that truly describes the behaviour of the event-triggered control system, and whose stability can be analysed using linear matrix inequalities. We provided bounds on the inter-event time and we formally proved that by using an impulsive systems approach, stability and performance can be guaranteed for event-triggered controllers with larger inter-event times than existing results in literature. Using two numerical examples, we illustrated the presented theory.

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