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Abstract. In this paper we study a single-server system with Erlang-r distributed service times and arbitrarily distributed interarrival times. It is shown that the waiting-time distribution can be expressed as a finite sum of exponentials, the exponents of which are the roots of an equation. Under certain conditions for the interarrival-time distribution, this equation can be transformed to r contraction equations, the roots of which can be easily found by successive substitutions. The conditions are satisfied for several practically relevant arrival processes. The resulting numerical procedures are easy to implement and efficient and appear to be remarkably stable, even for extreme high values of r and for values of the traffic load close to 1. Numerical results are presented.

Key words. equilibrium distribution, Markov chain, queues

AMS subject classifications. 60K25, 90B22

1. Introduction. In this paper we study a single-server system with Erlang-r distributed service times and arbitrarily distributed interarrival times. By using the embedded Markov chain approach we show that the waiting-time distribution can be expressed as a finite sum of exponentials, the exponents of which are the roots of an equation, involving the Laplace-Stieltjes transform of the interarrival-time distribution. In many cases of interest, the service times have a low coefficient of variation, which implies that the shape parameter r is large. Then standard methods to exactly compute the roots of an equation are expensive and may be not reliable, so that in those cases the numerical applicability of the analytical result is questionable. In the present paper it is investigated whether the explicit expression for the waiting time distribution is numerically useful, in particular when r is large and the traffic load is close to 1. Under certain conditions for the interarrival-time distribution, it is shown that the equation mentioned above can be transformed to r contraction equations, the roots of which can be easily found by successive substitutions. The conditions are satisfied for several practically relevant arrival processes, such as deterministic, mixed Erlang and hyper-exponential arrivals. The resulting numerical procedures are easy to implement and efficient and appear to be remarkably stable, even for extreme large values of r and for values of the traffic load close to 1.

In the literature on single-server queues many exact and approximate solution approaches have been proposed to find the waiting-time distribution.

An exact solution can be found in Cohen [4] for the $GI/K_n/1$ queue in which the Laplace-Stieltjes transform of the service time distribution is a rational function. In this reference it is shown that the Laplace-Stieltjes transform of the waiting time distribution is also a rational function, which implies that the waiting time distribution can be written as a sum of exponentials. A double series expression for the mean waiting time of a $\Gamma_a/\Gamma_b/1$ queue with gamma interarrival and service times is derived by Ikeda [7].

It is well-known that the steady state probability distribution of the embedded Markov chain for the $GI/Ph/1$ queue with phase-type services is matrix-geometric,

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see e.g. Chapter 4 in Neuts [9]. This aspect is exploited by Ramaswami and Lucantoni [10], who develop efficient schemes for the computation of the rate matrix R and provide an easy computable series expression in terms of R for the waiting time distribution.

Accurate approximations for the waiting time probabilities of the $Ph/Ph/1$ queue are given in Seelen and Tijms [12] and Seelen [11]. These approximations use that for the $Ph/Ph/1$ system the waiting time distribution has an exponential tail (see e.g. Takahashi [13]). For the special case of the $D/G/1$ queue, Fredericks [6] gives approximations for the delay probability and the mean waiting time. De kok [8] introduces a moment-iteration method for the $GI/G/1$ queue, which produces accurate approximations for the waiting time distribution.

Bux [1] presents a method for the numerical analysis of the embedded Markov chain of the $GI/E_{r,s}/1$ queue with mixed Erlang service times, which is based on truncation of the infinite state space of the Markov chain. Tijms and Van de Coevering [15] propose a clever truncation of the state space by exploiting the geometric-tail behavior of the equilibrium probabilities, if such behavior exists. In the latter reference the approach is applied to the $D/E_{k-1,k}/1$ queue yielding an efficient algorithm for the waiting time probabilities in this model. For an excellent survey of exact and approximation methods for queueing models, the reader is referred to Tijms [14].

The determination of roots of an equation plays an important role in many queueing problems. So the idea of reducing an equation to one or more contraction equations may also be useful in other queueing problems, such as e.g. the bulk service queue with deterministic service times (see Downton [5], eq. (25)) and the $E_m/D/1$ queue (see Chaudry [3], eq. (21)).

This paper is organized as follows. In section 2 the model is introduced and the balance equations are formulated. In the next section these equations are solved and an expression for the waiting time distribution in the form of a sum of exponentials is derived. The exponents of the exponentials are roots of equation. In section 4 conditions for the interarrival time distribution are formulated under which these roots can be easily found. In section 5 it is shown that these conditions hold for several important distributions. Numerical results are presented in section 6.

2. Model and equations. Consider a single-server system with Erlang-r distributed service times with mean r/μ and an arbitrary interarrival distribution $A(t)$ with mean $1/\lambda$. The system behavior will be analyzed at arrival instants. Let X_n be the number of uncompleted service stages in the system just prior to the n-th arrival. Then the sequence $\{X_n\}$ forms a Markov chain with state space $\{0, 1, \ldots\}$ and transition probabilities p_{ij} given by

$$p_{ij} = \begin{cases} \sum_{k=i+r}^{\infty} \beta_k \,, & i \geq 0\,,\ j = 0\,, \\ \beta_{i+r-j}\,, & i \geq 0\,,\ 1 \leq j \leq i+r\,, \\ 0\,, & i \geq 0\,,\ j \geq i+r+1\,, \end{cases}$$

where β_k is defined as the probability that k service stages are completed during an interarrival time, so

$$\beta_k = \int_0^\infty \frac{(\mu t)^k}{k!} e^{-\mu t} dA(t)\,, \qquad k \geq 0\,.$$

Note that the Markov chain $\{X_n\}$ is irreducible and aperiodic. Henceforth it will be assumed that the offered load ρ, defined by

$$\rho = \lambda r/\mu \, ,$$

is less than 1. In this case the Markov chain $\{X_n\}$ has an equilibrium distribution $\{\pi_i, i = 0, 1, \ldots\}$ which is the unique normalized solution of the balance equations:

$$(1) \qquad \pi_0 = \sum_{k=r}^{\infty} \beta_k \pi_0 + \sum_{k=r+1}^{\infty} \beta_k \pi_1 + \sum_{k=r+2}^{\infty} \beta_k \pi_2 + \ldots \, ;$$

$$(2) \qquad \pi_i = \beta_{r-i}\pi_0 + \beta_{r-i+1}\pi_1 + \beta_{r-i+2}\pi_2 + \ldots \, , \qquad 1 \le i \le r-1 \, ;$$

$$(3) \qquad \pi_i = \beta_0 \pi_{i-r} + \beta_1 \pi_{i-r+1} + \beta_2 \pi_{i-r+2} + \ldots \, , \qquad i \ge r \, .$$

In the next section we will show that the solution can be expressed as a linear combination of r geometric distributions, i.e.,

$$(4) \qquad \pi_i = \sum_{k=1}^{r} c_k (1 - \sigma_k) \sigma_k^i \, , \qquad i \ge 0 \, ,$$

where the factors σ_k are the roots of an equation and the coefficients c_k can be explicitly expressed in terms of these roots.

3. Analysis of the equations. The idea to solve the balance equations is to seek solutions of equation (3) of the form $\pi_i = \sigma^i$ and then to linearly combine these solutions to also satisfy the boundary conditions (1)–(2). Inserting this form into (3) and then dividing by σ^{i-r} yield

$$(5) \qquad \sigma^r = \beta_0 + \beta_1 \sigma + \beta_2 \sigma^2 + \ldots = \tilde{A}(\mu(1-\sigma)) \, ,$$

where $\tilde{A}(s)$ is the Laplace-Stieltjes transform of $A(t)$. Clearly, only solutions with $|\sigma| < 1$ are useful. The next lemma states that equation (5) has exactly r roots inside the unit circle. The lemma can be proved by applying Rouché's Theorem.

LEMMA 3.1. *Equation (5) has exactly r roots with $|\sigma| < 1$.*

For convenience, we assume that the roots of (5) with $|\sigma| < 1$ are simple. In fact, in the next section this is proved under some mild condition. Hence, we find r solutions of the form $\pi_i = \sigma^i$ with $|\sigma| < 1$ satisfying equation (3). These solutions are labeled σ_k^i, $k = 1, \ldots, r$. It now remains to determine coefficients c_k such that the linear combination (4) also satisfies the boundary conditions (1)–(2). Since the balance equations are dependent, equation (1) may be omitted. Substitution of (4) into (2) and using that the σ_k's satisfy (5) yield the following set of equations for the coefficients c_k.

$$c_1(1-\sigma_1)\tau_1^i + \ldots + c_r(1-\sigma_r)\tau_r^i = 0 \, , \qquad i = 1, \ldots, r-1 \, ,$$

where, by convention, $\tau_k = 1/\sigma_k$. These equations are of a VanderMonde-type and therefore can be solved explicitly using Cramer's rule. Then we get

$$c_k = \frac{C}{\prod_{j=1}^{r}(1-\tau_j)} \frac{\prod_{j \ne k}(1-\tau_j)}{\prod_{j \ne k}(\tau_k - \tau_j)} \, , \qquad k = 1, \ldots, r \, ,$$

for some constant C. This constant is determined from the normalization equation, which, by using Lagrange's interpolation formula, leads to $C = \prod_{j=1}^{r}(1-\tau_j)$. These findings are summarized in the following theorem.

THEOREM 3.2. *For all* $i \geq 0$,

$$\pi_i = \sum_{k=1}^{r} c_k (1-\sigma_k)\sigma_k^i ,$$

where $\sigma_1, \ldots, \sigma_r$ *are the roots of equation (5) inside the unit circle and for* $k = 1, \ldots, r$,

$$c_k = \frac{\prod_{j \neq k}(1-\tau_j)}{\prod_{j \neq k}(\tau_k - \tau_j)}$$

with $\tau_j = 1/\sigma_j$ *for* $j = 1, \ldots, r$.

By conditioning on the uncompleted service stages in the system found by an arrival and using the expression for the probabilities π_i, we immediately get an expression for the (complementary) distribution of the waiting time W_q in the queue.

COROLLARY 3.3. *For* $t \geq 0$,

$$P(W_q > t) = \sum_{k=1}^{r} c_k \sigma_k e^{-\mu(1-\sigma_k)t} . \tag{6}$$

Based on (6) it is easy to find expressions for the moments of the (conditional) waiting time. Theorem (3.2) can be easily extended to the case that the service time distribution is a mixture of Erlang distributions with the same scale parameters. This theorem reduces the problem of solving the equilibrium distribution to that of finding roots of an equation. For (pure) Erlang services we will be able to show that, under certain conditions, these roots can be found very efficiently.

Note: For Erlang-k interarrival times equation (5) can be solved explicitly yielding

$$\sigma_m = (1+\rho - \sqrt{(1+\rho)^2 - 4\rho\phi_m})/2$$

where $\phi_m = e^{2\pi i m/k}$ for $m = 1, \ldots, k$ with $i = \sqrt{-1}$.

4. Root finding. In this section we show that equation (5) can be transformed to r fixed point equations and formulate conditions under which these equations can be solved by successive substitutions. The same idea of transforming a single equation of the form (5) for r roots to r equations for a single root is employed in Chaudry *et al.* [2]. In this reference Muller's method is proposed to find the root of each of the r equations.

Raising both sides of (5) to the power $1/r$ yields

$$\sigma = \phi F(\sigma) , \tag{7}$$

where ϕ satisfies $\phi^r = 1$ and

$$F(\sigma) = \sqrt[r]{\tilde{A}(\mu(1-\sigma))} .$$

Since we are interested in roots with $|\sigma| < 1$, it is convenient if $F(\sigma)$ is analytic inside the unit circle. Therefore we require that the following condition is satisfied.

CONDITION 4.1. *$\tilde{A}(s)$ has no zeros in the half plane $Re\, s > 0$.*

This condition implies that $\tilde{A}(\mu(1-\sigma))$ has no zeros inside the unit circle, so that $F(\sigma)$ has no branch points inside the unit circle and thus is analytic in this region. This condition is, of course, not always satisfied. For instance, for $\alpha > 2$ the transform

$$\tilde{A}(s) = 1 - p + p\frac{1}{(1+s)^\alpha},$$

where $0 \leq p \leq 1$, has zeros in the half plane $Re\, s > 0$ if p is sufficiently close to 1. By applying Rouché's Theorem it can be proved for each feasible ϕ that equation (7) has exactly one root with $|\sigma| < 1$. This implies that the roots mentioned in Theorem (3.2) are simple. For given ϕ we can try to find the root of (7) with $|\sigma| < 1$ by using the iteration scheme

$$\sigma^{(k+1)} = \phi F(\sigma^{(k)}), \qquad k = 0, 1, \ldots \tag{8}$$

starting with $\sigma^{(0)} = 0$. Below it is investigated under which conditions the sequence $\sigma^{(0)}, \sigma^{(1)}, \ldots$ indeed converges to the desired root.

Since $\tilde{A}(x)$ is log-convex, it follows that $\tilde{A}'(x)/\tilde{A}(x)$ is (non-positive and) non-decreasing for $x \geq 0$. Hence,

$$\frac{F'(x)}{F(x)} = -\frac{\mu}{r}\frac{\tilde{A}'(\mu(1-x))}{\tilde{A}(\mu(1-x))} \tag{9}$$

is (non-negative and) non-decreasing, so the function $F(x)$ is (non-decreasing and) log-convex and thus convex for $0 \leq x \leq 1$. This implies, together with $F(0) > 0$, $F(1) = 1$ and $F'(1) = 1/\rho > 1$, that there is a unique χ with $0 < \chi < 1$ satisfying

$$\chi = F(\chi).$$

Now we try to show that $\phi F(s)$ is a contraction on $|s| \leq \chi$. For $|s| \leq \chi$ it follows that

$$|\phi F(s)| \leq F(|s|) \leq F(\chi) = \chi,$$

so, ϕF maps $|s| \leq \chi$ into itself. Further, for $|\tilde{s}|, |\hat{s}| < \chi$ we have

$$\begin{aligned} |\phi F(\tilde{s}) - \phi F(\hat{s})| &= \left| \int_0^1 F'(\hat{s} + t(\tilde{s} - \hat{s}))(\tilde{s} - \hat{s})dt \right| \\ &\leq |\tilde{s} - \hat{s}| \max_{0 \leq t \leq 1} |F'(\hat{s} + t(\tilde{s} - \hat{s}))|. \end{aligned} \tag{10}$$

If for all $|s| \leq \chi$,

$$|F'(s)| \leq F'(|s|), \tag{11}$$

so

$$|F'(s)| \leq F'(|s|) \leq F'(\chi) < 1,$$

then we get from (10) that

$$|\phi F(\tilde{s}) - \phi F(\hat{s})| \leq |\tilde{s} - \hat{s}| F'(\chi).$$

Hence, ϕF is a contraction, if Condition (11) is satisfied. Condition (11) involves $\tilde{A}(s)$, but also the service parameters μ and r. It is convenient, however, to have a condition in terms of $\tilde{A}(s)$ only. Such a condition is formulated below. By using relation (9) it is easily seen that this condition implies (11).

CONDITION 4.2. *For all s with Re s > 0,*

$$\left| -\frac{\tilde{A}'(s))}{\tilde{A}(s)} \right| \leq -\frac{\tilde{A}'(Re\, s)}{\tilde{A}(Re\, s)} .$$

The findings above are summarized in the following theorem.

THEOREM 4.3. *$\phi F(\sigma)$ is a contraction on $|\sigma| \leq \chi$ if Conditions (4.1)–(4.2) hold.*

Now we can conclude that for given ϕ satisfying $\phi^r = 1$ the root of equation (7) with $|\sigma| < 1$ can be found by using the iteration scheme (8), provided Conditions (4.1) and (4.2) hold. The sequence $\sigma^{(0)}, \sigma^{(1)}, \ldots$ converges exponentially fast to the desired root, at least with rate $F'(\chi)$. A question that now arises is: are there relevant distributions satisfying the Conditions (4.1) and (4.2)? In the next section it will be shown for an important class of distributions that these conditions are satisfied.

5. Distributions satisfying the conditions. The following theorem states that the two conditions introduced in the previous section hold for several useful interarrival time distributions, namely for deterministic, shifted exponential, gamma, mixed Erlang and hyper-exponential interarrival times.

THEOREM 5.1. *Conditions (4.1) and (4.2) are satisfied for:*

(i) $\tilde{A}(s) = e^{-sD}, \quad (D > 0)$;

(ii) $\tilde{A}(s) = e^{-sD}\eta/(\eta + s), \quad (D > 0, \eta > 0)$;

(iii) $\tilde{A}(s) = (\eta/(\eta + s))^\alpha, \quad (\eta > 0, \alpha > 0)$;

(iv) $\tilde{A}(s) = p(\eta/(\eta + s))^{k-1} + (1 - p)(\eta/(\eta + s))^k, \quad (0 \leq p \leq 1, \eta > 0, k \geq 1)$;

(v) $\tilde{A}(s) = p\mu_1/(\mu_1 + s) + (1 - p)\mu_2/(\mu_2 + s), \quad (0 \leq p \leq 1, 0 < \mu_1 < \mu_2)$.

Proof. It is easily seen that transform (i) satisfies the two conditions. Transform (ii) is the product of transform (i) and (iii) with $\alpha = 1$, so this transform satisfies the Conditions (4.1) and (4.2), once these conditions have been established for the transforms (i) and (iii). To prove that the two conditions hold for (iii)–(v), the transforms (iv) and (v) are first written in a form similar to transform (iii) given by

$$\tilde{A}(s) = \left(\frac{\eta}{\eta + s}\right)^\alpha . \tag{12}$$

Transform (iv) may be rewritten as

$$\tilde{A}(s) = \left(\frac{\eta}{\eta + s}\right)^k \left(\frac{\xi}{\xi + s}\right)^{-1} , \tag{13}$$

with $\xi = \eta/p$. Similarly, for transform (v) we get

$$\tilde{A}(s) = \left(\frac{\mu_1}{\mu_1 + s}\right) \left(\frac{\mu_2}{\mu_2 + s}\right) \left(\frac{\mu_3}{\mu_3 + s}\right)^{-1} , \tag{14}$$

with

$$\mu_3 = \frac{\mu_1\mu_2}{p\mu_1 + (1 - p)\mu_2} ,$$

so $\mu_1 < \mu_3 < \mu_2$. It is readily verified that transform (12) has no zeros, transform (13) vanishes for $s = -\xi < 0$ and transform (14) vanishes for $s = -\mu_3 < 0$. So these transforms satisfy Condition (4.1). Further, from (12)–(14) it follows that

$$\tilde{A}(s) = \exp\left(\int_0^\infty \frac{e^{-xs}-1}{x} h(x)dx\right),$$

where $h(x) = \alpha e^{-\eta x}$ for transform (12), $h(x) = ke^{-\eta x} - e^{-\xi x}$ for transform (13) and, finally, $h(x) = e^{-\mu_1 x} + e^{-\mu_2 x} - e^{-\mu_3 x}$ for transform (14). In each case, $h(x) > 0$. So

$$-\tilde{A}'(s)/\tilde{A}(s) =: \tilde{h}(s) = \int_0^\infty e^{-xs} h(x)dx\,.$$

Hence, for s with $Re\, s > 0$ we get

$$|\tilde{h}(s))| \leq \int_0^\infty |e^{-xs}| h(x)dx = \int_0^\infty e^{-xRe\, s} h(x)dx = \tilde{h}(Re\, s)\,,$$

from which we may conclude that the transforms (12)–(14) satisfy Condition (4.2). □

Hence we are able to efficiently and accurately compute the roots $\sigma_1, \ldots, \sigma_r$ for several interarrival time distributions. However, even if the roots can be accurately computed, it is conceivable that the *form* of the solution for the waiting-time distribution easily causes loss of significance when implemented. For instance, Chaudry [3] derives an expression for the waiting-time distribution of the $E_m/D/1$ queue, which appears to be numerically unstable, although the parameters in this expression can be accurately computed. Luckily, in our case, the resulting procedures appear to be stable, even for large values of r and traffic loads close to 1. For the important $D/E_r/1$ model the results appear to be in full agreement with the ones produced by using the approach of Tijms and Van de Coevering [15] for values of r up to 100 and values of ρ up to 0.99. Numerical results are reported in the next section.

Note: Condition (4.2) does not hold for uniform distributed interarrival times, but numerical evidence suggests that in this case the sequence $\sigma^{(0)}, \sigma^{(1)}, \ldots$ converges, which implies that Condition (4.2) may not be necessary for the convergence of this sequence.

6. Numerical results. To illustrate the stability and accuracy of the resulting procedures, we present some examples. In table 1, we list for several $GI/E_r/1$ queues with a traffic load $\rho = 0.99$ the delay probability $\Pi_{W_q} = 1 - \pi_0$ and the mean $E[W_q|W_q > 0]$ and the squared coefficient of variation $c^2_{W_q|W_q>0}$ of the conditional waiting time $W_q|W_q > 0$ in the queue for a range of values for the squared coefficients of variation c_a^2 and c_s^2 of the interarrival-time of the respective service-time distribution. We also computed the α-percentiles w_α of the conditional waiting time, where α runs through the values 0.8, 0.95 and 0.99. Note that w_α for $0 \leq \alpha < 1$ is the unique t satisfying

$$P(W_q \leq t | W_q > 0) = \alpha\,.$$

In each example we have set $\mu = r$, so that the mean service time is equal to 1. All values of parameters in the distributions used here are uniquely determined by the coefficients of variation c_a^2 and c_s^2. For the hyper-exponential distribution, we use the balanced means.

TABLE 1
Performance characteristics. In each example we have set $\rho = 0.99$ and $\mu = r$.

	c_a^2	c_s^2	Π_{W_q}	$E[W_q \mid W_q > 0]$	$c^2_{W_q \mid W_q > 0}$	α-Percentiles w_α of Cond. Waiting Time 0.8	0.95	0.99
$D/E_r/1$	0	0.1	0.9508	5.046	0.9886	8.105	15.06	23.13
	0	0.01	0.8615	.5134	0.9577	.8199	1.515	2.323
	0	0.001	0.6229	.0542	0.8832	.0856	.1551	.2359
$E_k/E_r/1$	0.1	0.1	0.9686	10.05	0.9902	16.15	30.01	46.10
	0.1	0.01	0.9616	5.533	0.9831	8.876	16.48	25.31
	0.1	0.001	0.9605	5.081	0.9818	8.149	15.13	23.23
	0.01	0.1	0.9538	5.547	0.9888	8.909	16.55	25.43
	0.01	0.01	0.9036	1.018	0.9670	1.628	3.014	4.624
	0.01	0.001	0.8754	.5625	0.9534	.8975	1.658	2.541
$H_2/E_r/1$	1.5	0.1	0.9928	79.67	0.9959	128.1	238.3	366.3
	1.5	0.01	0.9929	75.16	0.9951	120.9	224.8	345.5
	1.5	0.001	0.9929	74.71	0.9951	120.1	223.4	343.4
	2	0.1	0.9944	104.4	0.9967	167.9	312.3	480.1
	2	0.01	0.9945	99.86	0.9961	160.6	298.8	459.2
	2	0.001	0.9945	99.41	0.9961	159.9	297.4	457.1
$\Gamma_a/E_r/1$	1.5	0.1	0.9924	79.95	0.9954	128.6	239.1	367.5
	1.5	0.01	0.9924	75.45	0.9947	121.3	225.6	346.7
	2	0.1	0.9838	104.9	0.9961	168.7	313.8	482.3
	2	0.01	0.9938	100.4	0.9955	161.4	300.3	461.5

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