A compound Poisson EOQ model for perishable items with intermittent high and low demand periods

Citation for published version (APA):
Boxma, O., Perry, D., Stadje, W., & Zacks, S. (2022). A compound Poisson EOQ model for perishable items with intermittent high and low demand periods. *Annals of Operations Research*, *317*(2), 439-459. https://doi.org/10.1007/s10479-015-2031-1

DOI:
10.1007/s10479-015-2031-1

Document status and date:
Published: 01/10/2022

Document Version:
Accepted manuscript including changes made at the peer-review stage

Ann Oper Res
DOI 10.1007/s10479-015-2031-1

A compound Poisson EOQ model for perishable items with intermittent high and low demand periods

Onno Boxma[1] · **David Perry**[2] · **Wolfgang Stadje**[3] · **Shelley Zacks**[4]

Abstract We consider a stochastic EOQ-type model, with demand operating in a two-state random environment. This environment alternates between exponentially distributed periods of high demand and generally distributed periods of low demand. The inventory level starts at some level q, and decreases according to different compound Poisson processes during the periods of high demand and of low demand. Refilling of the inventory level to level q is required when level 0 is hit or when an expiration date is reached, whichever comes first. If such an event occurs during a high demand period, an order is instantaneously placed; otherwise, ordering is postponed until the beginning of the next high demand period. We determine various performance measures of interest, like the distribution of the inventory level at time t and of the inventory demand up to time t, the distribution of the time until refilling is required, the expected time between two refillings, the expected amount of discarded material and the expected total amount of material held in between two refillings, and the expected values of various kinds of shortages. For a given cost/revenue structure, we can thus determine the long-run average profit.

✉ David Perry
dperry@stat.haifa.ac.il

Onno Boxma
o.j.boxma@tue.nl

Wolfgang Stadje
wolfgang@mathematik.uni-osnabrueck.de

Shelley Zacks
shelly@math.binghamton.edu

1 Eurandom and Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

2 Department of Statistics, University of Haifa, 31909 Haifa, Israel

3 Department of Mathematics and Computer Science, University of Osnabrück, 49069 Osnabrück, Germany

4 Department of Mathematical Sciences, Binghamton University, Binghamton, NY 13902-6000, USA

Published online: 18 March 2016

Keywords EOQ model · Perishable inventory · Outdatings · Unsatisfied demands · Regenerative process · Compound Poisson process

1 Introduction

The simple Economic Order Quantity (EOQ) model, cf. Nahmias (2011), is one of the most fundamental models in Operations Research. Its objective function lucidly describes the trade-off between set-up costs and holding costs. The well-known *Wilson root formula* (Wilson 1934) for optimizing this objective function is remarkably robust, and still serves as an approximation for much more complex inventory models.

Until the sixties, the main focus of inventory research was on optimization problems like determining when to place an order and for how much. The landmark book of Prabhu (1965) has led to another main stream of research, with a focus on the stochastic analysis of inventory systems in which the decision variables are introduced as given parameters. In this stream, the stochastic models typically are stylized representations of very complex systems, and one aims to find expressions for the stochastic functionals that are the components of the objective function. Our study belongs to this latter stream of research.

We consider a stochastic EOQ model in which demand occurs in a two-state random environment. This environment alternates between periods of high demand and periods of low demand according to a continuous-time semi-Markov process. The demand is represented by a compound Poisson process $\{Y_H(t),\ t \geq 0\}$ during high demand periods and another compound Poisson process $\{Y_L(t),\ t \geq 0\}$ during low demand periods. We assume that the mean increment per time unit during high demand periods is higher than during low demand periods, but this assumption is not necessary for the analysis. The high demand periods and the low demand periods alternate according to an alternating renewal process as follows: the high demand periods are independent and exponentially distributed random variables with rate μ and the low demand periods are *i.i.d.* random variables with some distribution G.

We denote by $\mathbf{Q} = \{Q(t) : t \geq 0\}$, with $Q(0) = 0$, the cumulative inventory demand process. $\mathbf{Q}$ initially increases according to $\mathbf{Y_H} = \{Y_H(t),\ t \geq 0\}$, subsequently according to $\mathbf{Y_L} = \{Y_L(t),\ t \geq 0\}$, etc. We assume that the goods under consideration are *perishable*, with a fixed expiration date. At some time t_0, the outdating or expiration time of the perishable goods, the remaining inventory is discarded. Reordering is required when the first of two events occurs: either the inventory becomes outdated, or the inventory level reaches zero. If such an event occurs during a high demand period, the buffer is instantaneously refilled with fresh perishable goods up to level q (all goods of the same batch have common shelf life). If such an event occurs during a low demand period, then ordering is postponed until the end of the low demand period. The inventory level process $X(t) := q - Q(t)$ now stochastically repeats itself, again starting at level q with a period of high demand. Thus the inventory level process is a *regenerative process* whose cycles start at moments of replenishments. In Remark 3 below we comment on the assumption to postpone the order until the beginning of the next high demand period.

A controller may wish to maximize the long-run average profit, by controlling the order quantity q. Accordingly, he would wish to select the optimal q so as to properly balance revenues and costs. Revenues are earned by selling units. The costs are composed of setup costs, which are incurred each time an order is placed, holding costs, the costs for discarded units (outdating) and the costs for unsatisfied demand (shortage costs).

Remark 1 The two-state random environment model reflects situations in which the demand rate for a certain commodity undergoes periodically recurring changes. The demands for many goods change according to changes in the interest rate or due to fashion or other recurring seasonal/external effects. Models including a multi-state random environment can be suitable for such situations; the two-state case presented in this article could serve as a first approximation.

Remark 2 The rationale behind having exponentially distributed high demand periods is that low demand periods may represent some form of slowdown, whose starting times (in first approximation) may form a Poisson process. Due to a lack of specific information about these recessive forces, the assumption of Poisson arrivals (based on independence of the number of slowdowns in disjoint time intervals and temporal stationarity) seems natural to start with. This yields exponentially distributed high demand periods. Compound slowdowns within slowdown periods may give rise to extended low demand periods; it hence seems less natural to assume that low demand periods are exponentially distributed, and therefore we take them generally distributed. The treatment of non-exponential low-demand periods is a major step beyond the Markovian framework.

We determine various performance measures of interest, in particular: (i) the distribution of the inventory demand $Q(t)$ up to time t and of the inventory level $X(t) = q - Q(t)$; (ii) the distribution of the time until refilling is required; (iii) the expected time between two refillings; (iv) the expected amount of discarded material; (v) the expected total amount of material held in between two refillings (inventory); and (vi) the expected values of various kinds of shortages. That in principle allows one to analyze the following cost optimization problem: given the profit of selling one unit of stock, and given setup costs, holding costs, costs for discarding outdated units and costs for unsatisfied demand, choose q so as to maximize the long-run average profit. The study of this optimization problem does not fall in the scope of the present paper, but in Sect. 8 we indicate how it can be tackled numerically.

This is a companion paper of Boxma et al. (2015). The latter paper considers a stochastic fluid EOQ model with a similar underlying semi-Markov process, the key difference to the present model being that demand there is constant, with a high demand rate β_H during high demand periods and a demand rate $\beta_L < \beta_H$ during low demand periods.

Remark 3 If, in our compound Poisson model, we let the arrival rates tend to infinity, and take exponential demand sizes with rates that also go to infinity, such that the average increase rate equals β_H during high demand periods and β_L during low demand periods, then in the limit one arrives at the fluid EOQ model that was studied in Boxma et al. (2015). Actually, both the fluid EOQ model of Boxma et al. (2015) and the compound Poisson EOQ model of the present paper are special cases of an EOQ model that alternates between two different non-decreasing Lévy demand processes. It would be interesting to provide an analysis for that, more general and more complicated, model.

1.1 Literature review

We refer to Boxma et al. (2015) for an extensive literature review. Here we restrict ourselves to the following observations. The case of compound Poisson demand was introduced in Baron et al. (2010), but without a random environment. See also the surveys Giri and Chaudhuri (1998), Karaesmen et al. (2010), Nahmias (1982), Nahmias (2011), Raafat (1991) on Perishable Inventory Systems (PIS). Our paper is closest to the subject area of Karaesmen et al. (2010), which focuses on the stochastic analysis of PIS that operate under certain heuristic control policies.

According to Karaesmen et al. (2010) continuous review inventory models can be classified into three categories: those without fixed ordering cost or lead times, those without fixed ordering cost having positive lead times, and those with fixed ordering cost (typically with zero lead times). The first category was originated by Graves (1982), who assumed that items are continuously produced and perish after a deterministic time, and that demand follows a compound Poisson process with either a single-unit or an exponential demand at each arrival. The second category goes back to Pal (1989), who investigated the performance of an $(S-1, S)$ control policy. The third category, originated by Weiss (1980), is of relevance to our model; Deniz et al. (2010), Lian and Liu (2001), Lian et al. (2005), Liu and Lian (1999) and Perry et al. (2005) made significant contributions to models in this category. In particular, Lian et al. (2005) consider discrete demand for items and perishability times that are either fixed (and known) or follow a phase-type distribution.

A large variety of inventory models are presented in detail in the monograph (Zipkin 2000). The stochastic models are based on point processes to represent the demand arrivals in a random environment. The fluid systems in Zipkin (2000) are deterministic EOQ models with the classical extensions such as planned backorders, limited capacity, quantity discounts, and imperfect quality. In the deterministic setting, time-varying demands are considered also, but without multiple order quantities. We also refer the reader to the two recent papers Shi et al. (2014) and Shi et al. (2013). In Shi et al. (2014), a production-inventory system with a constant replenishment rate, compound Poisson demands and lost sales is considered. Two cost objective functions are minimized w.r.t. the replenishment rate. In Shi et al. (2013), inventory penalty pricing is considered for a production–inventory system with constant replenishment rate and a compound renewal demand stream, which is subject to underage and overage penalties.

1.2 Main contributions of the paper

We present a detailed performance analysis of a fundamental extension of the classical EOQ model, taking into account a random environment as well as perishability, and not restricting ourselves to a Markovian setting. We determine distributions and moments of key performance measures, which may form the basis for cost optimization. We believe that the method employed is of considerable value in inventory theory. It is based on a careful study of the total time in $(0, t)$ spent in high demand periods, and various properties of compound Poisson processes.

1.3 Organization of the paper

Section 2 contains a detailed model description. We study the distribution of the total time $W(t)$ in $(0, t)$ of high demand periods in Sect. 3. In Sects. 4–7 we analyze the key performance measures of the model. The main results for each of them are formulated as theorems. The results on $W(t)$ are used in Sect. 4 to determine the distribution of total demand $Q(t)$ and of the time until either complete stock depletion or stock discarding. In Sect. 5 we consider the expected amount of material discarded (because the deadline expires) and the expected total amount of material held until a refilling is required. Section 6 is devoted to determining the expected length of an inventory cycle; the mean shortages are derived in Sect. 7. Combining the various results from Sects. 4–7 enables us to determine the expected profit per time unit. In Sect. 8 we present numerical examples, evaluating the formulae for cost functionals and exploring the effect of several parameters on the cost function.

2 The model

Consider an inventory system filled with perishable items, which have to be discarded after t_0 time units. The demand follows intermittent periods of high demand (HD) followed by low demand (LD). Let $\{H_i, i = 1, 2, \dots\}$ and $\{L_i, i \geq 1\}$ denote two independent sequences of i.i.d. positive random variables, representing the lengths of the HD-periods and of the LD-periods, respectively. The intermittent sequence of high and low demand periods is then represented by the alternating renewal process associated to $\{H_1, L_1, H_2, L_2, \dots\}$. We denote by HD the union of all high demand periods. In the present paper we assume that $H_i \sim \text{Expo}(\mu)$ (exponential distribution with mean $1/\mu$) while the L_i have a general common absolutely continuous distribution G, with density g.

To describe the demand process we need *compound Poisson processes* (CPP(λ, F))

$$Y(t) = \sum_{n=0}^{N(t)} X_n, \quad X_0 \equiv 0, \tag{2.1}$$

where $\{N(t), t \geq 0\}$ is an ordinary Poisson process with intensity λ and $\{X_n, n \geq 1\}$ is a sequence of i.i.d. positive random variables with common distribution function F and independent of $\{N(t)\}$. We denote by $Y_H^{(i)}(t)$, $i \geq 1$, independent processes of the type CPP(λ_H, F_H) giving the total quantity demanded during the i-th HD-period (where t is the time elapsed since the beginning of that period). Similarly, let $Y_L^{(i)}(t)$, $i \geq 1$, be independent processes of type CPP(λ_L, F_L), also independent from the $Y_H^{(i)}(t)$, giving the total quantity demanded during the first time units of the i-th LD-period. Of course the pairs (λ_H, F_H) and (λ_L, F_L) of intensities and distribution functions can be different. Let $Q(t)$ denote the total demand up to time t. Formally, let $M(t)$ be the number of completed HD-periods in $(0, t)$. Then $Q(t) = Y_H^{(1)}(t)$ if $M(t) = 0$, while if $M(t) \geq 1$ and $t \in \text{HD}$,

$$Q(t) = \sum_{i=1}^{M(t)} \left[Y_H^{(i)}(H_i) + Y_L^{(i)}(L_i) \right] + Y_H^{(M(t)+1)} \left(t - \sum_{i=1}^{M(t)} [H_i + L_i] \right)$$

and if $M(t) \geq 1$, $t \in (0, \infty) \backslash \text{HD}$,

$$\begin{aligned} Q(t) = &\sum_{i=1}^{M(t)-1} \left[Y_H^{(i)}(H_i) + Y_L^{(i)}(L_i) \right] + Y_H^{(M(t))}(H_{M(t)}) \\ &+ Y_L^{(M(t))} \left(t - \sum_{i=1}^{M(t)-1} [H_i + L_i] - H_{M(t)} \right). \end{aligned}$$

In Fig. 1 we display a possible sample path of $Q(t)$.
In the present illustration the total demand at the *outdating* time t_0, $Q(t_0)$, is smaller than q. The quantity *discarded* is $q - Q(t_0)$. q is the amount of material replenished; i.e. the stock level at any replenishment time.

Define the stopping times

$$\tau = \inf\{t > 0 : Q(t) \geq q\}, \tag{2.2}$$

and

$$\tau^* = \min\{\tau, t_0\}. \tag{2.3}$$

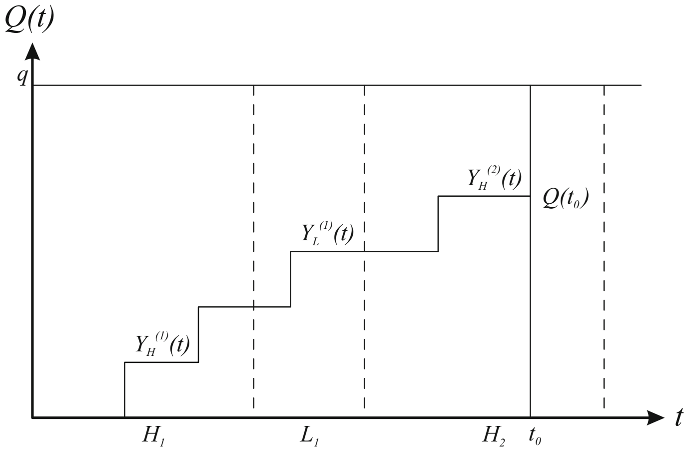

Fig. 1 A possible sample path of $Q(t)$

τ^* is the instant of complete stock depletion or stock discarding. Stock replenishment takes place at τ^*, if τ^* is in an HD-period. If τ^* falls in an LD-period, replenishment is deferred till the end of this LD-period. Accordingly, the inventory cycle has length

$$C = \tau^* + R.I\left(\tau^* \in LD\right), \tag{2.4}$$

where R is the remaining time, after τ^*, in LD, and where $I(\cdot)$ denotes an indicator function.

Remark 4 The above control policy guarantees that the cycle always starts with an HD-period. The following example shows that it is often natural to wait for the beginning of the next high demand period before ordering. HD-periods will typically be periods in which trade is relatively brisk. If the holding costs are relatively high compared to the costs of unsatisfied demand, entrepreneurs will not start new initiatives during LD-periods, since if they place orders during LD-periods the content level will stochastically increase and the holding cost component will cause the long-run average costs to go up. If they wait till the beginning of the next HD-period, the costs due to lost sales are low and the holding costs will also be low, since the content level will be depleted quickly. Hence in this case it may be worthwhile to pay the costs of unsatisfied demands by waiting with the content at level 0 rather than placing an order and for a long time paying considerable holding costs while the content level is near its maximum q.

Let $W(t)$ denote the total time in $(0, t)$ spent in HD-periods, i.e.,

$$W(t) = \int_0^t I(s \in \text{HD})ds. \tag{2.5}$$

Since $Y_L^{(i)}(t)$ and $Y_H^{(i)}(t)$, $i = 1, 2, \ldots$, are independent stochastic copies of Lévy processes and independent of the process $W(t)$, the strong Markov property yields the distributional equality

$$Q(t) =_d Y_H(W(t)) + Y_L(t - W(t)), \tag{2.6}$$

where $Y_H(t) =_d Y_H^{(1)}(t)$ and $Y_L(t) =_d Y_L^{(1)}(t)$. The inventory level $X(t)$ equals $q - Q(t)$, for $0 \leq t < \tau^*$; so its distribution is known once the distribution of $Q(t)$ is known. In the following sections we present the distribution of $W(t)$, leading to the distributions of

$Q(t)$ and $X(t)$; we subsequently study τ^*, R, the expected quantity of discarded material $ED(q) = q - EQ(\tau^*)$, and the expected total amount of material held. The study of these performance measures is highly relevant because a natural objective is to determine the level q such that the expected net *profit* per time unit is maximized, and the following profit objective function is a natural one:

$$O(q) := \frac{c_p q - K - c_d E\{D(q)\} - c_{sh} E\{\text{Sh}\} - c_h E\{T(\tau^*)\}}{E\{C\}}. \tag{2.7}$$

Here c_p are the earnings per item, K the set-up costs per cycle, and C the cycle time. c_d, c_{sh} and c_h are the costs of a discarded unit, the costs for shortage of one unit, and the costs of holding the inventory, while $E\{D(q)\}$ is the expected quantity of discarded material per cycle, $E\{\text{Sh}\}$ the expected total shortage per cycle, and $E\{T(\tau^*)\}$ the expected total amount of material held per cycle.

3 The distribution of $W(t)$

For $w > 0$, let $N^*(w) = \max\{n \geq 0 : \sum_{j=0}^{n} H_j \leq w\}$, where $H_0 \equiv 0$. Since $\{H_j\}$ are Expo(μ), $\{N^*(w), w \geq 0\}$ is an ordinary Poisson process, with intensity μ. Consider the CPP(μ, G) given by

$$Y^*(w) = \sum_{n=0}^{N^*(w)} L_n, \quad L_0 \equiv 0. \tag{3.1}$$

The c.d.f. of $Y^*(w)$ is

$$H^*(y; w) = \sum_{n=0}^{\infty} p(n; \mu w) G^{(n)}(y), \tag{3.2}$$

where $p(n; \eta) = e^{-\eta}\eta^n/n!$ denotes the Poisson p.d.f. with mean η. Notice that $H^*(0; w) = e^{-\mu w}$ (atom of (3.2)). We denote by $h^*(y; w)$ the p.d.f. of $Y^*(w)$, for $0 < y < \infty$, namely:

$$h^*(y; w) = \sum_{n=1}^{\infty} p(n; \mu w) g^{(n)}(y). \tag{3.3}$$

Upon reflection (see Fig. 2) we realize that $W(t)$ is the stopping time

$$W(t) = \inf\left\{w > 0 : Y^*(w) \geq t - w\right\}. \tag{3.4}$$

Since $Y^*(w)$ is non-decreasing with probability 1, (3.4) yields:

Theorem 3.1 *For* $0 < w < t$,

$$P\{W(t) > w\} = H^*(t - w; w); \tag{3.5}$$

and

$$P\{W(t) = t\} = e^{-\mu t}. \tag{3.6}$$

As shown in Fig. 2, if $W(t) = w$ and $Y^*(w) > t - w$, the alternating renewal process (ARP) is at an LD-period at t; and if $Y^*(w) = t - w$ the ARP is at an HD-period at t.

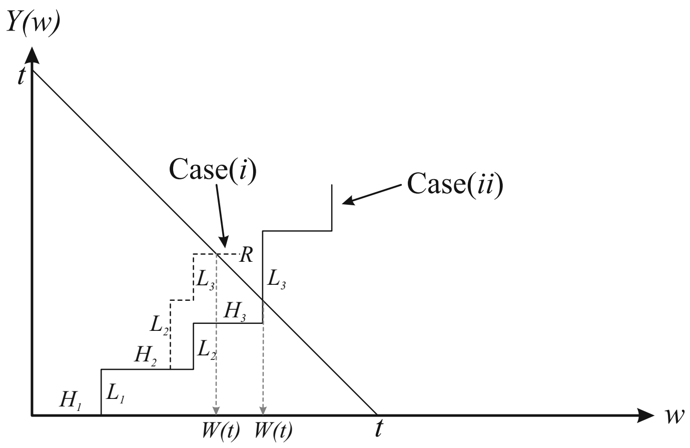

Fig. 2 The process $Y^*(w)$ and $W(t)$

We now derive the density of $W(t)$. The p.d.f. of $W(t)$, on $(0, t)$, is $\psi_{W(t)}(w; t) = -\dfrac{d}{dw}H^*(t-w; w)$, which is

$$\psi_{W(t)}(w; t) = \mu e^{-\mu w} + \mu \sum_{n=1}^{\infty}(p(n; \mu w) - p(n-1; \mu w))G^{(n)}(t-w) + \sum_{n=1}^{\infty} p(n; \mu w) g^{(n)}(t-w), \quad 0 < w < t. \tag{3.7}$$

Thus the density of $W(t)$ can be written as a sum of two components, i.e. $\psi_{W(t)}(w; t) = \psi_{W(t)}^{(\mathrm{LD})}(w; t) + \psi_{W(t)}^{(\mathrm{HD})}(w; t)$ where

$$\psi_{W(t)}^{(\mathrm{LD})}(w; t) = \mu e^{-\mu w} + \mu \sum_{n=1}^{\infty}(p(n; \mu w) - p(n-1; \mu w))G^{(n)}(t-w) \tag{3.8}$$

and

$$\psi_{W(t)}^{(\mathrm{HD})}(w; t) = h^*(t-w; w). \tag{3.9}$$

From the Bayes theorem we obtain

$$P\{t \in \mathrm{LD} \mid W(t) = w\} = \frac{\psi_{W(t)}^{(\mathrm{LD})}(w; t)}{\psi_{W(t)}(w; t)}, \quad 0 < w < t. \tag{3.10}$$

4 The distribution of $Q(t)$ and τ^*

Let $H^{(i)}(y; t)$ and $h^{(i)}(y, t)$ be the c.d.f. and p.d.f. of $Y_i(t)$, $i \in \{L, H\}$. These are

$$H^{(i)}(y; t) = \sum_{n=0}^{\infty} p(n; \lambda_i t) F_i^{(n)}(y), \quad y \geq 0, \tag{4.1}$$

and

$$h^{(i)}(y;t) = \sum_{n=1}^{\infty} p(n;\lambda_i t) f_i^{(n)}(y), \quad y > 0. \tag{4.2}$$

The *conditional* c.d.f. of $Y_H(W(t))$, given $\{W(t) = w\}$, $0 \le w \le t$, is $H^{(H)}(y;w)$, and that of $Y_L(t - W(t))$ is $H^{(L)}(y;t-w)$.

4.1 Distribution of $Q(t)$

The conditional distribution of $Q(t)$, given $W(t) = w$ is, according to (2.6),

$$\begin{aligned} H_Q(y;w,t) = {} & I(y=0, w<t)e^{-\lambda_H w - \lambda_L(t-w)} \\ & + I(0<y, 0<w<t)[e^{-\lambda_H w} H^{(L)}(y;t-w) \\ & + \int_0^y h^{(H)}(x;w) H^{(L)}(y-x;t-w)dx] \\ & + I(w=t)e^{-\mu t} H^{(H)}(y;t). \end{aligned} \tag{4.3}$$

Notice that the first term of (4.3) is $P\{Q(t) = 0 \mid W(t) = w\}$.

The corresponding conditional density of $Q(t)$, for $0 < y$, is for $W(t) = w$,

$$\begin{aligned} h_Q(y;w,t) = {} & e^{-\lambda_H w} h^{(L)}(y;t-w) + e^{-\lambda_L(t-w)} h^{(H)}(y;w) \\ & + \int_0^y h^{(H)}(x;w) h^{(L)}(y-x;t-w)dx. \end{aligned} \tag{4.4}$$

We obtain

Theorem 4.1 *The c.d.f. of $Q(t)$ is*

$$H_Q^*(y;t) = e^{-\mu t} H^{(H)}(y;t) + \int_0^t \psi_{W(t)}(w;t) H_Q(y;w,t)dw. \tag{4.5}$$

The corresponding density is, for $y > 0$,

$$h_Q^*(y;t) = e^{-\mu t} h^{(H)}(y;t) + \int_0^t \psi_{W(t)}(w;t) h_Q(y;w,t)dw. \tag{4.6}$$

The m-th moment of the truncated cumulative demand $Q_q(t) = \min(q, Q(t))$ is

$$E\{Q_q^m(t)\} = q^m - m\int_0^q x^{m-1} H_Q^*(x;t)dx, \quad m \ge 1. \tag{4.7}$$

As observed before, the distribution of the inventory level $X(t)$ at time t immediately follows from that of $Q(t)$, since $X(t) = q - Q(t)$ for $0 \le t < \tau^*$.

4.2 The distribution of τ^*

In this subsection we consider the distribution of the stopping time τ^*, i.e., the minimum of the time until the inventory level reaches zero and the time until the inventory becomes outdated.

Theorem 4.2

$$P\{\tau^* > t\} = P\{Q(t) < q\} = H_Q^*(q;t), \quad 0 < t < t_0 \tag{4.8}$$

and

$$P\{\tau^* = t_0\} = H_Q^*(q; t_0). \tag{4.9}$$

Here

$$\begin{aligned} H_Q^*(q;t) &= e^{-\mu t} H^{(H)}(q;t) \\ &\quad + \int_0^t \psi_{W(t)}(w;t) \Big[e^{-\lambda_H w} H^{(L)}(q; t-w) \\ &\quad + \int_0^q h^{(H)}(x;w) H^{(L)}(q-x; t-w) dx \Big] dw. \end{aligned} \tag{4.10}$$

Proof Since $Q(t)$ is a non-decreasing process, and $\tau^* = \min(\tau, t_0)$ where $\tau = \inf\{t > 0 : Q(t) \geq q\}$, (4.8) and (4.9) follow. (4.10) follows from (4.3) and (4.5). □

We next consider the density of τ^*, to be denoted by $p_{\tau^*}(t;q)$, for $0 < t < t_0$. Since $p_{\tau^*}(t;q) = -\frac{d}{dt} P\{\tau^* > t\}$, for $0 < t < t_0$, inserting (4.3) yields, for $0 < t < t_0$,

$$\begin{aligned} p_{\tau^*}(t;q) &= \mu e^{-\mu t} H^{(H)}(q;t) + e^{-\mu t} \left(-\frac{\partial}{\partial t} H^{(H)}(q;t) \right) \\ &\quad + \int_0^t \left(-\frac{\partial}{\partial t} \psi_{W(t)}(w;t) \right) H_Q(q; w, t) dw \\ &\quad + \int_0^t \psi_W(w;t) \left(-\frac{\partial}{\partial t} H_Q(q; w, t) \right) dw. \end{aligned} \tag{4.11}$$

In Lemmas 4.1 and 4.2 we give the two missing ingredients from (4.11), viz., $\frac{\partial}{\partial t} \psi_{W(t)}(w;t)$ and $\frac{\partial}{\partial t} H_Q(q; w, t)$. Since $\frac{d}{dt} G^{(n)}(t-w) = g^{(n)}(t-w)$ and

$$\frac{d}{dt} g^{(n)}(t-w) = \int_0^{t-w} g^{(n-1)}(x) \frac{d}{dt} g(t-w-x) dx \tag{4.12}$$

we obtain

Lemma 4.1

$$\begin{aligned} \frac{\partial}{\partial t} \psi_{W(t)}(w;t) &= \mu \sum_{n=1}^{\infty} (p(n;\mu w) - p(n-1;\mu w)) g^{(n)}(t-w) \\ &\quad + \sum_{n=1}^{\infty} p(n;\mu w) \left(\frac{d}{dt} g^{(n)}(t-w) \right). \end{aligned} \tag{4.13}$$

As an example, if $G(t) = 1 - e^{-\zeta t}, t \geq 0$ (exponential case), then $g^{(n)}(t) = \zeta p(n-1; \zeta t)$ and

$$\begin{aligned} -\frac{\partial}{\partial t} \psi_{W(t)}(w;t) &= \mu\zeta \sum_{n=1}^{\infty} (p(n-1;\mu w) - p(n;\mu w)) p(n-1; \zeta(t-w)) \\ &\quad - \zeta^2 \sum_{n=1}^{\infty} p(n;\mu w)(p(n-2;\zeta(t-w)) - p(n-1;\zeta(t-w))), \end{aligned} \tag{4.14}$$

where $p(-1; \cdot) \equiv 0$.

We derive now a formula for $\frac{\partial}{\partial t} H_Q(y; w, t)$. We start with the following observation. For $y > 0$, if $H(y; t)$ is the c.d.f. of a CPP(λ, F), then

$$-\frac{\partial}{\partial t} H(y;t) = \lambda e^{-\lambda t} \bar{F}(y) + \lambda \int_0^y h(x;t) \bar{F}(y-x) dx, \tag{4.15}$$

where $h(x; t)$ is the corresponding p.d.f. and $\bar{F}(\cdot) = 1 - F(\cdot)$.

Indeed, for the CPP(λ, F),

$$H(y;t) = \sum_{n=0}^{\infty} p(n; \lambda t) F^{(n)}(y) \quad \text{and} \quad h(y;t) = \sum_{n=1}^{\infty} p(n; \lambda t) f^{(n)}(y).$$

Thus,

$$\begin{aligned} -\frac{\partial}{\partial t} H(y;t) &= \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial t} p(n; \lambda t) \right) F^{(n)}(y) \\ &= \lambda e^{-\lambda t} + \lambda \sum_{n=1}^{\infty} (p(n; \lambda t) - p(n-1; \lambda t)) F^{(n)}(y) \\ &= \lambda \sum_{n=0}^{\infty} p(n; \lambda t) F^{(n)}(y) - \lambda \sum_{n=0}^{\infty} p(n; \lambda t) F^{(n+1)}(y) \\ &= \lambda e^{-\lambda t} \bar{F}(y) + \lambda \sum_{n=1}^{\infty} p(n; \lambda t) [F^{(n)}(y) - F^{(n+1)}(y)]. \end{aligned} \tag{4.16}$$

Moreover

$$\begin{aligned} F^{(n)}(y) - F^{(n+1)}(y) &= \int_0^y f^{(n)}(x)(1 - F(y-x)) dx \\ &= \int_0^y f^{(n)}(x) \bar{F}(y-x) dx. \end{aligned} \tag{4.17}$$

Substituting (4.17) in (4.16) we get (4.15).

Remark 4 A probabilistic interpretation of (4.15) is readily obtained when one realizes the following. $H(y; t)$ not only is the probability that the CPP(λ, F) at t is below y, but also the probability that a crossing of level y has only taken place after t. Minus the derivative of the latter probability gives the density of the crossing time of level y, which is easily seen to be the righthand side of (4.15).

We use this lemma in the following derivation. First, according to (4.15),

$$-\frac{\partial}{\partial t} H^{(H)}(q;t) = \lambda_H e^{-\lambda_H t} \bar{F}_H(q) + \lambda_H \int_0^q h^{(H)}(x;t) \bar{F}_H(q-x) dx. \tag{4.18}$$

Moreover, by (4.10),

$$-\frac{\partial}{\partial t} H^{(L)}(q;t-w) = \lambda_L e^{-\lambda_L (t-w)} \bar{F}_L(q) + \lambda_L \int_0^q h^{(L)}(u;t-w) \bar{F}_L(q-u) du. \tag{4.19}$$

Collecting terms we get, for $w < t$,

Lemma 4.2 *For* $0 < w < t < t_0$

$$-\frac{\partial}{\partial t} H_Q(q; w, t) = \lambda_L e^{-\lambda_H w - \lambda_L (t-w)} \bar{F}_L(q) + \lambda_L e^{-\lambda_H w} \int_0^q h^{(L)}(x; t-w) \bar{F}_L(q-x) dx + \lambda_L e^{-\lambda_L (t-w)} \int_0^q h^{(H)}(x; w) \bar{F}_L(q-x) dx + \lambda_L \int_0^q h^{(H)}(x; w) \int_0^{q-x} h^{(L)}(u; t-w) \bar{F}_L(q-x-u) du\, dx. \tag{4.20}$$

An explicit formula for $p_{\tau^*}(t; q)$ is obtained by substituting (4.13), (4.18) and (4.20) into (4.11).

Theorem 4.3 *The* m-*th moment of* τ^* *is*

$$E\left\{(\tau^*)^m\right\} = m \int_0^{t_0} t^{m-1} H_Q^*(q; t) dt, \quad m = 1, 2, \ldots \tag{4.21}$$

Proof Immediate from $E\{(\tau^*)^m\} = m \int_0^{t_0} t^{m-1} P\{\tau^* > t\} dt, \quad m = 1, 2, \ldots$ and $P\{\tau^* > t\} = H_Q^*(q; t)$. □

From the above theorem we get, as a special case,

$$E\left\{\tau^*\right\} = \int_0^{t_0} H_Q^*(q; t) dt. \tag{4.22}$$

Notice that $E\{\tau^*\}$ is an increasing function of q with a limit t_0.

5 The expected amount of discarded material and the expected total amount of material held

In this section we determine the expected amount of discarded material, $E\{D(q)\}$ (Theorem 5.1) and the expected total amount of material held during $[0, \tau^*]$, $E\{T(\tau^*)\}$ (Theorem 5.2).

Material is discarded only if $Q(t_0) < q$, which is the case when $\tau^* = t_0$.

Theorem 5.1 *The expected amount of discarded material is*

$$E\{D(q)\} = \int_0^q H_Q^*(x; t_0) dx. \tag{5.1}$$

Proof

$$E\{D(q)\} = \int_0^q P\{waste\ at\ t_0 > q - x\} dx = \int_0^q H_Q^*(q-x; t_0) dx = \int_0^q H_Q^*(x; t_0) dx. \tag{5.2}$$

□

We next turn to the expected total amount of material held during an inventory cycle.

Theorem 5.2 *The expected total amount of material held per cycle is*

$$\begin{aligned}E\{T(\tau^*)\} &= qt_0P\{Q(t_0)=0\} \\ &\quad + \int_0^{t_0} p_{\tau^*}(t;q)\int_0^t E\{q-Q(s)\mid \tau^*=t\}ds\,dt \\ &\quad + \int_0^q h_Q^*(y;t_0)\int_0^{t_0} E\{q-Q(s)\mid Q(t_0)=y\}dy\,ds, \qquad (5.3)\end{aligned}$$

where $E\{Q(s)\mid \tau^*=t\}$ *is given by*

$$E\{Q(s)\mid \tau^*=t\} = \frac{\int_0^q yh_Q^*(y;s)h_Q^*(q-y;t-s)dy}{\int_0^q h_Q^*(y;s)h_Q^*(q-y;t-s)dy}, \qquad (5.4)$$

and $E\{Q(s)\mid Q(t_0)=y\}$ *by*

$$E\{Q(s)\mid Q(t_0)=y\} = \frac{\int_0^y xh_Q^*(x;s)h_Q^*(y-x;t_0-s)dx}{\int_0^y h_Q^*(x;s)h_Q^*(y-x;t_0-s)dx}. \qquad (5.5)$$

Proof The second and third lines of (5.3) correspond to the cases $\tau^* < t_0$ and $\tau^* = t_0$, respectively. In both cases we integrate the conditional expectation of $X(s) = q - Q(s)$ from 0 till τ^*. In order to compute (5.3) we have to develop formulae for the two yet unknown terms $E\{Q(s)\mid \tau^*=t\}$ when $0<t<t_0$, and for $E\{Q(s)\mid Q(t_0)=y\}$ for $0<s<t_0$. Notice that

$$\int_0^{t_0} p_{\tau^*}(t;q)\int_0^t q\,ds\,dt + \int_0^q h_Q^*(y;t_0)\int_0^{t_0} q\,ds\,dy = qE\{\tau^*\}. \qquad (5.6)$$

The conditional expectation of $Q(s)$, given $\tau^*=t$, for $s<t<t_0$ is now easily seen to be given by (5.4), and the conditional expectation of $Q(s)$, given $Q(t_0)=y$ is given by (5.5). These functions are substituted in (5.3) to obtain the expected total amount held, $E\{T(\tau^*)\}$. □

6 The expected length of an inventory cycle

The inventory cycles are the periods between replenishing epochs. If $\tau^* \in$ HD, replenishing is done at τ^*. On the other hand, if $\tau^* \in$ LD, replenishing takes place at the beginning of the next HD-period. That is, if C denotes the length of an inventory cycle then C is given by (2.4), where R is the remaining length of the LD-period, after stopping. The expected length of an inventory cycle is thus

$$E\{C\} = E\{\tau^*\} + E\{R.I(\tau^* \in \text{LD})\}. \qquad (6.1)$$

We have to derive a formula for $E\{R.I(\tau^* \in \text{LD})\}$.

Generally, for a renewal process $\{V_i\}_{i=1}^{\infty}$, if G is the c.d.f. of V, then the remaining length of the last renewal cycle, at time t, has the c.d.f. (see p. 109 of Kao (1997))

$$F_R(x;t) = G(t+x) - \int_0^t \bar{G}(t+x-y)m(y)dy, \qquad (6.2)$$

where $\bar{G}(\cdot) = 1 - G(\cdot)$ and $m(y)$ is the renewal density

$$m(y) = \sum_{n=1}^{\infty} g^{(n)}(y), \quad y > 0. \tag{6.3}$$

Of course, if G is exponential with rate ζ, one has $m(y) \equiv \zeta$ and it is immediately verified that $F_R(x;t)$ is also exponential with rate ζ.

In the ARP, the conditional distribution of R, given $\{t \in \text{LD}\}$ and $\{W(t) = w\}$ is

$$P\{R \le x \mid t \in \text{LD}, W(t) = w\} = F_R(x; t - w). \tag{6.4}$$

Accordingly,

$$\begin{aligned} E\{R \mid t \in \text{LD}, W(t) = w\} &= \int_0^\infty \bar{F}_R(x; t - w)dx \\ &= \int_0^\infty \left[\bar{G}(t + x - w) + \int_0^{t-w} \bar{G}(t + x - w - y)m(y)dy\right] dx. \end{aligned} \tag{6.5}$$

Notice that if G is exponential with mean $1/\zeta$ then from (6.5) we obtain $E\{R|t \in LD, W(t) = w\} = E\{R\} = \frac{1}{\zeta}$, as expected. In this case $E\{R.I(\tau^* \in LD)\} = \frac{1}{\zeta}P\{\tau^* \in LD\}$. We now develop formulas for $P\{\tau^* \in LD\}$ and for $E\{R.I(\tau^* \in LD)\}$ in the general case. The conditional density of τ^*, given that $\tau^* \in HD$ and $W(\tau^*) = w$, is

$$\begin{aligned} p_{\tau^*}(t|\tau^* \in HD, W(t) = w) &= \lambda_H \mathrm{e}^{-(\lambda_H+\mu)t} \bar{F}_H(q) \\ &\quad + \lambda_H \mathrm{e}^{-\lambda_H w - \lambda_L(t-w)} \bar{F}_H(q) \\ &\quad + \lambda_H \int_0^q h_Q(y; w, t)\bar{F}_H(q - y)\mathrm{d}y, \end{aligned} \tag{6.6}$$

where $h_Q(y; w, t) = \frac{d}{dy} H_Q(y; w, t)$. Similarly,

$$\begin{aligned} p_{\tau^*}(t|\tau^* \in LD, W(t) = w) &= \lambda_L \mathrm{e}^{-\lambda_H w - \lambda_L(t-w)} \bar{F}_L(q) \\ &\quad + \lambda_L \int_0^q h_Q(y; w, t)\bar{F}_L(q - y)\mathrm{d}y. \end{aligned} \tag{6.7}$$

It follows that

Theorem 6.1 *In the general case,*

$$\begin{aligned} &E\left\{R.I\left(\tau^* \in LD\right)\right\} \\ &\quad = \int_0^{t_0} \int_0^t E\{R|\tau^* = t \in LD, W(t) = w\} p_{\tau^*}(t|\tau^* \in LD, W(t) = w\}\psi^{(LD)}(w;t)\mathrm{d}w\mathrm{d}t \\ &\quad + \int_0^{t_0} E\{R|t_0 \in LD, W(t_0) = w\} H_Q(q; w, t_0)\psi^{(LD)}(w; t_0)\mathrm{d}w. \end{aligned} \tag{6.8}$$

Corollary 1 *It follows from Theorem 6.1 that, if the distribution of the LD periods is exponential,*

$$\begin{aligned} P\{\tau^* \in LD\} &= \int_0^{t_0} \int_0^t p_{\tau^*}(t|\tau^* \in LD, W(t) = w\}\psi^{(LD)}(w;t)\mathrm{d}w\mathrm{d}t \\ &\quad + \int_0^{t_0} H_Q(q; w, t_0)\psi^{(LD)}(w; t_0)\mathrm{d}w. \end{aligned} \tag{6.9}$$

7 Mean shortages

There are three kinds of shortages in the present problem:

(i) The first kind of shortage, Sh_1, is the one when $\tau^* < t_0$ and $\tau^* \in \mathrm{HD}$. Observe that the level q is now exceeded via a jump of the CPP during an HD-period. In this case

$$\mathrm{Sh}_1 = \left(Q\left(\tau^*\right) - q\right) I\left(\tau^* < t_0, \tau^* \in \mathrm{HD}\right). \tag{7.1}$$

(ii) The second kind of shortage occurs when $\tau^* < t_0$ and $\tau^* \in \mathrm{LD}$. In this case

$$\mathrm{Sh}_2 = \left(Q\left(\tau^*\right) - q + Y_L\left(R\right)\right) I\left(\tau^* < t_0, \tau^* \in \mathrm{LD}\right). \tag{7.2}$$

(iii) The third kind of shortage occurs when $\tau^* = t_0$ and $\tau^* \in$ LD. In this case

$$\mathrm{Sh}_3 = Y_L(R) I(\tau^* = t_0 \in \mathrm{LD}). \tag{7.3}$$

We first derive formulas for $E\{\mathrm{Sh}_1\}$ and $E\{\mathrm{Sh}_2\}$. Recall that $Q(\tau^*) - q$ when $W(\tau^*) = w$, is the overshoot of $Y_H(w)$ at $\{\tau^* \in \mathrm{HD}\}$. The joint density of τ^* and $S = Q(\tau^*) - q$, given $W(t) = w$ and $\{\tau^* \in \mathrm{HD}\}$, is given in (7.4).

Let $h_Q(y; w, t) = \dfrac{d}{dy} H_Q(y; w, t)$. Then, with f_H (f_L) denoting the density of F_H (F_L),

$$\begin{aligned} p_{\tau^*,S}(t, s \mid \tau^* \in \mathrm{HD}, W(t) = w) &= \lambda_H e^{-\lambda_H w - \lambda_L (t-w)} f_H(q+s) \\ &\quad + \lambda_H \int_0^q h_Q(y; w, t) f_H(q+s-y) dy. \end{aligned} \tag{7.4}$$

Now use that

$$\int_0^\infty s f_H(q+s) ds = \int_q^\infty \bar{F}_H(u) du, \tag{7.5}$$

which follows from

$$\int_0^\infty s f_H(q+s) ds = \int_q^\infty (u-q) f_H(u) du = \int_q^\infty f_H(u) \int_q^u dy du = \int_q^\infty \bar{F}_H(u) du. \tag{7.6}$$

Thus,

$$\begin{aligned} E\{S \mid \tau^* \in \mathrm{HD}, W(t) = w\} &= \lambda_H \mathrm{e}^{-(\lambda_H+\mu)t} + \lambda_H e^{-\lambda_H w - \lambda_L (t-w)} \int_q^\infty \bar{F}_H(u) du \\ &\quad + \lambda_H \int_0^q h_Q(y; w, t) \int_{q-y}^\infty \bar{F}_H(u) du dy. \end{aligned} \tag{7.7}$$

Finally, integrating (7.7) with respect to $W(t)$ and t we get

Theorem 7.1

$$\begin{aligned} E\{Sh_1\} &= E\{S I(\tau^* \in HD)\} \\ &= \left[\int_q^\infty \bar{F}_H(u) du\right] \left[\frac{\lambda_H}{\lambda_H + \mu} (1 - e^{-(\lambda_H+\mu)t_0}) \right. \\ &\quad \left. + \lambda_H \int_0^{t_0} \int_0^t e^{-\lambda_H w - \lambda_L (t-w)} \psi_{W(t)}^{(HD)}(w; t) dw dt \right] \\ &\quad + \lambda_H \int_0^q \left(\int_0^{t_0} \int_0^t h_Q(y; w, t) \psi_{W(t)}^{(HD)}(w; t) dw dt \right) \\ &\quad \cdot \left(\int_{q-y}^\infty \bar{F}_H(u) du \right) dy. \end{aligned} \tag{7.8}$$

Similarly we get

Theorem 7.2

$$
\begin{aligned}
E\{Sh_2\} = \lambda_L & \left(\int_q^\infty \bar{F}_L(u)du\right) \\
& \cdot \int_0^{t_0}\int_0^t e^{-\lambda_H w-\lambda_L(t-w)}\psi_{W(t)}^{(LD)}(w,t)dwdt \\
& +\lambda_L \int_0^q \left(\int_0^{t_0}\int_0^t h_Q(y;w,t)\psi_{W(t)}^{(LD)}(w,t)dwdt\right) \\
& \cdot \left(\int_{q-y}^\infty \bar{F}_L(u)du\right)dy \\
& +\lambda_L\xi_L \int_0^{t_0}\int_0^t E\{R \mid t\in LD, W(t)=w\} \\
& \times p_{\tau^*}(t|t\in LD, W(t)=w)\psi_{W(t)}^{(LD)}(w;t)dwdt. \qquad (7.9)
\end{aligned}
$$

Finally we turn to the third kind of shortage, which occurs when $\tau^* < t_0$ and $\tau^* \in LD$.

Theorem 7.3 *The expected value of* Sh$_3$ *is*

$$E\{Sh_3\} = \lambda_L\xi_L E\{R.I(\tau^* = t_0 \in LD)\}, \qquad (7.10)$$

where $E\{R.I(\tau^* = t_0 \in LD)\}$ *is the second term on the right hand side of (6.8), and* $\xi_L = \int_0^\infty x dF_L(x)$.

Proof First, since the process $Y_L(t)$ is conditionally independent of $W(t)$,

$$E\{Y_L(R) \mid R\} = \lambda_L\xi_L R. \qquad (7.11)$$

Accordingly

$$
\begin{aligned}
E\{Y_L(R)I(\tau^* = t_0 \in \text{LD})\} &= \lambda_L\xi_L E\{R.I(\tau^* = t_0 \in \text{LD})\} \\
&= \lambda_L\xi_L \int_0^{t_0} E\{R \mid t_0 \in \text{LD}, W(t_0)=w\} \\
&\quad \times H_Q(q;w,t_0)\psi_{W(t)}^{(\text{LD})}(w;t_0)dw. \qquad (7.12)
\end{aligned}
$$

□

8 Cost functionals for exponential distributions and numerical examples

In the present section we specialize the formulae for the various cost functionals assuming that

$$
\begin{aligned}
G(t) &= 1 - e^{-\zeta t}, \quad t \ge 0, \\
F_i(t) &= 1 - e^{-\kappa_i t}, \quad t \ge 0,
\end{aligned}
$$

where $\kappa_i = \dfrac{1}{\xi_i}$, $i \in \{L, H\}$. The main purposes of the section are to show how the formulae for the various key performance indicators simplify in this case, and how one may evaluate them numerically. At the end of the section we provide some tables with numerical values of cost functionals. In principle, one may thus perform optimization; e.g., one may determine the q value that maximizes profit. That is outside the scope of the present paper; we refer the interested reader to Boxma et al. (2015), where considerable attention has been given to the profit optimization problem in the fluid demand case.

The c.d.f. of $Y_i(t)$ is

$$H^{(i)}(y;t) = \sum_{j=0}^{\infty} p(j;\kappa_i y) P(j;\lambda_i t), \quad i = L, H \tag{8.1}$$

and

$$H^*(y;t) = \sum_{j=0}^{\infty} p(j;\zeta y) P(j;\mu t). \tag{8.2}$$

Lemma 8.1 *The p.d.f. of $H^*(y,t)$ is for $y > 0$*

$$h^*(y,t) = \zeta \sum_{j=0}^{\infty} p(j;\zeta y) p(j+1;\mu t). \tag{8.3}$$

Proof

$$\begin{aligned} h^*(y;t) &= \frac{d}{dy} H^*(y;t) \\ &= \zeta \left[-\sum_{j=0}^{\infty} p(j;\zeta y) P(j;\mu t) + \sum_{j=1}^{\infty} p(j-1;\zeta y) P(j;\mu t) \right] \\ &= \zeta \left[\sum_{j=0}^{\infty} p(j;\zeta y)(P(j+1;\mu t) - P(j;\mu t)) \right]. \end{aligned}$$

This implies (8.3). □

Notice that the p.d.f. of $H^{(i)}(y;t)$ is similarly, for $y > 0$,

$$h^{(i)}(y;t) = \kappa_i \sum_{j=0}^{\infty} p(j;\kappa_i y) p(j+1;\lambda_i t), \quad i = L, H.$$

The densities of $W(t)$ at HD- and LD-periods are given in the next lemma.

Lemma 8.2

$$\begin{aligned} \psi^{(HD)}_{W(t)}(w;t) &= h^*(t-w;w) \\ &= \zeta \sum_{j=0}^{\infty} p(j;\zeta(t-w)) p(j+1;\mu w). \end{aligned} \tag{8.4}$$

$$\psi^{(LD)}_{W(t)}(w;t) = \mu \sum_{n=0}^{\infty} p(n;\mu w) p(n;\zeta(t-w)). \tag{8.5}$$

Proof

$$\psi^{(\mathrm{LD})}_{W(t)}(w;t) = \psi_{W(t)}(w;t) - \psi^{(\mathrm{HD})}_{W(t)}(w;t).$$

Table 1 Values of $P\{Q(10) < y\} = H_Q^*(y; 10)$

y	$H_Q^*(y, 10)$	y	$H_Q^*(y; 10)$
4	0.00267	24	0.56143
8	0.02677	28	0.70666
12	0.09970	32	0.81713
16	0.22877	36	0.89294
20	0.39346	40	0.94071

Table 2 $P\{\tau^* > t\}$

t	5	10	12	15	20
$P(\tau^* > t)$	0.9837	0.7665	0.6036	0.3559	0.0968

Moreover,

$$\begin{aligned}\psi_{W(t)}(w; t) &= -\frac{d}{dw} H^*(t-w; w)\\ &= -\frac{d}{dw}\sum_{j=0}^{\infty} p(j; \zeta(t-w))P(j; \mu w)\\ &= \zeta \sum_{j=0}^{\infty} p(j; \zeta(t-w))p(j+1; \mu w)\\ &\quad + \mu \sum_{j=0}^{\infty} p(j; \zeta(t-w))p(j; \mu w).\end{aligned}$$

□

An important function is the c.d.f of $Q(t)$, $H_Q^*(y; t)$, given by (4.5). For numerical computations it is convenient to use numerical integration to evaluate $H_Q^*(y; t)$. For example

$$\begin{aligned}&\int_0^t e^{-\lambda_H w} H_L(y; t-w)\psi_{W(t)}(w; t)dw\\ &\quad = t\int_0^1 e^{-\lambda_H tz} H_L(y; t(1-z))\psi_{W(t)}(tz; t)dz\\ &\quad \cong t\sum_{i=1}^{8} e^{-\lambda_H t z_i} H_L(y; t(1-z_i))\psi_{W(t)}(tz_i; t)\cdot w_i,\end{aligned} \tag{8.6}$$

where z_i and w_i are the abscissas and weight factors of the Gaussian integration (see p. 921 of Abramowitz and Stegun (1968)).

In the following table we present the values of $H_Q^*(y; t)$ for various values of y, at $t = 10$, $\lambda_H = 1.5$, $\lambda_L = 1.0$, $\kappa_H = 0.5$, $\kappa_L = 1$, $\mu = 1$, $\zeta = 2$ (Table 1).

For a system with $q = 30$ and $t_0 = 20$, the survival function of τ^*, i.e., $P\{\tau^* > t\} = H_Q^*(30; t)$ is displayed in Table 2 for the above values of λ_H, λ_L etc.

The expected value of τ^*, $E(\tau^*) = \int_0^{20} H_Q^*(30; t)dt = 13.31$. Also $P\{\tau^* = t_0\} = 0.0968$. The median of τ^* is $\tau_{0.5}^* = 13.26$. The expected amount of discarded material is $E\{D(30)\} = \int_0^{30} H_Q^*(x; 20)dx = 0.4554$.

Table 3 $E\{Q(s) \mid \tau^* = t\}$

s	2	4	6	8	10	12	14	15
$E\{Q(s) \mid \tau^* = t\}$	41.42	86.87	136.44	188.10	238.56	286.14	328.88	350.00

Since $G(t) = 1 - e^{-\zeta t}$, $t \geq 0$, $R \sim \text{Expo}(\zeta)$, independently of τ^*. Thus, for $E\{\text{Sh}_3\}$,

$$E\left\{RI\left(\tau^* = t_0 \in \text{LD}\right)\right\} = \frac{1}{\zeta} P\left\{\tau^* = t_0 \in \text{LD}\right\}, \tag{8.7}$$

and

$$P\{\tau^* = t_0 \in \text{LD}\} = \int_0^{t_0} H_Q(q; w, t_0)\psi^{(\text{LD})}_{W(t_0)}(w; t_0)dw. \tag{8.8}$$

For the above system, with $t_0 = 20$, $q = 30$, $\lambda_H = 1.5$, $\lambda_L = 1$, $\kappa_H = 0.5$, $\kappa_L = 1$, $\mu = 1$, $\zeta = 2$ we get $P\{\tau^* = 20 \in \text{LD}\} = 0.03408$, and $E\{\text{Sh}_3\} = 0.01704$.

For Sh_1 we have

$$\begin{aligned} E\left\{\text{Sh}_1 I(\tau^* = t \in \text{HD})\right\} = {} & \frac{\lambda_H}{\kappa_H} e^{-\kappa_H q}\left(\frac{1}{\lambda_H + \mu}(1 - e^{-(\lambda_H+\mu)t_0})\right. \\ & \left. + \int_0^{t_0}\int_0^t e^{-\lambda_H w - \lambda_L(t-w)}\psi^{(\text{HD})}_{W(t)}(w; t)dwdt\right) \\ & + \frac{\lambda_H}{\kappa_H}\int_0^q e^{-\kappa_H(q-y)}\left(\int_0^{t_0}\int_0^t h_Q(y; w, t)\psi^{(\text{HD})}_{W(t)}(w; t)dwdt\right)dy, \end{aligned} \tag{8.9}$$

and

$$E\{\text{Sh}_1\} = \int_0^{t_0} E\left\{\text{Sh}_1 \mid \tau^* = t \in \text{HD}\right\} P\left\{\tau^* = t \in \text{HD}\right\}dt. \tag{8.10}$$

For example, for a system with $t_0 = 20$, $q = 30$, $\lambda_H = 1.5$, $\lambda_L = 1$, $\kappa_H = 0.5$, $\kappa_L = 1$, $\mu = 1$, $\zeta = 2$ we get $E\{\text{Sh}_1\} = 1.4432$. In a similar manner, using eq. (7.9), we get $E\{\text{Sh}_2\} = 0.36152$. Finally, according to (5.3), the expected total material held is $E\{T(\tau^*)\} = 245.037$.

In Table 3 we present a few values of $E\{Q(s) \mid \tau^* = t\}$ for the case of $\lambda_H = 5$, $\lambda_L = 2$, $\kappa_H = 1/6$, $\kappa_L = 1/5$, $\mu = 0.1$, $\zeta = 0.2$, $t = 15$ and $q = 350$, computed according to (5.4).

In Table 4 we present the values of $E\{\tau^*\}$, $E\{R.I(\tau^* \in \text{LD})\}$, E {Sh} (which is defined as the sum of the three shortage terms $E\{\text{Sh}_1\}$, $E\{\text{Sh}_2\}$, and $E\{\text{Sh}_3\}$), $E\{D(q)\}$ and $E\{T(\tau^*)\}$ for various q values. The parameters are those of Table 3.

A reasonable objective function is the long-run average net profit per time unit, which is given by (2.7):

$$O(q) = \frac{c_p q - K - c_d E\{D(q)\} - c_{sh} E\{\text{Sh}\} - c_h E\left\{T\left(\tau^*\right)\right\}}{E\{C\}}. \tag{8.11}$$

Here c_p are the earnings per item, K the set-up costs per cycle, and c_d, c_{sh} and c_h are the costs of a discarded unit, the costs for shortage of one unit, and the costs of holding the inventory. It should be noted that, if not all items are sold, the profit from selling in one cycle is less than $c_p q$; however, that is taken into account by the $D(q)$ term. We compute the expected profit per cycle for the parameters of Table 3, when the cost values are $c_p = 5$, $K = 10$, $c_d = 10$, $c_{sh} = 2$, and varying values of c_h (Profit1); we do the same for $c_p = 10$, not changing the other parameters (Profit2). These are presented in the following table.

Table 4 Various performance measures

q	$E\{\tau^*\}$	$E\{R.I(\tau^* \in \text{LD})\}$	$E\{\text{Sh}\}$	$E\{D(q)\}$	$E\{T(\tau^*)\}$
90	3.77	0.615	1.492	0.00003	170.56
100	4.17	0.638	1.600	0.0001	208.67
110	4.58	0.656	1.786	0.0003	254.68
125	5.21	0.673	1.972	0.0014	326.65
150	6.29	0.685	2.244	0.0093	477.60
200	8.41	0.705	2.781	0.133	849.35
250	10.50	0.725	3.162	0.783	1340.09
300	12.52	0.760	3.145	2.805	1955.51
350	14.39	0.836	3.181	7.486	2664.45
400	16.05	0.952	3.037	16.458	3457.77
450	17.43	1.105	2.759	31.404	4342.10
500	18.49	1.264	2.405	53.889	5424.51

Table 5 The mean profit per cycle for a wide range of q-values

q	Profit1 $c_h = 0$	Profit1 $c_h = 0.1$	Profit1 $c_h = 0.2$	Profit2 $c_h = 0$	Profit2 $c_h = 0.1$	Profit2 $c_h = 0.2$
90	96.039	92.317	88.594	193.937	190.214	186.491
100	101.310	96.968	89.625	205.701	201.359	197.016
110	102.546	97.678	92.809	208.029	203.160	198.292
125	103.906	98.352	92.797	210.521	204.967	199.412
150	105.406	98.567	91.715	213.224	206.377	199.532
200	107.869	98.549	89.231	217.898	208.578	199.259
250	109.220	97.280	85.341	220.824	208.934	196.994
300	109.609	94.884	80.160	222.794	208.070	193.345
350	108.951	91.450	73.950	224.302	206.602	189.101
400	107.006	86.670	66.332	224.817	204.480	184.143
450	103.612	80.185	56.759	225.153	201.727	178.300
500	98.525	71.069	43.605	225.200	197.740	170.281

In the columns of Table 5 it is seen that the optimal value q^* of q lies in the interior of the search region ($q^* \approx 300$ in column 1, $q^* \approx 150$ in column 2 etc.) and that there would be a substantially smaller profit for too small or too large values of q. The profit seems to be a unimodal function of q.

From the managerial point of view, this paper leads to two main insights:

(1) In the presented complex (non-Markovian) EOQ model all relevant ingredients of the profit function are available in closed form, and this may be the basis to maximize the profit by numerically tackling the analytic results.
(2) The examples indicate that it is worthwhile to determine the optimal refilling level carefully, because a wrong choice of q can lead to a significantly lower profit.

Acknowledgments The research of David Perry and Wolfgang Stadje was supported in part by grant No. I-1184-31.4/2012 from the German-Israel Science Foundation.

References

Abramowitz, M., & Stegun, I. A. (1968). *Handbook of mathematical functions*. New York: Dover.

Baron, O., Berman, O., & Perry, D. (2010). Continuous review inventory models for perishable items ordered in batches. *Mathematical Methods of Operations Research*, *72*, 217–247.

Berk, E., & Gürler, U. (2008). Analysis of the (Q, r) inventory model for perishables with positive lead times and lost sales. *Operations Research*, *56*(5), 1238–1246.

Boxma, O. J., Perry, D., & Zacks, S. (2015). A fluid EOQ model of perishable items with intermittent high and low demand rates. *Mathematics of Operations Research*, *40*, 390–402.

Deniz, B., Karaesmen, I., & Scheller-Wolf, A. (2010). Managing perishables with substitution: Issuance and replenishment heuristics. *Manufacturing & Service Operations Management*, *12*, 319–329.

Giri, B. C., & Chaudhuri, K. S. (1998). Deterministic models of perishable inventory with stock-dependent demand rate and nonlinear holding cost. *European Journal of Operational Research*, *105*, 467–474.

Graves, S. (1982). The application of queueing theory to continuous perishable inventory systems. *Management Science*, *28*, 400–406.

Kao, E. P. C. (1997). *An introduction to stochastic processes*. New York: Duxbury Press.

Karaesmen, I., Scheller-Wolf, A., & Deniz, B. (2010). Managing perishable and aging inventories: Review and future research directions. In Handbook of production planning, Kempf, K., Keskinocak, P., Uzsoy, P. (eds.) Kluwer International Series in Operations Research and Management Science, Advancing the State-of-the-Art Subseries.

Lian, Z., & Liu, L. (2001). Continuous review perishable inventory systems: Models and heuristics. *IIE Transactions*, *33*, 809–822.

Lian, Z., Liu, L., & Neuts, M. (2005). A discrete-time model for common lifetime inventory systems. *Mathematics of Operations Research*, *30*, 718–732.

Liu, L., & Lian, Z. (1999). $(s; S)$ continuous review models for products with fixed lifetimes. *Operations Research*, *47*, 150–158.

Nahmias, S. (1982). Perishable inventory theory: A review. *Operations Research*, *30*, 680–708.

Nahmias, S. (2011). *Perishable inventory systems*. New York: Springer.

Pal, M. (1989). The $(S - 1, S)$ inventory model for deteriorating items with exponential lead time. *Calcutta Statistical Association Bulletin*, *38*, 83–91.

Perry, D., Stadje, W., & Zacks, S. (2005). Sporadic and continuous clearing policies for a production/inventory system under $M/G/$ demand. *Mathematics of Operations Research*, *30*(2), 354–368.

Prabhu, N. U. (1965). *Queues and inventories*. New York: Wiley.

Raafat, F. (1991). Survey of literature on continuously deteriorating inventory models. *Journal of the Operational Research Society*, *42*, 27–37.

Shi, J., Katehakis, M. N., & Melamed, B. (2013). Martingale methods for pricing inventory penalties under continuous replenishment and compound renewal demands. *Annals of Operations Research*, *208*, 593–612.

Shi, J., Katehakis, M. N., Melamed, B., & Xia, Y. (2014). Production-inventory systems with lost sales and compound Poisson demands. *Operations Research*, *62*, 1048–1063.

Weiss, H. (1980). Optimal ordering policies for continuous review perishable inventory models. *Operations Research*, *28*, 365–374.

Wilson, R. H. (1934). A scientific routine for stock control. *Harvard Business Review*, *13*, 116–128.

Zipkin, P. H. (2000). *Foundations of inventory management*. New York: McGraw-Hill.