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QUASIHOMOMORPHISMS FROM THE INTEGERS INTO HAMMING METRICS

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Abstract. A function $f : \mathbb{Z} \to \mathbb{Q}^n$ is a *c-quasihomomorphism* if the Hamming distance between $f(x+y)$ and $f(x)+f(y)$ is at most c for all $x, y \in \mathbb{Z}$. We show that any c-quasihomomorphism has distance at most some constant $C(c)$ to an actual group homomorphism; here $C(c)$ depends only on c and not on n or f. This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

1. Introduction

A c-quasihomomorphism from a group G to a group H with a left-invariant metric d is a map $f : G \to H$ such that $d(f(xy), f(x)f(y)) \leq c$ for all x, y in G. A central question in geometric group theory, first raised by Ulam in [Ula60, Chapter 6], is whether there exists an actual homomorphism $f' : G \to H$ such that $d(f(x), f'(x))$ is at most some constant C for all x. Different versions of this question are of interest: for instance, C may be allowed to depend on $c, G, (H, d)$ but not on f; or $G, (H, d)$ may be restricted to certain classes and or C is only allowed to depend on c.

A well-known example where the answer to this question is negative is the case where $G = H = \mathbb{Z}$ with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [A'C21], and the answer is yes only for those quasihomomorphisms that correspond to integers.

Much literature in this area focusses on *quasimorphisms*, which are quasihomomorphisms into the real numbers $\mathbb{R}$ with the standard metric; we refer to [Kot04] for a brief introduction. In another branch of the research on quasihomomorphisms H is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan's theorem on ε-representations of amenable groups [Kaz82]. For more recent results on quasihomomorphisms into nonabelian groups we refer to [FK16] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [KZ18].

Question 1.1. *Let $c \in \mathbb{N}$. Does there exist a constant $C = C(c)$ such that the following holds: For all $n \in \mathbb{N}$ and all functions $f : \mathbb{Z} \longrightarrow \mathbb{C}^{n\times n}$ such that*

$$\forall x, y \in \mathbb{Z} : \qquad \mathrm{rk}(f(x+y) - f(x) - f(y)) \leq c,$$

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there exists a matrix g such that

$$\forall x \in \mathbb{Z}: \qquad \mathrm{rk}(f(x) - x \cdot g) \leq C(c)?$$

Here, G equals $\mathbb{Z}$ and H equals $\mathbb{C}^{n\times n}$, both with addition, and the metric on H is defined by $d(A, B) := \mathrm{rk}(A-B)$. Our main result is an affirmative answer to this question in the special case where all matrices $f(x)$ are assumed to be *diagonal*.

Definition 1.2. Let $(Q, +)$ be an abelian group. For an element $v \in Q^n$, the *Hamming weight* $w_H(v)$ is the number of nonzero entries of v. For a pair of elements $u, v \in Q^n$, their *Hamming distance* is $w_H(v-u)$. This metric is clearly left-invariant.

Definition 1.3. Let A be another abelian group. A function $f : A \to Q^n$ is called a *c-quasihomomorphism* if

$$\forall x, y \in A: \qquad w_H(f(x+y) - f(x) - f(y)) \leq c.$$

Remark 1.4. The map diag : $\mathbb{C}^n \to \mathbb{C}^{n\times n}$ is an isometric embedding from $\mathbb{C}^n$ with the Hamming metric to $\mathbb{C}^{n\times n}$ with the rank metric. This connects Definition 1.3 to Question 1.1. ♢

Definition 1.5. Let $C \in \mathbb{N}$ and let $f : A \to Q^n$ be a c-quasihomomorphism. A group homomorphism $h : A \to Q^n$ is a *C-approximation of f* if the Hamming distance between f and h satisfies

$$\forall x \in A: \qquad w_H(f(x) - h(x)) \leq C.$$

We are ready to state our main result.

Theorem 1.6 (Main Theorem)**.** *Let $c \in \mathbb{N}$. Then there exists a constant $C = C(c) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and c-quasihomorphisms $f : \mathbb{Z} \to \mathbb{Q}^n$, we have:*

$$\forall x \in \mathbb{Z}: \qquad w_H(f(x) - x \cdot f(1)) \leq C.$$

Moreover, we can take $C = 28c$.

Remark 1.7. The coefficient 28 is probably not optimal. However, we certainly have that $C(c) \geq c$. Indeed, any map $f : \mathbb{Z} \to Q^n$ for which the only nonzero entries of $f(x)$ are among the first c, is automatically a c-quasimorphism. ♢

Corollary 1.8. *Theorem 1.6 also holds with $\mathbb{Q}$ replaced by any torsion-free abelian group Q, with the same value of $C = C(c)$.*

Proof. Suppose, for a contradiction, that we have a c-quasihomomorphism $f : \mathbb{Z} \to Q^n$ but $w_H(f(y) - y \cdot f(1)) > C$ for some $y \in \mathbb{Z}$. Since Q is torsion-free, the natural map ι from Q into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} Q$ is injective. Consequently, $g := \iota^n \circ f$ is a c-quasihomomorphism $\mathbb{Z} \to V^n$ with $w_H(g(y) - y \cdot g(1)) > C$. Now choose any $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ that is nonzero on the nonzero entries of $g(y) - y \cdot g(1)$. Then $h := \xi^n \circ g$ is a c-quasihomomorphism $\mathbb{Z} \to \mathbb{Q}^n$ with $w_H(h(y) - y \cdot h(1)) > C$, a contradiction to Theorem 1.6. □

Theorem 1.6 shows that for a c-quasihomomorphism $f : \mathbb{Z} \to \mathbb{Q}^n$, the group homomorphism $\tilde{f} : \mathbb{Z} \to \mathbb{Q}^n$ defined by $\tilde{f}(x) = x \cdot f(1)$ gives a C-approximation for some constant $C \in \mathbb{N}$ independent on n. However, $\tilde{f}$ need not be the homomorphism closest to f, as the next example shows.

Example 1.9. Let $c = 1$ and $n \geq 3$. Define $f : \mathbb{Z} \to \mathbb{Q}^n$ to be

$$f(x) = \left(\left\lfloor \frac{2x+2}{5} \right\rfloor, \left\lfloor \frac{x+2}{5} \right\rfloor, \alpha_x, 0, \dots, 0 \right),$$

where α_x is arbitrary if $5 \mid x$, and $\alpha_x = 0$ otherwise. This is a 1-quasihomomorphism.

Note that $w_H(f(x)-x\cdot f(1)) \leq 3$ where equality is sometimes achieved. However, there also exist 2-approximations of f. For instance, letting $v = (\frac{2}{5}, \frac{1}{5}, 0, \dots, 0) \in \mathbb{Q}^n$, one verifies that

$$w_H(f(x) - x \cdot v) \leq 2 \qquad \forall x \in \mathbb{Z}.$$

In fact, we can show that for every 1-quasihomomorphism $f : \mathbb{Z} \to \mathbb{Q}^n$ there exists a 2-approximation, as claimed in [KZ18]. The proof will appear in future work. ♢

On the other hand, the following shows that the best possible approximation of a given qusimorphism f is at most twice as close as the homomorphism $x \mapsto x \cdot f(1)$.

Remark 1.10. Suppose that a map $f : \mathbb{Z} \to \mathbb{Q}^n$ has a C'-approximation h. Then $h(x) = x \cdot v$ for some $v \in \mathbb{Q}^n$, and

$$w_H(f(x) - x \cdot v) \leq C' \qquad \forall x \in \mathbb{N}.$$

Substituting $x = 1$ yields $w_H(f(1) - v) \leq C'$. Thus

$$w_H(f(x) - x \cdot f(1)) \leq w_H(f(x) - x \cdot v) + w_H(x \cdot v - x \cdot f(1)) \leq 2C'. \qquad ♢$$

Remark 1.11. A result similar to Theorem 1.6 is easily proven in finite characteristic if we allow the constant C to depend on the characteristic. Let K be a field of characteristic $p > 0$, and let $f : \mathbb{Z} \to K^n$ be a c-quasihomomorphism. Then there exists a constant $C = C(p, c)$ such that $w_H(f(x) - x \cdot f(1)) \leq C$, for all $x \in \mathbb{Z}$.

To see this, we observe that for all $u, v \in \mathbb{Z}$ with $u \geq 1$, we have

$$w_H(f(uv) - uf(v)) \leq (u-1)c.$$

This follows by repeatedly applying the inequality $w_H(f(uv)-f((u-1)v)-f(v)) \leq c$ if $u > 1$; the case $u = 1$ is trivial.

For $x = kp + r$ with $k \in \mathbb{Z}$ and $0 \leq r \leq p-1$, we have

$$w_H(f(x) - xf(1)) = w_H(f(kp+r) - rf(1));$$

here we have used that $p \cdot f(1) = 0$. We rewrite the latter as

$$w_H(f(kp+r) - f(kp) - f(r) + f(kp) + f(r) - rf(1)).$$

We have $w_H(f(kp+r) - f(kp) - f(r)) \leq c$; $w_H(f(kp)) \leq (p-1)c$ using our observation with $u = p, v = k$; and also $w_H(f(r) - rf(1)) \leq (p-2)c$ (in the case $r > 0$) using our main observation. In total, this gives $w_H(f(x)-xf(1)) \leq 2(p-1)c$, so we can take $C = 2(p-1)c$. ♢

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3, we apply this auxiliary result to the component functions of a c-quasimorphism $\mathbb{Z} \to \mathbb{Q}^n$ to prove the Main Theorem.

2. Almost homomorphisms are almost zero

Let A be a finite abelian group and let H be a torsion-free abelian group. The only homomorphism $A \to H$ is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

Proposition 2.1. *Let a be a positive integer, A an abelian group of order a, H a torsion-free abelian group, $q \in [0,1]$, and $f : A \to H$ a map. Suppose that the* zero set

$$Z(f) := \{b \in A \mid f(b) = 0\}$$

has cardinality at most qa. Then the problem set

$$P(f) := \{(b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c)\}$$

has cardinality at least $\left(\frac{a-qa}{2}\right) \cdot \left(\frac{a-qa}{2} + 1\right)$.

The contraposition of this statement says that if $P(f)$ is a small fraction of a^2, so that f can be thought of as an (additive) "almost homomorphism" $A \to H$, then q must be close to 1 so that f is essentially zero.

Proof. Since H is torsion-free, it embeds into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} H$. By basic linear algebra, there exists a $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ such that $\xi(f(b)) \neq 0$ for all $b \notin Z(f)$, so that $Z(\xi \circ f) = Z(f)$. Since $P(\xi \circ f) \subseteq P(f)$, it suffices to prove the proposition for $\xi \circ f$ instead of f. In other words, we may assume from the beginning that $H = \mathbb{Q}$.

Set

$$B := \{b \in A \mid f(b) > 0\}.$$

Let $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$ be the distinct values in $f(B)$, and for each $i = 1, \ldots, k$ set

$$B_i := \{b \in B \mid f(b) = \lambda_i\} \text{ and } n_i := |B_i|;$$

as well as $n := n_1 + \cdots + n_k = |B|$.

Now for each $c \in B_1$ and each $b \in B$ we have

$$f(b) + f(c) = f(b) + \lambda_1 > \lambda_1$$

so that the left-hand side is not in $f(B)$ and in particular not equal to $f(b+c)$. We have thus found $n_1 \cdot (n_1 + \cdots + n_k)$ pairs $(b,c) \in P(f)$ with $c \in B_1$.

Next, suppose (b,c) is a pair with $c \in B_2$, $b \in B$, and $(b,c) \notin P(f)$. Then

$$f(b+c) = f(b) + f(c) > f(c) = \lambda_2$$

and hence $b + c \in B_1$. But given c, there are at most n_1 values of b with $b + c \in B_1$. (Note that here we have used that A is a group.) Hence we have at least $n_2 \cdot (n_2 + \cdots + n_k)$ pairs $(b,c) \in P(f)$ with $c \in B_2$.

Similarly, we find at least $n_i \cdot (n_i + \cdots + n_k)$ pairs $(b,c) \in P(f)$ with $c \in B_i$. In total, we have therefore found at least

$$\sum_{i=1}^{k} n_i \cdot (n_i + \cdots + n_k) \geq \frac{n(n+1)}{2} \tag{2.1}$$

pairs in $P(f)$; see Figure 1.

Let $B' := \{b' \in A \mid f(b') < 0\}$ and $n' := |B'|$. Repeating the same argument above with B' and n', we find at least $n'(n'+1)/2$ further pairs in $P(f)$, disjoint from those found above. Since $|Z(f)| \leq qa$, we have $n + n' \geq a(1-q)$. Therefore

$$|P(f)| \geq \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2+n'^2}{2} + \frac{n+n'}{2} \geq \left(\frac{n+n'}{2}\right)^2 + \frac{n+n'}{2},$$

where the second inequality is the Cauchy-Schwarz inequality

$$(n^2+n'^2)\left(\frac{1}{2^2}+\frac{1}{2^2}\right) \geq \left(\frac{n}{2}+\frac{n'}{2}\right)^2.$$

Since $n + n' \geq a(1-q)$, we conclude that

$$|P(f)| \geq \left(\frac{a-qa}{2}\right) \cdot \left(\frac{a-qa}{2}+1\right). \qquad \square$$

Remark 2.2. The lower bound in Proposition 2.1 is sharp. Let $a = 2k+1 \in \mathbb{Z}$, consider $A := \mathbb{Z}/a\mathbb{Z}$ and define $f : A \to \mathbb{Z}$ as $f(x) :=$the representative of $x + a\mathbb{Z}$ in $\{-k, \dots, 0, \dots, k\}$. We take $q = \frac{Z(f)}{a} = \frac{1}{2k+1}$. Then $f(x+y) = f(x) + f(y)$ if and only if the right-hand side is still inside the interval $\{-k, \dots, k\}$, and a straightforward count shows that this is the case for $3k^2 + 3k + 1$ pairs $(x, y) \in A^2$. Hence $P(f)$ has size $k \cdot (k+1)$, which equals $\left(\frac{a-qa}{2}\right) \cdot \left(\frac{a-qa}{2}+1\right)$. A similar construction for $a = 2k$ yields a problem set of size $\frac{a^2}{4} = k^2$, which equals the ceiling of the lower bound $\frac{a^2}{4} - \frac{1}{4}$. $\diamondsuit$

Below, we will use the following strengthening of Proposition 2.1:

Proposition 2.3. *Let a, A, H, q and f be as in Proposition 2.1. Furthermore, let $p \in [0, \frac{1-q}{2})$ and let $S \subseteq A$ be a subset of cardinality at most pa. Then the set*

$$P_S(f) := \{(b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \text{ and } b + c \notin S\}.$$

has cardinality at least $\left(\frac{a(1-q-2p)}{2}\right) \cdot \left(\frac{a(1-q-2p)}{2}+1\right)$.

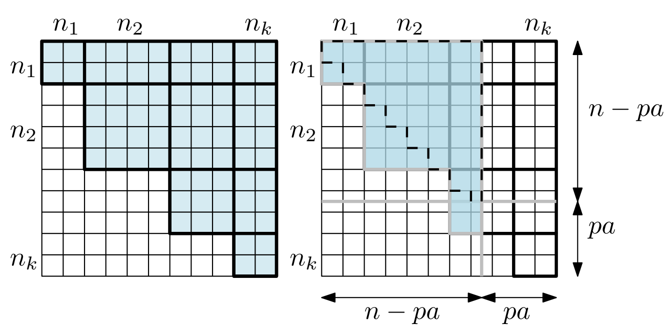

FIGURE 1. On the left, a graphical proof of the inequality (2.1): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (2.2): the left-hand side is the area of the shaded region, the right-hand side, the area enclosed by the dashed line.

Proof. Keep the notation from the proof of Proposition 2.1. Recall $n = |B|$ and $n' = |B'|$. Note that for a fixed b, there can be at most pa choices of c with $b + c \in S$. We then find at least $n_i \cdot (n_i + \cdots + n_k - pa)$ pairs $(b, c) \in P_S(f)$ with $b \in B_i$. Letting $k' \leq k$ be the largest index for which the second factor $(n_{k'} + \cdots + n_k - pa)$ is nonnegative, as in the proof of Proposition 2.1, we find that B contributes at least

$$\sum_{i=1}^{k'} n_i \cdot (n_i + \cdots + n_k - pa) = \sum_{i=1}^{k'} n_i \cdot (n - n_1 - \cdots - n_{i-1} - pa) \geq (n - pa)(n - pa + 1)/2 \quad (2.2)$$

to $P_S(f)$; see Figure 1. Similarly, B' contributes at least $(n' - pa)(n' - pa + 1)/2$, and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with n, n' replaced by $n - pa, n' - pa$. □

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write $[a]$ for the set $\{1, 2, \ldots, a\}$.

Corollary 2.4. *Let $p, q \in [0, 1]$ such that $p < \frac{1-q}{2}$. Let $f : [2a] \to \mathbb{Q}$ such that:*

(1) $|Z(f)| \leq qa$, where $Z(f) := \{x \in [a] \mid f(x) = 0\}$ is the zero set.
(2) $|NP(f)| \leq pa$, where $NP(f) := \{x \in [a] \mid f(x + a) \neq f(x)\}$ is the nonperiodicity set.

Then

$$|P(f)| \geq \frac{(1 - q - 2p)^2}{4} a^2 + \frac{(1 - q - 2p)}{2} a,$$

where

$$P(f) = \{(x, y) \in [a] \times [a] \mid f(x + y) \neq f(x) + f(y)\}.$$

Proof. Let $\tilde{f}$ be the restriction of f to the interval $[a]$, and identify $\mathbb{Z}/a\mathbb{Z}$ with $[a]$ with the group operation $*$ defined by $x * y := x + y \pmod{a}$.

Let $S = NP(f)$, and apply Proposition 2.3 to $\tilde{f}$. We find that

$$P_S(\tilde{f}) = \{(b, c) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \mid \tilde{f}(b * c) \neq \tilde{f}(b) + \tilde{f}(c) \text{ and } b * c \notin S\}$$

has cardinality at least $\frac{(1-q-2p)^2}{4} a^2 + \frac{(1-q-2p)}{2} a$. Since $b * c \notin S$ implies that $\tilde{f}(b * c) = f(b + c)$, this set is contained in the problem set $P(f)$. □

3. Proof of the Main Theorem

The main goal of this section is to prove Theorem 1.6. We start with some definitions.

Definition 3.1. Let $1 < a$, and $f : [2a] \to \mathbb{Q}$. We define the following *problem sets of f*:

$$P(f) := \{(x, y) \in [a] \times [a] \mid f(x + y) \neq f(x) + f(y)\},$$

and

$$P_1(f) := \{x \in [a] \mid f(x + 1) \neq f(x) + f(1)\},$$

and

$$P_a(f) := \{x \in [a] \mid f(x + a) \neq f(x) + f(a)\}.$$

Furthermore, we recall that the *zero set* of f is defined as

$$Z(f) := \{x \in [a] \mid f(x) = 0\}.$$

The following proposition says that $P_1(f), P_a(f), P(f)$ cannot be simultaneously small.

Proposition 3.2. *Let $p, q \in (0,1)$ such that $p < \frac{1-q}{2}$, $a \in \mathbb{N}$ with $1 < a$, and let $f : [2a] \to \mathbb{Q}$ such that $f(a) \neq af(1)$. If*

$$|P_1(f)| \leq qa \quad \text{and} \quad |P_a(f)| \leq pa$$

then

$$|P(f)| \geq F(p,q) \cdot a^2,$$

where

$$F(p,q) = \frac{(1-q-2p)^2}{4}.$$

Proof. Without loss of generality we can assume $f(a) = 0$ and hence $f(1) \neq 0$. Indeed, suppose we have shown the statement for every $\tilde{f}$ with $\tilde{f}(a) = 0$. Then for any $f : [2a] \to \mathbb{Q}$ with $f(a) \neq af(1)$, we take $\tilde{f} : [2a] \to \mathbb{Q}$ to be $\tilde{f}(x) = af(x) - xf(a)$. Now we observe that $\tilde{f}(a) = 0 \neq a\tilde{f}(1)$, and that $P(f) = P(\tilde{f})$, $P_1(f) = P_1(\tilde{f})$, $P_a(f) = P_a(\tilde{f})$.

We write $Z(f) = \{x_1 < \cdots < x_m\}$. Note that for $1 \leq i < m$, one of the elements $x_i, x_i + 1, \ldots, x_{i+1} - 1$ needs to be in $P_1(f)$ since $f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i)f(1)$. Likewise, at least one of the elements $1, 2, \ldots, x_1 - 1$ needs to be in $P_1(f)$. Thus we have

$$|Z(f)| \leq |P_1(f)| \leq qa,$$

and by assumption we have $|NP(f)| = |P_a(f)| \leq pa$. Now we can apply Corollary 2.4 to conclude. □

We now prove Theorem 1.6.

Proof of the Main Theorem. Consider a c-quasihomomorphism $f = (f_1, \ldots, f_n) : \mathbb{Z} \to \mathbb{Q}^n$. Our goal is to show that for every $a \in \mathbb{Z}$ we have $w_H(f(a) - a \cdot f(1)) \leq C$ for some constant C depending only on c. We start with the case $a > 0$.

Write $M := w_H(f(a) - af(1))$. Without loss of generality, we have $f_1(a) \neq af(1)$, $\ldots$, $f_M(a) \neq af(1)$. We will show that $M \leq C'$ for some constant C' depending on c only. To this end, fix small parameters $p, q \in (0,1)$ (to be optimized over later) and consider the restrictions $f_i : [2a] \to \mathbb{Q}$ of the components of f. By Proposition 3.2, for every i, we have that either

(i) $|P_1(f_i)| > qa$, or
(ii) $|P_a(f_i)| > pa$, or
(iii) $|P(f_i)| \geq F(p,q)a^2$.

Let m_0 be the number of coordinates i such that (iii) holds. We define m_1 and m_2 analogously, for (i) and (ii) respectively.

By counting the number of triples $(i, x, y) \in [M] \times [a] \times [a]$ such that $f_i(x+y) - f_i(x) - f_i(y) \neq 0$ in two ways, we see that

$$\sum_{x=1}^{a} \sum_{y=1}^{a} w_H\left(f(x+y) - f(x) - f(y)\right) = \sum_{i=1}^{M} |P(f_i)|.$$

Because f is a c-quasihomomorphism, the left hand side is at most a^2c. On the other hand, the right hand side is at least $m_0F(p,q)a^2$, so

$$a^2c \geq \sum_{x=1}^{a}\sum_{y=1}^{a} w_H\left(f(x+y)-f(x)-f(y)\right) = \sum_{i=1}^{M}|P(f_i)| \geq m_0F(p,q)a^2.$$

So we obtain $m_0 \leq \frac{c}{F(p,q)}$. Similarly we find

$$ac \geq \sum_{x=1}^{a} w_H\left(f(x+1)-f(x)-f(1)\right) = \sum_{i=1}^{M}|P_1(f_i)| > m_1qa,$$

so that $m_1 < \frac{c}{q}$. Finally,

$$ac \geq \sum_{x=1}^{a} w_H\left(f(x+a)-f(x)-f(a)\right) = \sum_{i=1}^{M}|P_a(f_i)| > m_2pa.$$

So $m_2 < \frac{c}{p}$. But now $M = m_0 + m_1 + m_2 < c(\frac{1}{F(p,q)} + \frac{1}{q} + \frac{1}{p}) =: C'$.

The case $a = 0$ is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \leq c.$$

Finally, let us consider the case $a < 0$. Then

$$\begin{aligned} w_H(f(a) - a \cdot f(1)) \leq & w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) \\ & + w_H(f(-a) - (-a) \cdot f(1)) \\ \leq & 2c + C' =: C. \end{aligned}$$

This completes the proof. □

To get the explicit bound $28c$ from Theorem 1.6, we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p,q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1-q-2p)^2}.$$

Note that this function is strictly convex for $(p,q) \in \mathbb{R}^2_{>0}$, so that it has at most one minimum in the positive orthant. We find this by setting the partial derivatives equal to zero and solving for p, q. The minimum is ≈ 27.6817 and attained at $(p,q) \approx (0.1167, 0.16500)$.

References

[A'C21] N. A'Campo. A natural construction for the real numbers. *Elem. Math.*, 76(3):89–105, 2021.

[FK16] K. Fujiwara and M. Kapovich. On quasihomomorphisms with noncommutative targets. *Geom. Funct. Anal.*, 26(2):478–519, 2016.

[Kaz82] D. Kazhdan. On ε-representations. *Israel J. Math.*, 43(4):315–323, 1982.

[Kot04] D. Kotschick. What is... a quasi-morphism? *Notices Amer. Math. Soc.*, 51(2):208–209, 2004.

[KZ18] D. Kazhdan and T. Ziegler. Approximate cohomology. *Selecta Math. (N.S.)*, 24(1):499–509, 2018.

[Ula60] S. M. Ulam. *A collection of mathematical problems*. Interscience Tracts in Pure and Applied Mathematics, no. 8. Interscience Publishers, New York-London, 1960.

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