Quasihomomorphisms from the integers into Hamming metrics

Citation for published version (APA):

DOI:
10.48550/arXiv.2204.08392

Document status and date:
Published: 18/04/2022

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
QUASIHOMOMORPHISMS FROM THE INTEGERS INTO HAMMING METRICS

JAN DRAISMA, ROB H. EGGERMONT, TIM SEYNNAEVE, NAFIE TAIRI, AND EMANUELE VENTURA

Abstract. A function $f : \mathbb{Z} \to \mathbb{Q}^n$ is a $c$-quasihomomorphism if the Hamming distance between $f(x + y)$ and $f(x) + f(y)$ is at most $c$ for all $x, y \in \mathbb{Z}$. We show that any $c$-quasihomomorphism has distance at most some constant $C(c)$ to an actual group homomorphism; here $C(c)$ depends only on $c$ and not on $n$ or $f$. This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

1. Introduction

A $c$-quasihomomorphism from a group $G$ to a group $H$ with a left-invariant metric $d$ is a map $f : G \to H$ such that $d(f(xy), f(x)f(y)) \leq c$ for all $x, y$ in $G$. A central question in geometric group theory, first raised by Ulam in [Ula60, Chapter 6], is whether there exists an actual homomorphism $f' : G \to H$ such that $d(f(x), f'(x))$ is at most some constant $C$ for all $x$. Different versions of this question are of interest: for instance, $C$ may be allowed to depend on $c, G, (H, d)$ but not on $f$; or $G, (H, d)$ may be restricted to certain classes and or $C$ is only allowed to depend on $c$.

A well-known example where the answer to this question is negative is the case where $G = H = \mathbb{Z}$ with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [A'C21], and the answer is yes only for those quasihomomorphisms that correspond to integers.

Much literature in this area focusses on quasimorphisms, which are quasihomomorphisms into the real numbers $\mathbb{R}$ with the standard metric; we refer to [Kot04] for a brief introduction. In another branch of the research on quasihomomorphisms $H$ is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan’s theorem on $\varepsilon$-representations of amenable groups [Kaz82]. For more recent results on quasihomomorphisms into nonabelian groups we refer to [FK16] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [KZ18].

**Question 1.1.** Let $c \in \mathbb{N}$. Does there exist a constant $C = C(c)$ such that the following holds: For all $n \in \mathbb{N}$ and all functions $f : \mathbb{Z} \to \mathbb{C}^{n \times n}$ such that

$$\forall x, y \in \mathbb{Z} : \quad \text{rk}(f(x + y) - f(x) - f(y)) \leq c,$$
there exists a matrix $g$ such that

$$\forall x \in \mathbb{Z} : \quad \text{rk}(f(x) - x \cdot g) \leq C(c)?$$

Here, $G$ equals $\mathbb{Z}$ and $H$ equals $\mathbb{C}^{n \times n}$, both with addition, and the metric on $H$ is defined by $d(A, B) := \text{rk}(A - B)$. Our main result is an affirmative answer to this question in the special case where all matrices $f(x)$ are assumed to be diagonal.

Definition 1.2. Let $(Q, +)$ be an abelian group. For an element $v \in Q^n$, the Hamming weight $w_H(v)$ is the number of nonzero entries of $v$. For a pair of elements $u, v \in Q^n$, their Hamming distance is $w_H(v - u)$. This metric is clearly left-invariant.

Definition 1.3. Let $A$ be another abelian group. A function $f : A \to Q^n$ is called a $c$-quasihomomorphism if

$$\forall x, y \in A : \quad w_H(f(x + y) - f(x) - f(y)) \leq c.$$  

Remark 1.4. The map $\text{diag} : \mathbb{C}^n \to \mathbb{C}^{n \times n}$ is an isometric embedding from $\mathbb{C}^n$ with the Hamming metric to $\mathbb{C}^{n \times n}$ with the rank metric. This connects Definition 1.3 to Question 1.1. 

Definition 1.5. Let $C \in \mathbb{N}$ and let $f : A \to Q^n$ be a $c$-quasihomomorphism. A group homomorphism $h : A \to Q^n$ is a $C$-approximation of $f$ if the Hamming distance between $f$ and $h$ satisfies

$$\forall x \in A : \quad w_H(f(x) - h(x)) \leq C.$$  

We are ready to state our main result.

Theorem 1.6 (Main Theorem). Let $c \in \mathbb{N}$. Then there exists a constant $C = C(c) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $c$-quasihomomorphisms $f : \mathbb{Z} \to Q^n$, we have:

$$\forall x \in \mathbb{Z} : \quad w_H(f(x) - x \cdot f(1)) \leq C.$$  

Moreover, we can take $C = 28c$.

Remark 1.7. The coefficient 28 is probably not optimal. However, we certainly have that $C(c) \geq c$. Indeed, any map $f : \mathbb{Z} \to Q^n$ for which the only nonzero entries of $f(x)$ are among the first $c$, is automatically a $c$-quasihomomorphism. 

Corollary 1.8. Theorem 1.6 also holds with $Q$ replaced by any torsion-free abelian group $Q$, with the same value of $C = C(c)$.

Proof. Suppose, for a contradiction, that we have a $c$-quasihomomorphism $f : \mathbb{Z} \to Q^n$ but $w_H(f(y) - y \cdot f(1)) > C$ for some $y \in \mathbb{Z}$. Since $Q$ is torsion-free, the natural map $\iota$ from $Q$ into the $\mathbb{Q}$-vector space $V := Q \otimes_{\mathbb{Z}} Q$ is injective. Consequently, $g := \iota^n \circ f$ is a $c$-quasihomomorphism $\mathbb{Z} \to V^n$ with $w_H(g(y) - y \cdot g(1)) > C$. Now choose any $\mathbb{Q}$-linear function $\xi : V \to \mathbb{Q}$ that is nonzero on the nonzero entries of $g(y) - y \cdot g(1)$. Then $h := \xi^n \circ g$ is a $c$-quasihomomorphism $\mathbb{Z} \to Q^n$ with $w_H(h(y) - y \cdot h(1)) > C$, a contradiction to Theorem 1.6. 

Theorem 1.6 shows that for a $c$-quasihomomorphism $f : \mathbb{Z} \to Q^n$, the group homomorphism $\tilde{f} : \mathbb{Z} \to Q^n$ defined by $\tilde{f}(x) = x \cdot f(1)$ gives a $C$-approximation for some constant $C \in \mathbb{N}$ independent on $n$. However, $\tilde{f}$ need not be the homomorphism closest to $f$, as the next example shows.
Example 1.9. Let \( c = 1 \) and \( n \geq 3 \). Define \( f : \mathbb{Z} \to \mathbb{Q}^n \) to be

\[
f(x) = \left( \left\lfloor \frac{2x + 2}{5} \right\rfloor, \left\lceil \frac{2x + 2}{5} \right\rceil, \alpha_x, 0, \ldots, 0 \right),
\]

where \( \alpha_x \) is arbitrary if \( 5 \mid x \), and \( \alpha_x = 0 \) otherwise. This is a 1-quasihomomorphism.

Note that \( w_H(f(x) - x \cdot f(1)) \leq 3 \) where equality is sometimes achieved. However, there also exist 2-approximations of \( f \). For instance, letting \( v = \left( \frac{2}{5}, \frac{1}{5}, 0, \ldots, 0 \right) \in \mathbb{Q}^n \), one verifies that

\[
w_H(f(x) - x \cdot v) \leq 2 \quad \forall x \in \mathbb{Z}.
\]

In fact, we can show that for every 1-quasihomomorphism \( f : \mathbb{Z} \to \mathbb{Q}^n \) there exists a 2-approximation, as claimed in [KZ18]. The proof will appear in future work. \( \Box \)

On the other hand, the following shows that the best possible approximation of a given quasimorphism \( f \) is at most twice as close as the homomorphism \( x \mapsto x \cdot f(1) \).

Remark 1.10. Suppose that a map \( f : \mathbb{Z} \to \mathbb{Q}^n \) has a \( C' \)-approximation \( h \). Then \( h(x) = x \cdot v \) for some \( v \in \mathbb{Q}^n \), and

\[
w_H(f(x) - x \cdot v) \leq C' \quad \forall x \in \mathbb{N}.
\]

Substituting \( x = 1 \) yields \( w_H(f(1) - v) \leq C' \). Thus

\[
w_H(f(x) - x \cdot f(1)) \leq w_H(f(x) - x \cdot v) + w_H(x \cdot v - x \cdot f(1)) \leq 2C'. \quad \Box
\]

Remark 1.11. A result similar to Theorem 1.6 is easily proven in finite characteristic if we allow the constant \( C \) to depend on the characteristic. Let \( K \) be a field of characteristic \( p > 0 \), and let \( f : \mathbb{Z} \to K^n \) be a \( c \)-quasihomomorphism. Then there exists a constant \( C = C(p, c) \) such that \( w_H(f(x) - x \cdot f(1)) \leq C \), for all \( x \in \mathbb{Z} \).

To see this, we observe that for all \( u, v \in \mathbb{Z} \) with \( u \geq 1 \), we have

\[
w_H(f(uv) - uf(v)) \leq (u - 1)c.
\]

This follows by repeatedly applying the inequality \( w_H(f(uv) - f((u-1)v) - f(v)) \leq c \) if \( u > 1 \); the case \( u = 1 \) is trivial.

For \( x = kp + r \) with \( k \in \mathbb{Z} \) and \( 0 \leq r \leq p - 1 \), we have

\[
w_H(f(x) - x f(1)) = w_H(f(kp + r) - r f(1));
\]

here we have used that \( p \cdot f(1) = 0 \). We rewrite the latter as

\[
w_H(f(kp + r) - f(kp) - f(r) + f(kp) + f(r) - f(1)).
\]

We have \( w_H(f(kp + r) - f(kp) - f(r)) \leq c \); \( w_H(f(kp)) \) is our observation with \( u = p, v = k \); and also \( w_H(f(r) - r f(1)) \leq (p - 2)c \) (in the case \( r > 0 \)) using our main observation. In total, this gives \( w_H(f(x) - x f(1)) \leq 2(p-1)c \), so we can take \( C = 2(p-1)c \). \( \Box \)

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3 we apply this auxiliary result to the component functions of a \( c \)-quasimorphism \( \mathbb{Z} \to \mathbb{Q}^n \) to prove the Main Theorem.
2. Almost homomorphisms are almost zero

Let \( A \) be a finite abelian group and let \( H \) be a torsion-free abelian group. The only homomorphism \( A \to H \) is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

**Proposition 2.1.** Let \( a \) be a positive integer, \( A \) an abelian group of order \( a \), \( H \) a torsion-free abelian group, \( q \in [0,1] \), and \( f : A \to H \) a map. Suppose that the zero set

\[
Z(f) := \{ b \in A \mid f(b) = 0 \}
\]

has cardinality at least \( qa \). Then the problem set

\[
P(f) := \{(b,c) \in A \times A \mid f(b+c) \neq f(b) + f(c)\}
\]

has cardinality at least \( (\frac{a-qa}{2} \cdot (\frac{a-qa}{2} + 1)) \).

The contraposition of this statement says that if \( P(f) \) is a small fraction of \( a^2 \), so that \( f \) can be thought of as an (additive) “almost homomorphism” \( A \to H \), then \( q \) must be close to 1 so that \( f \) is essentially zero.

**Proof.** Since \( H \) is torsion-free, it embeds into the \( \mathbb{Q} \)-vector space \( V := \mathbb{Q} \otimes \mathbb{Z} \. H \). By basic linear algebra, there exists a \( \mathbb{Q} \)-linear function \( \xi : V \to \mathbb{Q} \) such that \( \xi(f(b)) \neq 0 \) for all \( b \notin Z(f) \), so that \( Z(\xi \circ f) = Z(f) \). Since \( P(\xi \circ f) \subseteq P(f) \), it suffices to prove the proposition for \( \xi \circ f \) instead of \( f \). In other words, we may assume from the beginning that \( H = \mathbb{Q} \).

Set

\[
B := \{ b \in A \mid f(b) > 0 \}.
\]

Let \( \lambda_1 > \lambda_2 > \ldots > \lambda_k > 0 \) be the distinct values in \( f(B) \), and for each \( i = 1, \ldots, k \) set

\[
B_i := \{ b \in B \mid f(b) = \lambda_i \} \text{ and } n_i := |B_i|;
\]

as well as \( n := n_1 + \cdots + n_k = |B| \).

Now for each \( c \in B_1 \) and each \( b \in B \) we have

\[
f(b) + f(c) = f(b) + \lambda_1 > \lambda_1
\]

so that the left-hand side is not in \( f(B) \) and in particular not equal to \( f(b+c) \). We have thus found \( n_1 \cdot (n_1 + \cdots + n_k) \) pairs \((b,c) \in P(f)\) with \( c \in B_1 \).

Next, suppose \((b,c)\) is a pair with \( c \in B_2 \), \( b \in B \), and \((b,c) \notin P(f)\). Then

\[
f(b+c) = f(b) + f(c) > f(c) = \lambda_2
\]

and hence \( b + c \in B_1 \). But given \( c \), there are at most \( n_1 \) values of \( b \) with \( b+c \in B_1 \). (Note that here we have used that \( A \) is a group.) Hence we have at least \( n_2 \cdot (n_2 + \cdots + n_k) \) pairs \((b,c) \in P(f)\) with \( c \in B_2 \).

Similarly, we find at least \( n_i \cdot (n_i + \cdots + n_k) \) pairs \((b,c) \in P(f)\) with \( c \in B_i \). In total, we have therefore found at least

\[
\sum_{i=1}^{k} n_i \cdot (n_i + \cdots + n_k) \geq \frac{n(n+1)}{2} \tag{2.1}
\]

pairs in \( P(f) \); see Figure 1.
Let \( B' := \{ b' \in A \mid f(b') < 0 \} \) and \( n' := |B'| \). Repeating the same argument above with \( B' \) and \( n' \), we find at least \( n'(n' + 1)/2 \) further pairs in \( P(f) \), disjoint from those found above. Since \( |Z(f)| \leq qa \), we have \( n + n' \geq a(1 - q) \). Therefore

\[
|P(f)| \geq \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2 + n'^2}{2} + \frac{n + n'}{2} \geq \left(\frac{n + n'}{2}\right)^2 + \frac{n + n'}{2},
\]

where the second inequality is the Cauchy-Schwarz inequality

\[
(n^2 + n'^2)\left(\frac{1}{2^2} + \frac{1}{2^2}\right) \geq \left(\frac{n}{2} + \frac{n'}{2}\right)^2.
\]

Since \( n + n' \geq a(1 - q) \), we conclude that

\[
|P(f)| \geq \left(\frac{a - qa}{2}\right) \cdot \left(\frac{a - qa}{2} + 1\right).
\]

\(\square\)

**Remark 2.2.** The lower bound in Proposition 2.1 is sharp. Let \( a = 2k + 1 \in \mathbb{Z} \), consider \( A := \mathbb{Z}/a\mathbb{Z} \) and define \( f : A \to \mathbb{Z} \) as \( f(x) := \) the representative of \( x + a\mathbb{Z} \) in \( \{-k, \ldots, 0, \ldots, k\} \). We take \( q = \frac{Z(f)}{n} = \frac{1}{2k+1} \). Then \( f(x + y) = f(x) + f(y) \) if and only if the right-hand side is still inside the interval \( \{-k, \ldots, k\} \), and a straightforward count shows that this is the case for \( 3k^2 + 3k + 1 \) pairs \( (x, y) \in A^2 \). Hence \( P(f) \) has size \( k \cdot (k + 1) \), which equals \( (\frac{a - qa}{2^2}) \cdot (\frac{a - qa}{2} + 1) \). A similar construction for \( a = 2k \) yields a problem set of size \( \frac{a^2}{4} = k^2 \), which equals the ceiling of the lower bound \( \frac{a^2}{4} - \frac{1}{4} \). \(\diamond\)

Below, we will use the following strengthening of Proposition 2.1.

**Proposition 2.3.** Let \( a, A, H, q \) and \( f \) be as in Proposition 2.1. Furthermore, let \( p \in \left(0, \frac{1-q}{2}\right) \) and let \( S \subseteq A \) be a subset of cardinality at most \( pa \). Then the set

\[
P_S(f) := \{(b, c) \in A \times A \mid f(b + c) \neq f(b) + f(c) \text{ and } b + c \notin S\}.
\]

has cardinality at least \( \left(\frac{a(1-q-2p)}{2}\right) \cdot \left(\frac{a(1-q-2p)}{2} + 1\right) \).

![Figure 1](image-url)  

**Figure 1.** On the left, a graphical proof of the inequality (2.1): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (2.2): the left-hand side is the area of the shaded region, the right-hand side, the area enclosed by the dashed line.
Let 1

**Definition 3.1**. Recall $n = |B|$ and $n' = |B'|$. Note that for a fixed $b$, there can be at most $pa$ choices of $c$ with $b + c ∈ S$. We then find at least $n_i · (n_i + ⋯ + n_k - pa)$ pairs $(b, c) ∈ P_S(f)$ with $b ∈ B_i$. Letting $k' ≤ k$ be the largest index for which the second factor $(n_k + ⋯ + n_k - pa)$ is nonnegative, as in the proof of Proposition 2.1, we find that $B$ contributes at least

$$\sum_{i=1}^{k'} n_i · (n_i + ⋯ + n_k - pa) = \sum_{i=1}^{k'} n_i · (n - n_1 - ⋯ - n_i - pa)$$

$$≥ (n - pa)(n - pa + 1)/2$$

(2.2)

to $P_S(f)$; see Figure 1. Similarly, $B'$ contributes at least $(n' - pa)(n' - pa + 1)/2$, and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with $n, n'$ replaced by $n - pa, n' - pa$.

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write $[a]$ for the set $\{1, 2, ..., a\}$.

**Corollary 2.4.** Let $p, q ∈ [0, 1]$ such that $p < \frac{1-q}{2}$. Let $f : [2a] → \mathbb{Q}$ such that:

1. $|Z(f)| ≤ qa$, where $Z(f) := \{x ∈ [a] | f(x) = 0\}$ is the zero set.
2. $|NP(f)| ≤ pa$, where $NP(f) := \{x ∈ [a] | f(x + a) ≠ f(x)\}$ is the nonperiodicity set.

Then

$$|P(f)| ≥ \frac{(1 - q - 2p)^2}{4} a^2 + \frac{(1 - q - 2p)}{2} a,$$

where

$$P(f) = \{(x,y) ∈ [a] × [a] | f(x + y) ≠ f(x) + f(y)\}.$$

**Proof.** Let $f'$ be the restriction of $f$ to the interval $[a]$, and identify $\mathbb{Z}/a\mathbb{Z}$ with $[a]$ with the group operation $*$ defined by $x * y := x + y (mod a)$.

Let $S = NP(f)$, and apply Proposition 2.3 to $f'$. We find that

$$P_S(\tilde{f}) = \{(b,c) ∈ \mathbb{Z}/a\mathbb{Z} × \mathbb{Z}/a\mathbb{Z} | \tilde{f}(b * c) ≠ \tilde{f}(b) + \tilde{f}(c) and b * c ∉ S\}$$

has cardinality at least $\frac{(1-q-2p)^2}{4} a^2 + \frac{(1-q-2p)}{2} a$. Since $b * c ∉ S$ implies that $f(b * c) = f(b + c)$, this set is contained in the problem set $P(f)$.

3. **Proof of the Main Theorem**

The main goal of this section is to prove Theorem 1.6. We start with some definitions.

**Definition 3.1.** Let $1 < a$, and $f : [2a] → \mathbb{Q}$. We define the following **problem sets** of $f$:

$$P(f) := \{(x,y) ∈ [a] × [a] | f(x + y) ≠ f(x) + f(y)\},$$

and

$$P_1(f) := \{x ∈ [a] | f(x + 1) ≠ f(x) + f(1)\},$$

and

$$P_a(f) := \{x ∈ [a] | f(x + a) ≠ f(x) + f(a)\}.$$
The following proposition says that \( P_1(f), P_a(f), P(f) \) cannot be simultaneously small.

**Proposition 3.2.** Let \( p, q \in (0,1) \) such that \( p < \frac{1-a}{2} \), \( a \in \mathbb{N} \) with \( 1 < a \), and let \( f : [2a] \to \mathbb{Q} \) such that \( f(a) \neq af(1) \). If

\[
|P_1(f)| \leq qa \quad \text{and} \quad |P_a(f)| \leq pa
\]

then

\[
|P(f)| \geq F(p,q) \cdot a^2,
\]

where

\[
F(p,q) = \frac{(1-q-2p)^2}{4}.
\]

**Proof.** Without loss of generality we can assume \( f(a) = 0 \) and hence \( f(1) \neq 0 \). Indeed, suppose we have shown the statement for every \( \tilde{f} \) with \( \tilde{f}(a) = 0 \). Then for any \( f : [2a] \to \mathbb{Q} \) with \( f(a) \neq af(1) \), we take \( \tilde{f} : [2a] \to \mathbb{Q} \) to be \( \tilde{f}(x) = af(x) - xf(a) \). Now we observe that \( \tilde{f}(a) = 0 \neq af(1) \), and that \( P(f) = P(\tilde{f}) \), \( P_1(f) = P_1(\tilde{f}) \), \( P_a(f) = P_a(\tilde{f}) \).

We write \( Z(f) = \{ x_1 < \cdots < x_m \} \). Note that for \( 1 \leq i < m \), one of the elements \( x_i, x_{i+1}, \ldots, x_{i+1} - 1 \) needs to be in \( P_1(f) \) since \( f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i) \cdot f(1) \). Likewise, at least one of the elements \( 1, 2, \ldots, x_1 - 1 \) needs to be in \( P_1(f) \). Thus we have

\[
|Z(f)| \leq |P_1(f)| \leq qa,
\]

and by assumption we have \( |NP(f)| = |P_a(f)| \leq pa \). Now we can apply Corollary 2.4 to conclude. \( \square \)

We now prove Theorem 1.6

**Proof of the Main Theorem.** Consider a \( c \)-quasihomomorphism \( f = (f_1, \ldots, f_n) : \mathbb{Z} \to \mathbb{Q}^n \). Our goal is to show that for every \( a \in \mathbb{Z} \) we have \( w_H(f(a) - a \cdot f(1)) \leq C \) for some constant \( C \) depending only on \( c \). We start with the case \( a > 0 \).

Write \( M := w_H(f(a) - af(1)) \). Without loss of generality, we have \( f_1(a) \neq af(1), \ldots, f_M(a) \neq af(1) \). We will show that \( M \leq C' \) for some constant \( C' \) depending on \( c \) only. To this end, fix small parameters \( p, q \in (0,1) \) (to be optimized over later) and consider the restrictions \( f_i : [2a] \to \mathbb{Q} \) of the components of \( f \). By Proposition 3.2 for every \( i \), we have that either

(i) \( |P_1(f_i)| > qa \), or

(ii) \( |P_a(f_i)| > pa \), or

(iii) \( |P(f_i)| \geq F(p,q)a^2 \).

Let \( m_0 \) be the number of coordinates \( i \) such that (iii) holds. We define \( m_1 \) and \( m_2 \) analogously, for (i) and (ii) respectively.

By counting the number of triples \( (i,x,y) \in [M] \times [a] \times [a] \) such that \( f_i(x+y) - f_i(x) - f_i(y) \neq 0 \) in two ways, we see that

\[
\sum_{x=1}^{a} \sum_{y=1}^{a} w_H(f(x+y) - f(x) - f(y)) = \sum_{i=1}^{M} |P(f_i)|.
\]
Because $f$ is a $c$-quasihomomorphism, the left hand side is at most $a^2 c$. On the other hand, the right hand side is at least $m_0 F(p,q) a^2$, so

$$a^2 c \geq \sum_{x=1}^{a} \sum_{y=1}^{a} w_H (f(x+y) - f(x) - f(y)) = \sum_{i=1}^{M} |P(f_i)| \geq m_0 F(p,q) a^2.$$ 

So we obtain $m_0 \leq \frac{c}{F(p,q)}$. Similarly we find

$$ac \geq \sum_{x=1}^{a} w_H (f(x+1) - f(x) - f(1)) = \sum_{i=1}^{M} |P_1(f_i)| > m_1 qa,$$

so that $m_1 < \frac{c}{q}$. Finally,

$$ac \geq \sum_{x=1}^{a} w_H (f(x+a) - f(x) - f(a)) = \sum_{i=1}^{M} |P_a(f_i)| > m_2 pa.$$

So $m_2 < \frac{c}{p}$. But now $M = m_0 + m_1 + m_2 < c\left(\frac{1}{F(p,q)} + \frac{1}{q} + \frac{1}{p}\right) =: C'$. The case $a = 0$ is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \leq c.$$ 

Finally, let us consider the case $a < 0$. Then

$$w_H(f(a) - a \cdot f(1)) \leq w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) + w_H(f(-a) - (-a) \cdot f(1)) \leq 2c + C' =: C.$$

This completes the proof. 

To get the explicit bound $28c$ from Theorem 1.6, we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p,q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1-q-2p)^2}.$$ 

Note that this function is strictly convex for $(p,q) \in \mathbb{R}_{>0}^2$, so that it has at most one minimum in the positive orthant. We find this by setting the partial derivatives equal to zero and solving for $p,q$. The minimum is $\approx 27.6817$ and attained at $(p,q) \approx (0.1167, 0.16500)$.

References


MATHEMATICAL INSTITUTE, UNIVERSITY OF BERN, SIDLERSTRASSE 5, 3012 BERN, SWITZERLAND; AND DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EINDHOVEN UNIVERSITY OF TECHNOLOGY, P.O. BOX 513, 5600MB, EINDHOVEN, THE NETHERLANDS

Email address: jan.draisma@unibe.ch

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EINDHOVEN UNIVERSITY OF TECHNOLOGY, P.O. BOX 513, 5600MB, EINDHOVEN, THE NETHERLANDS

Email address: r.h.eggermont@tue.nl

MATHEMATICAL INSTITUTE, UNIVERSITY OF BERN, ALPENEGGSTRASSE 22, 3012 BERN, SWITZERLAND

Email address: tim.seynnaeve@unibe.ch

MATHEMATICAL INSTITUTE, UNIVERSITY OF BERN, ALPENEGGSTRASSE 22, 3012 BERN, SWITZERLAND

Email address: nafie.tairi@unibe.ch

POLITECNICO DI TORINO, DIPARTIMENTO DI SCIENZE MATEMATICHE “G.L. LAGRANGE”, CORSO DUCA DEGLI ABRUZZI 24 10129 TORINO, ITALY

Email address: emanuele.ventura@polito.it, emanueleventura.sw@gmail.com