Ultrametricity in the Edwards-Anderson model : Reply to a comment by T. Jörg and F. Krzakala

Contucci *et al.* Reply: In a recent comment [1] to our Letter [2] T. Jörg and F. Krząkała have investigated the properties of the two-dimensional Edwards-Anderson model. They found, by numerical methods, the interesting result that some ultrametric features hold for the probability distributions of the link overlap Q for square lattices of side 16 and 32. Namely, defining X (resp. Y) as the difference between the medium and the smallest (resp. the largest and the medium) for a triplet of link overlaps sampled with respect to the equilibrium quenched measure, the probability distributions coincide with those predicted by the replica symmetry breaking (RSB) theory [3], i.e., $\rho_X(x) = \delta(x)$ and $\rho_Y(y) = \frac{1}{4}\delta(y) + \frac{3}{2}\theta(y) \times \int_y^1 P(a)P(a-y)da$; see Fig. 1 of [1]. Since at positive temperature in two dimensions the RSB picture cannot hold, they conclude that the results presented in [2] are not sufficient to dismiss the droplet picture in the three-dimensional Edwards-Anderson model. Our answer can be summarized as follows. The conclusions obtained in [2] are mainly based on the analysis of the scaling properties of the variance for the two random variables X and Y. Our statement in favor of ultrametricity comes from the observation (see Figs. 1 and 3 in [2]) that for increasing volumes the variance of the variable X is shrinking to zero with a suitable scaling law while the variance of the variable Y is not. The subsequent analysis of the distribution shape for some finite volume (see Fig. 2 in [2]) was indeed proposed as a further support of the main result. Since the study in [1] is only concerned with a finite volume study of the overlap distributions with no asymptotic analysis, it cannot be used to weaken the conclusions obtained in [2]. In order to parallel the approach followed in [2] one should have performed the asymptotic analysis of the variances of the variables X and Y. This can be done indeed with a modest computational effort and gives the result shown in Fig. 1. One immediately sees that both the variances are shrinking to zero and by consequence ultrametricity does not hold. This shows that the method developed in [2] is robust. Still the result of [1] is interesting and deserves a proper explanation. Here we notice that in the (T, d) plane—dimension vs temperature—there is a curve which separates in the thermodynamic limit the region with broken symmetry from the paramagnetic one. At $T = 0$ the curve crosses the d axis at the lower critical dimension $d_l = 2.5$. For a finite volume system, if one is outside the spin glass region but close enough to this curve one might still observe some features of an ultrametric overlap distribution. The point investigated in [1] ($T = 0.2$, $d = 2$) is just below the critical curve and not far enough to observe, for the volumes they test which in $d = 2$ are quite small, the thermodynamic properties. To support our claim it is enough to study the point $d = 1$ and $T = 0$. Since we are now really away from the critical curve ultrametricity cannot hold anymore. We analyzed the probability distribution for a triplet of standard overlap (in $d = 1$ the link overlap is 1 with probability 1). An explicit computation shows that the quenched expectation of $S = \mathrm{sign}(q_{1,2}q_{2,3}q_{3,1})$ is equal to $1/2$, while for an ultrametric topology, as the one predicted by RSB theory, one should have $\langle S \rangle = 1$. Moreover, for the distribution of $\tilde{X} = \tilde{q}_{\rm med} - \tilde{q}_{\rm min}$ and $\tilde{Y} = \tilde{q}_{\rm max} - \tilde{q}_{\rm med}$, where $\tilde{q}_{\rm max} = \max_{i,j}\{|q_{i,j}|\}$, $\tilde{q}_{\rm med} = \mathrm{med}_{i,j}\{|q_{i,j}|\}$ and $\tilde{q}_{\rm min} = S\min_{i,j}\{|q_{i,j}|\}$, in the large volume limit we have $\rho_{\tilde{X}}(x) = -\frac{9}{2}x + 3$ for $0 \le x \le \frac{2}{3}$ and 0 otherwise, while $\rho_{\tilde{Y}}(y) = -2y + 2$. Clearly the previous formulas cannot satisfy the RSB ultrametric relation.

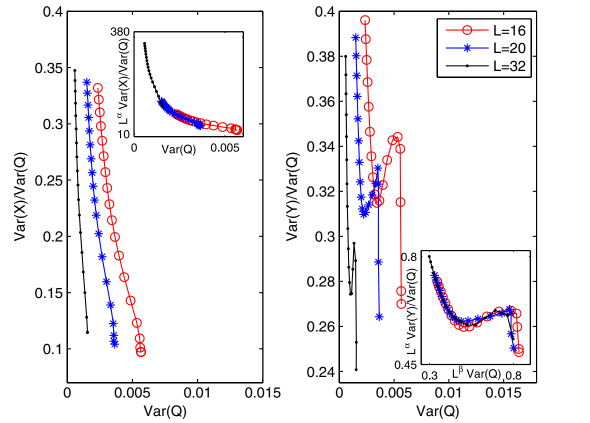

FIG. 1 (color online). Normalized variances of X (left) and Y (right) vs Var(Q) for 2D $+/-1$. The insets show the scaling laws $L^\alpha\mathrm{Var}(X)/\mathrm{Var}(Q)$ for $\alpha = 2$ and $L^\alpha\mathrm{Var}(Y)/\mathrm{Var}(Q)$ vs $L^\beta\mathrm{Var}(Q)$ for $\alpha = 0.22$ and $\beta = 1.8$. In both cases the scaled normalized variances are L independent.

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