Onset of chaotic advection in open flows

Citation for published version (APA):

DOI:
10.1103/PhysRevE.78.016317

Document status and date:
Published: 01/01/2008

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
Onset of chaotic advection in open flows

J. J. Benjamin Biemond, 1 Alessandro P. S. de Moura, 2 György Károlyi, 2 Celso Grebogi, 2 and Henk Nijmeijer 1

1 Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
2 College of Physical Sciences, King’s College, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom

(Received 21 February 2008; revised manuscript received 2 June 2008; published 25 July 2008)

In this paper we investigate the transition to chaos in the motion of particles advected by open flows with obstacles. By means of a topological argument, we show that the separation points on the surface of the obstacle imply the existence of a saddle point downstream from the obstacle, with an associated heteroclinic orbit. We argue that as soon as the flow becomes time periodic, these orbits give rise to heteroclinic tangles, causing passively advected particles to experience transient chaos. The transition to chaos thus coincides with the onset of time dependence in open flows with stagnant points, in contrast with flows with no stagnant points.

We also show that the nonhyperbolic nature of the dynamics near the walls causes anomalous scalings in the vicinity of the transition. These results are confirmed by numerical simulations of the two-dimensional flow around a cylinder.

DOI: 10.1103/PhysRevE.78.016317 PACS number(s): 47.52.+j, 05.45.—a, 47.10.Fg

Two-dimensional flow around an obstacle is stationary for low Reynolds numbers, and the dynamics of passively advected particles is integrable. For higher Reynolds numbers, the flow displays a time-periodic shedding of vortices, leading to the formation of the von Kármán vortex street [1]. In this nonstationary regime, particles commonly experience transient chaotic motion [2,3]. Chaotic advection is present in many important systems, especially in environmental flows such as the atmosphere and the oceans [4]. It is relevant for key environmental phenomena such as the depletion of the ozone layer [5] and plankton blooms [6–9].

A natural question to ask is whether the transition to chaos in open flows coincides with the transition from stationary to time-dependent flow; does chaotic advection appear as soon as the flow becomes time dependent? In this paper, the transition from the regular to the chaotic regime is investigated with a simplified analytical flow model, which incorporates the main features of the flow dynamics near the transition from stationarity to time dependence. The main goal is to determine if and under what conditions the transition from stationary to time-dependent fluid motion, and to identify the dynamical mechanisms governing this transition.

Our main result is that the nature of the transition to chaotic advection, and when it takes place, depend on whether or not there are stagnant points in the flow—points where the flow velocity is zero, such as on the surface of walls and obstacles. In the absence of stagnant points, the advection dynamics remains regular immediately after the transition to nonstationarity. This means that there is a range of Reynolds numbers, above the critical value for the onset of nonstationarity, for which the advection remains regular, even though the flow is time dependent. In this case, chaotic advection only sets in when the strength of time-dependent perturbation of the velocity field exceeds a certain threshold. If, on the other hand, stagnant points are present in the flow, we show that the motion of particles becomes chaotic as soon as the time dependence sets in. We argue in this work that this qualitative difference between the two cases is due to the presence of the stagnant points, resulting in the existence of heteroclinic orbits. These orbits break up and produce a heteroclinic tangle even for arbitrarily small time-dependent perturbations of the stationary flow, leading to chaos. This situation is very common in realistic flows, since stagnant points always exist in the presence of walls.

We also study in this work the effects of the nonhyperbolic nature of the advection on the dynamics, a result from the presence of walls in the flow. Due to the no-slip boundary conditions imposed on the Navier-Stokes equations, the surface of an obstacle consists of a set of fixed points with zero eigenvalues. Therefore the advection dynamics is nonhyperbolic when obstacles or other boundaries are present. The heteroclinic orbits which exist before the time dependence sets in break into heteroclinic tangles. Once this breakup occurs, particles coming from the outside can access the chaotic region present on the wake of the obstacle. In hyperbolic systems, the number of particles entering the region is limited by the previously existing separatrix, which is expected to depend on a bifurcation parameter $p$ (such as the Reynolds number) as $N_c \sim (p-p_c)^{-y}$, with $p_c$ being the critical parameter value (it can be thought of as the Reynolds number for which the flow becomes time dependent). Here, $y$ is a critical exponent, which depends on the eigenvalues of the fixed points [10,11]. We find that due to the presence of the surface consisting of fixed points and the nonhyperbolic dynamics close to this surface, this relationship does not hold in our case. We have instead an anomalous law of the form $N_v \sim \exp[k(p-p_c)^\eta]$, where $k$ and $\eta$ are constants.

In order to analyze the transition from a stationary to a time-dependent flow, we use an analytic model for a two-dimensional incompressible flow with a cylindrical obstacle, adapted from a model introduced in [2]. This model can serve as a prototype for the flow of other bluff-body obstacles in a uniform background flow. For two-dimensional (2D) incompressible flows, the velocity $(\dot{x}, \dot{y})$ of an advected particle is given by a time-dependent stream function $\psi(x,y,t)$, such that

$$
\dot{x}(t) = \frac{\partial}{\partial y} \psi(x,y,t), \quad \dot{y}(t) = -\frac{\partial}{\partial x} \psi(x,y,t).
$$

We use for $\psi$ the form introduced in Ref. [2], which is an analytical approximation to the time-dependent, periodic re-
FIG. 1. Part of a numerically calculated streakline for \( w=2 \) in the wake of the obstacle. Ten thousand particles were injected per period at position \((x,y) = (-3,0)\) into the flow, and their subsequent positions were plotted every period, at times given by \( t \mod T=0 \), until they leave the time-dependent region and travel downstream. The solid black area on the left is the cylindrical obstacle.

FIG. 2. Ratio of the number \( N_v \) of particles entering the wake to all \( N \) particles injected into the flow at time \( t=0 \), as a function of the bifurcation parameter, with random initial positions distributed uniformly in the region \(-16.0<x<1.1, -0.05<y<0.05\). The function \( \log() \) denotes the natural logarithm. In (a) the original model with bifurcation parameter \( w \) is investigated by injecting \( N=1 \times 10^6 \) particles; the fitted curve corresponds to \( N_v/N \sim \exp(kw^\beta) \), where \( k=-15.44 \) and \( \beta=1.046 \). In (b) the alternative model with bifurcation parameter \( \beta \) is investigated; the fitted curve corresponds to \( N_v/N \sim \exp(k\beta^\eta) \), where \( k=-2.807 \) and \( \eta=0.101 \). In these computations, \( N=1 \times 10^6 \) is used for parameters \( \beta > 2 \times 10^{-6} \), whereas \( N=5 \times 10^6 \) is used for smaller \( \beta \). We verified that, even though the particular value of the ratio \( N_v/N \) depends on where the particles are introduced in the flow, the scaling coefficient \( \eta \) is independent of this.

The flow model, which has a limiting behavior (for \( w \to 0 \)) closer to that of the realistic system, we find the same results.

In order to investigate the dynamics, we continuously inject particles at a fixed position and plot a superposition of all their trajectories in a stroboscopic manner with the period \( T=1 \) (i.e., we take snapshots of their positions at integer times). In this way we get a streakline. This is shown in Fig. 1 for \( w=2 \). We clearly see that the motion near the cylinder’s surface on the back of the obstacle is chaotic, with the presence of prominent Kolmogorov-Arnol’d-Moser (KAM) islands. Separation points \( S_\ast \), \( S_\ast \) and accumulation point \( S_0 \) on the cylinder’s surface are also noticeable.
ONSET OF CHAOTIC ADVECTION IN OPEN FLOWS

FIG. 3. Schematic picture of the manifolds escaping from the separation points in the flow around an obstacle. In (a), the manifolds are shown for an autonomous system, such that they form separatrices. In (b), the manifolds for a time-periodic flow are depicted, showing a heteroclinic tangle. Images (c) and (d) show the manifolds and streamlines in the case of persistent vortices.

In the region close to the KAM islands the particles whose trajectories get close to the surface of the cylinder exhibit transient chaotic motion. The particles that remain far away from the cylinder do not penetrate this chaotic region and are washed away rapidly by the background flow. To study the probability of particles getting in the chaotic region and are washed away rapidly by the background flow, we find that the relation \( \frac{N_v}{N} \) to scale as a power law, which depends: as soon as the flow ceases to be stationary, the \( w \) approaches zero as \( v \) becomes chaotic as soon as the flow becomes time dependent; this is indeed what we observe in our simulations. Our reasoning above depends purely on the existence of stagnation points, and our conclusion is a result of the interplay of

FIG. 4. Visualization of the heteroclinic tangle for \( w=2 \). Unstable manifolds are visualized by injecting 100×100 particles close to the separation points and plotting their subsequent positions stroboscopically for 18 periods. The stable manifolds are visualized by injecting 800×800 particles uniformly in the region \( 0.9<x<1.6, -0.5<y<0.5 \) and checking if they leave downstream (reach \( x=+10 \)) or pass through the region \( 1.25<x<1.35, -0.15<y<0.1; \) the boundary between these two outcomes marks the stable manifold of the saddle point.

The only way that incompressibility and the downstream boundary condition can be simultaneously satisfied is by the unstable manifolds of \( S_0 \) on the downstream surface of the cylinder, depicted in Fig. 3(a). The lateral points \( S_+ \) and \( S_- \) have unstable manifolds emanating from them, which act as separatrices at the transition parameter \( w=0 \), when the flow is stationary. Because of the incompressibility of the flow, there must be a similar point with a stable manifold, and that is the central point \( S_0 \). In this open flow the velocity has to be positive far downstream (\( x\rightarrow\infty \)). This implies that the stable manifold of \( S_0 \) cannot extend infinitely far in the downstream direction, otherwise the boundary condition would be violated, as there would be regions of negative flow velocity arbitrarily far downstream.

These heteroclinic orbits act as separatrices, insulating the inner region near the wall from the outer region. When the flow becomes time dependent for \( w>0 \), we expect the breakup of the separatrices and the formation of a heteroclinic tangle, which is schematically depicted in Fig. 3(b).

From these considerations we thus expect that the scattering becomes chaotic as soon as the flow becomes time dependent; this is indeed what we observe in our simulations. Our reasoning above depends purely on the existence of stagnation points, and our conclusion is a result of the interplay of
the incompressible character of the fluid with the openness of
the flow with its associated boundary condition. We thus ex-
pect the general conclusion to be valid for other flows with
stagnant points (or regions). For example, we should have a
similar behavior for stagnant or trapped fluid bodies as well,
such as the structures depicted in Figs. 3.

We calculated the heteroclinic tangle numerically. For the
parameter value \( w = 2 \), the manifolds are depicted in Fig. 4.
We found that the heteroclinic tangle is present for all \( w > 0 \),
consistent with our conclusion described above, that transi-
tent chaotic advection appears as soon as the flow be-
comes time dependent. This is also confirmed in Fig. 2(a),
where the ratio of particles accessing the chaotic region be-
hind the cylinder is positive for all \( w > 0 \). As argued above,
the fact that the transitions to nonstationarity and to chaotic
scattering coincide is a consequence of the presence of stag-
nant points on the surface of the obstacle. This behavior is
qualitatively different from that of flows without stagnant
points, which have no separatrices like those shown in Fig. 3.

To verify that our findings are not an artifact of the par-
ticular flow model we use, we investigated a different model.
In this modified model, there are two stationary vortices
present in the time-independent case, as expected in real
flows, in contrast to the previous model. This is achieved by
replacing Eq. (4) with

\[
g(x, y, t) = \beta[ - w_1 g_1(x, y, t) + w_2 g_2(x, y, t)] + (1 - \beta) \\
\times [ - w_3 g_1(x, y) + w_4 g_2(x, y)] + u_0 y f(x, y),
\]

where the two new functions are

\[
g_1(x, y) = \exp\{ - R_0[(x - x_s)^2 + (y - y_s)^2] \},
\]

\[
g_2(x, y) = \exp\{ - R_0[(x - x_s)^2 + (y + y_s)^2] \},
\]
describing the contribution to the stream function of two
stationary vortices. These vortices are positioned at \( x_s = 1,0 \)
and \( y_s = \pm 0.3 \). The parameters are chosen to be \( \alpha = 2.0, \ w = 6.0 \), such that the stationary vortices behave qualita-
tively as expected in real flows. The value \( w = 24 \) is used in
accordance with [2]. The parameter \( \beta \) is used in this model
as the bifurcation parameter. For \( \beta = 0 \), the flow is station-
ary, whereas for \( \beta = 1 \) the model is the same as the one used in
[2]. The streamlines of this flow for \( \beta = 0 \) are shown in Fig.
5(a).

For all values of \( \beta \), we find again a heteroclinic tangle, as
depicted in Fig. 5(b) for \( \beta = 0.001 \). This region is accessible
for particles injected upstream from the obstacle. By inject-
ing many particles into the flow upstream, the number \( N_i \) of
particles entering the chaotic region is shown in Fig. 2(b) for
different values of \( \beta \). We see that the ratio of chaotic to all
particle trajectories, again, follows \( N_i/N = \exp(k^{-\gamma}) \).

The above analysis is only valid for flows with stagnation
points. In flows with no boundaries we may have no stagna-
tion points at the stationary time-dependent transition, and in
this situation we expect that advection will be regular imme-
diately after the onset of time dependence. An example of
boundaryless flows consists of a flow with a uniform velocity field, on which a time-dependent component is superimposed. If the time-dependent part has small amplitude, a particle will be washed away by the constant background flow, and there will be no chaotic advection. Transition to chaos only happens for sufficiently high amplitudes, in contrast to the case with boundaries. An example of such a flow is given by the stream function

\[ \psi = V_y + \mu \exp\left(-\frac{(x-L)^2+y^2}{2\sigma^2}\right) \cos[k(y-ut)], \]

where \( V \) is the constant background velocity, \( \mu \) is the strength of the periodic perturbation (so the flow is stationary for \( \mu = 0 \)), \( \sigma \) is the characteristic length scale of the perturbation, \( k \) is its wave number, and \( u \) is its propagation velocity; the period of the perturbation is given by \( 2\pi/ku \). This flow is a modification of the one described in Ref. [6], used for modeling ecological flows. Based on the above reasoning, we predict that there is a range of positive \( \mu \) for which the flow is time dependent, but the advection is regular, in contrast with the flow with boundaries. A simple way to test this is to plot the escape time of a particle as a function of its initial starting point; this plot should be smooth in the case of regular advection, while it has a Cantor set of singularities in the case with boundaries. An example of such a flow is given in Fig. 6(a) shows that for a small value of \( \mu (\mu = 2) \), the escape time plot is perfectly regular, and the advection is regular, even though the flow is time dependent; this supports our predictions. As \( \mu \) increases, this flow eventually becomes chaotic; this can be seen in Fig. 6(b) (with \( \mu = 10 \), which shows the patterns of peaks characteristic of chaotic scattering. Magnifications of regions with peaks in this figure reveal an infinitely fine structure of ever higher peaks, a result of the fact that the peaks lie on a Cantor set. We found that for this system, and for the parameters we used (see caption of Fig. 6), the transition from regular to chaotic advection happens at around \( \mu = 5 \).

In conclusion, we established that fluid flows with stagnant points or regions contain separatrices that are barriers to transport in the stationary case. These separatrices break up as soon as the flow becomes time dependent, resulting in a chaotic sea that can trap for a transient time the particles coming from upstream. Hence there is an immediate transition to chaotic advection in these systems. This behavior is to be contrasted with systems without such stagnant regions: there the lack of separatrices imply that a small time-dependent velocity component is not enough to compensate for the background velocity that washes out the particles, and hence time periodicity does not necessarily imply chaos. Due to the presence of the surface and the KAM islands the dynamics around an obstacle is nonhyperbolic, which yields a nontrivial scaling near the bifurcation. As a final remark, we note that, although the stagnant points with zero velocity can only be distinguished for an obstacle with fixed position, a system with a moving obstacle will experience the same dynamics in a comoving coordinate system.

We would like to thank CNPq for partial support of this project. Gy.K. is indebted for financial support from OTKA No. NK 72037.