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# On double extremes of Gaussian stationary processes

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## Abstract

We consider a Gaussian stationary process with Pickands' conditions and evaluate an exact asymptotic behavior of probability of two high extremes on two disjoint intervals.

## 1 Introduction. Main results.

Let  $X(t)$ ,  $t \in \mathbb{R}$ , be a zero mean stationary Gaussian process with unit variance and covariance function  $r(t)$ . An object of our interest is the asymptotic behaviour of the probability

$$P_d(u; [T_1, T_2], [T_3, T_4]) = \mathbf{P} \left( \max_{t \in [T_1, T_2]} X(t) > u, \max_{t \in [T_3, T_4]} X(t) > u \right)$$

as  $u \rightarrow \infty$ , where  $[T_1, T_2]$  and  $[T_3, T_4]$  are disjoint intervals. To evaluate the asymptotic behaviour we develop an analogue of Pickands' theory of high extremes of Gaussian processes, see [1] and extensions in [2]. We follow main steps of the theory. First we assume an analogue of the Pickands' conditions.

**A1** For some  $\alpha \in (0, 2)$ ,

$$\begin{aligned} r(t) &= 1 - |t|^\alpha + o(|t|^\alpha) \text{ as } t \rightarrow 0, \\ |r(t)| &< 1 \text{ for all } t > 0. \end{aligned}$$

Then, we specify covariations between values of the process on intervals  $[T_1, T_2]$  and  $[T_3, T_4]$ . We assume that there is an only domination point of correlation between the values. This makes some similarity with Pirabarg&Prisyazhn'uck's extension of the Pickands' theory to non-stationary Gaussian processes.

**A2** In the interval  $S = [T_3 - T_2, T_4 - T_1]$  there exists only point  $t_m = \arg \max_{t \in S} r(t) \in (T_3 - T_2, T_4 - T_1)$ ,  $r(t)$  is twice differentiable in a neighbourhood of  $t_m$  with  $r''(t_m) \neq 0$ .

As an alternative of assumption **A2** one can suppose that the point of maximum of  $r(t)$  is one of the end points of  $S$ ,  $T_3 - T_2$  is more natural candidate.

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**A3**  $r(t)$  is continuously differentiable in a neighbourhood of the point  $t_m = T_3 - T_2$ ,  $r'(t_m) < 0$  and  $r(t_m) > r(t)$  for all  $t \in (T_3 - T_2, T_4 - T_1]$ .

**A3'**  $r(t)$  is continuously differentiable in a neighbourhood of the point  $t_m = T_4 - T_1$ ,  $r'(t_m) > 0$  and  $r(t_m) > r(t)$  for all  $t \in [T_3 - T_2, T_4 - T_1)$ .

Denote by  $B_\alpha(t)$ ,  $t \in \mathbb{R}$ , a normed fractional Brownian motion with the Hurst parameter  $\alpha/2$ , that is a Gaussian process with a.s. continuous trajectories,  $B_\alpha(0) = 0$  a.s.,  $\mathbf{E}B_\alpha(t) \equiv 0$ , and  $\mathbf{E}(B_\alpha(t) - B_\alpha(s))^2 = 2|t - s|^\alpha$ . For any set  $T \subset \mathbb{R}$  we denote

$$H_\alpha(T) = \mathbf{E} \exp \left( \sup_{t \in T} B_\alpha(t) - |t|^\alpha \right).$$

It is known, [1], [2], that there exists a positive and finite limit

$$H_\alpha := \lim_{T \rightarrow \infty} \frac{1}{T} H_\alpha([0, T]), \quad (1)$$

the Pickands' constant. Further, for a number  $c$  denote

$$H_1^c(T) = \mathbf{E} \exp \left( \sup_{t \in T} B_1(t) - |t - ct| \right).$$

It is known, [2], that for any positive  $c$ , the limit  $H_1^c := \lim_{T \rightarrow \infty} H_1^c([0, T])$  exists and is positive. We stand  $a \vee b$  for  $\max(a, b)$  and  $a \wedge b$  for  $\min(a, b)$ . Denote

$$p_2(u, r) = \frac{(1+r)^2}{2\pi u^2 \sqrt{1-r^2}} e^{-\frac{u^2}{1+r}}$$

and notice that for a Gaussian vector  $(\xi, \eta)$  where the components are standard Gaussian and correlation between them is  $r$ ,  $\mathbf{P}(\xi > u, \eta > u) = p_2(u, r)(1 + o(1))$  as  $u \rightarrow \infty$ .

**Theorem 1** *Let  $X(t)$ ,  $t \in \mathbb{R}$ , be a Gaussian centred stationary process with a.s. continuous trajectories. Let assumptions **A1** and **A2** be fulfilled for its covariance function  $r(t)$ . Then*

$$\begin{aligned} P_d(u; [T_1, T_2], [T_3, T_4]) \\ = K \sqrt{\pi A^{-1}} (1 + r(t_m))^{-4/\alpha} H_\alpha^2 u^{-3+4/\alpha} p_2(u, r(t_m))(1 + o(1)) \end{aligned}$$

as  $u \rightarrow \infty$ , where  $K = T_2 \wedge (T_4 - t_m) - T_1 \vee (T_3 - t_m) > 0$ ,

$$A = -\frac{1}{2} \frac{r''(t_m)}{(1 + r(t_m))^2}.$$

**Theorem 2** *Let  $X(t)$ ,  $t \in \mathbb{R}$ , be a Gaussian centred stationary process with a.s. continuous trajectories. Let assumptions **A1** and **A3** or **A3'** be fulfilled for its covariance function  $r(t)$ . Then,*

(i) for  $\alpha > 1$ ,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = p_2(u, r(t_m))(1 + o(1))$$

as  $u \rightarrow \infty$ .

(ii) For  $\alpha = 1$ ,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = \left( H_1^{|r'(t_m)|} \right)^2 p_2(u, r(t_m))(1 + o(1))$$

as  $u \rightarrow \infty$ .

(iii) For  $\alpha < 1$ ,

$$P_d(u; [T_1, T_2], [T_3, T_4]) = B^{-2}(1 + r(t_m))^{-4/\alpha} H_\alpha^2 u^{-6+4/\alpha} p_2(u, r(t_m))(1 + o(1))$$

as  $u \rightarrow \infty$ , where

$$B = \frac{r'(t_m)}{(1 + r(t_m))^2}.$$

## 2 Lemmas

For a set  $A \subset \mathbb{R}$  and a number  $a$  we write  $aA = \{ax : x \in A\}$  and  $a + A = \{a + x : x \in A\}$ .

**Lemma 1** Let  $X(t)$  be a Gaussian process with mean zero and covariance function  $r(t)$  satisfying assumptions **A1**, **A2**. Let a time moment  $\tau = \tau(u)$  tends to  $t_m$  as  $u \rightarrow \infty$  in such a way that  $|\tau - t_m| \leq C\sqrt{\log u}/u$ , for some positive  $C$ . Let  $T_1$  and  $T_2$  be closures of two bounded open subsets of  $\mathbb{R}$ . Then

$$\begin{aligned} \mathbf{P} \left( \max_{t \in u^{-2/\alpha} T_1} X(t) > u, \max_{t \in \tau + u^{-2/\alpha} T_2} X(t) > u \right) &= \\ &= \frac{(1 + r(\theta))^2}{2\pi u^2 \sqrt{1 - r^2(\theta)}} e^{-\frac{u^2}{1+r(\theta)}} H_\alpha \left( \frac{T_1}{(1 + r(\theta))^{2/\alpha}} \right) H_\alpha \left( \frac{T_2}{(1 + r(\theta))^{2/\alpha}} \right) (1 + o(1)), \end{aligned} \quad (2)$$

as  $u \rightarrow \infty$ , where  $\theta = t_m$ .

**Lemma 2** Let  $X(t)$  be a Gaussian process with mean zero and covariance function  $r(t)$  satisfying assumptions **A1**, **A2** with  $\alpha < 1$ . Let  $T_1$  and  $T_2$  be closures of two bounded open subsets of  $\mathbb{R}$ . Then, for any (fixed)  $\tau > 0$  the asymptotic relation of Lemma 1 holds true with  $\theta = \tau$ .

**Lemma 3** Let  $X(t)$  be a Gaussian process with mean zero and covariance function  $r(t)$  satisfying assumptions **A1**, **A2** with  $\alpha = 1$ . Let  $T_1$  and  $T_2$  be closures of two bounded open subsets of  $\mathbb{R}$ . Then

$$\begin{aligned} \mathbf{P} \left( \max_{t \in u^{-2} T_1} X(t) > u, \max_{t \in \tau + u^{-2} T_2} X(t) > u \right) &= \\ &= H_1^{r'(\tau)} \left( \frac{T_1}{(1 + r(\tau))^2} \right) H_1^{-r'(\tau)} \left( \frac{T_2}{(1 + r(\tau))^2} \right) p_2(u, r(\tau))(1 + o(1)), \end{aligned} \quad (3)$$

as  $u \rightarrow \infty$ .

**Proof of Lemmas 1 - 3.** We prove the three lemmas simultaneously, computations of conditional expectation (4) and related evaluations are performed in parallel, separately for each lemma. We have for  $u > 0$ ,

$$\begin{aligned} \mathbf{P} &= \mathbf{P} \left( \max_{t \in u^{-2/\alpha} T_1} X(t) > u, \max_{t \in \tau + u^{-2/\alpha} T_2} X(t) > u \right) = \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{P} \left( \max_{t \in u^{-2/\alpha} T_1} X(t) > u, \max_{t \in \tau + u^{-2/\alpha} T_2} X(t) > u \mid X(0) = a, X(\tau) = b \right) \mathbf{P}_{0\tau}(a, b) da db, \end{aligned}$$

where

$$\mathbf{P}_{0\tau}(a, b) = \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \exp \left( -\frac{1}{2} \cdot \frac{a^2 - 2r(\tau)ab + b^2}{1-r^2(\tau)} \right).$$

Now we change variables,  $a = u - x/u$ ,  $b = u - y/u$ ,

$$\begin{aligned} \mathbf{P}_{0\tau}(x, y) &= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \times \\ &\quad \times \exp \left( -\frac{1}{2} \cdot \frac{(u-x/u)^2 - 2r(\tau)(u-x/u)(u-y/u) + (u-y/u)^2}{1-r^2(\tau)} \right) \\ &= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \exp \left( -\frac{u^2}{1+r(\tau)} \right) \times \\ &\quad \times \exp \left( -\frac{1}{2} \cdot \frac{\frac{x^2+y^2}{u^2} - 2x - 2y + 2r(\tau)(x+y) - 2r(\tau)\frac{xy}{u^2}}{1-r^2(\tau)} \right) \\ &= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \exp \left( -\frac{u^2}{1+r(\tau)} \right) \cdot \tilde{\mathbf{P}}(u, x, y). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{P} &= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \frac{1}{u^2} \exp \left( -\frac{u^2}{1+r(\tau)} \right) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{P} \left( \max_{t \in u^{-2/\alpha} T_1} X(t) > u, \right. \\ &\quad \left. \max_{t \in \tau + u^{-2/\alpha} T_2} X(t) > u \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{\mathbf{P}}(u, x, y) dx dy. \end{aligned}$$

Consider the following families of random processes,

$$\begin{aligned} \xi_u(t) &= u \left( X(u^{-2/\alpha} t) - u \right) + x, \quad t \in T_1, \\ \eta_u(t) &= u \left( X(\tau + u^{-2/\alpha} t) - u \right) + y, \quad t \in T_2. \end{aligned}$$

We have,

$$\begin{aligned} \mathbf{P} &= \frac{1}{2\pi\sqrt{1-r^2(\tau)}} \frac{1}{u^2} \exp \left( -\frac{u^2}{1+r(\tau)} \right) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{P} \left( \max_{t \in T_1} \xi_u(t) > x, \right. \\ &\quad \left. \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{\mathbf{P}}(u, x, y) dx dy. \end{aligned}$$

Compute first two conditional moments of Gaussian random vector process  $(\xi_u(t), \eta_u(t))^\top$ . We have

$$\mathbf{E} \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \Big| \begin{matrix} X(0) \\ X(\tau) \end{matrix} = \mathbf{E} \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} + \mathbf{A} \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix},$$

where

$$\mathbf{A} = \text{cov} \left( \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix}, \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix} \right) \left[ \mathbf{E} \left( \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix} \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix}^\top \right) \right]^{-1},$$

or

$$\mathbf{A} = \frac{u}{1-r^2(\tau)} \begin{pmatrix} r(u^{-2/\alpha t}) - r(\tau)r(\tau - u^{-2/\alpha t}) & r(\tau - u^{-2/\alpha t}) - r(\tau)r(u^{-2/\alpha t}) \\ r(\tau + u^{-2/\alpha t}) - r(\tau)r(u^{-2/\alpha t}) & r(u^{-2/\alpha t}) - r(\tau)r(\tau + u^{-2/\alpha t}) \end{pmatrix}.$$

We denote  $\text{cov}X$ , the matrix of covariances of a vector  $X$  and  $\text{cov}(X, Y)$ , the matrix of cross-covariances between components of  $X$  and  $Y$ . Substituting the values  $X(0) = u - x/u$ ,  $X(\tau) = u - y/u$ , of the conditions, we get from here that

$$\mathbf{E} \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} = \begin{pmatrix} \frac{1}{1-r^2(\tau)} (r(u^{-2/\alpha t}) (u^2 - x - r(\tau)(u^2 - y)) + r(\tau - u^{-2/\alpha t}) (u^2 - y - r(\tau)(u^2 - x))) - u^2 + x \\ \frac{1}{1-r^2(\tau)} (r(u^{-2/\alpha t}) (u^2 - y - r(\tau)(u^2 - x)) + r(\tau + u^{-2/\alpha t}) (u^2 - x - r(\tau)(u^2 - y))) - u^2 + y \end{pmatrix}. \quad (4)$$

In conditions of every lemma 1-3 we have

$$\mathbf{E} \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} = \begin{pmatrix} -\frac{1}{1+r(\tau)} |t|^\alpha + o(1) + u^2 \frac{r(\tau - u^{-2/\alpha t}) - r(\tau)}{1+r(\tau)} + (y - xr(\tau)) \frac{r(\tau) - r(\tau - u^{-2/\alpha t})}{1-r^2(\tau)} \\ -\frac{1}{1+r(\tau)} |t|^\alpha + o(1) + u^2 \frac{r(\tau + u^{-2/\alpha t}) - r(\tau)}{1+r(\tau)} + (y - xr(\tau)) \frac{r(\tau) - r(\tau + u^{-2/\alpha t})}{1-r^2(\tau)} \end{pmatrix} \quad (5)$$

as  $u \rightarrow \infty$ .

Now, let conditions of the Lemma 1 be fulfilled. Since  $\alpha < 2$  and  $r'(\tau) = O(\sqrt{\log u}/u)$  uniformly in  $|\tau - t_m| \leq C\sqrt{\log u}/u$ , we have,

$$\left| u^2 \frac{r(\tau - u^{-2/\alpha t}) - r(\tau)}{1+r(\tau)} \right| \leq \max_{|\tau - t_m| \leq C\sqrt{\log u}/u} \left| u^2 (-u^{-2/\alpha t}) \frac{r'(\tau)}{1+r(\tau)} \right| = o(1). \quad (6)$$

Thus

$$\mathbf{E} \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} = \begin{pmatrix} -\frac{1}{1+r(t_m)} |t|^\alpha + o(1) \\ -\frac{1}{1+r(t_m)} |t|^\alpha + o(1) \end{pmatrix} \quad (7)$$

as  $u \rightarrow \infty$ .

Let now the conditions of Lemma 2 be fulfilled, that is  $\alpha < 1$ . In this situation even for fixed  $\tau$ , by Taylor, the third terms in the column array of right-hand part of (5) tend to zero as  $u \rightarrow \infty$ , hence (7) takes place, with  $\theta = \tau$ .

Next, let  $\alpha = 1$ , by differentiability of  $r$ ,

$$u^2(r(\tau - u^{-2}t) - r(\tau)) \rightarrow -tr'(\tau) \quad \text{and} \quad u^2(r(\tau + u^{-2}t) - r(\tau)) \rightarrow tr'(\tau)$$

as  $u \rightarrow \infty$ , therefore in conditions of Lemma 3,

$$\mathbf{E} \begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} = \begin{pmatrix} -\frac{|t|+tr'(\tau)}{1+r(\tau)} + o(1) \\ -\frac{|t|-tr'(\tau)}{1+r(\tau)} + o(1) \end{pmatrix} \quad (8)$$

It is clear that

$$\begin{aligned} \mathbf{E} \begin{pmatrix} \xi_u(0) \\ \eta_u(0) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \mathbf{E} \begin{pmatrix} \xi_u^2(0) \\ \eta_u^2(0) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (9)$$

Computing conditional covariance matrix, we have,

$$\mathbf{cov} \left( \begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix} \Big| \begin{matrix} X(0) \\ X(\tau) \end{matrix} \right) = \mathbf{cov} \begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix} - \mathbf{B} \mathbf{cov} \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix} \mathbf{B}^\top,$$

where

$$\mathbf{B} = \mathbf{cov} \left( \begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix}, \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix} \right) \left[ \mathbf{E} \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix} \begin{pmatrix} X(0) \\ X(\tau) \end{pmatrix}^\top \right]^{-1}.$$

Using expressions for  $\xi_u(t)$  and  $\eta_u(t)$ ,

$$\mathbf{B} = \frac{u}{1-r^2(\tau)} \begin{pmatrix} r(u^{-2/\alpha t}) - r(\tau)r(\tau - u^{-2/\alpha t}) - & r(\tau - u^{-2/\alpha t}) - r(\tau)r(u^{-2/\alpha t}) - \\ -r(u^{-2/\alpha s}) + r(\tau)r(\tau - u^{-2/\alpha s}) & -r(\tau - u^{-2/\alpha s}) + r(\tau)r(u^{-2/\alpha s}) \\ r(\tau + u^{-2/\alpha t}) - r(\tau)r(u^{-2/\alpha t}) - & r(u^{-2/\alpha t}) - r(\tau)r(\tau + u^{-2/\alpha t}) - \\ -r(\tau + u^{-2/\alpha s}) + r(\tau)r(u^{-2/\alpha s}) & r(u^{-2/\alpha s}) + r(\tau)r(\tau + u^{-2/\alpha s}) \end{pmatrix}.$$

Letting now  $u \rightarrow \infty$ , we get

$$\mathbf{cov} \begin{pmatrix} \xi_u(t) - \xi_u(s) \\ \eta_u(t) - \eta_u(s) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix} = \begin{pmatrix} 2|t-s|^\alpha(1+o(1)) & o(1) \\ o(1) & 2|t-s|^\alpha(1+o(1)) \end{pmatrix}, \quad (10)$$

where  $o(1)$ s are uniform of  $x$  and  $y$ , moreover they do not depends of values of conditions  $X(0)$  and  $X(\tau)$ . Note that (10) holds true for all  $\alpha \in (0, 2)$ . From (10) it also followed that for some  $C > 0$  all  $t, s$  and all sufficiently large  $u$ ,

$$\mathbf{var} (\xi_u(t) - \xi_u(s) | (X(0), X(\tau)) = (u - x/u, u - y/u)) \leq C|t - s|^\alpha, \quad (11)$$

$$\mathbf{var} (\eta_u(t) - \eta_u(s) | (X(0), X(\tau)) = (u - x/u, u - y/u)) \leq C|t - s|^\alpha. \quad (12)$$

Thus from (7-11) it follows that the family of conditional Gaussian distributions

$$\mathbf{P} \begin{pmatrix} \xi_u(\cdot) \\ \eta_u(\cdot) \end{pmatrix} \Big| \begin{matrix} X(0) = u - x/u \\ X(\tau) = u - y/u \end{matrix}, \quad (13)$$

is weakly compact in  $C(T_1) \times C(T_2)$  and converges weakly, under conditions of Lemmas 1 and 2, to the distribution of the random vector process

$$(\xi(t), \eta(t))^\top = (B_\alpha(t) - |t|^\alpha/(1+r(\tau)), \bar{B}_\alpha(t) - |t|^\alpha/(1+r(\tau)))^\top,$$



$t \in \mathbb{R}$ , where  $\tilde{B}$  is an independent copy of  $B$ . If the conditions of Lemma 3 are fulfilled, the family of Gaussian conditional distributions converges to the distribution of

$$(\xi(t), \eta(t))^\top = (B_1(t) - (|t| + tr'(\tau))/(1 + r(\tau)), \tilde{B}_1(t) - (|t| - tr'(\tau))/(1 + r(\tau)))^\top.$$

Thus

$$\begin{aligned} & \lim_{u \rightarrow \infty} \mathbf{P} \left( \max_{t \in T_1} \xi_u(t) > x, \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \\ &= \mathbf{P} \left( \max_{t \in T_1} \xi(t) > x, \max_{t \in T_2} \eta(t) > y \right). \end{aligned}$$

In order to prove a convergence of the integral

$$\begin{aligned} \mathbf{I}(T_1, T_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{P} \left( \max_{t \in T_1} \xi_u(t) > x, \right. \\ & \quad \left. \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{\mathbf{P}}(u, x, y) dx dy \end{aligned}$$

as  $u \rightarrow \infty$ , we construct an integrable dominating function, which have different representation in different quadrants of the plane.

1. For the quadrant  $(x < 0, y < 0)$  we bound the probability by 1, and the  $\tilde{\mathbf{P}}(u, x, y)$  by  $\exp(\frac{x+y}{1+r(t_m)})$ , using relations  $|r(t)| \leq 1$  and  $x^2 + y^2 \geq 2xy$ . The last function is integrable in the considered quadrant, so it is a desirable dominating function.

2. Within the quadrant  $(x > 0, y < 0)$  we bound the probability by

$$\mathbf{P} \left( \max_{t \in T_1} \xi_u(t) > x, \mid X(0) = u - x/u, X(\tau) = u - y/u \right)$$

and, using arguments similar the above, we bound  $\tilde{\mathbf{P}}(u, x, y)$  by

$$\exp \left( \frac{y}{1 + r(t_m)} + \frac{x}{0.9 + r(t_m)} \right),$$

for sufficiently large  $u$ . The function  $p(x)$  can be bounded by a function of type  $C \exp(-\epsilon x^2)$ ,  $\epsilon$  is positive, using, for example the Borel inequality with relations (7 - 10). Similar arguments one can find in [2].

3. Considerations in the quarter-plane  $(x < 0, y > 0)$  are similar, the dominating function is

$$C \exp(-\epsilon y^2) \exp \left( \frac{x}{1 + r(t_m)} + \frac{y}{0.9 + r(t_m)} \right).$$

4. In the quarter-plane  $(x > 0, y > 0)$  we bound  $\tilde{\mathbf{P}}$  by

$$\exp \left( \frac{x}{0.9 + r(t_m)} + \frac{y}{0.9 + r(t_m)} \right)$$

and the probability by

$$\mathbf{P} \left( \max_{(t,s) \in T_1 \times T_2} \xi_u(t) + \eta_u(s) > x + y \mid X(0) = u - x/u, X(\tau) = u - y/u \right).$$

Again, for the probability we can apply the Borel inequality, just in the same way, to get the bound  $C \exp(-\epsilon(x+y)^2)$ , for a positive  $\epsilon$ .

Thus we have the desirable domination on the hole plane and therefore we have,

$$\begin{aligned} & \lim_{u \rightarrow \infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{P} \left( \max_{t \in T_1} \xi_u(t) > x, \right. \\ & \quad \left. \max_{t \in T_2} \eta_u(t) > y \mid X(0) = u - x/u, X(\tau) = u - y/u \right) \tilde{\mathbf{P}}(u, x, y) dx dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\frac{x+y}{1+r(tm)}} \mathbf{P} \left( \max_{t \in T_1} \xi(t) > x, \max_{t \in T_2} \eta(t) > y \right) dx dy \\ &= \int_{-\infty}^{+\infty} e^{\frac{x}{1+r(tm)}} \mathbf{P} \left( \max_{t \in T_1} \xi(t) > x \right) dx \int_{-\infty}^{+\infty} e^{\frac{y}{1+r(\tau)}} \mathbf{P} \left( \max_{t \in T_2} \eta(t) > y \right) dy. \end{aligned}$$

Then we proceed,

$$\begin{aligned} & \int_{-\infty}^{+\infty} e^{\frac{x}{1+r(\theta)}} \mathbf{P} \left( \max_{t \in T_1} \xi(t) > x \right) dx = \\ &= (1+r(\theta)) \mathbf{E} \exp \left[ \frac{\max_{T_1} \xi(t)}{1+r(\theta)} \right] = (1+r(\theta)) \mathbf{E} \exp \left[ \frac{\max_{T_1} B_\alpha(t) - \frac{|t|^\alpha}{1+r(\theta)}}{1+r(\theta)} \right] = \\ &= (1+r(\theta)) \mathbf{E} \exp \left[ \max_{T_1} B_\alpha \left( \frac{t}{(1+r(\theta))^{2/\alpha}} \right) - \left( \frac{t}{(1+r(\theta))^{2/\alpha}} \right)^\alpha \right] = \\ &= (1+r(\theta)) \mathbf{E} \exp \left[ \max_{T_1/(1+r(\theta))^{2/\alpha}} B_\alpha(s) - s^\alpha \right] = (1+r(\theta)) H_\alpha \left( \frac{T_1}{(1+r(\theta))^{2/\alpha}} \right), \end{aligned}$$

where we use self-similarity properties of Fractional Brownian Motion. Similarly for  $\eta(t)$ ,  $t \in T_2$ . Similarly for  $H_1^{\pm r'(\tau)}$ . Thus Lemmas follow.

The following lemma is proved in [2] in multidimensional case. We formulate it here for one-dimensional time.

**Lemma 4** *Suppose that  $X(t)$  is a Gaussian stationary zero mean process with covariance function  $r(t)$  satisfying assumption **A1**. Let  $\epsilon, \frac{1}{2} > \epsilon > 0$  be such that*

$$1 - \frac{1}{2}|t|^\alpha \geq r(t) \geq 1 - 2|t|^\alpha$$

for all  $t \in [0, \epsilon]$ . Then there exists an absolute constant  $F$  such that the inequality

$$\mathbf{P} \left( \max_{t \in [0, Tu^{-2/\alpha}]} X(t) > u, \max_{t \in [t_0 u^{-2/\alpha}, (t_0+T)u^{-2/\alpha}]} X(t) > u \right) \leq FT^2 u^{-1} e^{-\frac{1}{2}u^2 - \frac{1}{8}(t_0-T)^\alpha}$$

holds for any  $T$ ,  $t_0 > T$  and for any  $u \geq (4(T + t_0)/\varepsilon)^{\alpha/2}$ .

The following two lemmas are straightforward consequences of Lemma 6.1, [2].

**Lemma 5** *Suppose that  $X(t)$  is a Gaussian stationary zero mean process with covariance function  $r(t)$  satisfying assumption **A1**. Then*

$$\mathbf{P} \left( \max_{t \in [0, Tu^{-2/\alpha}] \cup [t_0 u^{-2/\alpha}, (t_0 + T) u^{-2/\alpha}]} X(t) > u \right) = H_\alpha([0, T] \cup [t_0, t_0 + T]) \frac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2} (1 + o(1))$$

as  $u \rightarrow \infty$ , where

$$H_\alpha([0, T] \cup [t_0, t_0 + T]) = \mathbf{E} \exp \left( \max_{t \in [0, T] \cup [t_0, t_0 + T]} (B_\alpha(t) - |t|^\alpha) \right).$$

**Lemma 6** *Suppose that  $X(t)$  is a Gaussian stationary zero mean process with covariance function  $r(t)$  satisfying assumption **A1**. Then*

$$\begin{aligned} \mathbf{P} \left( \max_{t \in [0, Tu^{-2/\alpha}]} X(t) > u, \max_{t \in [t_0 u^{-2/\alpha}, (t_0 + T) u^{-2/\alpha}]} X(t) > u \right) \\ = H_\alpha([0, T], [t_0, t_0 + T]) \frac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2} (1 + o(1)) \end{aligned}$$

as  $u \rightarrow \infty$ , where

$$H_\alpha([0, T], [t_0, t_0 + T]) = \int_{-\infty}^{\infty} e^s \mathbf{P} \left( \max_{t \in [0, T]} B_\alpha(t) - |t|^\alpha > s, \max_{t \in [t_0, t_0 + T]} B_\alpha(t) - |t|^\alpha > s \right) ds.$$

**Proof.** Write

$$\begin{aligned} & \mathbf{P} \left( \max_{t \in [0, Tu^{-2/\alpha}]} X(t) > u, \max_{t \in [t_0 u^{-2/\alpha}, (t_0 + T) u^{-2/\alpha}]} X(t) > u \right) \\ &= \mathbf{P} \left( \max_{t \in [0, Tu^{-2/\alpha}]} X(t) > u \right) + \mathbf{P} \left( \max_{t \in [t_0 u^{-2/\alpha}, (t_0 + T) u^{-2/\alpha}]} X(t) > u \right) \\ & \quad - \mathbf{P} \left( \max_{t \in [0, Tu^{-2/\alpha}] \cup [t_0 u^{-2/\alpha}, (t_0 + T) u^{-2/\alpha}]} X(t) > u \right) \end{aligned}$$

and apply Lemma 6.1, [2] and Lemma 3 to the right-hand part.

From Lemmas 4 and 2 we get,

**Lemma 7** *For any  $t_0 > T$ ,*

$$H_\alpha([0, T], [t_0, t_0 + T]) \leq F\sqrt{2\pi}T^2 e^{-\frac{1}{8}(t_0 - T)^\alpha}.$$

When  $t_0 = T$  the Lemma holds true, but the bound is trivial. A non-trivial bound for  $H_\alpha([0, T], [T, 2T])$  one can get from the proof of Lemma 7.1, [2], see page 107, inequalities (7.5) and the previous one. These inequalities, Lemma 6.8, [2] and Lemma 5 give the following,

**Lemma 8** *There exists a constant  $F_1$  such that for all  $T \geq 1$ ,*

$$H_\alpha([0, T], [T, 2T]) \leq F_1 \left( \sqrt{T} + T^2 e^{-\frac{1}{8}T^{\alpha/2}} \right).$$

Applying Lemma 1 to the sets  $T_1 = [0, T] \cup [t_0, t_0 + T]$ ,  $T_2 = [0, T] \cup [t_1, t_1 + T]$  and combining probabilities similarly as in the proof of Lemma 4, we get,

**Lemma 9** *Let  $X(t)$  be a Gaussian process with mean zero and covariance function  $r(t)$  satisfying conditions of Theorem 1. Let  $\tau = \tau(u)$  tends to  $t_m$  as  $u \rightarrow \infty$  in such a way that  $|\tau - t_m| \leq C\sqrt{\log u}/u$ , for some positive  $C$ . Then for all  $T > 0$ ,  $t_0 \geq T$ ,  $t_1 \geq T$*

$$\begin{aligned} & \mathbf{P} \left( \max_{t \in [0, u^{-2/\alpha}T]} X(t) > u, \max_{t \in [u^{-2/\alpha}t_0, u^{-2/\alpha}(t_0+T)]} X(t) > u, \right. \\ & \quad \left. \max_{t \in [\tau, \tau+u^{-2/\alpha}T]} X(t) > u, \max_{t \in [\tau+u^{-2/\alpha}t_1, \tau+u^{-2/\alpha}(t_1+T)]} X(t) > u \right) \\ &= \frac{(1+r(t_m))^2}{2\pi\sqrt{1-r^2(t_m)}} \cdot \frac{1}{u^2} e^{-\frac{u^2}{1+r(\tau)}} \\ & \times H_\alpha \left( \left[ 0, \frac{T}{(1+r(t_m))^{2/\alpha}} \right], \left[ \frac{t_0}{(1+r(t_m))^{2/\alpha}}, \frac{t_0+T}{(1+r(t_m))^{2/\alpha}} \right] \right) \\ & \times H_\alpha \left( \left[ 0, \frac{T}{(1+r(t_m))^{2/\alpha}} \right], \left[ \frac{t_1}{(1+r(t_m))^{2/\alpha}}, \frac{t_1+T}{(1+r(t_m))^{2/\alpha}} \right] \right) (1+o(1)), \end{aligned}$$

as  $u \rightarrow \infty$ .

### 3 Proofs

#### 3.1 Proof of Theorem 1

We denote  $\Pi = [T_1, T_2] \times [T_3, T_4]$ ,  $\delta = \delta(u) = C\sqrt{\log u}/u$ , the value of the positive  $C$  we specify later on.  $D = \{(t, s) \in \Pi : |t - s - t_m| \leq \delta\}$ . We have,

$$\begin{aligned} & \mathbf{P} \left( \max_{t \in [T_1, T_2]} X(t) > u, \max_{t \in [T_3, T_4]} X(t) > u \right) = \mathbf{P} \left( \bigcup_{(s,t) \in \Pi} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ &= \mathbf{P} \left( \left\{ \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right\} \cup \left\{ \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u\} \cap \{X(s) > u\} \right\} \right) \\ &\leq \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) + \mathbf{P} \left( \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u\} \cap \{X(s) > u\} \right). \quad (14) \end{aligned}$$

From the other hand,

$$\begin{aligned}
\mathbf{P} \left( \max_{t \in [T_1, T_2]} X(t) > u, \max_{t \in [T_3, T_4]} X(t) > u \right) &= \mathbf{P} \left( \bigcup_{(s,t) \in \Pi} \{X(t) > u\} \cap \{X(s) > u\} \right) \\
&= \mathbf{P} \left( \left\{ \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right\} \cup \left\{ \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u\} \cap \{X(s) > u\} \right\} \right) \\
&\geq \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right). \tag{15}
\end{aligned}$$

The second term in the right-hand part of (14) we estimate as following,

$$\mathbf{P} \left( \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u\} \cap \{X(s) > u\} \right) \leq \mathbf{P} \left( \max_{(s,t) \in \Pi \setminus D} X(t) + X(s) > 2u \right). \tag{16}$$

Making use of Theorem 8.1, [2], we get that the last probability does not exceed

$$const \cdot u^{-1+2/\alpha} \exp \left( -\frac{u^2}{1 + \max_{(t,s) \in \Pi \setminus D} r(t-s)} \right). \tag{17}$$

Further, for  $\epsilon = 1/6$  and all sufficiently large  $u$ ,

$$\max_{(t,s) \in \Pi \setminus D} r(t-s) \leq r(t_m) + \left(\frac{1}{2} - \epsilon\right) r''(t_m) \delta^2 = r(t_m) + \frac{1}{3} C^2 r''(t_m) \log u/u.$$

Hence,

$$\mathbf{P} \left( \bigcup_{(s,t) \in \Pi \setminus D} \{X(t) > u\} \cap \{X(s) > u\} \right) \leq const \cdot u^{-1+2/\alpha} \exp \left( -\frac{u^2}{1 + r(t_m)} \right) u^{-G}, \tag{18}$$

where

$$G = \frac{-2C^2 r''(t_m)}{3(1 + r(t_m))^2}.$$

Now we deal with the first probability in the right-hand part of (14). It is equal to the probability in right-hand part of (15). We are hence in a position to bound the probability from above and from below getting equal orders for the bounds. Denote  $\Delta = Tu^{-2/\alpha}$ ,  $T > 0$ , and define the intervals

$$\begin{aligned}
\Delta_k &= [T_1 + k\Delta, T_1 + (k+1)\Delta], \quad 0 \leq k \leq N_k, \quad N_k = [(T_2 - T_1)/\Delta], \\
\Delta_l &= [T_3 + l\Delta, T_3 + (l+1)\Delta], \quad 0 \leq l \leq N_l, \quad N_l = [(T_4 - T_3)/\Delta],
\end{aligned}$$

where  $[\cdot]$  stands for the integer part of a number. In virtue of Lemma 1,

$$\begin{aligned}
& \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \\
& \leq \mathbf{P} \left( \bigcup_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \bigcup_{t \in \Delta_k, s \in \Delta_l} \{X(t) > u\} \cap \{X(s) > u\} \right) \\
& \leq \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \\
& \leq \frac{(1 + \gamma(u))}{2\pi u^2 \sqrt{1 - r^2(t_m)}} H_\alpha^2 \left( \frac{T}{(1 + r(t_m))^{2/\alpha}} \right) \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{u^2}{1 + r(\tau_{k,l})} \right), \quad (19)
\end{aligned}$$

where  $\gamma(u) \downarrow 0$  as  $u \rightarrow \infty$  and  $\tau_{k,l} = T_3 - T_1 + (l - k)\Delta$ . For the last sum we get,

$$\begin{aligned}
S &= \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{u^2}{1 + r(\tau_{k,l})} \right) \\
&= \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -u^2 \frac{r(t_m) - r(\tau_{k,l})}{(1 + r(\tau_{k,l}))(1 + r(t_m))} \right).
\end{aligned}$$

Define  $\theta$  by  $t_m = T_3 - T_1 + \Delta\theta$ , we obtain,

$$\begin{aligned}
\frac{r(t_m) - r(\tau_{k,l})}{(1 + r(\tau_{k,l}))(1 + r(t_m))} &\leq (\geq) \frac{-\frac{1}{2}r''(t_m)(\tau_{k,l} - t_m)^2}{(1 + r(t_m))^2} (1 + (-)\gamma_1(u)) \\
&= -A((k - l)\Delta - \theta\Delta)^2 (1 + (-)\gamma_1(u)),
\end{aligned}$$

where  $\gamma_1(u) \downarrow 0$  as  $u \rightarrow \infty$ . In the last sum, index  $k$  variates between  $(T_{\min} + O(\delta(u)))/\Delta$  and  $(T_{\max} + O(\delta(u)))/\Delta$ , as  $u \rightarrow \infty$ , where  $T_{\min} = T_1 \vee (T_3 - t_m)$  and  $T_{\max} = T_2 \wedge (T_4 - t_m)$ . Indeed, for the co-ordinate  $x$  of the left end of a segment of length  $t_m$  which variates having left end inside  $[T_1, T_2]$  and right end inside  $[T_3, T_4]$ , we have the restrictions  $T_1 < x < T_2$ , and  $T_3 < x + t_m < T_4$ , so that  $x \in (T_{\min}, T_{\max})$ . The index  $m = k - l - \theta$  variates thus between  $-\delta(u)/\Delta + O(\Delta)$  and  $\delta(u)/\Delta + O(\Delta)$  as  $u \rightarrow \infty$ . Note that  $u\Delta \rightarrow 0$  as  $u \rightarrow \infty$ . Using this, we continue,

$$\begin{aligned}
S &= (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{T_{\max} - T_{\min}}{\Delta} \sum_{m = -\delta(u)/\Delta + O(\Delta)}^{\delta(u)/\Delta + O(\Delta)} \exp(-A(mu\Delta)^2) \\
&= (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{T_{\max} - T_{\min}}{u\Delta^2} \sum_{mu\Delta = -u\delta(u) + O(u\Delta^2)}^{u\delta(u) + O(u\Delta^2)} \exp(-A(mu\Delta)^2) u\Delta \\
&= (1 + o(1)) \exp \left( -\frac{u^2}{1 + r(t_m)} \right) \frac{T_{\max} - T_{\min}}{u\Delta^2} \int_{-\infty}^{\infty} e^{-Ax^2} dx.
\end{aligned}$$

Compute the integral and substitute this in right-hand part of (19), we get,

$$\begin{aligned} & \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \leq \frac{(1+r(t_m))^2(1+\gamma_2(u))(T_{\max} - T_{\min})u^{-3+4/\alpha}}{2\sqrt{A\pi}(1-r^2(t_m))} \frac{1}{T^2} H_\alpha^2 \left( \frac{T}{(1+r(t_m))^{2/\alpha}} \right) \exp \left( -\frac{u^2}{1+r(t_m)} \right), \end{aligned} \quad (20)$$

where  $\gamma_2(u) \downarrow 0$  as  $u \rightarrow \infty$ .

Now we bound from below the probability in the right-hand part of (15). We have

$$\begin{aligned} & \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \geq \mathbf{P} \left( \bigcup_{(k,l): \Delta_k \subset D, \Delta_l \subset D} \bigcup_{t \in \Delta_k, s \in \Delta_l} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \geq \sum_{(k,l): \Delta_k \subset D, \Delta_l \subset D} \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \\ & \quad - \sum \sum \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u, \max_{t \in \Delta_{k'}} X(t) > u, \max_{t \in \Delta_{l'}} X(t) > u \right), \end{aligned} \quad (21)$$

where the double-sum is taken over the set

$$\{(k, l, k', l') : (k', l') \neq (k, l), \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset, \Delta_{k'} \cap D \neq \emptyset, \Delta_{l'} \cap D \neq \emptyset\}.$$

The first sum in the right-hand part of (21) can be bounded from below exactly by the same way as the previous sum, thus we have,

$$\begin{aligned} & \sum_{(k,l): \Delta_k \subset D, \Delta_l \subset D} \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \\ & \geq \frac{(1+r(t_m))^2(1-\gamma_2(u))(T_{\max} - T_{\min})u^{-3+4/\alpha}}{2\sqrt{A\pi}(1-r^2(t_m))} \frac{1}{T^2} H_\alpha^2 \left( \frac{T}{(1+r(t_m))^{2/\alpha}} \right) \exp \left( -\frac{u^2}{1+r(t_m)} \right), \end{aligned} \quad (22)$$

where  $\gamma_2(u) \downarrow 0$  as  $u \rightarrow \infty$ . We are now able to select the constant  $C$ . We take it as large as  $G > 2 - 2/\alpha$  to get that left-hand part of (18) is infinitely smaller than left-hand part of (22) as  $u \rightarrow \infty$ .

Consider the second sum (the double-sum) in the right-hand part of (21). For sakes of simplicity we denote

$$H(m) = H_\alpha \left( \left[ 0, \frac{T}{(1+r(t_m))^{2/\alpha}} \right], \left[ \frac{mT}{(1+r(t_m))^{2/\alpha}}, \frac{(m+1)T}{(1+r(t_m))^{2/\alpha}} \right] \right)$$

and notice that

$$H(0) = H_\alpha \left( \left[ 0, \frac{T}{(1+r(t_m))^{2/\alpha}} \right] \right).$$

In virtue of Lemma 9 we have for the double-sum in (21), taking into account only different  $(k, l)$  and  $(k', l')$ ,

$$\begin{aligned}
\Sigma_2 &:= \sum \sum \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u, \max_{t \in \Delta_{k'}} X(t) > u, \max_{t \in \Delta_{l'}} X(t) > u \right) \\
&\leq \frac{(1+r(t_m))^2(1+\Gamma(u))}{2\pi u^2 \sqrt{1-r^2(t_m)}} \sum \sum H(|k-k'|)H(|l-l'|) \exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right) \\
&= \frac{2(1+r(t_m))^2(1+\Gamma(u))}{2\pi u^2 \sqrt{1-r^2(t_m)}} \sum_{n=1}^{\infty} H(n) \left( H(0) + 2 \sum_{m=1}^{\infty} H(m) \right) \\
&\quad \times \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp\left(-\frac{u^2}{1+r(\tau_{k,l})}\right),
\end{aligned}$$

where  $\Gamma(u) \downarrow 0$  as  $u \rightarrow \infty$ . The last sum is already bounded from above, therefore by (19) and (20) we have,

$$\begin{aligned}
\Sigma_2 &\leq \frac{2}{T^2} \sum_{n=1}^{\infty} H(n) \left( H(0) + 2 \sum_{m=1}^{\infty} H(m) \right) \\
&\quad \times \frac{(1+r(t_m))^2(1+\Gamma_2(u))(T_{\max} - T_{\min})u^{-3+4/\alpha}}{2\sqrt{A\pi}(1-r^2(t_m))} \exp\left(-\frac{u^2}{1+r(t_m)}\right).
\end{aligned}$$

By Lemmas 6.8, [2], 7 and 8 we get that  $H(0) \leq \text{const} \cdot T$ ,  $H(1) \leq \text{const} \cdot \sqrt{T}$  and for  $m > 1$ ,

$$H(m) \leq \text{const} \cdot e^{-\frac{1}{8}m^{\alpha/2}T^{\alpha/2}},$$

hence

$$\sum_{n=1}^{\infty} H(n) \left( H(0) + 2 \sum_{m=1}^{\infty} H(m) \right) \leq \text{const} \cdot T^{3/2}.$$

Thus

$$\Sigma_2 \leq \text{const} \cdot T^{-1/2}u^{-3+4/\alpha} \exp\left(-\frac{u^2}{1+r(t_m)}\right). \quad (23)$$

Now since by (1),

$$\lim_{T \rightarrow \infty} \frac{1}{T} H_{\alpha} \left( \frac{T}{(1+r(t_m))^{2/\alpha}} \right) = (1+r(t_m))^{-2/\alpha} H_{\alpha},$$

we get that the double sum can be made infinitely smaller by choosing large  $T$ . Thus Theorem 1 follows.

### 3.2 Proof of Theorem 2.

We prove the theorem for the case  $t_m = T_3 - T_2$ , another case can be considered similarly. First, as in the proof of Theorem 1 put  $D = \{(t, s) \in \Pi : |t - s - t_m| \leq \delta\}$ , but with



$\delta = \delta(u) = C\sqrt{\log u/u^2}$ , for sufficiently large  $C$ . The evaluations (14), (16) and (17) still hold true. Further we have for  $\epsilon = 1/6$  and all sufficiently large  $u$ ,

$$\max_{(t,s) \in \Pi \setminus D} r(t-s) \leq r(t_m) + \left(\frac{1}{2} - \epsilon\right)r'(t_m)\delta = r(t_m) + \frac{1}{3}C^2r'(t_m)\log u/u^2.$$

Hence, (18) holds true with

$$G = \frac{-2C^2r'(t_m)}{3(1+r(t_m))^2}.$$

Let now  $\alpha > 1$ . For any positive arbitrarily small  $\epsilon$  we have for all sufficiently large  $u$  that,  $\epsilon u^{-2/\alpha} > \delta(u)$ , hence for such values of  $u$ ,

$$\begin{aligned} & \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \leq \mathbf{P} \left( \max_{t \in [T_2 - \epsilon u^{-2/\alpha}, T_2]} X(t) > u, \max_{t \in [T_3, T_3 + \epsilon u^{-2/\alpha}]} X(t) > u \right). \end{aligned} \quad (24)$$

We wish to apply Lemma 1 to the last probability for the intervals  $[-\epsilon, 0]$  and  $[t_m, t_m + \epsilon]$ . To this end we turn to (5). Since for a sufficiently small  $\epsilon$ ,  $r'(t_m) < 0$ , we have that

$$\frac{r(\tau - u^{-2/\alpha}t) - r(\tau)}{1 + r(\tau)} < 0 \text{ for all } t \in [-\epsilon, 0]$$

and

$$\frac{r(\tau + u^{-2/\alpha}t) - r(\tau)}{1 + r(\tau)} < 0 \text{ for all } t \in [t_m, t_m + \epsilon],$$

hence

$$\limsup_{u \rightarrow \infty} \mathbf{E}(\xi_u(t) | X(0) = u - x/u, X(\tau) = u - y/u) \leq -\frac{1}{1 + r(t_m)}|t|^\alpha,$$

for all  $t \in [-\epsilon, 0]$ , and

$$\limsup_{u \rightarrow \infty} \mathbf{E}(\eta_u(t) | X(0) = u - x/u, X(\tau) = u - y/u) \leq -\frac{1}{1 + r(t_m)}|t|^\alpha,$$

for all  $t \in [t_m, t_m + \epsilon]$ . All other arguments in the proof of Lemma 1 still hold true, therefore, using time-symmetry of the fractional Brownian motion, we have,

$$\begin{aligned} & \limsup_{u \rightarrow \infty} u^2 e^{\frac{u^2}{1+r(t_m)}} \mathbf{P} \left( \max_{t \in [T_2 - \epsilon u^{-2/\alpha}, T_2]} X(t) > u, \max_{t \in [T_3, T_3 + \epsilon u^{-2/\alpha}]} X(t) > u \right) \\ & \leq \frac{(1 + r(t_m))^2}{2\pi\sqrt{1 - r^2(t_m)}} H_\alpha^2 \left( \frac{[0, \epsilon]}{(1 + r(t_m))^{2/\alpha}} \right) \end{aligned} \quad (25)$$

Using Fatou monotone convergence we have  $\lim_{\epsilon \downarrow 0} H_\alpha(\epsilon) = 1$ , therefore

$$\begin{aligned} & \limsup_{u \rightarrow \infty} u^2 e^{\frac{u^2}{1+r(t_m)}} \mathbf{P} \left( \max_{t \in [T_2 - \epsilon u^{-2/\alpha}, T_2]} X(t) > u, \max_{t \in [T_3, T_3 + \epsilon u^{-2/\alpha}]} X(t) > u \right) \\ & \leq \frac{(1 + r(t_m))^2}{2\pi\sqrt{1 - r^2(t_m)}} \end{aligned} \quad (26)$$

But

$$P_d(u; [T_1, T_2], [T_3, T_4]) \geq \mathbf{P}(X(T_2) > u, X(T_3) > u) = \frac{(1 + r(t_m))^2}{2\pi u^2 \sqrt{1 - r^2(t_m)}} e^{-\frac{u^2}{1+r(t_m)}} (1 + o(1))$$

as  $u \rightarrow \infty$ . Thus (i) follows.

Let now  $\alpha = 1$ . From now on, we redefine  $\Delta_k$  and  $\Delta_l$ , by

$$\begin{aligned} \Delta_k &= [T_2 - (k+1)\Delta, T_2 - k\Delta], \quad 0 \leq k \leq N_k, \quad N_k = \lfloor (T_2 - T_1)/\Delta \rfloor, \\ \Delta_l &= [T_3 + l\Delta, T_3 + (l+1)\Delta], \quad 0 \leq l \leq N_l, \quad N_l = \lfloor (T_4 - T_3)/\Delta \rfloor, \end{aligned}$$

for the case of  $\Delta_k$ ,  $k = 0$ , we denote  $\Delta_0 = \Delta_{-0}$ , indicating difference with  $\Delta_0$  for the case  $\Delta_l$ ,  $l = 0$ . Recall that now  $\Delta = Tu^{-2/\alpha} = Tu^{-2}$ . We have for sufficiently large  $u$ ,

$$\mathbf{P}\left(\bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\}\right) \geq \mathbf{P}\left(\max_{t \in \Delta_{-0}} X(t) > u, \max_{t \in \Delta_0} X(t) > u\right), \quad (27)$$

and

$$\begin{aligned} \mathbf{P}\left(\bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\}\right) &\leq \mathbf{P}\left(\max_{t \in \Delta_{-0}} X(t) > u, \max_{t \in \Delta_0} X(t) > u\right) + \\ &+ \sum_{k=0, l=0, k+l>0}^{\lfloor \log u/T \rfloor + 1} \mathbf{P}\left(\max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u\right). \end{aligned} \quad (28)$$

First probability in right-hand parts of the inequalities is already considered by Lemma 3. We set  $\tau = t_m = T_3 - T_2$ ,  $T_1 = [-T, 0]$ ,  $T_2 = [0, T]$ , by time-symmetry of Brownian motion, we have that

$$H_1^{r'(\tau)}([-T, 0]) = H_1^{-r'(\tau)}([0, T]). \quad (29)$$

In order to estimate the sum, we observe, that for all sufficiently large  $u$  and all  $t \in [T_3, T_3 + \delta(u)]$ ,  $s \in [T_2 - \delta(u), T_2]$ ,

$$r(t-s) \leq r(t_m) + \frac{1}{3}r'(t_m)(t-s-t_m) \quad \text{and} \quad r(t-s) \geq r(t_m) + \frac{2}{3}r'(t_m)(t-s-t_m). \quad (30)$$

Hence

$$\begin{aligned} \frac{-u^2}{1 + r(t_m + (k+l)\Delta)} &\leq \frac{-u^2}{1 + r(t_m) + \frac{1}{3}r'(t_m)(k+l)Tu^{-2}} \\ &\leq \frac{-u^2}{1 + r(t_m)} + \frac{r'(t_m)(k+l)T}{6(1 + r(t_m))^2} = \frac{-u^2}{1 + r(t_m)} - a(k+l)T, \end{aligned}$$

where  $a > 0$ . Now, in Lemma 3 let  $\tau = t_m + (k+l)\Delta$ ,  $T_1 = [-T, 0]$ ,  $T_2 = [0, T]$ , using the above mentioned property of the constants  $H_1^c(T)$ , we get, that for all sufficiently large  $u$  and  $T$ ,

$$\mathbf{P}\left(\max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u\right) \leq Cp_2(u, r(\tau_m))e^{-a(k+l)T},$$

From here we get,

$$\sum_{k=0, l=0, k+l>0}^{\lfloor \log u/T \rfloor + 1} \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \leq Cp_2(u, r(\tau_m)) e^{-a(k+l)T},$$

Applying now Lemma 3 to first summands in right-part hands of (27, 28) and letting  $T \rightarrow \infty$ , we get the assertion (ii) of Theorem.

Let now  $\alpha < 1$ . Proof of the Theorem in this case is similar to the proof of Theorem 1. We have to consider a sum of small almost equal probabilities and a double sum. Using the more recent definition of  $\Delta_k$  and  $\Delta_l$ , we have by Lemma 2,

$$\begin{aligned} & \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \leq \mathbf{P} \left( \bigcup_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \bigcup_{t \in \Delta_k, s \in \Delta_l} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \leq \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \mathbf{P} \left( \max_{t \in \Delta_k} X(t) > u, \max_{t \in \Delta_l} X(t) > u \right) \\ & \leq \frac{(1+r(t_m))^2(1+\gamma(u))}{2\pi u^2 \sqrt{1-r^2(t_m)}} H_\alpha^2 \left( \frac{T}{(1+r(t_m))^{2/\alpha}} \right) \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{u^2}{1+r(\tau_{k,l})} \right), \quad (31) \end{aligned}$$

where  $\gamma(u) \downarrow 0$  as  $u \rightarrow \infty$  and now  $\tau_{k,l} = T_3 - T_2 + (l+k)\Delta$ . For the last sum we get,

$$\begin{aligned} S &= \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -\frac{u^2}{1+r(\tau_{k,l})} \right) \\ &= \exp \left( -\frac{u^2}{1+r(t_m)} \right) \sum_{(k,l): \Delta_k \cap D \neq \emptyset, \Delta_l \cap D \neq \emptyset} \exp \left( -u^2 \frac{r(t_m) - r(\tau_{k,l})}{(1+r(\tau_{k,l}))(1+r(t_m))} \right). \end{aligned}$$

Next,

$$\begin{aligned} \frac{r(t_m) - r(\tau_{k,l})}{(1+r(\tau_{k,l}))(1+r(t_m))} &\leq (\geq) \frac{-r'(t_m)(t_m - \tau_{k,l})}{(1+r(t_m))^2} (1 + (-)\gamma_1(u)) \\ &= -B(k+l)\Delta(1 + (-)\gamma_1(u)), \end{aligned}$$

where  $\gamma_1(u) \downarrow 0$  as  $u \rightarrow \infty$ . Remind that now  $u^2\Delta \rightarrow 0$  as  $u \rightarrow \infty$ . Using this, and denoting  $m = k+l$ , we continue,

$$\begin{aligned} S &= (1 + o(1)) \exp \left( -\frac{u^2}{1+r(t_m)} \right) \sum_{m=0}^{\delta(u)/\Delta + O(\Delta)} m \exp(-Bu^2m\Delta) \\ &= (1 + o(1)) \exp \left( -\frac{u^2}{1+r(t_m)} \right) \frac{1}{(\Delta u^2)^2} \sum_{m=0}^{\delta(u)/\Delta + O(\Delta)} m \Delta u^2 \exp(-Bm\Delta u^2) (\Delta u^2) \end{aligned}$$

$$= (1 + o(1)) \exp\left(-\frac{u^2}{1 + r(t_m)}\right) \frac{1}{u^4 \Delta^2} \int_0^\infty x e^{-Bx} dx = (1 + o(1)) \exp\left(-\frac{u^2}{1 + r(t_m)}\right) \frac{1}{B^2 u^4 \Delta^2}.$$

Substitute this in right-hand part of (31), we get,

$$\begin{aligned} & \mathbf{P} \left( \bigcup_{(s,t) \in D} \{X(t) > u\} \cap \{X(s) > u\} \right) \\ & \leq \frac{(1 + r(t_m))^2 (1 + \gamma_2(u)) u^{-6+4/\alpha}}{2\pi B^2 \sqrt{(1 - r^2(t_m))}} \frac{1}{T^2} H_\alpha^2 \left( \frac{T}{(1 + r(t_m))^{2/\alpha}} \right) \exp\left(-\frac{u^2}{1 + r(t_m)}\right), \end{aligned} \quad (32)$$

where  $\gamma_2(u) \downarrow 0$  as  $u \rightarrow \infty$ .

Estimation the probability from below repeats the corresponding steps in the proof of Theorem 1, see (21) and followed. Thus Theorem 2 follows.

## References

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