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Jon Aaronson
Manfred Denker

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ON EXACT GROUP EXTENSIONS

Jon Aaronson and Manfred Denker

Abstract. We give conditions for the exactness of $\mathbb{R}^d$-extensions.

§0 Introduction

A nonsingular transformation $(X, \mathcal{B}, m, T)$ of a standard probability space is called a *fibred system* if there is a generating measurable partition α such that $T : a \to Ta$ is invertible, nonsingular for $a \in \alpha$, and a *Markov map* (or Markov fibred system) if in addition, $Ta \in \sigma(\alpha) \mod m \;\; \forall\, a \in \alpha$.

Write $\alpha = \{a_s : s \in S\}$ and endow $S^{\mathbb{N}}$ with its canonical (Polish) product topology. Let

$$\Sigma = \{s = (s_1, s_2, \dots) \in S^{\mathbb{N}} :\; m(\bigcap_{k=1}^{n} T^{-k} a_{s_k}) > 0 \;\; \forall\, n \geq 1\},$$

then Σ is a closed, shift invariant subset of $S^{\mathbb{N}}$, and there is a measurable map $\phi : \Sigma \to X$ defined by $\{\phi(s_1, s_2, \dots)\} := \bigcap_{k=1}^{\infty} T^{-(k-1)} a_{s_k}$.

The closed support of the probability $m' = m \circ \phi^{-1}$ is Σ, and ϕ is a conjugacy of $(X, \mathcal{B}, m, T)$ with $(\Sigma, \mathcal{B}(\Sigma), m', \text{shift})$. Thus we may, and sometimes do, assume that $X = \Sigma$, T is the shift, and $\alpha = \{[s] : s \in S\}$.

For $n \geq 1$, there are m-nonsingular inverse branches of T denoted $v_a : T^n a \to a$ and with Radon Nikodym derivatives denoted

$$v_a' := \frac{dm \circ v_a}{dm}.$$

Let $(X, \mathcal{B}, m, R)$ be a nonsingular transformation of a standard probability space. The *Frobenius-Perron* operators $P_{R^n} = P_{R^n, m} : L^1(m) \to L^1(m)$ are defined by

$$\int_X P_{R^n} f \cdot g dm = \int_X f \cdot g \circ R^n dm$$

and for the locally invertible $(X, \mathcal{B}, m, T, \alpha)$ (as above) have the form

$$P_{T^n} f = \sum_{a \in \alpha_0^{n-1}} 1_{T^n a} v_a' \cdot f \circ v_a.$$

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A locally invertible map $(X,\mathcal{B},m,T,\alpha)$ has:

the *Renyi property* if $\exists\ C>1$ such that $\forall\ n\ge 1,\ a\in\alpha_0^{n-1},\ m(a)>0$: $\left|\frac{v_a'(x)}{v_a'(y)}\right|\le C$ for $m\times m$-a.e. $(x,y)\in T^na\times T^na$.

It is well known (a proof is recalled in [A-D-U]) that any topologically mixing probability preserving Markov map with the Renyi property is *exact* in the sense that $\bigcap_{n\ge 1}T^{-n}\mathcal{B}=\{\emptyset,X\}\mod m$.

Examples include:

- topological Markov shifts equipped with Gibbs measures ([Bo],[Bo-Ru]) and
- uniformly expanding, piecewise onto C^2 interval maps $T:[0,1]\to[0,1]$ satisfying Adler's condition $\sup_{x\in[0,1]}\frac{|T''(x)|}{T'(x)^2}<\infty$ ([Ad]);

or, more generally,

- Gibbs-Markov maps as in [A-D1].

Now let $\phi:X\to\mathbb{R}^d$ be measurable and consider the skew product $T_\phi:X\times\mathbb{R}^d\to X\times\mathbb{R}^d$ defined by $T_\phi(x,y):=(Tx,y+\phi(x))$ with respect to the (invariant) product measure $m\times m_{\mathbb{R}^d}$ where $m_{\mathbb{R}^d}$ denotes Lebesgue measure.

We say that ϕ is *aperiodic* if $\gamma(\phi)=z\overline{h}h\circ T$ has no nontrivial solution in $\gamma\in\hat{\mathbb{R}}^d$, $z\in S^1$ and $h:X\to S^1$ measurable. It is not hard to show that if T_ϕ is ergodic, and T is weakly mixing, then T_ϕ is weakly mixing iff ϕ is aperiodic.

We're interested in the exactness of T_ϕ.

We establish two (partial) results in this direction.

Theorem 1.

Suppose that $(X,\mathcal{B},m,T,\alpha)$ is a probability preserving Markov map with the Renyi property. Let $N\ge 1$ and $\phi:X\to\mathbb{R}^d$ be α_0^{N-1}-measurable (i.e. $\phi(x)=\phi(\alpha_0^{N-1}(x))$ where $x\in\alpha_0^{N-1}(x)\in\alpha_0^{N-1}$).

If T_ϕ is topologically mixing, then T_ϕ is exact.

For the other result, we assume that $(X,\mathcal{B},m,T,\alpha)$ is an exact probability preserving locally invertible map with the property that for some Banach space $(L,\|\cdot\|_L)$ of functions with $\|\cdot\|_2\le\|\cdot\|_L$, such that $P_T:L\to L$ and $\exists M>0,\ \theta\in(0,1)$ such that

$$\|P_{T^n}f-\int_X fdm\|_L\le M\theta^n\|f\|_L\ \forall\ f\in L.$$

This property can be obtained as a consequence of the quasi compactness of Doeblin-Fortet operators, see [D-F], [IT-M]).

Given $\phi:X\to\mathbb{R}^d$ measurable, we define the *characteristic function operators* $P_t(f)=P_T(e^{i\langle t,\phi\rangle}f)\ \ (t\in\mathbb{R}^d)$.

We assume also that $P_t:L\to L\ \ (t\in\mathbb{R}^d)$ and that $t\mapsto P_t$ is continuous $(\mathbb{R}^d\to\operatorname{Hom}(L,L)$.

It is shown in [Nag] (see also theorem 4.1 of [A-D1]) that

(i) there are constants $\epsilon>0,\ K>0$ and $\theta\in(0,1)$; and continuous functions $\lambda:B(0,\epsilon)\to B_{\mathbb{C}}(0,1),\ g:B(0,\epsilon)\to L$ such that

$\|P_t^nh-\lambda(t)^ng(t)\int_X hdm\|_L\le K\theta^n\|h\|_L\quad\forall\ |t|<\epsilon,\ n\ge 1,\ h\in L;$

and

(*ii*) in case ϕ is aperiodic, then $\forall\ 0 < \delta < M < \infty,\ \exists\ K > 0,\ 0 < \rho < 1$ such that

$$\|P_\gamma^n h\|_L \le K\rho^n \quad \forall\ h \in L,\ n \ge 1,\ \delta \le |\gamma| \le M.$$

Examples include:

• (see [A-D1]), $(X, \mathcal{B}, m, T, \alpha)$ a Gibbs-Markov maps and $\phi : X \to \mathbb{R}^d$ uniformly Hölder continuous on partition sets. Here L is a space of Hölder continuous functions $f : X \to \mathbb{C}$.

• (see [Rou], [Ry]), $X = [0,1]$, m Lebesgue measure, α a partition of $X \mod m$ into open intervals, and $T : a \to Ta$ an invertible, m-nonsingular homeomorphism for each $a \in \alpha$ with $\inf |T'| > 1$ and $\frac{1}{T'}$ of bounded variation on X; and $\phi : X \to \mathbb{R}^d$ either: of bounded variation on X; or constant on each $a \in \alpha$.

Set $\phi_n = \phi + \phi \circ T + ... + \phi \circ T^{n-1}$.

Theorem 2.
Suppose that

$$(\diamond) \qquad \forall\ \lambda > 1\ \exists\ n_k \to \infty \text{ such that } \frac{\phi_{n_k}}{\lambda^{n_k}} \to 0 \text{ a.e. as } k \to \infty$$

and that ϕ is aperiodic;
then T_ϕ is exact.

Remarks.
1) Theorem 2 generalises the corresponding theorem on page 443 in [G].
2) The condition $(\diamond)$ is satisfied if m-dist(ϕ) is in the domain of attraction of a stable law.
3) The condition $(\diamond)$ is not satisfied iff $\exists\ \lambda > 1$ and $\epsilon > 0$ such that $m([|\phi_n| > \lambda^n]) \ge \epsilon\ \ \forall\ n \ge 1$ and there are independent processes like this.

§1 Frobenius-Perron operators, exactness and relative exactness

Let $(X, \mathcal{B}, m, R)$ be a nonsingular transformation of a standard probability space. The *tail σ-algebra* of $(X, \mathcal{B}, m, R)$ is $\mathcal{T}(R) := \bigcap_{n=1}^\infty R^{-n}\mathcal{B}$ and the nonsingular transformation R is called *exact* if $= \{\emptyset, X\} \mod m$.

Theorem 1.1 [D-L].

$$\|P_{R^n} f\|_1 \to \|E(f|\mathcal{T}(R))\|_1 \text{ as } n \to \infty\ \forall\ f \in L^1(m).$$

In particular (see [L]), R is exact iff $\|P_{R^n} f\|_1 \to 0\ \forall\ f \in L^1(m),\ \int_X f dm = 0$.

Proof.
First note that $|P_T f| \le P_T|f|$ whence $\|P_{R^n} f\|_1 \downarrow$ and $\exists\ \lim_{n\to\infty} \|P_{R^n} f\|_1$. Next, $\forall\ n \ge 1\ \exists\ g_n \in L^\infty(\mathcal{B})$ with $\int_X (P_{R^n} f) g_n dm = \|P_{R^n} f\|_1$, whence

$$\|P_{R^n} f\|_1 = \int_X f g_n \circ R^n dm.$$

By weak $*$ compactness, $\exists\ n_k \to \infty$ and $g \in L^\infty(\mathcal{B})$ such that $g_{n_k} \circ R^{n_k} \rightharpoonup g$ weak $*$ in $L^\infty(\mathcal{B})$.

It follows that $g \in L^\infty(\mathcal{T}(R))$, $\|g\|_\infty \le 1$ and $\lim_{n\to\infty} \|P_{R^n} f\|_1 = \int_X fgdm$. Thus

$$\lim_{n\to\infty} \|P_{R^n} f\|_1 \le \sup\{\int_X fhdm : h \in L^\infty(\mathcal{T}(R)),\ \|h\|_\infty \le 1\} = \|E(f|\mathcal{T}(R))\|_1.$$

To show the converse inequality, note that $\exists\ g \in L^\infty(\mathcal{T}(R))$, $\|g\|_\infty = 1$ such that

$$\|E(f|\mathcal{T}(R))\|_1 = \int_X E(f|\mathcal{T}(R))gdm = \int_X fgdm$$

whence $\forall\ n \ge 1$, $\exists\ g_n \in L^\infty(\mathcal{B}), g = g_n \circ R^n$ and

$$\|E(f|\mathcal{T}(R))\|_1 = \int_X fgdm = \int_X fg_n \circ R^n dm = \int_X (P_{R^n} f) g_n dm \le \|P_{R^n} f\|_1.$$

□

Let $(X, \mathcal{B}, m, R)$ and $(Y, \mathcal{C}, \mu, S)$ be nonsingular transformations of standard probability spaces. A *factor map* is a function $\pi : X \to Y$ satisfying $\pi^{-1}\mathcal{C} \subset \mathcal{B}$, $\pi \circ T = S \circ \pi$, $m \circ \pi^{-1} = \mu$.

The *fibre expectation* of the factor map $\pi : X \to Y$ is an operator

$$f \mapsto E(f|\pi),\ L^1(X, \mathcal{B}, m) \to L^1(Y, \mathcal{C}, \mu)$$

defined by $\int_Y E(f|\pi) gd\mu = \int_X fg \circ \pi dm$.

The factor map $\pi : X \to Y$ is called *relatively exact* if

$$f \in L^1(\mathcal{B}),\ E(f|\pi) = 0 \text{ a.e.} \implies \|P_{R^n} f\|_1 \to 0.$$

The corollary below appears in [G]. For the convenience of the reader, we supply a (possibly different) proof.

Proposition 1.2. *Suppose that $\pi : X \to Y$ is relatively exact, then $\mathcal{T}(R) = \pi^{-1}\mathcal{T}(S) \mod m$.*

Proof.

Evidently, $\pi^{-1}\mathcal{T}(S) \subseteq \mathcal{T}(R)$. We show that $\pi^{-1}\mathcal{T}(S) \supseteq \mathcal{T}(R)$.

By relative exactness and theorem 1.1, if $f \in L^1(\mathcal{B})$ and $E(f|\pi) = 0$ a.e., then $\int_X fgdm = 0\ \forall\ g \in L^\infty(\mathcal{T}(R))$.

Thus if $f \in L^2(\mathcal{B}) \ominus L^2(\pi^{-1}\mathcal{C})$, then $E(f|\pi) = 0$ a.e. and so

$$\int_X fgdm = 0\ \forall\ g \in L^\infty(\mathcal{T}(R)), \implies f \perp L^2(\mathcal{T}(R)).$$

Thus $L^2(\mathcal{B}) \ominus L^2(\pi^{-1}\mathcal{C}) \subset L^2(\mathcal{B}) \ominus L^2(\mathcal{T}(R))$ whence $L^2(\mathcal{T}(R)) \subset L^2(\pi^{-1}\mathcal{C})$ and $\mathcal{T}(R) \subset \pi^{-1}\mathcal{C} \mod m$.

To see that in fact $\mathcal{T}(R) \subseteq \pi^{-1}\mathcal{T}(S) \mod m$, fix $N \ge 1$, then

$$\mathcal{T}(R) = \bigcap_{n\ge1} R^{-n}\mathcal{B} = \bigcap_{n\ge N+1} R^{-n}\mathcal{B}$$
$$= R^{-N}\mathcal{T}(R) \subset R^{-N}\pi^{-1}\mathcal{C} = \pi^{-1}S^{-N}\mathcal{C}.$$

Taking the intersection over N shows the claim. □

Corollary 1.3 ([G], proposition 1).

If S is exact and $\pi : X \to Y$ is relatively exact, then T is exact.

§2 Proof of theorem 1

For a nonsingular transformation $(X,\mathcal{B},m,R)$, define the *tail relation* of R:

$$\mathfrak{T}(R) := \{(x,y)\in X\times X:\ \exists\, n\ge 0,\ R^nx=R^ny\}.$$

Evidently $\mathfrak{T}(R)$ is an equivalence relation and if $(X,\mathcal{B},m)$ is standard, then $\mathfrak{T}(R)\in\mathcal{B}(X\times X)$.

If R is locally invertible, then $\mathfrak{T}(R)$ has countable equivalence classes and is nonsingular in the sense that $m(\mathfrak{T}(R)(A)) = 0\ \forall\ A\in\mathcal{B},\ m(A)=0$ where $\mathfrak{T}(R)(A) := \{y\in X:\ \exists\, x\in A\ (x,y)\in\mathfrak{T}(R)\}$.

A set $A\in\mathcal{B}(X)$ is *invariant* under the equivalence relation $\mathfrak{T}\in\mathcal{B}(X\times X)$ if $\mathfrak{T}(A)=A$ and the equivalence relation $\mathfrak{T}$ is called *ergodic* if $\mathfrak{T}$-invariant sets have either zero, or full measure.

The collection of invariant sets under $\mathfrak{T}(R)$ is the tail σ-algebra $\mathcal{T}(R)$ (whence the name "tail relation").

In order to prove theorem 1, it suffices to show that $\mathfrak{T}(T_\phi)$ is ergodic.

The tail relation of T_ϕ is given by

$$\begin{aligned}&\mathfrak{T}(T_\phi)\\ &=\{((x,s),(y,t))\in(X\times G)^2:\ \exists\, n\ge 0,\ T^nx=T^ny,\ s-t=\phi_n(y)-\phi_n(x)\}\\ &=\{((x,s),(y,t))\in(X\times G)^2:\ (x,y)\in\mathfrak{T}(T),\ \tilde\phi(x,y)=s-t\}\end{aligned}$$

where $\tilde\phi:\mathfrak{T}(T)\to\mathbb{R}^d$ is defined by $\tilde\phi(x,y):=\sum_{n=0}^\infty(\phi(T^ny)-\phi(T^nx))$.

We prove that $\mathfrak{T}(T_\phi)$ is ergodic by the method of Schmidt (explained in [S]), by showing that $\forall\ t\in\mathbb{R}^d$, U a neighbourhood of t and $A\in\mathcal{B}\ m(A)>0$, $\exists\ B\in\mathcal{B}\ B\subset A$ and $\tau:B\to B$ nonsingular such that $(x,\tau(x))\in\mathfrak{T}(T)$ and $\tilde\phi(x,\tau(x))\in U\ \forall\ x\in B$.

This boils down to showing that

$$\begin{aligned}&\forall\ A\in\mathcal{B}_+\ g_0\in\mathbb{R}^d\ \eta>0,\ \exists\ B\in\mathcal{B}_+\ B\subset A,\ n\ge 1\\ &\qquad\text{and } \tau:B\to\tau B\subset A \text{ nonsingular such that}\\ (\ddagger)\qquad &\qquad T^n\circ\tau\equiv T^n \text{ and } \|\phi_n\circ\tau-\phi_n-g_0\|<\eta \text{ on } B.\end{aligned}$$

The proof of (‡) will be written as a sequence of minor claims, ¶0,¶1, ...

¶0 We first claim that there is no loss in generality in assuming that $N=1$ (i.e. that $\phi: X\to\mathbb{R}^d$ is α-measurable). This is because $(X,\mathcal{B},m,T,\alpha_0^{N-1})$ is also a probability preserving Markov map with the Renyi property and inducing the same (shift) topology on X as $(X,\mathcal{B},m,T,\alpha)$.

¶1 $\forall\ s,t\in S,\ \exists\ \kappa=\kappa_{s,t}\ge 1$ and $a=a_{s,t}=[a_1,\dots a_\kappa],\ b=b_{s,t}=[b_1,\dots b_\kappa]\in\alpha_0^{\kappa-1},\ a_1=b_1=s\ a_\kappa=b_\kappa=t$ such that $\|\phi_\kappa(b)-\phi_\kappa(a)-g_0\|<\eta$.

This follows from topological mixing of T_ϕ.

By the Renyi property, $\exists\ M>1$ such that

$$M^{-1}m(u)m(v)\le m(u\cap T^{-k}v)\le Mm(u)m(v)\ \ \forall\, u\in\alpha_0^{k-1},\ v\in\alpha_0^{\ell-1},\ [v_1]\subset T[u_k].$$

Given $u = [u_1, \ldots, u_n] \in \alpha_0^{n-1}$ with $u_n = t$, define $\tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ by

$$\tau(u_1, \ldots, u_n, a_1, \ldots a_\kappa, y) := \tau(u_1, \ldots, u_n, b_1, \ldots b_\kappa, y).$$

¶2 $\tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ is invertible nonsingular and $\frac{dm\circ\tau}{dm} = M^{\pm 4}\frac{m(b)}{m(a)}$.
PROOF

$$\begin{aligned}\int_{u\cap T^{-n}a\cap c} \frac{dm\circ\tau}{dm} dm &= m(u \cap T^{-n}b \cap c)\\ &= M^{\pm 2}\frac{m(b)}{m(a)} m(u)m(b)m(c)\\ &= M^{\pm 4}\frac{m(b)}{m(a)} m(u \cap T^{-n}a \cap c).\end{aligned}$$

□

¶3 PROOF OF ‡
Fix $0 < \epsilon < M^{-1}\min\{m(a_{s,t}),\ m(b_{s,t})\}$, then

$$m(u \cap T^{-n}a_{s,t}),\ m(u \cap T^{-n}b_{s,t}) \geq \epsilon m(u)\ \forall\ u \in \alpha_0^{n-1},\ [s] \subset T[u_n].$$

Let $\delta > 0$ be so small that $\delta < \frac{m(b)(\epsilon-\delta)}{M^4 m(a)}$.

$\exists\ n \geq 1$ and $u \in \alpha_0^{n-1}$ such that $m(A \cap u) \geq (1-\delta)m(u)$ and $[s] \subset T[u_n]$.

Consider $\tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ as in ¶2. Evidently $T^{n+\kappa} \circ \tau \equiv T^{n+\kappa}$ and $\|\phi_{n+\kappa} \circ \tau - \phi_{n+\kappa} - g_0\| < \eta$ on $u \cap T^{-n}a$.

To complete the proof we claim that $\exists\ B \in \mathcal{B}_+\ B \subset A \cap u \cap T^{-n}a$ such that $\tau B \subset A$.

To see this, note that

$$m(u \cap T^{-n}a \cap A) \geq m(u \cap T^{-n}a) - m(u \setminus A) \geq (\epsilon - \delta)m(u),$$

whence using ¶2,

$$m(\tau(u \cap T^{-n}a \cap A)) \geq \frac{m(b)}{M^4 m(a)} m(u \cap T^{-n}a \cap A) \geq \frac{m(b)(\epsilon-\delta)}{M^4 m(a)} m(u).$$

Since $\tau(u \cap T^{-n}a \cap A) \subset u$, the condition on $\delta > 0$ ensures that $m(\tau(u \cap T^{-n}a \cap A) \cap A) > 0$ whence $m(B) > 0$ where $B := \tau^{-1}\Big(\tau(u \cap T^{-n}a \cap A) \cap A\Big) \subset A$. □

§3 PROOF OF THEOREM 2

We prove theorem 2 via corollary 1.3. To do this, we must consider T_ϕ as a nonsingular transformation with respect to some probability $P \sim m \times m_{\mathbb{R}^d}$.

Let $p : \mathbb{R}^d \to \mathbb{R}_+$ be continuous with $\int_{\mathbb{R}^d} p(y)dy = 1$ and define a probability P on $X \times \mathbb{R}^d$ by $dP(x,y) := p(y)dm(x)dy$; then $(X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P, T_\phi)$ is a nonsingular transformation with Frobenius-Perron operators given by

$$P_{T_\phi^n,P}f(x,y)=\frac{1}{p(y)}P_{T_\phi^n}(f\cdot 1\otimes p)(x,y)$$

where $P_{T_\phi^n}:=P_{T_\phi^n,m\times m_{\mathbb{R}^d}}$.

Consider the map $\pi : X\times\mathbb{R}^d\to X$ defined by $\pi(x,y)=x$. This is a factor map as it satisfies $\pi^{-1}\mathcal{B}(X)\subset\mathcal{B}(X\times\mathbb{R}^d)$, $\pi\circ T_\phi=T\circ\pi$, $P\circ\pi^{-1}=m$.

The fibre expectation of π is given by

$$E(f|\pi)(x)=\int_{\mathbb{R}^d}f(x,y)p(y)dy\quad (f\in\ L^1(X\times\mathbb{R}^d,\mathcal{B}(X\times\mathbb{R}^d),P)).$$

By corollary 1.3 and exactness of T, it suffices to show that π is relatively exact. To do this, we show that

$$\int_{\mathbb{R}^d}f(x,y)p(y)dy=0\text{ a.e.}\quad\Longrightarrow$$
$$\int_{X\times\mathbb{R}^d}|P_{T_\phi^n,P}f|dP=\int_{X\times\mathbb{R}^d}|P_{T_\phi^n}(f\cdot 1\otimes p)|d(m\times m_{\mathbb{R}^d})\to 0$$

as $n\to\infty$; equivalently (taking $F(x,y):=f(x,y)p(y)$),

$$(\star)\qquad \int_{\mathbb{R}^d}F(x,y)dy=0\text{ a.e.}\quad\Longrightarrow\quad\int_{X\times\mathbb{R}^d}|P_{T_\phi^n}F|d(m\times m_{\mathbb{R}^d})\to 0$$

as $n\to\infty$.

To prove $(\star)$, we first claim that

¶1 for $\lambda>1$, $h\in L^1(m)$ and $f\in L^1(\mathbb{R}^d)$,

$$\|P_{T_\phi^{n_k}}(h\otimes f)\|_1\le C\lambda^{\frac{n_k d}{2}}\|P_{T_\phi^{n_k}}(h\otimes f)\|_2+o(1)$$

as $k\to\infty$ where $C=2^{\frac{d}{2}}m(B(0,1))$ and $\frac{\phi_{n_k}}{\lambda^{n_k}}\to 0$ a.e..

PROOF As can be checked,

$$P_{T_\phi^n}(h\otimes f)(x,y)=P_{T^n}(h(\cdot)f(y-\phi_n(\cdot)))(x)\quad (h\in L^1(m),\ f\in L^1(\mathbb{R}^d)).$$

Denoting $E(H):=\int_X Hdm$ for $H\in L^1(m)$, we have

$$(2)\ \ \|P_{T_\phi^{n_k}}(h\otimes f)\|_1=\int_{\mathbb{R}^d}|E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))))|dy\le\int_{|y|\le 2\lambda^{n_k}}+\int_{|y|>2\lambda^{n_k}}.$$

By the Cauchy-Schwartz inequality,

$$(3)\qquad\int_{|y|\le 2\lambda^{n_k}}\le\sqrt{m_{\mathbb{R}^d}(B(0,2\lambda^{n_k}))}\|P_{T_\phi^{n_k}}(h\otimes f)\|_2=C\lambda^{\frac{n_k d}{2}}\|P_{T_\phi^{n_k}}(h\otimes f)\|_2$$

whereas

$$\int_{|y|>2\lambda^{n_k}} \le \int_{|y|>2\lambda^{n_k}} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))1_{[|\phi_{n_k}(\cdot))|\le\lambda^{n_k}]})|dy$$
$$+\int_{|y|>2\lambda^{n_k}} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))1_{[|\phi_{n_k}(\cdot)|>\lambda^{n_k}]}))|dy = I + II.$$

Here as $k\to\infty$:

(4) $$II \le \|f\|_1 E(|h|1_{[|\phi_{n_k}(\cdot)|>\lambda^{n_k}]}) \to 0$$

since $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e.; and

$$I \le \int_{|y|>2\lambda^{n_k}} E(|h||f(y-\phi_{n_k})|1_{[|\phi_{n_k}(\cdot)|\le\lambda^{n_k}]})dy$$
$$= E\Big(|h|1_{[|\phi_{n_k}|\le\lambda^{n_k}]}\int_{|y|>2\lambda^{n_k}} |f(y-\phi_{n_k})|dy\Big)$$
(5) $$\le E(|h|)\int_{|y|>\lambda^{n_k}} |f(y)|dy \to 0,$$

Substituting (3),(4) and (5) into (2) proves ¶1. □

To complete the proof of $(\star)$, let $F\in L^1(m\times m_{\mathbb{R}^d})$ satisfy $\int_{\mathbb{R}^d} F(x,y)dy = 0$ for m-a.e. $x\in X$ and fix $\epsilon>0$. We show that

$(\star_\epsilon)$ $$\limsup_{n\to\infty}\int_{X\times\mathbb{R}^d} |P_{T_\phi^n} F| d(m\times m_{\mathbb{R}^d}) < \epsilon.$$

Standard approximation techniques show that $\forall\ \epsilon>0,\ \exists\ N\in\mathbb{N},\ h_1,\dots,h_N\in L,\ g_1,\dots,g_N\in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g_k(y)dy = 0\ \ (1\le k\le N)$ and

$$\|F-\sum_{k=1}^N h_k\otimes g_k\|_{L^1(m\times m_{\mathbb{R}^k})} < \frac{\epsilon}{2}.$$

Next, it follows from theorems 1.6.3 and 1.6.4 in [Rud] that
$\exists\ f_1,\dots,f_N\in L^1\cap L^2$ such that
- $[\hat{f}_k\neq 0]$ is compact and bounded away from 0 $(1\le k\le N)$; and
- $\|f_k-g_k\|_{L^1(m_{\mathbb{R}^d})} < \frac{\epsilon}{2N\|h_k\|_{L^1(m)}}$ $(1\le k\le N)$, whence

$$\|\sum_{k=1}^N h_k\otimes f_k - \sum_{k=1}^N h_k\otimes g_k\|_{L^1(m\times m_{\mathbb{R}^d})} \le \sum_{k=1}^N \|h_k\|_{L^1(m)}\cdot\|f_k-g_k\|_{L^1(\mathbb{R}^d)} < \frac{\epsilon}{2},$$

$$\|F-\sum_{k=1}^N h_k\otimes f_k\|_{L^1(m\times m_{\mathbb{R}^d})} < \epsilon$$

where $h \in L$ and $f \in L^1 \cap L^2$ is such that $[\hat f \neq 0]$ is compact and bounded away from 0.

We claim

¶2 If $h \in L$ and $f \in L^1 \cap L^2$ is such that $[\hat f \neq 0]$ is compact and bounded away from 0, then $\exists\ 0 < \rho < 1$ such that

$$\|P_{T_\phi^n}(h \otimes f)\|_2 = O(\rho^n) \text{ as } n \to \infty. \tag{6}$$

Proof

Let $[\hat f \neq 0] \subset B(0, M) \setminus B(0, \delta)$. By (ii) (above), $\exists\ K > 0,\ 0 < \rho < 1$ such that

$$|P_\gamma^n h(x)| \le K\rho^n \quad \forall\ x \in X,\ n \ge 1,\ \delta \le |\gamma| \le M,$$

whence using the fact that the Fourier transform of $y \mapsto P_{T_\phi}^n(h \otimes f)(x, y)$ is $\gamma \mapsto \hat f(\gamma) P_\gamma^n h(x)$ and Plancherel's formula, we have

$$\begin{aligned}
\|P_{T_\phi^n}(h \otimes f)\|_2^2 &= \int_X \left(\int_{\mathbb{R}^d} |P_{T_\phi^n}(h \otimes f)(x, y)|^2 dy \right) dm(x) \\
&= \int_X \left(\int_{\mathbb{R}^d} |\hat f(\gamma)|^2 |P_\gamma^n h(x)|^2 d\gamma \right) dm(x) \\
&= \int_{\mathbb{R}^d} |\hat f(\gamma)|^2 \|P_\gamma^n h\|_2^2 d\gamma \le K^2 \rho^{2n} \int_{\mathbb{R}^d} |\hat f(\gamma)|^2 d\gamma
\end{aligned}$$

proving ¶2. □

To finish the proof of theorem 2, we claim

¶3 if (6) holds for $h \in L$ and $f \in L^1 \cap L^2$, then

$$\|P_{T_\phi^n}(h \otimes f)\|_1 \to 0. \tag{7}$$

Proof

Fix $\lambda > 1$ such that $\lambda^{\frac{d}{2}} \rho < 1$. Suppose that $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e.. Using (6), we have by ¶1,

$$\|P_{T_\phi^{n_k}}(h \otimes f)\|_1 \le C\lambda^{\frac{n_k d}{2}} \|P_{T_\phi^{n_k}}(h \otimes f)\|_2 + o(1) = O(\lambda^{\frac{n_k d}{2}} \rho^{n_k}) + o(1) \to 0$$

as $k \to \infty$; establishing (7) since $\|P_{T_\phi^n}(h \otimes f)\|_1 \downarrow$. □

This completes the proof of theorem 2.

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AARONSON: SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, 69978 TEL AVIV, ISRAEL.

E-mail address: aaro@math.tau.ac.il

DENKER: INSTITUT FÜR MATHEMATISCHE STOCHASTIK, UNIVERSITÄT GÖTTINGEN, LOTZESTR. 13, 37083 GÖTTINGEN, GERMANY

E-mail address: denker@math.uni-goettingen .de