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# On Infinity Norms as Lyapunov Functions: Alternative Necessary and Sufficient Conditions

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**Abstract**—This paper considers the synthesis of infinity norm Lyapunov functions for discrete-time linear systems. A proper conic partition of the state-space is employed to construct a finite set of linear inequalities in the elements of the Lyapunov weight matrix. Under typical assumptions, it is proven that the feasibility of the derived set of linear inequalities is equivalent with the existence of an infinity norm Lyapunov function. Furthermore, it is shown that the developed solution extends naturally to several relevant classes of discrete-time nonlinear systems.

## I. INTRODUCTION

Lyapunov functions (LFs) represent a powerful tool for stability analysis of dynamical systems [1], [2]. Among Lyapunov functions, quadratic and polyhedral LFs (more recently, also polynomial LFs) are very popular, as they can be searched for efficiently. In particular, polyhedral LFs are of interest because they yield a less conservative domain of attraction when polytopic constraints are present. A classical problem related to polyhedral LFs is the existence and synthesis of a Lyapunov function defined using a weighted infinity norm. For stable systems described by a linear polytopic differential or difference inclusion it is known [3]–[5] that existence of an infinity norm LF is a necessary condition. Moreover, it is also known [6], [7] that existence of an infinity norm LF is equivalent with existence of a 0-symmetric polyhedral contractive (invariant) set. As such, available methods for constructing an infinity norm LF for a linear equation or polytopic inclusion either search (i) for a matrix that satisfies the corresponding standard Lyapunov conditions, see, e.g., [3]–[5], [8]–[11] or, (ii) for a 0-symmetric polyhedral contractive (invariant) set, see, e.g., [7], [12]–[14]. For an overview the interested reader is referred to the excellent monograph [15]. For results valid for certain relevant classes of nonlinear systems, such as, e.g., hybrid systems or nonlinear quadratic systems, see [14], [16]–[22] and the references therein.

This paper focuses on the construction of infinity norm LFs for linear discrete-time systems *via* approach (i) indicated above. A novel set of linear inequalities in the elements of the Lyapunov weight matrix that involves a proper conic partition of the state-space is proposed. It is proven that the existence of a proper conic partition on which the developed set of inequalities admits a feasible solution is equivalent with the existence of a infinity norm Lyapunov function. A distinguishing feature of the developed method is that it does not involve an eigen value restriction or decomposition,

the strict diagonal dominance property or the necessary conditions of [3]–[5]. This facilitates a natural extension to several relevant classes of discrete-time nonlinear systems. The necessity of the developed conditions is preserved for linear polytopic difference inclusions, as it is also the case for the conditions of [3]–[5]. For a fixed proper conic partition of the state-space, evaluating the feasibility of the constructed set of inequalities requires solving a linear program.

## II. PRELIMINARIES

Let  $\mathbb{R}$ ,  $\mathbb{R}_+$ ,  $\mathbb{Z}$  and  $\mathbb{Z}_+$  denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every  $c \in \mathbb{R}$  and  $\Pi \subseteq \mathbb{R}$  we define  $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$  and similarly  $\Pi_{\leq c}$ ,  $\mathbb{R}_\Pi := \Pi$  and  $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$ . For two arbitrary sets  $\mathcal{S} \subseteq \mathbb{R}^n$  and  $\mathcal{P} \subseteq \mathbb{R}^n$ , let  $\mathcal{S} \oplus \mathcal{P} := \{x + y \mid x \in \mathcal{S}, y \in \mathcal{P}\}$  denote their Minkowski sum. For a set  $\mathcal{S} \subseteq \mathbb{R}^n$ , we denote by  $\text{int}(\mathcal{S})$  the interior of  $\mathcal{S}$ . A set  $\mathcal{S}$  is called 0-symmetric if for all  $x \in \mathcal{S}$  it holds that  $-x \in \mathcal{S}$ . For any  $Z \in \mathbb{R}^{l \times n}$  and  $\mathcal{S} \subseteq \mathbb{R}^n$ ,  $-\mathcal{S} := \{-x \mid x \in \mathcal{S}\}$  and  $Z\mathcal{S} := \{Zx \mid x \in \mathcal{S}\}$ . A C-set [6] is a compact set that contains the origin in its interior. A polyhedron (or a polyhedral set) in  $\mathbb{R}^n$  is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A polytope is a compact polyhedron. Given  $n + 1$  affinely independent points of  $\mathbb{R}^n$ , i.e.,  $\{\theta_l\}_{l \in \mathbb{Z}_{[0, n]}}$ , a simplex is defined as  $\mathbb{S} := \text{Co}(\{\theta_l\}_{l \in \mathbb{Z}_{[0, n]}})$ , where  $\text{Co}(\cdot)$  denotes the convex hull. For a vector  $x \in \mathbb{R}^n$ ,  $[x]_i$  denotes the  $i$ -th element of  $x$  and  $\|x\| := \|x\|_\infty = \max_{i=1, \dots, n} |[x]_i|$  denotes the infinity norm of  $x$ , where  $|\cdot|$  denotes the absolute value. For a matrix  $Z \in \mathbb{R}^{l \times n}$ ,  $[Z]_{ij} \in \mathbb{R}$  denotes the element in the  $i$ -th row and  $j$ -th column of  $Z$  and  $[Z]_{i\bullet} \in \mathbb{R}^{1 \times n}$  denotes the  $i$ -th row of  $Z$ . For a matrix  $Z \in \mathbb{R}^{l \times n}$  let  $\|Z\| := \sup_{x \neq 0} \frac{\|Zx\|}{\|x\|}$  denote its induced matrix infinity norm. It is well known (see, e.g., Proposition 9.4.9 in [23]) that  $\|Z\| = \max_{i \in \mathbb{Z}_{[1, l]}} \sum_{j=1}^n |[Z]_{ij}|$ .  $I_n \in \mathbb{R}^{n \times n}$  denotes the  $n$ -th dimensional identity matrix. For a symmetric matrix  $Z \in \mathbb{R}^{n \times n}$  let  $Z \succ 0$  ( $Z \succeq 0$ ) denote that  $Z$  is positive definite (semi-definite). For any  $x, y \in \mathbb{R}^n$ ,  $c \in \mathbb{R}_+$ ,  $x \leq y$ ,  $x < y$ ,  $x \geq y$  and  $x > y$  denote the corresponding set of component-wise inequalities and  $\pm x \leq c$  denotes the inequalities  $-c \leq x \leq c$ . A subset  $\mathcal{C}$  of  $\mathbb{R}^n$  is a convex cone if and only if  $c_1\mathcal{C} \oplus c_2\mathcal{C} = \mathcal{C}$  for any  $c_1, c_2 \in \mathbb{R}_+$ . A convex cone  $\mathcal{C}$  is salient if and only if  $\mathcal{C} \cap -\mathcal{C} = \{0\}$ . A convex cone  $\mathcal{C}$  is pointed if  $0 \in \mathcal{C}$ . A  $n$ -th dimensional cone  $\mathcal{C}$  in  $\mathbb{R}^n$  is called a proper cone if it is convex, salient, pointed and  $\text{int}(\mathcal{C}) \neq \emptyset$ . For any point  $y \in \mathbb{R}^n$ ,  $y \neq 0$ , the set  $r(y) := \{x \in \mathbb{R}^n \mid x = cy, c \in \mathbb{R}_+\}$  is called a ray.

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A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\varphi(0) = 0$ . A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}_\infty$  if  $\varphi \in \mathcal{K}$  and  $\lim_{s \rightarrow \infty} \varphi(s) = \infty$ .

Consider the discrete-time system

$$x(k+1) = \Phi(x(k)), \quad k \in \mathbb{Z}_+, \quad (1)$$

where  $x(k) \in \mathbb{R}^n$  is the state at the discrete-time instant  $k$  and  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an arbitrary map with  $\Phi(0) = 0$ .

**Definition II.1** Let  $\lambda \in \mathbb{R}_{(0,1]}$ . We call a set  $\mathcal{P} \subseteq \mathbb{R}^n$   $\lambda$ -contractive (or shortly, contractive) for system (1) if for all  $x \in \mathcal{P}$  it holds that  $\Phi(x) \in \lambda\mathcal{P}$ . When this property holds with  $\lambda = 1$  we call  $\mathcal{P}$  a *positively invariant (PI)* set.

**Definition II.2** Let  $\mathbb{X}$  with  $0 \in \text{int}(\mathbb{X})$  be a subset of  $\mathbb{R}^n$ . System (1) is *Lyapunov stable* if for all  $\varepsilon \in \mathbb{R}_{>0}$  there exists a  $\delta(\varepsilon) \in \mathbb{R}_{>0}$  such that  $\|x(0)\| \leq \delta(\varepsilon)$  implies  $\|x(k)\| \leq \varepsilon$  for all  $k \in \mathbb{Z}_+$ . The origin of (1) is *attractive in  $\mathbb{X}$*  if for any  $x(0) \in \mathbb{X}$  it holds that  $\lim_{k \rightarrow \infty} \|x(k)\| = 0$ . System (1) is *asymptotically stable in  $\mathbb{X}$*  if it is Lyapunov stable and attractive in  $\mathbb{X}$ . System (1) is *exponentially stable in  $\mathbb{X}$*  if for any  $x(0) \in \mathbb{X}$  it holds that  $\|x(k)\| \leq \theta \mu^k \|x(0)\|$  for some  $\theta \in \mathbb{R}_{\geq 1}$ ,  $\mu \in \mathbb{R}_{(0,1]}$ . System (1) is *globally asymptotically (exponentially) stable (GAS (GES))* if it is asymptotically (exponentially) stable in  $\mathbb{R}^n$ .

**Theorem II.3** Let  $\mathbb{X}$  be a PI set for (1) with  $0 \in \text{int}(\mathbb{X})$ . Furthermore, let  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\rho \in \mathbb{R}_{(0,1]}$  and let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a function such that:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}, \quad (2a)$$

$$V(\Phi(x)) \leq \rho V(x), \quad \forall x \in \mathbb{X}. \quad (2b)$$

Then system (1) is asymptotically stable in  $\mathbb{X}$ . If the above inequalities hold with  $\alpha_1(s) := c_1 s^\lambda$  and  $\alpha_2(s) := c_2 s^\lambda$  for some  $c_1, c_2, \lambda \in \mathbb{R}_{>0}$ , then system (1) is exponentially stable in  $\mathbb{X}$ .

A proof of the above theorem can be found in [24], [25].

**Definition II.4** A function  $V$  that satisfies (2) is called a *Lyapunov function in  $\mathbb{X}$* . A Lyapunov function in  $\mathbb{R}^n$  is called a *global Lyapunov function*.

Next, consider a linear discrete-time system, i.e.,

$$x(k+1) = Ax(k), \quad k \in \mathbb{Z}_+, \quad (3)$$

where  $A \in \mathbb{R}^{n \times n}$ . Let  $V_q(x) := x^\top P_q x$  with  $P_q \in \mathbb{R}^{n \times n}$  and  $V(x) := \|Px\|$  with  $P \in \mathbb{R}^{l \times n}$ ,  $l \in \mathbb{Z}_{\geq n}$  be a quadratic and polyhedral Lyapunov function candidate, respectively. Let us recall the classical results on stability of system (3).

**Theorem II.5** The following statements are equivalent.

- (i) System (3) is GES.
- (ii) For any  $\rho \in \mathbb{R}_{(0,1]}$  there exists a matrix  $P_q \succ 0$  such that  $\rho P_q - A^\top P_q A \succeq 0$ .

(iii) For any  $\rho \in \mathbb{R}_{(0,1]}$  there exists a number  $l \in \mathbb{Z}_{\geq n}$ , a matrix  $P \in \mathbb{R}^{l \times n}$  with  $\text{rank}(P) = n$  and a matrix  $Q \in \mathbb{R}^{l \times l}$  such that  $PA = QP$  and  $\|Q\| \leq \rho$ .

For the proof of (i)  $\Leftrightarrow$  (ii), see, for example, [26], and for the proof of (i)  $\Leftrightarrow$  (iii), see [3]–[5]. In [4] a direct relation between  $P_q$  and  $P$  was indicated as well. In the above statements  $\rho$  can be taken equal to one, if the non-strict conditions are replaced by strict ones.

Notice that in the quadratic case, exponential stability is obtained from the well known fact [23] that

$$\lambda_{\min}(P_q) \|x\|_2^2 \leq x^\top P_q x \leq \lambda_{\max}(P_q) \|x\|_2^2, \quad \forall x \in \mathbb{R}^n,$$

where  $\|\cdot\|_2$  denotes the 2-norm and  $\lambda_{\min(\max)}(P_q) \in \mathbb{R}_{>0}$  denotes the smallest (largest) eigenvalue of  $P_q \succ 0$ . For infinity norm Lyapunov functions, in [4] it was indicated that  $V(x) = \|Px\|$  is positive definite and radially unbounded, which is sufficient for GAS, but not for GES.

As such, in what follows it is proven that a direct relation between  $V(x) = \|Px\|$  and GES can be established as well. For any  $n \in \mathbb{Z}_{\geq 1}$  let  $l \in \mathbb{Z}_{\geq n}$  and  $P \in \mathbb{R}^{l \times n}$ .

**Fact II.6** The following statements are equivalent.

- (i)  $\text{rank}(P) = n$ .
- (ii) There exists a  $c \in \mathbb{R}_{>0}$  such that  $\|Px\| \geq c\|x\|$  for all  $x \in \mathbb{R}^n$ .

*Proof:* Let  $\sigma_{\min}(P)$  denote the smallest singular value of  $P$ . By Proposition 5.6.2 in [23] statement (i) is equivalent with  $\sigma_{\min}(P) > 0$ . Let us first prove that (i)  $\Rightarrow$  (ii). By Corollary 9.5.5 in [23] or the Courant-Fischer theorem it holds that

$$\frac{\|Px\|_2}{\|x\|_2} \geq \sigma_{\min}(P), \quad \forall x \neq 0.$$

Then, by the equivalence of norms it holds that

$$\frac{\|Px\|}{\|x\|} \geq \frac{\frac{1}{\sqrt{l}} \|Px\|_2}{\|x\|_2} \geq \frac{\sigma_{\min}(P)}{\sqrt{l}}, \quad \forall x \neq 0.$$

Hence, statement (ii) holds with  $c := \frac{\sigma_{\min}(P)}{\sqrt{l}} > 0$ . Next, we prove that (ii)  $\Rightarrow$  (i). By the Courant-Fischer theorem and the equivalence of norms we have that

$$\sigma_{\min}(P) = \min_{x \neq 0} \frac{\|Px\|_2}{\|x\|_2} \geq \min_{x \neq 0} \frac{\|Px\|}{\sqrt{n}\|x\|} \geq \frac{c}{\sqrt{n}} > 0.$$

Thus, statement (i) holds by Proposition 5.6.2 in [23].  $\blacksquare$

**Fact II.7** If  $\text{rank}(P) = n$ , then  $\|P\| > 0$ .

*Proof:* Suppose that  $\text{rank}(P) = n$  and  $\|P\| = 0$ . Then,  $\max_{i \in \mathbb{Z}_{[1,l]}} \sum_{j=1}^n |[P]_{ij}| = 0$ , which yields  $\text{rank}(P) = 0$ . Thus, we reached a contradiction.  $\blacksquare$

**Corollary II.8** The following statements are equivalent.

- (i)  $\text{rank}(P) = n$ .
- (ii) The function  $V(x) = \|Px\|$  satisfies (2a) with  $\alpha_1(s) := cs$  for some  $c \in \mathbb{R}_{>0}$  and  $\alpha_2(s) := \|P\|s$ .

The claim of Corollary II.8 is a direct consequence of Fact II.6 and Fact II.7.

Although the conditions specified by Theorem II.5-(iii) are non-conservative, finding a solution that satisfies these conditions is challenging, due to the rank constraint and the bilinear equality constraint. Several attempts were made to design a tractable algorithm that solves this problem, see, e.g., [8], [9] for some of the most important breakthroughs. More recently, an efficient computational procedure was presented for the discrete-time case in [11]. However, the latter procedure exploits an eigen value decomposition of the system matrix, which, apart from being sensitive to numerical errors [27], hampers an extension to any other system class. An extension of the results in [8], [9] to linear polytopic differential inclusions was presented in [10].

Motivated by this, the next section proposes an alternative set of necessary and sufficient conditions for the existence of an infinity norm Lyapunov function that leads to a tractable algorithm and allows a natural extension to several relevant classes of nonlinear systems.

### III. MAIN RESULTS

The idea is to start directly from the Lyapunov conditions (2) and transform them into a finite set of convex conditions on a specific conic partition of the state-space. To this end, let us define a proper conic partition of  $\mathbb{R}^n$ .

**Definition III.1** Let  $l \in \mathbb{Z}_{\geq n}$  and let  $\mathcal{L} := \mathbb{Z}_{[1,l]}$ . A finite set of cones  $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$  is called a *proper  $l$ -conic partition* of  $\mathbb{R}^n$  if  $\cup_{i \in \mathcal{L}} \{\mathcal{C}_i \cup -\mathcal{C}_i\} = \mathbb{R}^n$ ,  $\mathcal{C}_i$  is a proper cone for all  $i \in \mathcal{L}$  and  $\text{int}(\mathcal{C}_i) \cap \text{int}(\mathcal{C}_j) = \emptyset$  for all  $(i, j) \in \mathcal{L} \times \mathcal{L}$  with  $i \neq j$ .

The following straightforward facts that follow directly from the definition of the infinity norm will be instrumental in proving the main result.

**Fact III.2** Let  $x \in \mathbb{R}^n$ ,  $\Gamma \in \mathbb{R}_+$  and  $P \in \mathbb{R}^{l \times n}$ . The following statements are equivalent.

- (i)  $\|Px\| \leq \Gamma$ .
- (ii)  $\pm [P]_{i\bullet} x \leq \Gamma$  for all  $i \in \mathcal{L}$ .

**Fact III.3** Let  $x \in \mathbb{R}^n$ ,  $i \in \mathcal{L}$  and  $P \in \mathbb{R}^{l \times n}$ . The following statements are equivalent.

- (i)  $\|Px\| = [P]_{i\bullet} x$ .
- (ii)  $([P]_{i\bullet} \pm [P]_{j\bullet})x \geq 0$  for all  $j \in \mathcal{L} \setminus \{i\}$ .

Let  $E \in \mathbb{R}_{\geq n}$  and  $\mathcal{E} := \mathbb{Z}_{[1,E]}$ . For all  $i \in \mathcal{L}$  let  $\{x_i^e\}_{e \in \mathcal{E}}$  with  $x_i^e \in \mathbb{R}^n$ ,  $x_i^e \neq 0$  for all  $e \in \mathcal{E}$  be such that  $\mathcal{C}_i := \text{Co}(\{r(x_i^1), \dots, r(x_i^E)\})$  is a proper cone in  $\mathbb{R}^n$ . The following facts follow directly from the definition of a ray and cone, respectively.

**Fact III.4** Let  $h \in \mathbb{Z}_{\geq 1}$ ,  $i \in \mathcal{L}$ ,  $e \in \mathcal{E}$  and  $H \in \mathbb{R}^{h \times n}$ . If  $Hx_i^e \geq (\leq) 0$  then  $Hx \geq (\leq) 0$  for all  $x \in r(x_i^e)$ . Furthermore, if  $Hx_i^e \geq (\leq) 0$  for all  $e \in \mathcal{E}$ , then  $Hx \geq (\leq) 0$  for all  $x \in \mathcal{C}_i$ .

**Definition III.5** A set of points  $\{\{x_i^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}}$  with  $x_i^e \in \mathbb{R}^n$ ,  $x_i^e \neq 0$  for all  $(i, e) \in \mathcal{L} \times \mathcal{E}$  is said to induce a proper  $l$ -conic partition of  $\mathbb{R}^n$  if  $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$  with  $\mathcal{C}_i := \text{Co}(\{r(x_i^1), \dots, r(x_i^E)\})$  for all  $i \in \mathcal{L}$  is a proper  $l$ -conic partition of  $\mathbb{R}^n$ .

The main result is stated next.

**Theorem III.6** Let  $l \in \mathbb{Z}_{\geq n}$ ,  $\mathcal{N} := \mathbb{Z}_{[1,n]}$  and  $P \in \mathbb{R}^{l \times n}$ . The following statements are equivalent.

(i) The function  $V(x) = \|Px\|$  is a global Lyapunov function for system (3).

(ii) There exists a  $E \in \mathbb{R}_{\geq n}$ , a corresponding set of points  $\{\{x_i^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}}$  with  $x_i^e \in \mathbb{R}^n$ ,  $x_i^e \neq 0$  for all  $(i, e) \in \mathcal{L} \times \mathcal{E}$  that induces a proper  $l$ -conic partition of  $\mathbb{R}^n$  and a  $c \in \mathbb{R}_{>0}$  such that the following inequalities hold for all  $i \in \mathcal{L}$ :

$$([P]_{i\bullet} \pm c[I_n]_{j\bullet})x_i^e \geq 0, \quad \forall j \in \mathcal{N}, \forall e \in \mathcal{E}, \quad (4a)$$

$$([P]_{i\bullet} \pm [P]_{j\bullet})x_i^e \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \forall e \in \mathcal{E}, \quad (4b)$$

$$(\rho[P]_{i\bullet} \pm [P]_{j\bullet}A)x_i^e \geq 0, \quad \forall j \in \mathcal{L}, \forall e \in \mathcal{E}. \quad (4c)$$

*Proof:* Let us begin with the proof of (ii)  $\Rightarrow$  (i). Fact III.4 and (4a) yield

$$([P]_{i\bullet} \pm c[I_n]_{j\bullet})x \geq 0, \quad \forall j \in \mathcal{N}, \quad \forall x \in \mathcal{C}_i.$$

Letting  $\Gamma := [P]_{i\bullet}x$ , the above inequality and Fact III.2 yield  $[P]_{i\bullet}x \geq c\|x\|$  for all  $x \in \mathcal{C}_i$ . Similarly, (4b) and Fact III.4 imply

$$([P]_{i\bullet} \pm [P]_{j\bullet})x \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \quad \forall x \in \mathcal{C}_i.$$

Then, from Fact III.3 it follows that  $\|Px\| = [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ . This further yields that  $\|Px\| = -[P]_{i\bullet}x$  for all  $x \in -\mathcal{C}_i$ . As such, since (4) holds for all  $i \in \mathcal{L}$  and  $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$  is a proper  $l$ -conic partition of  $\mathbb{R}^n$ , we have that  $\|Px\| \geq c\|x\|$  for all  $x \in \mathbb{R}^n$ . Using Fact II.6 we obtain that  $\text{rank}(P) = n$  and hence, by Corollary II.8  $V(x) = \|Px\|$  satisfies (2a) for all  $x \in \mathbb{R}^n$ . Using a similar reasoning, from (4c) one obtains that  $\rho[P]_{i\bullet}x \geq \|PAx\|$  for all  $x \in \mathcal{C}_i$  and  $\rho(-[P]_{i\bullet})x \geq \|PAx\|$  for all  $x \in -\mathcal{C}_i$ . Therefore,  $\rho\|Px\| \geq \|PAx\|$  for all  $x \in \mathbb{R}^n$ , which is exactly inequality (2b) for the considered candidate Lyapunov function.

Let us proceed with the proof of (i)  $\Rightarrow$  (ii). If  $V(x) = \|Px\|$  with  $P \in \mathbb{R}^{l \times n}$  is a global Lyapunov function for system (3), then it induces [6] a family of 0-symmetric polytopic  $\lambda$ -contractive sets (with  $\lambda = \rho$ ) for system (3), i.e.,

$$\{\mathcal{P}_\Gamma\}_{\Gamma \in \mathbb{R}_{>0}}, \quad \mathcal{P}_\Gamma := \{x \in \mathbb{R}^n \mid \|Px\| \leq \Gamma\}.$$

For any  $\Gamma \in \mathbb{R}_{>0}$  let  $\mathbb{V}_\Gamma$  denote the set of vertices of  $\mathcal{P}_\Gamma$ . Each polytope that belongs to the family of sets  $\{\mathcal{P}_\Gamma\}_{\Gamma \in \mathbb{R}_{>0}}$  can be described as the union of  $2l$  proper *polyhedral cones* [28], i.e.,  $\cup_{i \in \mathcal{L}} \{\mathcal{C}_i^\Gamma \cup -\mathcal{C}_i^\Gamma\}$ , with each  $\mathcal{C}_i^\Gamma$  equal to the convex hull of the origin and a subset of  $\mathbb{V}_\Gamma$ . For example, each  $\mathcal{C}_i^\Gamma$  can be obtained as the  $n$ -pyramid [29] formed by the origin and one of the facets of  $\mathcal{P}_\Gamma$ . Then, for all  $x \in \mathcal{C}_j^\Gamma$ , it holds that  $\|Px\| = [P]_{i\bullet}x$  for some  $i \in \mathcal{L}$ , for all  $j \in \mathcal{L}$ , i.e., one of the rows of  $P$ , which defines one of the facets

of  $\mathcal{P}_\Gamma$ , is dominant in each polyhedral cone. For any  $\mathcal{C}_i^\Gamma$  let  $\{x_{i,\Gamma}^e\}_{e \in \mathcal{E}} \subset \mathbb{V}_\Gamma$  with<sup>1</sup>  $\mathcal{E} := \{1, \dots, E\}$ ,  $E \in \mathbb{Z}_{\geq n}$ , denote its corresponding set of non-zero vertices. Notice that  $\text{Co}(\{x_{i,\Gamma}^e\}_{e \in \mathcal{E}})$  defines one of the facets of  $\mathcal{P}_\Gamma$  for all  $i \in \mathcal{L}$ . Then, for any  $\Gamma \in \mathbb{R}_{>0}$ , the set of points  $\{x_{i,\Gamma}^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}}$  induces a proper  $l$ -conic partition of  $\mathbb{R}^n$ , i.e.,  $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$  with

$$\mathcal{C}_i := \text{Co}(\{r(x_{i,\Gamma}^1), \dots, r(x_{i,\Gamma}^E)\}), \quad \forall i \in \mathcal{L}.$$

Furthermore, it holds that  $\|Px\| = |[P]_{i\bullet}x|$  for some  $i \in \mathcal{L}$  for all  $x \in \mathcal{C}_i$ . Next, let us prove that either  $\|Px\| = [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$  or  $\|Px\| = -[P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ . Firstly, let  $\Gamma \in \mathbb{R}_{>0}$  and notice that  $|[P]_{i\bullet}x| = \Gamma$  for all  $x \in \text{Co}(\{x_{i,\Gamma}^e\}_{e \in \mathcal{E}})$ . Suppose that  $[P]_{i\bullet}x_{i,\Gamma}^e > 0$  for all  $e \in \mathcal{E}$ . Then, by Fact III.4 it follows that  $[P]_{i\bullet}x \geq 0$  for all  $x \in \mathcal{C}_i$ . Secondly, suppose that there exists one  $e^* \in \mathcal{E}$  such that  $[P]_{i\bullet}x_{i,\Gamma}^{e^*} < 0$ . Then, consider the point  $x^* := \sum_{e \in \mathcal{E}} \mu_e x_{i,\Gamma}^e$  with  $\mu_e \in \mathbb{R}_{\geq 0}$  for all  $e \in \mathcal{E}$ ,  $\sum_{e \in \mathcal{E}} \mu_e = 1$  and  $\sum_{e \in \mathcal{E}, e \neq e^*} \mu_e = \mu_{e^*}$ . As such,

$$\begin{aligned} |[P]_{i\bullet}x^*| &= \left| \sum_{e \in \mathcal{E}, e \neq e^*} \mu_e [P]_{i\bullet}x_{i,\Gamma}^e + \mu_{e^*} [P]_{i\bullet}x_{i,\Gamma}^{e^*} \right| \\ &= \left| \sum_{e \in \mathcal{E}, e \neq e^*} \mu_e \Gamma - \mu_{e^*} \Gamma \right| = 0. \end{aligned}$$

As  $x^* \in \text{Co}(\{x_{i,\Gamma}^e\}_{e \in \mathcal{E}})$ , we reached a contradiction. Similarly, it follows that if  $[P]_{i\bullet}x_{i,\Gamma}^e < 0$  for all  $e \in \mathcal{E}$  then  $[P]_{i\bullet}x \leq 0$  for all  $x \in \mathcal{C}_i$  and, the assumption that there exists one  $e^* \in \mathcal{E}$  such that  $[P]_{i\bullet}x_{i,\Gamma}^{e^*} > 0$  yields a contradiction. Thus, it is established that either  $\|Px\| = [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$  or  $\|Px\| = -[P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ .

Next, by taking  $l$  arbitrary cones  $\mathcal{C}_i$  that belong to the  $l$ -conic partition induced by  $\mathcal{P}_\Gamma$  for any  $\Gamma \in \mathbb{R}_{>0}$  (notice that  $\cup_{\Gamma \in \mathbb{R}_{>0}} \mathcal{C}_i^\Gamma \subseteq \mathcal{C}_i$ ) and using Fact II.6, Fact III.2 and Fact III.3 yields that the set of inequalities (4) is feasible for the corresponding set of points  $\{x_{i,\Gamma}^e\}_{e \in \mathcal{E}}$ ,  $[P]_{i\bullet}$ , if  $\|Px\| = [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ ,  $-[P]_{i\bullet}$ , if  $\|Px\| = -[P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ , and  $c = \frac{\sigma_{\min}(P)}{\sqrt{l}}$ . ■

Inequality (2b) and the lower bound in inequality (2a) for  $V(x) = \|Px\|$  are essentially non-convex inequalities. The crux of Theorem III.6 is the set of constraints (4b) that imposes  $V(x) = [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ . This enables the convexification of the lower bound in (2a), *via* the set of constraints (4a), and of inequality (2b), *via* the set of inequalities (4c) for each  $\mathcal{C}_i$ . In contrast, the conditions of Theorem II.5-(iii) employ the relation  $PA = QP$  to eliminate  $\|Px\|$  from (2b), which yields a condition of the induced norm of  $Q$ .

Several remarks about the complexity of testing the conditions of Theorem III.6 are in order. For each  $i \in \mathcal{L}$ , condition (4a) yields  $2nE$ , condition (4b) yields  $2(l-1)E$  and condition (4c) yields  $2lE$  linear inequalities in  $c$  and the elements of  $P$ , respectively. So, testing (4) for a fixed

<sup>1</sup>Without loss of generality, it was implicitly assumed that all facets of  $\mathcal{P}_\Gamma$  have the same number of vertices  $E$ . If this is not case, it is sufficient to further partition some of the facets of  $\mathcal{P}_\Gamma$  until all  $n$ -pyramids  $\mathcal{C}_i^\Gamma$  have as basis a polytope with the same number of vertices.

$l \in \mathbb{Z}_{\geq n}$  and  $E \in \mathbb{Z}_{\geq n}$  amounts to solving a single linear program with  $ln+1$  variables and  $2lE(2l+n-1)$  inequalities. This is a tractable problem for  $x \in \mathbb{R}^n$  with  $n$  reasonable large, as the number of inequalities and variables does not depend exponentially on the system dimension. The number of inequalities can be further reduced by replacing (4a) with  $[P]_{i\bullet}x^e \geq c\|x_i^e\|$  for all  $e \in \mathcal{E}$ , which yields a LP with  $ln+1$  variables and  $E(4l^2 - 2l + 1)$  inequalities. The number of inequalities in (4c) can be further reduced by selecting only the indexes  $j \in \mathcal{L}$  for which  $AC_i \cap C_j \neq \emptyset$ , while some of the inequalities in (4b) are redundant and can be removed, i.e.,  $([P]_{i\bullet} + [P]_{j\bullet})x_i^e = ([P]_{j\bullet} + [P]_{i\bullet})x_j^e$  if  $x_i^e = x_j^e$ .

**Remark III.7** The Farkas lemma [28] can be used to remove the points  $x_i^e$  from (4). However, this will yield a reduction in the number of linear inequalities at the price of an increase in the number of unknown variables. □

**Remark III.8** A continuous-time correspondent of the developed results require changing (2b) with  $\mathcal{D}^+V(x(t)) < 0$ , where  $\mathcal{D}^+$  denotes the upper right Dini derivative [7]. As such, the expression of the Dini derivative established in [7] can be employed to obtain a continuous-time correspondent of condition (4c). Establishing equivalent continuous-time results makes the object of further research. □

Choosing the number  $l$  amounts to the classical problem of finding an upper bound on the number of rows of the matrix  $P$ . A solution to this problem for continuous-time linear systems can be found in [30], [31]. Also, the results therein can be employed to choose the right number and position of ray directions that define each cone  $\mathcal{C}_i$ . In what follows we consider that the results therein apply *mutatis mutandis* to discrete-time linear systems, as well. The interested reader is referred to [11] for a conservative upper bound that is valid in the discrete-time case. However, notice that if the conditions (4) hold on a proper  $l^*$ -conic partition of  $\mathbb{R}^n$ , then they also hold on a finer, proper  $l$ -conic partition of  $\mathbb{R}^n$  with  $l \in \mathbb{Z}_{\geq l^*}$ .

As a proper  $l$ -conic partition of  $\mathbb{R}^n$  can always be constructed by partitioning a 0-symmetric polytope using simplicial polyhedral cones [28], one can always take  $E = n$ . However, this might result in a larger number of inequalities. Consider for example the case of a unit cub in  $\mathbb{R}^3$ . A proper  $l$ -conic partition is obtained for  $l = 3$  and  $E = 4$ , while a proper simplicial  $l$ -conic partition requires  $l = 6$  and  $E = 3$ . For a fixed  $l$  and  $E$ , suitable algorithms for constructing a proper  $l$ -conic partition of  $\mathbb{R}^n$  were indicated in [14]. Alternatively, as the vertices employed in the proof of Theorem III.6 were chosen arbitrarily, one can select an arbitrary candidate polytope (e.g., the infinity norm unit sphere in  $\mathbb{R}^n$ ) for inducing a suitable proper  $l$ -conic partition of  $\mathbb{R}^n$ . Then, one has to iterate on checking the conditions (4) by successively rotating the original fixed proper  $l$ -conic partition.

Note that the conditions (4) lead to a simple algebraic test for checking  $\lambda$ -contractiveness or positive invariance (for  $\rho = 1$ ) of a 0-symmetric polytope.

### A. Brief comparative remarks

The most detailed alternative solution for discrete-time linear systems can be found in [11]. The approach therein is based on the necessary and sufficient conditions of Theorem II.5-(iii) established in [3]–[5] and involves an eigenvalue decomposition of the  $A$  matrix and solving a finite sequence of so-called “feasibility LPs”. Moreover, an algebraic test is proposed for substituting the LPs, which yields an impressive computational efficiency. The method developed in this paper, which also requires solving a finite number of “feasibility LPs”, is tractable, but it may be less computationally efficient, especially for high dimensions, as it requires the construction of a proper  $l$ -conic partition. However, such a partition can be generated analytically for linear systems, see, e.g., [14]. More importantly, the proposed procedure does not require an eigenvalue decomposition of  $A$  or strict diagonal column dominance of the matrix  $P$ .

Another relevant contribution is the method for synthesis of piecewise linear (PWL) LFs presented in [18] for continuous-time PWL systems. The set of conditions proposed therein involve a proper conic partition of  $\mathbb{R}^n$  and amount to the “Farkas lemma equivalent” of (4a) and the continuous-time correspondent of (4c) (the condition on the derivative), along with a parametrization that guarantees continuity (for details see Theorem 4.3 and Lemma 4.6 in [18]). However, when applied to a linear system, the approach of [18] does not necessarily yield an infinity norm LF, as condition (4b) is missing. More precisely, if one defines a piecewise linear Lyapunov function  $V_{\text{PWL}}(x) := [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ , as done in [18], this does not necessarily imply that  $\|Px\| = [P]_{i\bullet}x$  for all  $x \in \mathcal{C}_i$ , as  $|[P]_{j\bullet}x| > [P]_{i\bullet}x$  can occur for some  $j \neq i$ . *Vice versa*, when the conditions in (4) are translated to PWL systems, they yield a piecewise infinity norm Lyapunov function, i.e.,  $V(x) = \|P_jx\|$  if  $x \in \mathcal{C}_j$ , where  $P_j \in \mathbb{R}^{l_j \times n}$  and  $l_j \in \mathbb{Z}_{\geq n}$  for each  $j$  in a finite set of indexes, rather than just a PWL LF.

## IV. DISCRETE-TIME NONLINEAR SYSTEMS

The goal of this section is to indicate several relevant classes of nonlinear discrete-time systems for which the conditions (4) can be formulated in a tractable way.

### A. Polytopic difference inclusions

Consider systems of the form

$$x(k+1) \in \Phi(x(k)), \quad k \in \mathbb{Z}_+, \quad (5)$$

where  $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ ,

$$\Phi(x) := \{Ax \mid A \in \text{Co}(\{A_w\}_{w \in \mathcal{W}})\},$$

$A_w \in \mathbb{R}^{n \times n}$  for all  $w \in \mathcal{W} := \{1, \dots, W\}$ ,  $W \in \mathbb{Z}_{\geq 1}$ . Notice that finding an infinity norm Lyapunov function for system (5) is equivalent with solving the same problem for a discrete-time switched linear system under arbitrary switching. As such, the following result applies to this class of hybrid systems as well.

**Theorem IV.1** *Let  $l \in \mathbb{Z}_{\geq n}$  and  $P \in \mathbb{R}^{l \times n}$ . The following statements are equivalent.*

(i) *The function  $V(x) = \|Px\|$  is a global common Lyapunov function for system (5).*

(ii) *There exists a  $E \in \mathbb{R}_{\geq n}$ , a corresponding set of points  $\{\{x_i^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}}$  with  $x_i^e \in \mathbb{R}^n$ ,  $x_i^e \neq 0$  for all  $(i, e) \in \mathcal{L} \times \mathcal{E}$  that induces a proper  $l$ -conic partition of  $\mathbb{R}^n$  and a  $c \in \mathbb{R}_{>0}$  such that the following inequalities hold for all  $i \in \mathcal{L}$ :*

$$([P]_{i\bullet} \pm c[I_n]_{j\bullet})x_i^e \geq 0, \quad \forall j \in \mathcal{N}, \forall e \in \mathcal{E}, \quad (6a)$$

$$([P]_{i\bullet} \pm [P]_{j\bullet})x_i^e \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \forall e \in \mathcal{E}, \quad (6b)$$

$$(\rho[P]_{i\bullet} \pm [P]_{j\bullet}A_w)x_i^e \geq 0, \quad \forall w \in \mathcal{W}, \forall j \in \mathcal{L}, \forall e \in \mathcal{E}. \quad (6c)$$

*Proof:* Observe that for each  $j \in \mathcal{L}$  and  $e \in \mathcal{E}$  condition (6c) is affine in  $A_w$ . Thus, as  $A \in \text{Co}(\{A_w\}_{w \in \mathcal{W}})$ , (6c) implies that

$$(\rho[P]_{i\bullet} \pm [P]_{j\bullet}A)x_i^e \geq 0, \quad \forall A \in \text{Co}(\{A_w\}_{w \in \mathcal{W}}), \\ \forall j \in \mathcal{L}, \forall e \in \mathcal{E}.$$

Then, the proof of (ii)  $\Rightarrow$  (i) follows by employing the same arguments used in the corresponding part of the proof of Theorem III.6. Next, suppose that  $V(x)$  is a global common Lyapunov function for system (5). Then, a proper  $l$ -conic partition of  $\mathbb{R}^n$  can be constructed as indicated in the proof of Theorem III.6, which yields that (6b) holds and thus, (6a) holds. Then, (2b) yields that

$$\rho[P]_{i\bullet}x \geq \|PAx\|, \quad \forall A \in \text{Co}(\{A_w\}_{w \in \mathcal{W}}), \forall x \in \mathcal{C}_i, \forall i \in \mathcal{L}.$$

Since for all  $i \in \mathcal{L}$  it holds that  $x_i^e \in \mathcal{C}_i$  for all  $e \in \mathcal{E}$ , from Fact III.2 and the above inequality one obtains that (6c) holds, which completes the proof.  $\blacksquare$

Notice that the LP that corresponds to checking feasibility of (6) has  $ln + 1$  variables and  $2lE(l(w + 1) + n - 1)$  inequalities.

### B. Quadratic nonlinear systems

Consider systems of the form

$$x(k+1) = Ax(k) + \Phi(x(k)), \quad k \in \mathbb{Z}_+, \quad (7)$$

where  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,

$$\Phi(x) := (B_1^\top x \quad B_2^\top x \quad \dots \quad B_n^\top x)^\top x,$$

$B_i \in \mathbb{R}^{n \times n}$  for all  $i \in \mathcal{N} = \mathbb{Z}_{[1, n]}$ . Recently, the continuous-time equivalent of this class of systems was considered in [20], where an iterative algorithm that requires the solution of a min-max optimization problem subject to a rank constraint was proposed. Stabilization of discrete-time quadratic nonlinear systems using polyhedral control Lyapunov functions was considered in [21], where sufficient conditions for positive invariance of a polyhedral C-set were established for system (7). These conditions are an extension of the results for linear discrete-time systems from [12] that make use of the results in [16].

Next, we state the extension of Theorem III.6.

**Theorem IV.2** Let  $l \in \mathbb{Z}_{\geq n}$  and  $P \in \mathbb{R}^{l \times n}$ . Suppose that there exists a  $E \in \mathbb{R}_{>n}$ , a corresponding set of points  $\{\{x_i^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}}$  with  $x_i^e \in \mathbb{R}^n$ ,  $x_i^e \neq 0$  for all  $(i, e) \in \mathcal{L} \times \mathcal{E}$  that induces a proper  $l$ -conic partition of  $\mathbb{R}^n$  and a  $c \in \mathbb{R}_{>0}$  such that the following inequalities hold for all  $i \in \mathcal{L}$ :

$$([P]_{i\bullet} \pm c[I_n]_{j\bullet})x_i^e \geq 0, \quad \forall j \in \mathcal{N}, \forall e \in \mathcal{E}, \quad (8a)$$

$$([P]_{i\bullet} \pm [P]_{j\bullet})x_i^e \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \forall e \in \mathcal{E}, \quad (8b)$$

$$\begin{aligned} &(\rho[P]_{i\bullet} \pm [P]_{j\bullet}A)x_i^{e_2} \geq 0, \\ &(\rho[P]_{i\bullet} \pm [P]_{j\bullet}(A + (B_1^\top x_i^{e_1} \ \dots \ B_n^\top x_i^{e_1})^\top))x_i^{e_2} \geq 0, \\ &\forall j \in \mathcal{L}, \forall (e_1, e_2) \in \mathcal{E} \times \mathcal{E}. \end{aligned} \quad (8c)$$

Let  $\mathcal{V} := \text{Co}(\{\{x_i^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}})$ ,  $\mathcal{P}_\lambda := \{x \in \mathbb{R}^n \mid \|Px\| \leq \lambda\}$  and let  $\lambda^* := \sup\{\lambda \in \mathbb{R}_{>0} \mid \mathcal{P}_\lambda \subseteq \mathcal{V}\}$ .

Then, the function  $V(x) = \|Px\|$  is a Lyapunov function in  $\mathcal{P}_{\lambda^*}$  for system (7).

*Proof:* Let  $\bar{A}(x) := A + (B_1^\top x \ \dots \ B_n^\top x)^\top$  and let  $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$  denote the proper  $l$ -conic partition that corresponds to  $\{\{x_i^e\}_{e \in \mathcal{E}}\}_{i \in \mathcal{L}}$ . Observe that  $x \in \mathcal{C}_i \cap \mathcal{V}$  implies  $x \in \text{Co}(\{0, \{x_i^e\}_{e \in \mathcal{E}}\})$  and  $\bar{A}(0) = A$ . As  $\bar{A}(x)$  is an affine function of  $x$ , it follows that for any fixed  $e_2 \in \mathcal{E}$ , (8c) implies that

$$(\rho[P]_{i\bullet} \pm [P]_{j\bullet}\bar{A}(x))x_i^{e_2} \geq 0, \quad \forall j \in \mathcal{L}, \forall x \in \mathcal{C}_i \cap \mathcal{V}. \quad (9)$$

Then, as by (8c) we also have that (9) holds for all  $e_2 \in \mathcal{E}$ , yields that

$$(\rho[P]_{i\bullet} \pm [P]_{j\bullet}\bar{A}(x))x \geq 0, \quad \forall j \in \mathcal{L}, \forall x \in \mathcal{C}_i \cap \mathcal{V}.$$

Then, observing that  $\mathcal{P}_{\lambda^*} \subseteq \mathcal{V}$  is a PI set for system (7), the proof follows by employing the same arguments used in the (ii)  $\Rightarrow$  (i) part of the proof of Theorem III.6. ■

The conditions of Theorem IV.2 yield a local infinity norm Lyapunov function for quadratic nonlinear discrete-time systems and lead to a LP with  $ln + 1$  variables and  $2lE(l(E + 2) + n - 1)$  inequalities. Note that the set  $\mathcal{P}_{\lambda^*}$  can be enlarged by enlarging the set  $\mathcal{V}$ , as long as the corresponding LP remains feasible.

## V. ILLUSTRATIVE EXAMPLES

In what follows the developed theory is illustrated using 2 examples taken from the literature. For each example the feasibility of the corresponding LP was successfully checked using 4 different solvers, including the Matlab *linprog* solver. The simulation results presented in the paper correspond to the *GLPK* solver. For any  $\lambda \in \mathbb{R}_{>0}$ ,  $\mathcal{P}_\lambda := \{x \in \mathbb{R}^n \mid V(x) \leq \lambda\}$ . In all cases  $\rho = 0.94$  was used and the constraint  $c \geq 0.1$  was imposed. The feasible solution  $c = 0.1$  was obtained for the first example and  $c = 0.5042$  was attained for the second example. It can be observed that the guaranteed contraction is obeyed non-trivially for both examples.

*Example 1* [12]: Consider the linear system (3) with  $A = \begin{pmatrix} -0.32 & 0.32 \\ -0.42 & -0.92 \end{pmatrix}$  and eigenvalues  $-0.62 \pm 0.2107i$ . The following values were chosen:  $l = 2$ ,  $E = 2$ ,  $x_1^1 = [1 \ 1]^\top$ ,  $x_1^2 = x_2^1 = [1 \ -1]^\top$ ,  $x_2^2 = [-1 \ -1]^\top$ . The resulting LP has 5 variables and 33 constraints. The Lyapunov function matrix

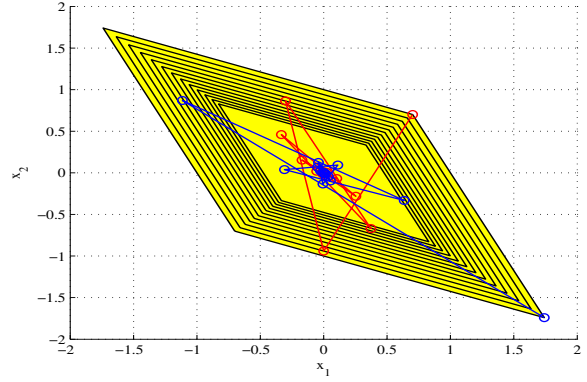


Fig. 1. Simulation results - Example 1.

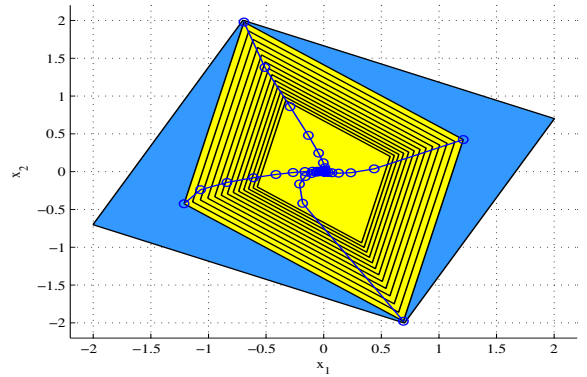


Fig. 2. Simulation results - Example 2.

is  $P = \begin{pmatrix} 1 & 0.4255 \\ -0.4255 & -1 \end{pmatrix}$  with singular values 1.4255 and 0.5745. The polytopes  $\{\rho^i \mathcal{P}_1\}_{i \in \mathbb{Z}_{[0,12]}}$  are plotted in Figure 1 in yellow. The red and blue circle-lines represent system trajectories for two of the vertices of  $\mathcal{P}_1$ .

*Example 2* [21]: Consider the nonlinear quadratic system (7) with  $A = \begin{pmatrix} 0.58 & 0.12 \\ -0.04 & 0.44 \end{pmatrix}$ ,  $B_1 = \begin{pmatrix} -0.099 & -0.171 \\ -0.099 & -0.171 \end{pmatrix}$  and  $B_2 = \begin{pmatrix} -0.066 & -0.114 \\ 0.066 & 0.114 \end{pmatrix}$ . The following values were chosen:  $l = 2$ ,  $E = 2$ ,  $x_1^1 = [-0.7006 \ 2.0023]^\top$ ,  $x_1^2 = x_2^1 = [2.0023 \ 0.7006]^\top$ ,  $x_2^2 = [0.7006 \ -2.0023]^\top$ . These points were obtained by rotating a fixed conic partition scaled by 1.5, which resulted in  $\lambda^* = 2.3$  and a corresponding domain of attraction  $\mathcal{P}_{\lambda^*}$  of a similar size with the set constructed in [21]. Notice that therein a control invariant set was obtained, i.e., in this example the closed-loop system of [21] was considered. The resulting LP has 5 variables and 65 constraints. The corresponding Lyapunov function matrix is  $P = \begin{pmatrix} 1.3293 & 1.6282 \\ 2.0545 & -0.4442 \end{pmatrix}$  with singular values 2.5350 and 1.5525. The polytopes  $\{\rho^i \mathcal{P}_1\}_{i \in \mathbb{Z}_{[0,12]}}$  are plotted in Figure 2 in yellow. The blue polytope denotes the set  $\mathcal{V}$  obtained as the convex hull of all points  $x_i^e$  given above. The blue circle-lines represent system trajectories for all vertices of  $\mathcal{P}_{\lambda^*}$ .

## VI. CONCLUSIONS

This paper considered the synthesis of infinity norm Lyapunov functions for discrete-time linear systems. A proper

conic partition of the state-space was employed to construct a finite set of linear inequalities in the elements of the Lyapunov weight matrix. Under typical assumptions, it was proven that the feasibility of the derived set of linear inequalities is equivalent with the existence of an infinity norm Lyapunov function. Furthermore, it was shown that the developed solution extends naturally to several relevant classes of discrete-time nonlinear systems.

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