# Singularities of Poisson structures and Hamiltonian bifurcations

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by

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# Singularities of Poisson structures and Hamiltonian bifurcations

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**Abstract**

Consider a Poisson structure on $C^\infty(\mathbb{R}^3, \mathbb{R})$ with bracket $\{,\}$ and suppose that $C$ is a Casimir function. Then $\{f, g\} =< \nabla C, (\nabla g \times \nabla f) >$ is a possible Poisson structure. This confirms earlier observations concerning the Poisson structure for Hamiltonian systems that are reduced to a one degree of freedom system and generalizes the Lie-Poisson structure on the dual of a Lie algebra and the KKS-symplectic form. The fact that the governing reduced Poisson structure is described by one function makes it possible to find a representation, called the energy-momentum representation of the Poisson structure, describing both the singularity of the Poisson structure and the singularity of the energy-momentum mapping and hence the bifurcation of relative equilibria for such systems. It is shown that Hamiltonian Hopf bifurcations are directly related to singularities of Poisson structures of type $\mathfrak{sl}(2)$.
**Key Words:** Poisson structure, Casimir, Bifurcation, Hamiltonian system, reduction, singularity, Hamiltonian Hopf bifurcation, relative equilibria.
**AMS Subject Classification:** 37J20; 37J15; 53D20; 53D05; 53D17; 70H33; 70H12; 34C14.

# 1 Introduction

In this paper we will start with showing that a Poisson structure on $\mathbb{R}^3$ with Casimir $C$ is determined by this Casimir. This concept is then used in considering bifurcations of relative equilibria of Hamiltonian systems on $\mathbb{R}^{2n}$ with $\mathbb{T}^{n-1}$ symmetry. When such a system can be reduced to a one degree of freedom system, the Poisson structure for the reduced system can be described by just one Casimir function. Choosing the right form of equivalence this Casimir function can be put into a normal form such that its singularity describes the bifurcation of relative equilibria. It is shown that this singularity is actually the same as the singularity of the energy-momentum mapping. The ideas are illustrated by several examples.

Consider a Hamiltonian system $(H, \mathbb{R}^{2n}, \omega)$ which is symmetric with respect to a Hamiltonian $G$-action, where the group $G$ is generated by the flows of $n-1$ independent integrals $S_1, \cdots, S_{n-1}$. In this situation the system can be reduced to a one-degree-of-freedom system by singular reduction (see [2]). Symplectic reduction was first introduced by Meyer [18] and Marsden and Weinstein [16]. The more constructive approach in [2] gives a general framework for the construction of reduced phase spaces as introduced in [14, 4, 20]. The reduction to a one-degree-of-freedom system is performed through the construction of an orbit map

$$\rho : \mathbb{R}^{2n} \to \mathbb{R}^k; x \to (\rho_1(x), \rho_2(x), \rho_3(x), \cdots, \rho_k(x)) ,$$

where the $\rho_i$ are invariants for the group action of $G$. Assume that by restricting to the surface given by $S_i = s_i$ the orbit map restricts to

$$\rho_s : \mathbb{R}^{2n} \to \mathbb{R}^3; x \to (\rho_1(x), \rho_2(x), \rho_3(x)) ,$$

where a relation $C(\rho_1, \rho_2, \rho_3; s) = 0$ among the invariants, and depending on parameters $s$, determines a two-dimensional algebraic variety in $\mathbb{R}^3$, which is the reduced phase space. The standard Poisson structure on $\mathbb{R}^{2n}$ is induces a Poisson structure on the image of the orbit map, which is assumed to be a restriction of a Poisson structure on $\mathbb{R}^k$. Note that not in all cases the Poisson structure on the orbit space extends to a Poisson structure on the ambient space (see [8]). The symplectic leaves of the Poisson structure on the orbit space are symplectic manifolds which can be identified with the smooth parts of the level sets of the Casimir functions. The same holds for the reduction map $\rho_s$ and the Casimir $C(\rho_1, \rho_2, \rho_3; s)$ in $\mathbb{R}^3$ obtained by fixing the $s$-level. These level sets are the reduced phase spaces (see[17]). The structure matrix for the induced Poisson structure on $\mathbb{R}^3$ is given by $W_{ij} = \{\rho_i, \rho_j\}_{\mathbb{R}^{2n}}$, where $\{,\}_{\mathbb{R}^{2n}}$ denotes the standard Poisson bracket on $\mathbb{R}^{2n}$. The fact that the induced bracket is $\{f, g\}_{\mathbb{R}^3} =< \nabla C, (\nabla g \times \nabla f) >$ was already observed in [2], but also in [6, 9, 11], where the above line of reduction to a one-degree-of-freedom system is applied to specific problems such as the spherical pendulum, the Lagrange top and the 3D Hénon-Heiles system.

In, for instance, [19, 17] it can be found that the above form for the bracket is the natural form on co-adjoint orbits, or corresponds to the Lie-Poisson structure on the dual of a Lie algebra, which corresponds to the Kirillov-Kostant-Souriau symplectic form on the symplectic leaves. However, in general the above reduction procedure will not map to a set of invariants that can be identified with a finite dimensional Lie algebra, more specifically because the induced Poisson structure is in general nonlinear. Thus the more general form $\{f, g\}_{\mathbb{R}^3} =< \nabla C, (\nabla g \times \nabla f) >$ generalizes these ideas to Poisson structures on reduced phase spaces embedded in $\mathbb{R}^3$. One may now put $C$ into a local normal form.

By choosing a special form for the Casimir function (see section (4)) related to the Hamiltonian function of the reduced system, one obtains what is called the energy-momentum representation of the Poisson structure. By choosing the appropriate form of equivalence (see def. (4.4)) it is shown that the singularity of the Poisson structure in energy-momentum representation is the same as the singularity of the energy-momentum mapping for the Hamiltonian system. Therefore the singularity of the Poisson structure is directly related to the bifurcation of relative equilibria of the Hamiltonian system of which the reduced Poisson structure was considered.

Finally in a number of examples it is shown what the consequences of this approach are for showing equivalence of systems and determining bifurcations. Especially Hamiltonian Hopf bifurcations are directly related to points where the local Poisson structure has a local normal form of type $\mathfrak{sl}(2)$.

# 2 Poisson structure on $\mathbb{R}^3$ with a given Casimir

Let $\{,\}$ denote a Poisson structure on $C^\infty(\mathbb{R}^3,\mathbb{R})$, that is, $\{,\}$ is a Lie bracket, making $C^\infty(\mathbb{R}^3,\mathbb{R})$ into a Lie algebra, which also satisfies the Leibnitz identity, that is, $\{fg,h\} = f\{g,h\}+g\{f,h\}$, for all $f,g,h \in C^\infty(\mathbb{R}^3,\mathbb{R})$. Such a Poisson structure can also be written as

$$\{f,g\} =< \nabla f, W\nabla g > ,$$

where $W$ is the structure matrix of the Poisson structure and $<,>$ is the standard inner product on $\mathbb{R}^3$. Due to the antisymmetry of the bracket $W$ must be a antisymmetric matrix.

A function $C$ is a Casimir for the Poisson structure if $\{C,f\} = 0$ for all $f \in C^\infty(\mathbb{R}^3,\mathbb{R})$.

**Theorem 2.1** *Let $\{,\}$ denote a Poisson bracket on $C^\infty(\mathbb{R}^3,\mathbb{R})$ with Casimir $C$, then $\{f,g\} =< \varphi\nabla C\, , (\nabla g \times \nabla f) >$, with $\varphi$ an arbitrary smooth function.*

**Proof :** An antisymmetric matrix

$$W = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}$$

can be identified with a vector $w \in \mathbb{R}^3$, such that $Wv = w \times v$, for a vector $v \in \mathbb{R}^3$, where $\times$ denotes the common vector product on $\mathbb{R}^3$. Consequently $\{f,g\} =< \nabla f, W\nabla g >=$ $=< \nabla f, w\times\nabla g >=< w, \nabla g\times\nabla f >$. If $C$ is a Casimir it follows that $< \nabla f, w\times\nabla C >= 0$ for all $f \in C^\infty(\mathbb{R}^3,\mathbb{R})$. Thus $w = \varphi\nabla C$, with $\varphi$ an arbitrary smooth function and $\{f,g\} = \varphi < \nabla C, (\nabla g \times \nabla f) >$. Consequently in the above expression for $W$ we have

$$w_i = \varphi(x_1,x_2,x_3)\frac{\partial C(x_1,x_2,x_3)}{\partial x_i} \; .$$

Note that the anti-symmetry is evident. The fact that this bracket satisfies the Jacobi identity can be shown by checking that the Schouten bracket

$$[W,W]_{ijk} = \sum_{\ell=1}^{3} \left( W_{\ell k} \frac{\partial W_{ij}}{\partial x_\ell} + W_{\ell i} \frac{\partial W_{jk}}{\partial x_\ell} + W_{\ell j} \frac{\partial W_{ki}}{\partial x_\ell} \right)$$

vanishes. **q.e.d.**

When the Poisson structure on $\mathbb{R}^3$ is a Poisson structure which is induced by an orbit map, then there is no freedom left for choosing an arbitrary function $\varphi$, because the invariants determining the orbit map completely determine the structure matrix for the Poisson structure, as well as the Casimir.

This theorem can also be obtained from a more general theorem in [10]. Where it is stated that, given smooth functions $f$, $f_1, \cdots, f_{n-2}$ on $\mathbb{R}^n$, one may associate to the $(n-2)$-form $\psi = f df_1 \wedge \cdots \wedge f_{n-2}$ a bivector field $\Lambda$ through $\psi = i_\Lambda \Omega$, where $\Omega$ is the standard volume form on $\mathbb{R}^n$. This bivector field $\Lambda$ is then a Poisson structure with Casimirs $f_1, \cdots, f_{n-2}$ and Poisson bracket

$$\{g,h\}\Omega = f\, dg \wedge dh \wedge df_1 \wedge \cdots \wedge f_{n-2} \; .$$

Using the Hodge $*$ operator the righthand side can also be written as

$$f < df_1 \wedge \cdots \wedge f_{n-2}, *\,(dg \wedge dh) > \Omega \; ,$$

which on $\mathbb{R}^3$ results in the theorem above if one identifies $*\,(dg \wedge dh)$ with the vector product $dg \times dh$.

# 3 Changing the Poisson structure with a given Casimir

Suppose we have a reduction through an orbit map

$$\rho : \mathbb{R}^{2n} \to \mathbb{R}^3; x \to (\rho_1(x), \rho_2(x), \rho_3(x)) \; ,$$

as described in the introduction. Let $C_\rho$ denote the Casimir and denote the bracket on the target space $\mathbb{R}^3$ with coordinates $\rho$ by $\{\ ,\ \}_\rho$. Thus $\{f,g\}_\rho =< \nabla_\rho C, (\nabla_\rho g \times \nabla_\rho f) >$. When a diffeomorphism $\psi : \rho(\mathbb{R}^{2n}) \to \mathbb{R}^3; \rho \to \psi(\rho)$ is considered, it follows that

$$\{f,g\}_\psi = \{f \circ \psi, g \circ \psi\}_\rho \; ,$$

where $\{\ ,\ \}_\psi$ denotes the induced bracket on the image of $\psi$. It follows that

$$\{f,g\}_\psi =< \nabla_\psi \tilde{C}, (\nabla_\psi g \times \nabla_\psi f) > \; ,$$

where $\tilde{C}(\psi(\rho)) = C(\rho)$.

**Definition 3.1** *Let $\mathcal{P}_C$ denote a Poisson structure on $\mathbb{R}^3$ with Casimir $C$. Two Poisson structures $\mathcal{P}_C$ and $\mathcal{P}_{\tilde{C}}$ are locally $\mathcal{D}_p^\infty$-equivalent at a point $p$ if there exists a $C^\infty$ diffeomorphism $\varphi : \rho(\mathbb{R}^{2n}) \to \mathbb{R}^3$ in a neighborhood of $p$, with $\varphi(p) = p$, such that $\tilde{C} \circ \varphi = C$.*

Considering $C$ as a singularity at a point $p$ it is possible to put $C$ locally in its singularity theoretic normal form by a local $p$ preserving diffeomorphism. Disregarding any parameters on which $C$ might depend, it follows that the only locally stable structures at the origin have $C(u) = u_1^2 + u_2^2 + u_3^2$ or $C(u) = u_1^2 + u_2^2 - u_3^2$. Consequently for the stable structures the image of the reduction map is locally near $p$ isomorphic to the Lie algebra $\mathfrak{u}(2)$ or $\mathfrak{u}(1,1)$. In [7] these are called of type $\mathfrak{so}(3)$ and type $\mathfrak{sl}(2)$ respectively. However, in reduction problems $C$ will depend on parameters and one has to take these parameters into account. When also unfolding parameters come into the problem the parameters introduced by the reduction process will have to be dealt with as distinguished parameters.

When a specific Hamiltonian system is reduced, besides the reduced phase space, there will be a reduced Hamiltonian function, which together with the the induced Poisson structure defines the reduced system. Let $C(\rho; s)$ denote the Casimir defining the Poisson structure for the reduced phase space. It depends also on parameters $s$ introduced by restricting to integral level sets. Let $H(\rho; s, \lambda)$ denote the reduced Hamiltonian, which besides the parameters $s$ might depend on some system parameters $\lambda$. If the map $\psi : (\rho_1, \rho_2, \rho_3) \to (\rho_1, \rho_2, H(\rho) : s, \lambda)$ is a global diffeomorphism on $\rho(\mathbb{R}^{2n})$, then we obtain a Poisson structure with Casimir $\tilde{C}(\rho_1, \rho_2, H; s, \lambda)$. We call this the *energy representation* of the reduced Poisson structure. In this representation the Casimir not only reflects properties of the Poisson structure but also of the reduced Hamiltonian system. It may now be viewed from a different singularity theoretic angle to deal with relative equilibria, as will be explained in the next section.

# 4 Relative equilibria and singularities of projections

Consider a Hamiltonian system $(H_\lambda, \mathbb{R}^{2n}, \omega)$, $H_\lambda$ being the Hamiltonian $H_\lambda : \mathbb{R}^{2n} \to \mathbb{R}$, with $\lambda$ a parameter, and $\omega$ the standard symplectic form. Suppose this system has $n-1$ functionally independent integrals $S_1, \cdots, S_{n-1}$ and an orbit mapping $\rho : \mathbb{R}^{2n} \to (X, Y, H_\lambda, S_1, \cdots, S_{n-1})$.

**Lemma 4.1** *If for the group $G$ generated by the flows of $S_1, \cdots, S_{n-1}$ we have an orbit map $\rho : \mathbb{R}^{2n} \to (X, Y, H_\lambda, S_1, \cdots, S_{n-1})$ then $\{S_i, S_j\} = 0$ for $i, j = 1, \cdots, n-1$, i.e the Lie algebra generated by the $S_i$ is abelian.*

**Proof :** If $\rho$ is an orbit map the $S_i$ are invariants for $G$ and consequently $\{S_i, S_j\} = 0$ for $i, j = 1, \cdots, n-1$. **q.e.d.**

In the following we will furthermore suppose that the group $G$ generated by the flows of the $S_i$ is compact. Consequently, the existence of a orbit map of the above form implies that $G = \mathbb{T}^{n-1}$.

Suppose that the image of $\rho$ is determined by $C(X, Y, H, S, \lambda) = 0$, and possibly some inequalities, as a subset of $\mathbb{R}^{n+2}$. If $C(X, Y, H, S, \lambda)$ is polynomial, then $\rho(\mathbb{R}^{2n}) \subset \mathbb{R}^{n+2}$ will be a semi-algebraic set $\hat{S}_\lambda$. Taking $S_1 = s_1, \cdots, S_{n-1} = s_{n-1}$ constant, this mapping is a reduction mapping reducing the system to a one-degree-of-freedom system on a set $\hat{S}_{s,\lambda} = \rho(S^{-1}(s))$ defined by $C(X, Y, H; s, \lambda) = 0$, and possibly some inequalities, in $X, Y, H$-space. $\hat{S}_{s,\lambda}$ is the reduced phase space. $\hat{S}_{s,\lambda}$ will again be a semi algebraic set. In view of section 2 $C$ is a Casimir for the induced Poisson structure on $\mathbb{R}^3$ and therefore defines the Poisson structure on $\mathbb{R}^3$. Similarly the Casimirs $C$, $S_1, \cdots, S_{n-1}$ determine the Poisson structure on $\mathbb{R}^{n+2}$ being the target space of the orbit map. Also on $\mathbb{R}^{n+2}$ the Poisson structure is completely determined by $C$ because $C$ determines the only nonzero $3 \times 3$ block in the structure matrix. We call this the *energy-momentum representation* for the Poisson structure on the orbit space. Taking $H = h$ a constant on the reduced phase space we obtain the trajectories of the reduced system. If this trajectory is a point $p$ its counter image $\rho^{-1}(p)$ consists of relative equilibria (using the definition in [1]). With respect to this property $\hat{S}_{s,\lambda}$ can be considered as a generalization of the graph of an amended potential. Consider the energy-momentum map

$$\mathcal{EM} : \mathbb{R}^{2n} \to \mathbb{R}^n; (x, y) \to (H, S_1, \cdots, S_{n-1}) .$$

The relative equilibria are also the critical points of $\mathcal{EM}$. That is, a critical point is a fixed point for at least one of the actions induced by the Hamiltonian flow of one of the integrals $H, S_1, \cdots, S_{n-1}$. Thus the singularity of the energy-momentum map describes the relative equilibria.

By factorizing the energy momentum map through the orbit map the following is now immediate

**Proposition 4.2** *The singularity of the map $\mathcal{EM}$ is the same as the singularity of the orthogonal projection $P : \hat{S}_\lambda \to (H, S_1, \cdots, S_{n-1})$.*

When $H$ depends on some system parameters $\lambda$, then so will the energy-momentum map, and $\lambda$ will unfold the singularity. The singularity is obtained as a surface in $(H, S_1, \cdots, S_{n-1}, \lambda)$-space by eliminating $X$ and $Y$ from the system of equations

$$\begin{cases} \frac{\partial}{\partial X} C(X, Y, H, S_1, \cdots, S_{n-1}, \lambda) = 0 , \\ \frac{\partial}{\partial y} C(X, Y, H, S_1, \cdots, S_{n-1}, \lambda) = 0 , \\ C(X, Y, H, S_1, \cdots, S_{n-1}, \lambda) = 0 , \end{cases}$$

taking into account any inequalities involving $X$ and $Y$ defining the orbit space.

To normalize the singularity of the projection locally, the group of diffeomorphisms on the image of the orbit map should be restricted to those diffeomorphisms that fiber over a diffeomorphism of the base space (see [3]).

**Definition 4.3** *Let $\hat{S}_\lambda$ and $\tilde{S}_\mu$ be orbit spaces as defined above, depending on parameters $\lambda \in \mathbb{R}^s$ and $\mu \in \mathbb{R}^t$. Then $\hat{S}_\lambda$ and $\tilde{S}_\mu$ are locally projection equivalent at the origin if there exist $\mu$ dependent origin preserving diffeomorphisms $\varphi_1$ and $\varphi_2$ on $\hat{S}$ and $\mathbb{R}^n$ respectively, and a smooth map $\chi$ between the parameter spaces, $\chi(\mu) = \lambda$, $\chi(0) = 0$, such that $P(\varphi_1(\chi^*(\hat{S}_\lambda))) = \varphi_2(P(\tilde{S}_\mu))$.*

**Definition 4.4** *We call two Poisson structures on $\mathbb{R}^{n+2}$ given by $\hat{C}$ and $\tilde{C}$ as above, i.e. in energy-momentum representation, locally projection equivalent if the corresponding orbit spaces are locally projection equivalent.*

The following now follows with Proposition (4.2).

**Theorem 4.5** *If two Poisson structures in energy-momentum representation are projection equivalent then the two corresponding energy-momentum mappings have diffeomorphic singularities.*

This theorem can be extended under certain conditions

**Theorem 4.6** *If each diffeomorphism on $\rho(\mathbb{R}^{2n})$ lifts to a G-equivariant diffeomorphism on $\mathbb{R}^{2n}$, two Poisson structures in energy-momentum representation are projection equivalent if and only if the two corresponding energy-momentum mappings are $\mathcal{A}_G$-equivalent.*

Here a diffeomorphism $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a lift of $\varphi$ if $\rho \circ \psi = \varphi \circ \rho$. Note that here $\mathcal{A}_G$-equivalence means right-left equivalence by diffeomophisms, where we consider $G$-equivariant diffeomorphisms on the domain and, because the $G$-action on the image is trivial, just diffeomorphisms on the target space. Furthermore the fact that the Poisson structures are in energy-momentum representation ensures that the energy-momentum map factorizes through the orbit map. If a Poisson structure is in energy-momentum representation and one wants to preserve the $S_j$ part of the representation one has to restrict to diffeomorphisms of the orbit space, respectively, to G-equivariant diffeomorphisms, that leave the $S_j$ fixed.

**Proof :** Let $\rho$ and $\tilde{\rho}$ be orbit maps such that the corresponding Poisson structures are in energy-momentum representation. This implies that there are energy-momentum maps $\mathcal{EM}$ and $\widetilde{\mathcal{EM}}$ such that $\mathcal{EM} = P \circ \rho$ and $\widetilde{\mathcal{EM}} = P \circ \tilde{\rho}$. Now projection equivalence gives that there exist a $\varphi \in Diff(\rho(\mathbb{R}^{2n}))$ and a $\theta \in Diff(\mathbb{R}^n)$ such that $P \circ \varphi = \theta \circ P$ on the images of $\rho$ and $\tilde{\rho}$, which is equivalent to saying that $P \circ \varphi \circ \rho = \theta \circ P \circ \tilde{\rho}$. If $\varphi$ lifts under $\rho$ to an $G$-equivariant diffeomorphism $\underset{\smile}{\psi}$ then this is equivalent to $P \circ \rho \circ \psi = \theta \circ P \circ \tilde{\rho}$, which is equivalent to $\mathcal{EM} \circ \psi = \theta \circ \widetilde{\mathcal{EM}}$. **q.e.d.**

**Remark 4.7** It is clear that any $G$-equivariant diffeomorphism on $\mathbb{R}^{2n}$ corresponds under $\rho$ to a diffeomorphism on $\mathbb{R}^{n+2}$. The condition of the theorem states the opposite, for a compact group $G$ any diffeomorphism on $\rho(\mathbb{R}^{2n})$ should lift to a $G$-equivariant diffeomorphism on $\mathbb{R}^{2n}$. This result is only known for finite groups (see [15]). Furthermore if we consider orientation preserving diffeomorphisms, i.e. diffeomorphisms from the identity component of $Diff(\rho(\mathbb{R}^{2n}))$, then the existence of a lift follows from the Schwarz isotopy lifting theorem (see [22]).

# 5 Example: The Hamiltonian Hopf bifurcation, the Lagrange top and the spherical pendulum

## 5.1 The Hamiltonian Hopf bifurcation

Consider a Hamiltonian system on $\mathbb{R}^4$ (with the standard symplectic form) given by

$$H(x_1, x_2, y_1, y_2) = X + \nu Y + aY^2 \ ,$$

with $X(x,y) = \frac{1}{2}(x_1^2 + x_2^2)$, $Y(x,y) = \frac{1}{2}(y_1^2 + y_2^2)$. The system has integral $S(x,y) = x_2 y_1 - x_1 y_2$, which generates an $S^1$-action. Set $Z = x_1 y_1 + x_2 y_2$. Then $X, Y, Z, S$ form a Hilbert basis for the invariants of this $S^1$-action. Reduction with respect to this $S^1$-action, and setting $S(x,y) = s$, gives the reduction map

$$\rho : \mathbb{R}^4 \to (X, Y, Z; s) \ .$$

The reduced phase spaces are determined by

$$C(X, Y, Z; s) = 4XY - Z^2 - s^2 = 0 \ , X \geqslant 0 \ , Y \geqslant 0 \ ,$$

where $S(x,y) = s$. Obviously, the singularity of the Poisson structure at the origin is of type $\mathfrak{sl}(2)$. The reduced Hamiltonian is

$$H(X, Y, Z; \nu) = X + \nu Y + aY^2 \ .$$

The Casimir $C$ defines the Poisson structure, i.e.

$$\{f, g\} = < \nabla C, \nabla g \times \nabla f > \ .$$

The image of the reduction map is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. The map

$$\psi : \mathbb{R}^3 \to \mathbb{R}^3; (X,Y,Z) \to (H,Y,Z)$$

is a diffeomorphism. It changes $C$ into

$$\tilde{C}(H,Y,Z;s,\nu) = 4YH - 4\nu Y^2 - 4aY^3 - Z^2 - s^2 \ .$$

The relation between the projection and the energy-momentum map is implicit in [20]. In fact $\tilde{C}$ can be considered as a local normal form for the Hamiltonian Hopf bifurcation.

The surface $\tilde{C}(H,Y,Z;s,\nu) = 0$ is a smooth surface for $s \neq 0$, which is the graph of the function $H(Y,Z;s,\nu) = \nu Y + aY^2 + \frac{Z^2+s^2}{4Y}$. When $s = 0$ the surface has a cone-like singularity at the origin. The relative equilibria are the critical points of $H(Y,Z;s,\nu)$ in the halfplane $Y > 0$. When $s = 0$ one has to add the origin.

## 5.2 The Lagrange top

In [6] the reduction of the Lagrange top is performed in detail (see also [5]). Reduction is performed with respect to a right and left $S^1$-action. The corresponding integrals are the corresponding angular momenta $J_\ell$ and $J_r$. The reduction map is

$$\sigma : TSO(3) \to (\sigma_1, \sigma_2, \sigma_3) \ .$$

The reduced phase space is determined by

$$C_L(\sigma;a,b) = -\sigma_2^2 - (a - b\sigma_1)^2 + (1-\sigma_1^2)\sigma_3 = 0 \ , \ |\sigma_1| \leqslant 1 \ , \ \sigma_3 \geqslant 0 \ ,$$

where $J_\ell = a$ and $J_r = b$. The reduced Hamiltonian is

$$H_L(\sigma;a,b) = \frac{1}{2I_1}\sigma_3 + \chi\sigma_1 \ ,$$

with $I = diag(I_1, I_2, I_3)$ the moment of inertia tensor of the top, and $\chi$ a constant. Changing time-scale by setting $t_{new} = I_1 t$, and length-scale by setting $\chi I_1 = 1$, the Hamiltonian becomes

$$H_L(\sigma;a,b) = \frac{1}{2}\sigma_3 + \sigma_1 \ .$$

The Casimir $C_L$ defines the Poisson structure, i.e.

$$\{f,g\} = < \nabla C_L, \nabla g \times \nabla f > \ .$$

The singularity at the origin of this Poisson structure is not an isolated singularity of one of the two basic types but the origin is part of a fold singularity. The map

$$\psi : \mathbb{R}^3 \to \mathbb{R}^3; (\sigma_1,\sigma_2,\sigma_3) \to (\sigma_1,\sigma_2,H_L)$$

is a diffeomorphism. It changes $C_L$ into

$$\tilde{C}_L(\sigma_1, \sigma_2, H_L; a, b) = -\sigma_2^2 - (a - b\sigma_1)^2 + 2(H_L - \sigma_1)(1 - \sigma_1^2) .$$

which gives an energy-momentum representation for the Poisson structure.

**Theorem 5.1** *The Poisson structure given by $C_L$ is at $(\sigma_1, \sigma_2, \sigma_3) = (1, 0, 0)$ locally of type $\mathfrak{sl}(2)$.*

**Proof :** Introduce new generators $N$ and $S$ for the action of $J_\ell$ and $J_r$ by setting $J_\ell = N + \frac{1}{2}S$ and $J_r = N - \frac{1}{2}S$. Furthermore apply the translation $\sigma_1 = 1 - \tilde{\sigma}_1$, $0 \leqslant \tilde{\sigma}_1 \leqslant 2$, then $C_L$ becomes

$$-\sigma_2^2 - S^2 + 2\tilde{\sigma}_1\sigma_3 - 2NS\tilde{\sigma}_1 + S^2\tilde{\sigma}_1 - \sigma_3\tilde{\sigma}_1^2 - N^2\tilde{\sigma}_1^2 + NS\tilde{\sigma}_1^2 - \frac{1}{4}S^2\tilde{\sigma}_1^2 \ , \ 0 \leqslant \tilde{\sigma}_1 \leqslant 2 \ , \ \sigma_3 \geqslant 0 \ ,$$

which is at zero locally equivalent to

$$-\sigma_2^2 - S^2 + 2\tilde{\sigma}_1\sigma_3 .$$

**q.e.d.**

**Theorem 5.2** *The Poisson structure given by $\tilde{C}_L(\sigma_1, \sigma_2, H_L, J_\ell, J_r)$ is at $(\sigma_1, \sigma_2, \sigma_3) = (1, 0, 0)$ locally projection equivalent to $\tilde{C}(H, Y, Z, S, \nu)$.*

**Proof :** Like in the previous proof set $J_\ell = N + \frac{1}{2}\tilde{S}$ and $J_r = N - \frac{1}{2}\tilde{S}$. Furthermore apply the translation $\sigma_1 = 1 - \tilde{\sigma}_1$, $0 \leqslant \tilde{\sigma}_1 \leqslant 2$. In addition apply the transformation $H_L = \tilde{H} + \frac{1}{2}NS$ then $\tilde{C}_L$ transforms to

$$-\tilde{S}^2 - \sigma_2^2 + 4\tilde{H}\tilde{\sigma}_1 + \tilde{S}^2\tilde{\sigma}_1 + 4\tilde{\sigma}_1^2 - 2\tilde{H}\tilde{\sigma}_1^2 - \frac{1}{4}\tilde{S}^2\tilde{\sigma}_1^2 - 2\tilde{\sigma}_1^3 .$$

Finally transform $S = \tilde{S} - \frac{1}{2}\tilde{S}\tilde{\sigma}_1$, and $H = \tilde{H} - \frac{1}{2}\tilde{H}\tilde{\sigma}_1$. Setting $\sigma_2 = Z$, $\tilde{\sigma}_1 = Y$, and $1 - \frac{1}{4}N^2 = \nu$ finally gives

$$4YH - 4\nu Y^2 - 2Y^3 - Z^2 - S^2 .$$

**q.e.d.**

Note that considering $\nu$ as a parameter actually means that one compares the Lagrange top to the Hamiltonian Hopf bifurcation after reduction with respect to $J_r$, and such that the origin corresponds to the sleeping top. This gives yet an other proof of the presence of a Hamiltonian Hopf bifurcation in the Lagrange top (cf. [12]), which is very simple and based completely on a singularity theoretic interpretation of the reduced geometry.

### 5.3 The spherical pendulum

In a similar fashion one can describe the spherical pendulum ([6]). The reduction is performed with respect to rotation about the $x_3$ axis in configuration space which is parallel to the gravitational force. Let $J$ denote the corresponding angular momentum. The reduction mapping is

$$\sigma : TS^2 \to (\sigma_1, \sigma_2, \sigma_3) \ .$$

The reduced phase space is determined by

$$C_S(\sigma; j) = \sigma_2^2 - (1 - \sigma_1^2)\sigma_3 + j^2 = 0 \ , \ |\sigma_1| \leqslant 1 \ , \ \sigma_3 - \sigma_2^2 \geqslant 0 \ ,$$

where $J = j$. The reduced Hamiltonian is

$$H_S(\sigma; j) = \frac{1}{2}\sigma_3 + \sigma_1 \ .$$

The Casimir $C_S$ defines the Poisson structure, i.e.

$$\{f, g\} =< \nabla C_S, \nabla g \times \nabla f > \ .$$

The map

$$\psi : \mathbb{R}^3 \to \mathbb{R}^3; (\sigma_1, \sigma_2, \sigma_3) \to (\sigma_1, \sigma_2, H_S)$$

is a diffeomorphism. It changes $C_S$ into

$$\tilde{C}_S(\sigma_1, \sigma_2, H_S; j) = \sigma_2^2 - 2(H_S - \sigma_1)(1 - \sigma_1^2) + j^2 \ .$$

**Theorem 5.3** *The spherical pendulum is equivalent to a subsystem of the Lagrange top.*

**Proof :** Let $\tilde{\sigma}_2 = \sqrt{\chi I_1}\sigma_2$, $\tilde{\sigma}_3 = \chi I_1 \sigma_3$, $\tilde{H} = \chi H_S$. We have

$$\tilde{C}_S(\sigma_1, \sigma_2, H; j) = \tilde{C}_L(\sigma_1, \sigma_2, H_L; j, 0) \ .$$

**q.e.d.**

# 6 Example: The normalized Hénon-Heiles Hamiltonian on $\mathbb{R}^4$

On $\mathbb{R}^4$ with coordinates $z = (x, y) = (x_1, x_2, y_1, y_2)$ consider the problem introduced by Hénon and Heiles in [13] given by the Hamiltonian system

$$\dot{z} = \{z, H\} \ .$$

where $\{\ ,\ \}$ is the standard Poisson bracket, and $H$ is the Hamiltonian

$$H(x,y) = \tfrac{1}{2}\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) + \frac{\varepsilon}{3}(x_1^3 - 3x_1x_2^2)\ .$$

The normalized system, truncated after terms of order 6, is

$$\bar{H}(w_1, w_2, w_3, w_4) = \bar{H}_2 + \varepsilon \bar{H}_4 + \varepsilon^2 \bar{H}_6\ , \tag{1}$$

with

$$\begin{aligned} \bar{H}_2 &= \frac{1}{2} w_4\ , \\ \bar{H}_4 &= \frac{1}{48}(7w_2^2 - 5w_4^2)\ , \\ \bar{H}_6 &= \frac{1}{64}(-\frac{67}{54}w_4^3 - \frac{7}{8}w_2^2 w_4 - \frac{28}{9}w_3^3 + \frac{28}{3}w_1^2 w_3)\ . \end{aligned} \tag{2}$$

Here the $w_i$, $i = 1, 2, 3, 4$, form a Hilbert basis for the polynomials invariant under the action of the one-parameter group given by the flow of $H_2(x,y) = \frac{1}{2}\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right)$. These invariants are

$$\begin{aligned} w_1(x,y) &= x_2 y_1 - x_1 y_2\ , \\ w_2(x,y) &= x_1 x_2 + y_1 y_2\ , \\ w_3(x,y) &= \tfrac{1}{2}\left(x_1^2 - x_2^2 + y_1^2 - y_2^2\right)\ , \\ w_4(x,y) &= \tfrac{1}{2}\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) = H_2(x,y)\ . \end{aligned} \tag{3}$$

These are also the invariants used in the reduction process which goes back on [14], [4]. More details on this example can be found in [21].

The reduction is a carried out by using the orbit map

$$\rho : (x,y) \to (w_1, w_2, w_3, w_4)\ .$$

The image of this map is determined by the relation

$$w_1^2 + w_2^2 + w_3^2 = w_4^2\ ,\ w_4 \geqslant 0\ .$$

The reduced phase space is obtained by setting $w_4 = r \geqslant 0$. On $\mathbb{R}^4$ this defines a 3-sphere $S_h^3$. Furthermore

$$\rho(S_r^3) = S_r^2\ ,$$

where $S_r^2$ is the 2-sphere in $\mathbb{R}^3$ given by $w_1^2 + w_2^2 + w_3^2 = r^2$. The orbital fibration is the Hopf fibration. The Poissonstructure is given by the table 1. This defines a Poisson

Table 1: The bracket relations $\{w_i, w_j\}$.

| . | $w_1$ | $w_2$ | $w_3$ |
|---|---|---|---|
| $w_1$ | 0 | $2w_3$ | $-2w_2$ |
| $w_2$ | $-2w_3$ | 0 | $2w_1$ |
| $w_3$ | $2w_2$ | $-2w_1$ | 0 |

structure

$$\{f,g\} =< \nabla C_{HH}, \nabla g \times \nabla f > ,$$

with Casimir

$$C_{HH}(w_1,w_2,w_3,w_4) = w_1^2 + w_2^2 + w_3^2 - w_4^2 \ .$$

The corresponding Poisson structure is locally at the origin of type $\mathfrak{so}(3)$.

Note that in this case there is no diffeomorphism putting the Poisson structure in an energy representation because $\bar{H}$ is not linear in $w_1$, $w_2$, or $w_3$. However, from (1) one may obtain

$$w_2^2 = \frac{\bar{H} - \frac{1}{2}w_4 + \varepsilon(\frac{5}{48}w_4^2) - \varepsilon^2(\frac{1}{64}(-\frac{67}{54}w_4^3 - \frac{28}{9}w_3^3 + \frac{28}{3}w_1^2 w_3))}{\frac{7}{48}\varepsilon - \varepsilon^2(\frac{7}{512}w_4)} \ .$$

Putting $w_4 = r$ equal to a constant, the substitution into $w_1^2 + w_2^2 + w_3^2 = r^2$ gives

$$w_1^2 + w_3^2 + \frac{\bar{H} - \frac{1}{2}r + \varepsilon(\frac{5}{48}r^2) - \varepsilon^2(\frac{1}{64}(-\frac{67}{54}r^3 - \frac{28}{9}w_3^3 + \frac{28}{3}w_1^2 w_3))}{\frac{7}{48}\varepsilon - \frac{7}{512}r\varepsilon^2} = r^2 \ .$$

Which gives a transformation of the Casimir such that its inverse image covers the interior of the disc $w_1^2 + w_3^2 \leqslant r^2$ twice. The relative equilibria are now the critical points of the function

$$\bar{H}(w_1,w_3;r) = (\frac{7}{48}\varepsilon - \frac{7}{512}r\varepsilon^2)(r^2 - w_1^2 - w_3^2) + \frac{1}{2}r - \varepsilon(\frac{5}{48}r^2) + \varepsilon^2(\frac{1}{64}(-\frac{67}{54}r^3 - \frac{28}{9}w_3^3 + \frac{28}{3}w_1^2 w_3)) \ ,$$

under the constraint $w_1^2 + w_3^2 \leqslant r^2$.

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# References

[1] Abraham, R, and Marsden, J.E. : *Foundations of Mechanics*, Benjamin/Cummings publishing Company, (1978).

[2] Arms, J.H., Cushman, R.H. and Gotay, M.J.: *A Universal Reduction Procedure for Hamiltonian Group Actions*, in: The Geometry of Hamiltonian Systems, Berkeley 1989, ed. T.S. Ratiu, MSRI series **22**, Springer Verlag,(1991), pp 26–38.

[3] Arnol'd, V.I.: *Singularities of caustics and wave fronts*, Kluwer Academic Publishers, (1990).

[4] Cushman, R.H. and Rod, D.L.: *Reduction of the semisimple 1:1 resonance*, Physica D **6**, (1982), pp 105–112.

[5] Cushman, R. and Van der Meer, J.C. : *The Hamiltonian Hopf bifurcation in the Lagrange top*. In : *Géométrie Symplectique et Mécanique, La Grande Motte 1988* (ed. C. Albert) LNM **1416**, Springer Verlag, Berlin etc. (1990), pp 26 – 38.

[6] Cushman, R.H. and Bates, L.M. : *Global Aspects of Classical Integrable Systems*, Birkhäuser Verlag, (1997).

[7] Dufour, J.P. and Zhitomirskii, M.: *Singularities and bifurcations of 3-dimensional Poisson structures*, Israel Journal of Mathematics **121**, (2001), pp 199–220.

[8] Egilsson, A.S.: *Newton polyhedra and Poisson structures from certain linear Hamiltonian circle actions*, arXiv:math/0411398v1 [math.SG]

[9] Ferrer, S., Hanßmann, H., Palacián, J. and Yanguas, P. : *On Perturbed Oscillators in 1-1-1 Resonance : The Case of Axially Symmetric Cubic Potentials*, J. of Geometry and Physics **40**, (2002), pp 320–369.

[10] Grabowski, J., Marmo, G. and Perelemov, A.M.: *Poisson structures: towards a classification*, Modern Phys. Lett. A **8**, (1993), pp 1719–1733.

[11] Hanßmann, H. and Van der Meer, J.C. : *On the Hamiltonian Hopf bifurcations in the 3D Hénon–Heiles family*, J. Dynamics Diff. Eq. **14**, (2002), pp 675–695.

[12] Hanßmann, H. and Van der Meer, J.C. : *On non-degenerate Hamiltonian Hopf bifurcations in 3DOF systems*. In: *EQUADIFF 2003, Proceedings of the International Conference on Differential Equations*, Hasselt, Belgium 22 - 26 July 2003, eds: F. Dumortier , H. Broer , J. Mawhin , A. Vanderbauwhede, S. Verduyn Lunel. World Scientific, Singapore, (2005), pp 476–481.

[13] Hénon, M. and Heiles, C.:*The applicability of the third integral of motion: some numerical experiments*, Astron. J. **69**, (1976), pp 73–79.

[14] Kummer, M.: *On the construction of the reduced phase space of a Hamiltonian systen with symmetry*, Indiana Univ. math. J. **30**, (1981), pp 281–291.

[15] Losik, M.V. : *Lifts of diffeomorphisms of orbit spaces for representations of compact Lie groups*, Geometriae Dedicata **88**, (2001), pp 21–36.

[16] Marsden, J.E. and Weinstein, A.: *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5**, (1974), pp 121–130.

[17] Marsden, J.E. and Ratiu,T.S.: *Introduction to Mechanics and Symmetry*, TAM series **17**, Springer-Verlag, (1999).

[18] Meyer, K.R.: *Symmetries and integrals in mechanics*, in: Dynamical Systems, ed. M.M. Peixoto, Academic Press, (1973), pp 259–272.

[19] Olver, P.J.: *Applications of Lie groups to Differential Equations*, GTM series **107**, Springer-Verlag, (1986).

[20] Van der Meer, J.C.: *The Hamiltonian Hopf bifurcation*, LNM 1160, Springer Verlag, (1985).

[21] Sadovskií,D. and Efstathiou, K.: No polar coordinates (R.H. Cushman); In: Geometric Mechanics and Symmetry: the Peyresq lectures, Editors J. Montaldi and T.S. Ratiu. London Math. Soc. Lect. Note Series **306**, Cambridge Univ. Press, (2005), pp 211-302.

[22] Schwarz, G.W.: *Lifting smooth homotopies of orbit spaces*, Publ. Math. IHES **51**, (1980), pp 37–136.

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