

Polynomial matrices and feedback

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EINDHOVEN UNIVERSITY OF TECHNOLOGY

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Polynomial Matrices and Feedback

by

Rikus Eising

Eindhoven, the Netherlands

January, 1984

Introduction

In this paper we describe the use of feedback with respect tot some polynomial matrix constructions.

Consider the following problem

- (1) Given a polynomial matrix $P(\lambda) \in \mathbb{R}[\lambda]^{p \times q}$ (the set of $p \times q$ -matrices with entries in $\mathbb{R}[\lambda]$ (the set of real polynomials in λ)) such that $q > p$ and that $P(\lambda)$ has full rank for all $\lambda \in \mathbb{C}$ (the set of complex numbers).

Construct a matrix $Q(\lambda) \in \mathbb{R}[\lambda]^{(q-p) \times q}$ such that

$$(2) \quad \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$$

is unimodular.

Of course this problem has a well-known solution. Algorithms providing us with $Q(\lambda)$ are mostly based on elementary row (column) operations, reducing $P(\lambda)$ to some simple form. Having obtained such a simple form (for instance lower triangular, Hermite form(like), Smith form(like)) the construction of $Q(\lambda)$ is straightforward.

Our construction is not based on elementary row (column) operations. We work on real matrices directly.

One of the main problems concerning the methods based on elementary operations, which, in turn, are based on the Euclidean algorithm, is their bad numerical behaviour.

Our method is based on a numerically reliable method for the construction of a feedback matrix solving the deadbeat control problem for a generalized state space system $Ex_{k+1} = Ax_k + Bu_k$. This method is closely related to [2]. The method we use, also gives the inverse of (2) in a straightforward way.

A number of applications of $Q(\lambda)$ in (1), (2) can be found in [1].

Preliminaries

In this section we describe the generalized deadbeat control problem. The solution of this problem will be needed for the construction of $Q(\lambda)$ in (1), (2).

This problem is the following

$$(3) \text{ Given } (E,A,B) , E \in \mathbb{R}^{n \times n} , A \in \mathbb{R}^{n \times n} , B \in \mathbb{R}^{n \times m} .$$

Construct a matrix $F \in \mathbb{R}^{m \times n}$ such that all generalized eigenvalues of the pencil $[\lambda E - (A + BF)]$ are zero.

Observe that for the solvability of (3) we must have that E is regular because generalized eigenvalues at infinity are not allowed.

This problem is called the generalized deadbeat control problem because it is a deadbeat control problem for the generalized state space system

$$Ex_{k+1} = Ax_k + Bu_k ; x_k \in \mathbb{R}^n , u_k \in \mathbb{R}^m , k = 0,1,2,\dots .$$

Problem (3) is equivalent to the usual deadbeat control problem for a system $(E^{-1}A, E^{-1}B)$. However, we will consider (3) because of the possible ill conditioning of E with respect to inversion.

An equivalent statement for all generalized eigenvalues of $[\lambda E - (A+BF)]$ being zero is

$[E - \lambda(A+BF)]$ is a unimodular matrix .

This can be seen as follows.

If $[E - \lambda(A+BF)]$ is unimodular we must have that E is regular. Therefore $[\lambda E - (A+BF)]$ does not have generalized eigenvalues at infinity. Now suppose that $[\lambda E - (A+BF)]$ has a nonzero eigenvalue λ_1 . Then $\lambda_1 E x - (A+BF)x = 0$ for some nonzero $x \in \mathbb{C}^n$. Therefore $E x - \frac{1}{\lambda_1}(A+BF)x = 0$, contradicting the unimodularity of $[E - \lambda(A+BF)]$. The other implication can be proved similarly.

Next we consider the solvability of problem (3).

It is well-known that there exists a matrix F such that $[E^{-1}A + E^{-1}BF]$ is nilpotent if and only if the noncontrollable eigenvalues of $E^{-1}A$ (for the system $(E^{-1}A, E^{-1}B)$) are zero. Therefore (3) is solvable if and only if all generalized eigenvalues λ_i of $[\lambda E - A]$, such that $[\lambda_i E - A, B]$ does not have full rank, are zero.

An equivalent condition for this is

$$(4) \quad [E - \lambda A, \lambda B]$$

is right invertible for $\lambda \in \mathbb{C}$ (or, equivalently, right invertible over $\mathbb{R}[\lambda]$). This can easily be seen (remember that E is regular).

We will need a solution for (1) for the case $P(\lambda) = [E - \lambda A, \lambda B]$. Therefore, let F be such that $[E^{-1}A + E^{-1}BF]$ is nilpotent. Such an F exists by the pole placement theorem because all nonzero eigenvalues of $E^{-1}A$ are controllable for the system $(E^{-1}A, E^{-1}B)$.

Now we have that

$$\begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix}$$

is unimodular because

$$(5) \quad \begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} = \begin{bmatrix} E - \lambda(A+BF) & \lambda B \\ 0 & I \end{bmatrix}.$$

Here $[E - \lambda(A+BF)]$ is unimodular because $[I - (E^{-1}A + E^{-1}BF)]$ is unimodular.

An explicit formula for $[E - (A+BF)]^{-1}$ is

$$[E - \lambda(A+BF)]^{-1} = [I + \lambda(E^{-1}A + E^{-1}BF) + \dots + \lambda^{n-1}(E^{-1} + E^{-1}BF)^{n-1}]E^{-1}.$$

Therefore a solution $Q(\lambda)$ to problem (1) for $P(\lambda) = [E - \lambda A, \lambda B]$ is

$$Q(\lambda) = [F \quad I].$$

In the next section we will show that problem (1) for general $P(\lambda)$, may be solved by solving (1) for a pencil $[E - \lambda A, \lambda B]$ where (E, A, B) is derived from $P(\lambda)$ in a straightforward way.

Results

Let $P(\lambda) \in \mathbb{R}[\lambda]^{p \times q}$ be such that $P(\lambda)$ has rank p for all $\lambda \in \mathbb{C}$ ($q > p$).

$P(\lambda)$ can be written as

$$P(\lambda) = P_0 + P_1\lambda + \dots + P_n\lambda^n,$$

where $P_i \in \mathbb{R}^{p \times q}$, $i = 0, \dots, n$.

Because $P(0)$ has rank p we have that P_0 has rank p .

We may assume that

$$(6) \quad P_0 = [P_e, 0]$$

where P_e is a regular $p \times p$ -matrix because we can solve problem (1) for $P(\lambda)$ if we can solve (1) for $P(\lambda)U$ where U is a regular matrix. Of course U may be taken to be unitary in order to obtain this particular form for P_0 .

Next we partition $P(\lambda)$ as

$$P(\lambda) = [P_e, 0] + [P_{a1}, P_{b1}] \lambda + \dots + [P_{an}, P_{bn}] \lambda^n$$

where

$$P_{ai} \in \mathbb{R}^{p \times p}, P_{bi} \in \mathbb{R}^{p \times (q-p)} \text{ for } i = 1, \dots, n.$$

Consider matrices E, A, B defined by

$$(7) \quad E = \begin{bmatrix} I & & & & \\ & \cdot & & \bigcirc & \\ & & & I & \\ & & & & P_e \\ \bigcirc & & & & \end{bmatrix}, \quad A = \begin{bmatrix} 0 & & & -P_{an} \\ I & \cdot & & \cdot \\ & \cdot & \cdot & \cdot \\ & & \cdot & 0 \\ \bigcirc & & & I \\ & & & & -P_{a1} \end{bmatrix}, \quad B = \begin{bmatrix} P_{bn} \\ \cdot \\ \cdot \\ \cdot \\ P_{b1} \end{bmatrix}$$

It can easily be seen that $[E - \lambda A, \lambda B]$ has full rank for all $\lambda \in \mathbb{C}$.

Let $F \in \mathbb{R}^{(q-p) \times np}$ be a feedback matrix such that

$$\begin{bmatrix} E - \lambda A & B \\ F & I \end{bmatrix}$$

is unimodular. (See previous section for the existence of F)

Next we observe that

$$\begin{bmatrix} I & 0 & \dots & \dots & \dots & 0 \\ \lambda I & I & \dots & \dots & \dots & \dots \\ \dots & \lambda I & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda^{n-1} I & \dots & \dots & \lambda I & I & 0 \\ 0 & \dots & \dots & \dots & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & \dots & 0 & \lambda P_{an} & \lambda P_{bn} \\ -\lambda I & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & I & \lambda P_{a2} & \dots \\ 0 & \dots & \dots & 0 & -\lambda I & P_e + \lambda P_{a1} & \lambda P_{b1} \end{bmatrix} = \begin{bmatrix} I & 0 & \dots & \dots & 0 & X_n(\lambda) & Y_n(\lambda) \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & I & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & X_1(\lambda) & Y_1(\lambda) \end{bmatrix} = \bar{P}(\lambda) .$$

Here $X_i(\lambda) = P_{ai} + P_{a,i+1}\lambda^2 + \dots + P_{a,n}\lambda^{n-i+1}$ for $i = 2, \dots, n$.

The matrices $Y_i(\lambda)$ are defined analogously for $i = 2, \dots, n$.

Futhermore

$$[X_1(\lambda), Y_1(\lambda)] = P(\lambda) .$$

Observe that $\bar{P}(\lambda)$ is unimodular (because both factors in the left hand side are unimodular).

Let $U = [U_1, U_2]$ be a unitary matrix such that

$$[U_1, U_2] \begin{bmatrix} I & & & \\ & \dots & & \\ & & \dots & \\ \hline & & & I \\ F_1 & & & \end{bmatrix} = \begin{bmatrix} & & & R \\ & & & \\ & & & \\ \hline & & & 0 \end{bmatrix}$$

where R is regular $\left(U_1 \in \mathbb{R}^{((n-1)p+q-p) \times (n-1)p}, U_2 \in \mathbb{R}^{((n-1)p+(q-p) \times (q-p))} \right)$.

Then

$$(8) \quad \begin{bmatrix} U_1 & 0 & U_2 \\ 0 & I & 0 \end{bmatrix} \bar{P}(\lambda) = \begin{bmatrix} R & * \\ 0 & P(\lambda) \\ 0 & Q(\lambda) \end{bmatrix}$$

for some polynomial matrix $Q(\lambda)$.

The matrix

$$\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$$

is unimodular because $\bar{P}(\lambda)$ is unimodular, R is regular and U is unitary.

This shows that $Q(\lambda)$ is a solution to problem (1).

Observe that the degree of $Q(\lambda)$ is less than the degree of $P(\lambda)$

The inverse of (2) (and also a right inverse of $P(\lambda)$) can easily be constructed using the polynomial matrix

$$\begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix}^{-1}$$

Up to now we have used two unitary transformations (6), (8) in order to obtain $Q(\lambda)$ where we assumed that F can be computed. If we show that a feedback matrix for the generalized deadbeat control problem can be constructed in a numerically reliable way we will have obtained a reliable method for the construction of a matrix $Q(\lambda)$ solving problem (1).

Algorithms

In this section we present a collection of numerically reliable algorithms for the construction of a feedback matrix F such that $[E - \lambda(A+BF)]$ is unimodular (F is a solution to problem (3)).

Let (E,A,B) be a matrix triple such that $[E - \lambda A, \lambda B]$ is right invertible over $\mathbb{R}[\lambda]$. Here $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Algorithm

$i := 1; n_i := n; s_i := 1; E_i := E; A_i := A; B_i := B;$

while $n_i > 0$ and $s_i > 0$ and $B_i \neq 0$ do

begin

Step 1: Using a minor modification of the singular value decomposition we have (some of the zero-matrices may be empty)

$$B_i = U_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{ib} & 0 \\ 0 & 0 & D_{ig} \end{bmatrix} V_i^T$$

where U_i, V_i are unitary matrices and D_{ib}, D_{ig} are diagonal matrices together containing the singular values of B_i . D_{ib} contains only "bad" singular values (think of "bad" as "too small") and D_{ig} contains g_i "good" singular values of B_i (think of "good" as "large enough").

We will assume that "good" implies positive.

If $g_i = 0$ then

begin

Test whether all generalized eigenvalues of $\lambda E_i - A_i$ are zero.

(this can be done by means of the QZ algorithm) If not all gene-

ralized eigenvalues of $\lambda E_i - A_i$ are zero we have to define "good" and

"bad" differently in order to obtain at least one "good" singular value. ($g_i > 0$).

end

Step 2: If $g_i > 0$ the following partitioned matrices are computed

$$\left(\begin{bmatrix} E_{ia} & E_{ib} \\ E_{if} & E_{ig} \end{bmatrix}, \begin{bmatrix} A_{ia} & A_{ib} \\ A_{if} & A_{ig} \end{bmatrix}, \begin{bmatrix} B_{ib} \\ B_{ig} \end{bmatrix} \right) := \left(U_i^T E_i, U_i^T A_i, U_i^T B_i \right).$$

Here

$$B_{ib} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{ib} & 0 \end{bmatrix} V_i^T, \quad B_{ig} = \begin{bmatrix} 0 & 0 & D_{ig} \end{bmatrix} V_i^T$$

$$E_{ig} \in \mathbb{R}^{g_i \times g_i}, \quad A_{ig} \in \mathbb{R}^{g_i \times g_i}.$$

The dimensions of the other matrices involved are chosen accordingly.

Step 3: If $g_i > 0$ a unitary matrix W_i is computed such that

$$\begin{bmatrix} A_{ia} & A_{ib} \\ A_{if} & A_{ig} \end{bmatrix} W_i = \begin{bmatrix} \bar{A}_{ia} & 0 \\ \bar{A}_{if} & \bar{A}_{ig} \end{bmatrix}, \quad \bar{A}_{ig} \in \mathbb{R}^{g_i \times g_i}.$$

$$F_i := [0, F_{ig}] := V_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D_{ig}^{-1} \bar{A}_{ig} \end{bmatrix}.$$

Now it is clear that

$$\begin{bmatrix} \bar{A}_{ia} & 0 \\ \bar{A}_{if} & \bar{A}_{ig} \end{bmatrix} + \begin{bmatrix} B_{ib} \\ B_{ig} \end{bmatrix} F_i = \begin{bmatrix} \bar{A}_{ia} & 0 \\ \bar{A}_{if} & 0 \end{bmatrix}.$$

Step 4: If $g_i > 0$ a unitary matrix Y_i is computed such that

$$Y_i^T \begin{bmatrix} E_{ia} & E_{ib} \\ E_{if} & E_{ig} \end{bmatrix} W_i = \begin{bmatrix} \bar{E}_{ia} & 0 \\ \bar{E}_{if} & \bar{E}_{ig} \end{bmatrix}, \quad \bar{E}_{ig} \in \mathbb{R}^{g_i \times g_i}.$$

Then we have

$$\left(Y_i^T U_i^T E_i W_i, Y_i^T U_i^T A_i W_i + Y_i^T U_i^T B F_i, Y_i^T U_i^T B \right) =$$

$$\left(\begin{bmatrix} E_{i+1} & 0 \\ * & \bar{E}_{ig} \end{bmatrix}, \begin{bmatrix} A_{i+1} & 0 \\ * & 0 \end{bmatrix}, \begin{bmatrix} B_{i+1} \\ * \end{bmatrix} \right)$$

for some

$$E_{i+1} \in \mathbb{R}^{(n_i - g_i) \times (n_i - g_i)}, \quad A_{i+1} \in \mathbb{R}^{(n_i - g_i) \times (n_i - g_i)}, \quad B_{i+1} \in \mathbb{R}^{(n_i - g_i) \times m}.$$

Step 5: $n_{i+1} := n_i - g_i$; $s_{i+1} := g_i$; $i := i + 1$.

end (of while-loop)

Observe that

- $(E_{i+1}, A_{i+1}, B_{i+1})$ does not depend on F_i
- $(E_{i+1}, A_{i+1}, B_{i+1})$ satisfies the solvability condition for (1) with

$$(E, A, B) = (E_{i+1}, A_{i+1}, B_{i+1}).$$

for each cycle i of this part of the algorithm.

It will be clear that termination of this while-loop will be obtained after n cycles at most.

We have termination because $n_i = 0$ or $s_i = 0$ or $B_i = 0$ for some i .

If $n_{i_0+1} = 0$ we have obtained (after i_0 cycles)

$$n_1, \dots, n_{i_0}; g_1, \dots, g_{i_0}; W_1, \dots, W_{i_0}; U_1, \dots, U_{i_0}; Y_1, \dots, Y_{i_0}; F_{1g}, \dots, F_{i_0g}$$

(W_{i_0} and Y_{i_0} may be taken to be identity matrices).

If $s_{i_1+1} = 0$ we have obtained (after i_1 cycles)

$$n_1, \dots, n_{i_1}; g_1, \dots, g_{i_1-1}; W_1, \dots, W_{i_1-1}; U_1, \dots, U_{i_1-1}; Y_1, \dots, Y_{i_1-1};$$

$$F_{1g}, \dots, F_{i_1,g}, E_{i_1} \in \mathbb{R}^{n_{i_1} \times n_{i_1}}, A_{i_1} \in \mathbb{R}^{n_{i_1} \times n_{i_1}}, B_{i_1} \in \mathbb{R}^{n_{i_1} \times m}$$

such that $\lambda E_{i_1} - A_{i_1}$ has generalized eigenvalues equal to zero.

If $B_{i_2} = 0$ we have obtained (after $i_2 - 1$ cycles)

$$n_1, \dots, n_{i_2}; g_1, \dots, g_{i_2-1}; W_1, \dots, W_{i_2-1}; U_1, \dots, U_{i_2-1}; Y_1, \dots, Y_{i_2-1};$$

$$F_{1g}, \dots, F_{i_2-1,g}, E_{i_2} \in \mathbb{R}^{n_{i_2} \times n_{i_2}}, A_{i_2} \in \mathbb{R}^{n_{i_2} \times n_{i_2}}$$

such that $\lambda E_{i_2} - A_{i_2}$ has generalized eigenvalues equal to zero because

(3) is solvable.

Next we compute unitary matrices $X_\ell, Z_\ell \in \mathbb{R}^{n \times n}$

$$X_\ell := U_1 Y_1 \begin{bmatrix} U_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} Y_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \dots \begin{bmatrix} U_\ell & 0 \\ 0 & I_{m_\ell} \end{bmatrix} \begin{bmatrix} Y_\ell & 0 \\ 0 & I_{m_\ell} \end{bmatrix}$$

$$Z_\ell := W_1 \begin{bmatrix} W_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \dots \begin{bmatrix} W_\ell & 0 \\ 0 & I_{m_\ell} \end{bmatrix}$$

for $\ell = i_0, i_1-1, i_2-1$ if $n_{i_0+1} = 0, s_{i_1+1} = 0, B_{i_2} = 0$ respectively.

Here I_{m_i} is the $m_i \times m_i$ identity matrix; $m_i = n - n_i, i = 2, \dots, \ell$.

The matrix $F_g \in \mathbb{R}^{m \times n}$ is formed as follows

$$F_g := [0 \quad F_{\ell g}, \dots, F_{1g}]$$

where the zero matrix is empty if $n_{i_0+1} = 0$, for some i_0 . Otherwise this

matrix is a $m \times n_{i_1}$ or $m \times n_{i_2}$ zero matrix depending on the termination condition

$s_{i_1+1} = 0$ or $B_{i_2} = 0$ respectively.

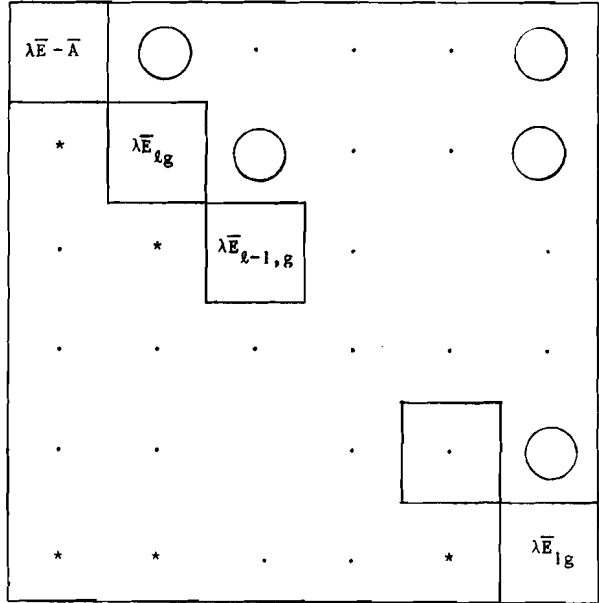
In the final step of the algorithm we compute $F \in \mathbb{R}^{m \times n}$

$$F := F_g Z_\ell^T$$

end of the algorithm.

In order to prove that the matrix F is a solution to problem (3) we observe that

$$(2) \quad \lambda X_{\ell}^T E Z_{\ell} - \left(X_{\ell}^T A Z_{\ell} + X_{\ell}^T B F \right) =$$



Here $\lambda \bar{E} - \bar{A} =$ "empty", $\lambda E_{i_1} - A_{i_1}$, $\lambda E_{i_2} - A_{i_2}$ for the termination condition $n_{i_0+1} = 0$, $s_{i_1+1} = 0$, $B_{i_2} = 0$ respectively.

This shows that $[\lambda E - (A+BF)]$ only has generalized eigenvalues equal to zero. Therefore $[E - \lambda(A+BF)]$ is unimodular.

Discussion of the algorithms

The algorithm described in the previous section in fact represents a collection of algorithms because any selection policy between "good" and "bad" with respect to singular values, generally results in a different algorithm.

Concerning termination of the algorithm we observe the following.

If the matrix triple (E,A,B) , where E is regular, represents a controllable generalized system (this means that $(E^{-1}A, E^{-1}B)$ is controllable) then the condition $B_i = 0$ will not terminate the while loop because in each cycle we

have that (E_i, A_i, B_i) is controllable.

Controllability of (E, A, B) is not really a restriction because any generalized system (E, A, B) may be transformed (using unitary transformations) into a generalized system of the form

$$(10) \quad \left(\left[\begin{array}{cc} E_b & 0 \\ * & E_g \end{array} \right], \left[\begin{array}{cc} A_b & 0 \\ * & A_g \end{array} \right], \left[\begin{array}{c} 0 \\ B_g \end{array} \right] \right)$$

where (E_g, A_g, B_g) is controllable (E_g is regular because E is regular).

Observe that solvability of (3) for (E, A, B) means that (3) is solvable for (E_g, A_g, B_g) and that $[E_b - \lambda A_b]$ is unimodular.

If the matrix triple (E, A, B) , where E is regular, represents a controllable generalized system such that all generalized eigenvalues are non zero ($E^{-1}A$ is regular) then the condition $n_i = 0$ for some i will terminate the while loop.

We may restrict our algorithm to this case because a controllable generalized system (E, A, B) having some zero generalized eigenvalues can be transformed into the following form (using the QZ algorithm)

$$(11) \quad \left(\left[\begin{array}{cc} E_z & * \\ 0 & E_n \end{array} \right], \left[\begin{array}{cc} A_z & * \\ 0 & A_n \end{array} \right], \left[\begin{array}{c} B_z \\ B_n \end{array} \right] \right)$$

where the generalized eigenvalues of $\lambda E_z - A_z$ all are zero and $\lambda E_n - A_n$ only has non zero generalized eigenvalues. Furthermore it will be clear that (E_n, A_n, B_n) is controllable.

If we apply our algorithm to (E_n, A_n, B_n) we will obtain a matrix F_n such that $[0, F_n]$ solves (3) for the controllable system (11). A solution to problem (3) for the system (10) can be obtained straightforwardly in this case.

We have chosen to describe the algorithm for the general case of a system (E, A, B) , where E is regular such that (3) is solvable, and not only for a controllable system or even a controllable system having only non zero generalized eigenvalues because we can deal with a larger class of cases in this way.

If we take the special version of the algorithm:

"good" = "non zero"; "bad" = "zero"

we obtain a feedback matrix F such that $(E^{-1}A + E^{-1}BF)^k = 0$ where k satisfies

$$k = \min_F \left\{ \ell_F \mid (E^{-1}A + E^{-1}BF)^{\ell_F} = 0 \right\} .$$

Furthermore F has minimum Frobenius norm (see [3]).

With respect to the numerical properties of the algorithm we observe that the construction of the matrix F_g may be postponed (it is not necessary to calculate F_{ig} within the while-loop) until the matrices X_ℓ and Z_ℓ have been computed. This can be seen as follows.

In (9) it can be seen that

$$X_\ell^T A Z_\ell + X_\ell^T B F_g = \begin{bmatrix} \boxed{\bar{A}} & & & & & & \bigcirc \\ & \boxed{0} & & & & & \\ & & \boxed{0} & & & & \\ & & & \boxed{0} & & & \\ & * & & & \boxed{0} & & \\ & & & & & \boxed{0} & \end{bmatrix} .$$

Let

$$X_{\ell}^T A Z_{\ell} = \begin{bmatrix} \bar{A} & A_{\ell+1,\ell} & \dots & A_{\ell+1,1} \\ A_{\ell,\ell+1} & \dots & \dots & A_{\ell,1} \\ \vdots & & & \vdots \\ A_{1,\ell+1} & \dots & \dots & A_{11} \end{bmatrix}, \quad X_{\ell}^T B = \begin{bmatrix} B_{\ell+1} \\ B_{\ell} \\ \vdots \\ B_1 \end{bmatrix}.$$

Here

$$A_{ij} \in \mathbb{R}^{g_i \times g_j} \quad i = 1, \dots, \ell; \quad j = 1, \dots, \ell$$

$$A_{i,\ell+1} \in \mathbb{R}^{g_i \times r} \quad i = 1, \dots, \ell$$

$$A_{\ell+1,j} \in \mathbb{R}^{r \times g_j} \quad j = 1, \dots, \ell$$

$$\bar{A} \in \mathbb{R}^{r \times r}$$

where

$$r = n - \sum_{i=1}^{\ell} g_i$$

and

$$B_i \in \mathbb{R}^{g_i \times m} \quad i = 1, \dots, \ell$$

$$B_{\ell+1} \in \mathbb{R}^{r \times m}.$$

Then we have

$$\begin{bmatrix} B_{\ell+1} \\ B_{\ell} \\ \vdots \\ B_i \end{bmatrix} F_{ig} = - \begin{bmatrix} A_{\ell+1,i} \\ A_{\ell,i} \\ \vdots \\ A_{i,i} \end{bmatrix}, \quad i = 1, \dots, \ell$$

which shows that the construction of $F_g = [0, F_{\ell,g}, \dots, F_{1,g}]$ merely consists of solving a set of linear equations.

Therefore this algorithm has the same numerical behaviour as the algorithms in [2] and [3]. This shows that the algorithm may be considered to be a numerically reliable algorithm.

Formally this generalized deadbeat control algorithm is not numerically stable (backward stability). It can be proved, as in [3], that the feedback matrix F is an exact solution to the generalized deadbeat control problem for

$$(E + \delta E, A + \delta A, B + \delta B)$$

where

$$\|\delta E, \delta A, \delta B\| = \psi(\varepsilon) \cdot \varphi(\|E\|, \|A\|, \|B\|, \|F\|) .$$

Here $\psi(\varepsilon)$ is of the order of the relative machine precision ε and φ is a bilinear function in $\|E\|, \|A\|, \|B\|$ and $\|F\|$. ($\| \cdot \|$ is the Frobenius norm).

It can also be proved that, analogously to [4], that there exists a perturbation $\delta P(\lambda)$ such that F is an exact solution to the generalized deadbeat control problem for $(E + \delta E, A + \delta A, B + \delta B)$ where this latter generalized system is obtained from $P(\lambda) + \delta P(\lambda)$ in the same way as (E, A, B) is obtained from $P(\lambda)$ (this perturbation has the same structure as (E, A, B)).

In (8) we obtain a polynomial matrix $Q(\lambda)$ such that there exists a perturbation $\delta Q(\lambda)$ having the property that

$$\begin{bmatrix} P(\lambda) + \delta P(\lambda) \\ Q(\lambda) + \delta Q(\lambda) \end{bmatrix}$$

is exactly unimodular. The perturbations $\delta P(\lambda), \delta Q(\lambda)$ consist of polynomials whose coefficients are of the order of ε . However, these coefficients are also

functions of $\|F\|$.

The existence of $\delta Q(\lambda)$ is proved using forward stability analysis and $\delta P(\lambda)$ is obtained using a backward stability argument.

Details of the proof will be omitted.

Observe that the choice of the identity matrix I in

$$\begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix}$$

is an arbitrary one. We could have chosen any regular matrix D instead because

$$\begin{bmatrix} E - \lambda A & \lambda B \\ DF & D \end{bmatrix}$$

also is unimodular if F is a solution to the generalized deadbeat control problem.

The choice of a "good" D and the exploitation of the (sometimes existing) freedom for F in order to obtain a "good" unimodular matrix

$$\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$$

is a point of current research.

Examples

The algorithm has been used to compute $Q(\lambda)$ for various matrices $P(\lambda)$. Here the coefficients of $P(\lambda)$ have been chosen uniformly distributed in $[-1,1]$. After having computed $Q(\lambda)$ the determinant of $\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$ has been computed for

$\lambda = 0, 0.1, 0.2, \dots, 0.9$. Let d denote the number of digits in the determinant of the computed unimodular matrix that does not depend on λ . The number of rows of $P(\lambda)$ is p , the number of columns of $P(\lambda)$ is q and the degree of $P(\lambda)$ is n .

The following table shows the "unimodularity" of the matrix $\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$ for various choices of p, q, n . If d is large then the matrix is highly unimodular.

<u>p</u>	<u>q</u>	<u>n</u>	<u>d</u>
1	7	5	11
3	7	2	9
6	7	5	8
3	7	8	9
1	15	25	9
10	25	5	9
1	2	25	8
1	2	35	7
20	30	2	8

unimodularity of $\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$.

References

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