Cluster-size decay in supercritical kernel-based spatial random graphs

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CLUSTER-SIZE DECAY IN SUPERCRITICAL KERNEL-BASED SPATIAL RANDOM GRAPHS

JOOST JORRITSMA\textsuperscript{1}, JÚLIA KOMJÁTHY\textsuperscript{2}, AND DIETER MITSCHE\textsuperscript{3}

ABSTRACT. We consider a large class of spatially-embedded random graphs that includes among others long-range percolation, continuum scale-free percolation and the age-dependent random connection model. We assume that the parameters are such that there is an infinite component. We identify the stretch-exponent $\zeta \in (0, 1)$ of the subexponential decay of the cluster-size distribution. That is, with $|C(0)|$ denoting the number of vertices in the component of the vertex at $0 \in \mathbb{R}^d$, we prove

$$\mathbb{P}(k \leq |C(0)| < \infty) = \exp \left(-k^{\zeta-o(1)}\right), \quad \text{as } k \to \infty.$$ 

The value of $\zeta$ undergoes several phase transitions with respect to three main model parameters: the Euclidean dimension $d$, the power-law tail exponent $\tau$ of the degree distribution and a long-range parameter $\alpha$ governing the presence of long edges in Euclidean space. In this paper we present the proof for the region in the phase diagram where the model is a generalization of continuum scale-free percolation and/or hyperbolic random graphs: $\zeta$ in this regime depends both on $\tau, \alpha$. We also prove that the second-largest component in a box of volume $n$ is of size $\Theta((\log(n))^{1/\zeta-o(1)})$ with high probability. We develop a deterministic algorithm as a new methodology: we allocate to each partially revealed cluster $C$ a spatial set $K(C)$ with volume linear in the number of vertices in $C$, so that not-yet-revealed vertices with spatial location in $K(C)$ have good connection probability to $C$. This algorithm makes it possible to prove that too large components $C$ – regardless of the spatial configuration of its vertices – cannot exist outside of the giant component, enabling us to treat both de-localized and locally dense components in space.

Keywords: Cluster-size distribution, second-largest component, spatial random graphs, scale-free networks. 
MSC Class: 05C80, 60K35.

1. Introduction

In this paper we study component sizes in a large class of spatially embedded random graph models. For nearest neighbor Bernoulli percolation on $\mathbb{Z}^d$ \cite{8}, it is a result of a sequence of works \cite{2, 4, 10, 29, 44, 51} that in a supercritical model, i.e., for $p > p_c(\mathbb{Z}^d)$ – the critical percolation probability on $\mathbb{Z}^d$ – and writing $|C(0)|$ for the number of vertices in the cluster containing the origin,

$$\mathbb{P}(k \leq |C(0)| < \infty) = \exp \left(-\Theta(k^{\zeta})\right), \quad (1.1)$$

with $\zeta = (d-1)/d$. Intuitively, the stretched exponential decay with exponent $(d-1)/d$ emanates from the fact that all the $\Omega(k^{(d-1)/d})$ edges on the boundary of a cluster $C$ with $|C| \geq k$ need to be absent: the tail decay in (1.1) is determined by surface tension. More recently, these results have been extended to Bernoulli percolation on general classes of transitive graphs \cite{12, 58}.

In the present paper and the accompanying paper \cite{43}, we consider $\mathbb{P}(k \leq |C(0)| < \infty)$ for a large class of spatial random graph models where the degree distribution and/or the edge-length distribution have heavy tails. We identify when this structural inhomogeneity changes the surface-tension driven behavior of finite clusters: in those cases, the cluster-size decay in (1.1) is still stretched exponential, but the exponent $\zeta$ changes. Its value (together with our proof techniques) reflects the structure of the infinite/giant component in spatial graph models: it describes the size and structure of a “backbone” decorated with “traps” – almost isolated peninsulas attached to a well-connected component. This topological description is of independent interest, e.g., it affects the behavior of random walk \cite{14}.

Our starting points are continuum and classical long-range percolation (CLRP and LRP) \cite{3, 63}, examples of models where the edge-length distribution has a heavy tail, while the degree distribution is light-tailed. Relating to the distribution of small clusters in supercritical settings as in (1.1), the authors of \cite{13} gave a polylogarithmic upper bound on the size of the second-largest component $|C_n^{(2)}|$ for some regimes of long-range percolation in a box of volume $n$, with unidentified exponent, which is the

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only known result in this direction. In long-range percolation, each potential edge \( \{u, v\} \in \mathbb{Z}^d \times \mathbb{Z}^d \) is independently present with probability proportional to \( \beta^{\alpha \|u-v\|^{-d}} \) for some \( \alpha, \beta > 0 \). Our combined results (below, and [42, 43]) show that, under the additional assumption that \( \beta \) is sufficiently large if \( (d-1)/d > 2 - \alpha \),

\[
\mathbb{P}(k \leq |C(0)| < \infty) = \exp \left(-k^{\max\{2-\alpha, (d-1)/d\}+\omega(1)}\right), \quad \text{and} \quad |C_n^{(2)}| = (\log n)^{1/\max\{2-\alpha, (d-1)/d\}+\omega(1)}.
\]

Our main focus is to prove (1.1) for models where both the degree- and the edge-length distribution are heavy-tailed. Recently, several such spatial random graph models gained significant interest: scale-free percolation (SFP) [16], geometric inhomogeneous random graphs (GIRG) [7], continuum scale-free percolation (CSFP) [17], hyperbolic random graphs (HRG) [50], the ultra-small scale-free geometric network [65]; the scale-free Gilbert model (SGM) [32], the Poisson Boolean model with random radii [22], the age- and the weight-dependent random connection models (ARCM) [24, 26]. Concerning sizes of small clusters in these models, the only known result concerns the asymptotic size of the second-largest component of HRGs [46]. Among the more classical models, for random geometric graphs and scale-free percolation (SFP) [17], hyperbolic random graphs (HRG) [50], the ultra-small scale-free geometric network [65]; the scale-free Gilbert model (SGM) [32], the Poisson Boolean model with random radii [22], the age- and the weight-dependent random connection models (ARCM) [24, 26]. Concerning sizes of small clusters in these models, the only known result concerns the asymptotic size of the second-largest component of HRGs [46]. Among the more classical models, for random geometric graphs and Bernoulli percolation on them, the papers [53, 62] identify \( |C_n^{(2)}| \).

In this paper we prove the stretched exponential decay and identify the values of \( \zeta \) in (1.1) for (regions of the parameter space of) supercritical continuum scale-free percolation, geometric inhomogeneous random graphs, and hyperbolic random graphs (CSFP, GIRG, HRG). We introduce a novel proof technique, the cover expansion, and give a quantitative backbone-and-traps description of the giant component which is novel for these models. Additionally, we unfold the general relation between the cluster-size decay and the size of the second-largest component. In [43], we establish (1.1) for the scale-free Gilbert model, for the age-dependent random connection model, for (continuum) long-range percolation, and other regions of the parameter space of CSFP, GIRG, and HRG. The results here and in the accompanying paper [43] together also determine the speed of the lower tail of large deviations for the size of the giant component \( |C_n^{(1)}| \) for all these models. With the same \( \zeta \) as the one in the cluster-size decay, for all sufficiently small \( \rho > 0 \),

\[
\mathbb{P}(|C_n^{(1)}| < \rho n) = \exp \left(-n^{\zeta(1+\omega(1))}\right). \quad (1.2)
\]

Here we present the lower bound for (1.2), and in [43] we present the upper bound.

We consider a framework unifying these models, which we call Kernel-Based Spatial Random Graphs (KSRG). Beyond the above models, our framework also contains classical models: nearest-neighbor bond percolation on \( \mathbb{Z}^d \) (NNP), random geometric graphs (RGG) [56, 60]. The KSRG framework was essentially present in [49], and then refined in [26]. The following general definition introduces the relevant parameters of the models that determine a phase diagram of the exponent \( \zeta \) in (1.1).

1.1. Kernel-based spatial random graphs (KSRG). A random graph model in the KSRG class requires four ingredients to be specified:

- a vertex set \( V \): any stationary ergodic point process embedded in a metric space with norm \( \| \cdot \| \); typically a Poisson point process (PPP) \( \Xi \) on \( \mathbb{R}^d \) or a lattice \( \mathbb{Z}^d \);
- a mark distribution \( W \) for vertices, the mark \( W_u \) of vertex \( u \) describing its ability to form connections;
- a profile function \( \rho \) describing the influence of spatial distance on the probability of an edge;
- a kernel \( \kappa \) that describes how marks rescale the spatial distance between vertices.

Conditionally on the vertex set and on the marks \( W_u = w_u, W_v = w_v \), two vertices \( u \) and \( v \) are connected by an edge independently with a probability given by the connectivity function, which is the composition of the kernel and the profile, and a parameter \( \beta > 0 \) that controls the edge density (high values of \( \beta \) correspond to higher edge density):

\[
\text{Prob}(\text{vertex } u \text{ is connected to vertex } v \text{ by an edge}) = \rho(\beta \kappa(w_u, w_v)\|u-v\|^{-d}), \quad (1.3)
\]

where \( \text{Prob} \) is a measure on \( V \times V \), conditionally on \( V \) and \( \{w_v\}_{v \in V} \). These components together define a random graph model \( G(V, E) \), where \( E \) is a random set of edges.

The role of marks. Throughout the paper, we will assume that the marks are power-law distributed with tail-exponent \( \tau - 1 \) for some \( \tau > 1 \), that is,

\[
\mathbb{P}(W \geq x) \sim x^{-(\tau-1)}. \quad (1.4)
\]

We shall see that the mark distribution regulates the tail of the degree-distribution. Marks have different names in the literature: in geometric inhomogeneous random graphs [7] and scale-free percolation
they are called ‘vertex-weight’ or ‘fitness’, while for hyperbolic random graphs [50] the mark corresponds to the ‘radius’ of the location of a vertex in the native representation of the hyperbolic plane. For the age-dependent random connection model [24] the mark corresponds to the rescaled ‘age’ of a vertex. For nearest-neighbor percolation, random geometric graphs and long-range percolation marks are not used for the formation of the graph: in this case we set \( W \equiv 1 \), and \( \tau = \infty \).

**The role of the profile function.** The profile function is generally either threshold or long-range type in the literature, and two typical choices are as follows:

\[
\varrho(t) = \mathbb{1}_{[|1/R,\infty|]}(t), \quad \varrho(t) \sim t^\alpha,
\]

(1.5)

where the latter is meant to hold close as \( t \) tends to 0. Since the inverse distance \( 1/\|u - v\|^d \) is used in (1.3), \( \varrho \) corresponds to connecting all pairs of vertices at distance at most \( R^{1/d} \) by an edge, while \( \varrho \sim \) originating from long-range percolation \([53]\) allows for long-range edges. The parameter \( \alpha > 1 \) in (1.5) calibrates this effect: the smaller \( \alpha \), the more long-range edges, and the condition \( \alpha > 1 \) ensures a locally finite graph. With slight abuse of notation we say that \( \alpha = \infty \) when considering a threshold profile. Both parameters \( \alpha, \tau \) regulate exceptional connectedness, but differently: \( \alpha \) in (1.5) controls the presence of exceptionally long edges, whereas \( \tau \) in (1.4) controls exceptionally high-degree vertices.

**The role of the kernel function.** The kernel function \( \kappa \) in (1.3) is a symmetric non-negative function that rescales the spatial distance so that high-mark vertices experience ‘shrinkage of spatial distance’ and hence, the connection probability increases towards high-mark vertices. Models in the literature commonly use the trivial, product, max, sum, min, and preferential attachment (PA) kernels, the last one mimicking the spatial preferential attachment model [11, 20], i.e., with \( \tau > 2 \) as in (1.4).

\[
\kappa_{\text{triv}}(x, y) \equiv 1, \quad \kappa_{\text{prod}}(x, y) = xy, \quad \kappa_{\text{max}}(x, y) = \max(x, y), \quad \kappa_{\text{sum}}(x, y) = x + y,
\]

\[
\kappa_{\text{min}} = \min(x, y)\tau, \quad \kappa_{\text{pa}}(x, y) = \max(x, y)\min(x, y)^{\tau-2}.
\]

(1.6)

This parametrization of the non-trivial kernels yields for the corresponding models a power-law degree distribution with the same tail exponent as the marks [29], establishing the scale-free property often desired in complex network modeling [6, 13, 59, 58]. The kernel \( \kappa \) controls the correlation between degrees of two endpoints of an edge, called assortativity, which, as we shall see, will also have an important role in determining the value of \( \zeta \) in (1.1), see Section 1.3. The trivial kernel \( \kappa_{\text{triv}} \) includes classical models as bond percolation on \( \mathbb{Z}^d \), the random geometric graph, and (continuum) long-range percolation to the class of KSRGs.

**Other graph properties, further related work.** We refer to Table 1 for an overview of models and their parameters. Many graph properties and processes on graphs have been studied on KSRGs with non-trivial kernels in the past years: the degree distribution in [7, 16, 24, 32, 35, 55, 65]; graph distances in the giant component in [6, 16, 17, 25, 32, 63]; first-passage percolation on the giant component in [37, 48, 49]; the random walk on the giant component [11, 26, 31, 45]; the contact process on the giant component [23, 54]; the clustering coefficient in [7, 15, 21, 36, 64]; bootstrap percolation in [9, 47], the existence of subcritical and supercritical phases in [16, 17, 27, 28], and subcritical phase in the more general model [39] that allows for dependencies between edges.

### 1.2. Four regimes of \( \zeta \) in the phase diagram

In KSRGs in general, one can determine the value of the exponent \( \zeta \) of the cluster-size decay via the following back-of-the-envelope calculation, that is based on the lower bound and aims to find a spatially localized component of size \( \Theta(k) \). Consider two neighboring boxes \( \Lambda_k^{(1)}, \Lambda_k^{(2)} \) of volume \( k^d \) in \( \mathbb{R}^d \), and compute, as a function of \( k \), the expected number of edges \( \mathbb{E}[|\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)})|] \) between vertices in these boxes, and also the expected number of vertices in \( \Lambda_k^{(1)} \) having at least one edge towards a vertex in \( \Lambda_k^{(2)} \), \( \mathbb{E}[|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|] \). If these two quantities are the same order, and are both \( \Theta(k^\zeta) \), then the probability that none of the potential edges is present is \( \exp(-\Theta(k^\zeta)) \). The non-presence of all these edges is necessary for a large isolated cluster of size \( \Theta(k) \) to be confined to its original box. If, however, we find \( \mathbb{E}[|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|] = o(\mathbb{E}[|\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)})|]) \), and the former quantity is \( \Theta(k^\zeta) \), then typically there are a few high-mark vertices that have many edges to the neighboring box. The most likely way to have a component of size \( \Theta(k) \) is then to simply not have any of these vertices present in the two boxes, again happening with probability \( \exp(-\Theta(k^\zeta)) \). Clearly, large finite clusters can be non-localized, so the rigorous version of this argument only gives lower bounds. Nevertheless, it describes the most likely way of connectivity towards and within the infinite/giant component and gives a conjecture for the value of \( \zeta \).
Based on this intuition, we distinguish four possible types of connectivity that describe how neighboring boxes are connected. We call the type of connectivity that produces the largest contribution to $\mathbb{E}[|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|]$ the dominant type. Changes of the dominant type give the phase diagram of $\zeta$ in the parameter space. We visualize these phases in Figure 1a for models using $\kappa_{\text{prod}}, \kappa_{\text{pa}},$ or $\kappa_{\text{max}}$. All arguments below are meant as $k \to \infty$. We say that the model shows dominantly

(II) low-low-type connectivity if the main contribution to $\mathbb{E}[|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|]$ is coming from pairs of low-mark vertices (vertices with mark $\Theta(1)$) at distance $\Theta(k^{1/d})$ from each other. We find

$$E(|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|) = \Theta(E(\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)}))) = \Theta(k^{\zeta_{\text{ll}}})$$

where

$$\zeta_{\text{ll}} := 2 - \alpha,$$

(1.7)

which neither depends on $\kappa$ nor on $\tau$ since low-mark vertices are not affected by those. Models with parameters falling in this regime behave qualitatively similarly to long-range percolation. We present proofs for parameter regimes where $\zeta_{\text{ll}}$ is dominant in [43]. In KSRGs with kernels in (1.6) this regime may occur if both $\alpha < \tau - 1$ and $\alpha < 2$.

(lh) low-high-type connectivity if $E(\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)})) \gg E(|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|)$ and the main contributions are coming from edges between a high-mark and a low-mark vertex ($\Theta(1)$) at distance $\Theta(k^{1/d})$. A vertex has high-mark here when its mark is $\Omega(k^{\gamma_{\text{hh}}})$, where

$$\gamma_{\text{lh}} := 1 - 1/\alpha$$

is the smallest exponent such that a linear proportion of vertices of mark $\Omega(k^{\gamma_{\text{hh}}})$ in $\Lambda_k^{(1)}$ is connected by an edge to a low-mark vertex in $\Lambda_k^{(2)}$. Vertices of mark $\Omega(k^{\gamma_{\text{hh}}})$ have to be absent to find a large but finite component confined to $\Lambda_k^{(1)}$, and since $E(|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|) = \Theta(k^{1-\gamma_{\text{hh}}(\tau-1)})$, we obtain

$$\zeta_{\text{lh}} := 1 - \gamma_{\text{lh}}(\tau - 1) = 1 - (\tau - 1)(1/\alpha).$$

(1.8)

We prove in our accompanying paper [43] that this type of behavior is dominant in KSRGs with kernels $\kappa_{\text{max}}, \kappa_{\text{sum}},$ and $\kappa_{\text{pa}}$ (e.g. for the age-dependent random connection model [24] and scale-free Gilbert model [32]) when both $\alpha \in (\tau - 1, (\tau - 1)/(\tau - 2))$, and $\tau \in (2, 3)$ holds.

(hh) high-high-type connectivity if $E(\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)})) \gg E(|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|)$, and the main contributions are coming from edges between high-mark vertices at distance $\Theta(k^{1/d})$. A vertex has high mark here when its mark is $\Omega(k^{\gamma_{\text{hh}}})$, where $\gamma_{\text{hh}}$ is the smallest exponent such that a linear proportion of vertices of mark $\Omega(k^{\gamma_{\text{hh}}})$ in $\Lambda_k^{(1)}$ is connected by an edge to a high-mark vertex in $\Lambda_k^{(2)}$. This leads in expectation to $\Theta(k^{\gamma_{\text{hh}}})$ many high-mark vertices in $\Lambda_k^{(1)}$ and $\Lambda_k^{(2)}$ (which have to be absent to find a large but finite component), where $\zeta_{\text{hh}} := 1 - \gamma_{\text{hh}}(\tau - 1)$. The formula of $\gamma_{\text{hh}}$ depends on the choice of the kernel $\kappa$, see [1.11d] below. In this paper we focus on this phase. We show that this type of connectivity can be dominant in KSRGs with kernel $\kappa_{\text{prod}}$: (continuum) scale-free percolation (SFP) [16, 17], geometric inhomogeneous random graphs (GIRG) [17] and hyperbolic random graphs (HRG) [30] when $\tau < 3$ and $\alpha \geq 1 - \tau$.

(nn) nearest-neighbor-type connectivity if $E(\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)})) = \Theta(E(|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|))$, and the main contributions are coming from edges of length $\Theta(1)$ between low-mark vertices close to the shared boundary of the two boxes. The expected number of such edges and vertices is $\Theta(k^{\zeta_{\text{nn}}})$ where

$$\zeta_{\text{nn}} := (d - 1)/d.$$  

(1.9)

KSRGs with kernel $\kappa_{\text{triv}}$ and threshold profile show this type of connectivity behavior in their entire parameter space, e.g., random geometric graphs and nearest-neighbor percolation on $\mathbb{Z}^d$. KSRGs with either non-trivial kernel or long-range profile also show this phase in a region of the parameter space, e.g. long-range percolation when $\alpha > 1 + 1/d$. The accompanying paper [42] treats sub-regions of this phase in the $(\alpha, \tau)$ parameter-plane.

In general, we conjecture that for any model belonging to the KSRG framework, the value of $\zeta$ in [1.11d] is determined by whichever connectivity type yields the largest contribution to $E(\mathcal{E}(\Lambda_k^{(1)}, \Lambda_k^{(2)}))$ and $E(|\mathcal{V}(\Lambda_k^{(1)} \to \Lambda_k^{(2)})|)$. Formally, we may compute

$$\max\{\zeta_{\text{ll}}, \zeta_{\text{lh}}, \zeta_{\text{hh}}, \zeta_{\text{nn}}\} =: \zeta,$$

yielding the conjectured exponent in [1.11]. See also Figure 1a.
1.3. An interpolating model (i-KSRG). We will study models belonging to the KSRG class in a unified manner, using a parameterized kernel that contains the kernels in (1.6) as special cases. We call this the interpolation kernel, and the corresponding random graph models interpolation KSRGs. Independently of our work, this kernel appeared recently in [27, 55] and will be used in [30]. Formally, we define the kernel as

$$\kappa_{\sigma_1, \sigma_2} := \max(x, y)^{\sigma_1} \min(x, y)^{\sigma_2}; \quad \text{and} \quad \kappa_{1, \sigma} := \max(x, y)^{\min(x, y)^{\sigma}},$$

(1.10)

for some $$\sigma_1 \geq 0$$ and $$\sigma_2 = \sigma \in \mathbb{R}$$. Non-negativity of $$\sigma_1$$ ensures the natural requirement that a higher mark corresponds to a higher expected degree. In models with kernels $$\kappa_{\max}, \kappa_{\text{prod}}, \kappa_{\text{sum}}$$ in (1.6) we set $$\sigma_1 = 1$$, while for $$\kappa_{\min}$$ we set $$\sigma_1 = 0$$ and $$\sigma_2 = 1$$. The kernel $$\kappa_{\text{sum}}$$ cannot be directly expressed using the interpolation kernel. However, since $$\max(x, y) \leq x + y \leq 2 \max(x, y)$$, and all known phase transitions regarding macroscopic behavior are the same for models with $$\kappa_{\text{sum}}$$ and with $$\kappa_{\text{max}}$$, we consider this a minor restriction. By appropriately changing the mark distribution, unless $$\sigma_1 = 0$$, any KSRG with kernel $$\kappa_{\sigma_1, \sigma_2}$$ can be re-parameterized to have $$\sigma_1 = 1$$. Models with $$\sigma_1 = 0$$ can be approximated with the kernel $$\kappa_{1, \sigma}$$ [41]. The parameter $$\sigma$$ affects the tail exponent of the degree distribution, but only when $$\sigma > \tau - 1$$ [55].

Hence, from now on we fix $$\sigma_1 = 1$$ and set $$\sigma := \sigma_2$$, i.e., we restrict to the second formula in (1.10). Any KSRG model with $$\kappa_{\text{equiv}}$$ has the same connection probability as a model with $$\kappa_{1, \sigma}$$ and all marks identical to 1, in this case we artificially set $$\tau := \infty$$, see (1.3) and below (1.4). The parameter $$\sigma$$ affects assortativity: a larger value of $$\sigma$$ increases $$\min(x, y)^{\sigma}$$ in (1.10), and in turn the connection probability in (1.3). In a natural coupling of these models using common edge-variables, edges incident to at least one low-mark vertex are barely affected by changing $$\sigma$$. However, edges between two high-mark vertices are created rapidly if $$\sigma$$ increases. Intuitively, this should imply that $$\zeta$$ depends on $$\sigma$$ only when the high-high-type connectivity is dominant. We prove this, and find

$$\gamma_{\text{hh}} = \begin{cases} \frac{1 - 1/\alpha}{\sigma + 1 - (\tau - 1)/\alpha}, & \text{if } \tau \leq 2 + \sigma, \\ \frac{1}{\sigma + 1}, & \text{if } \tau > 2 + \sigma, \end{cases}$$

(1.11)

leading to

$$\zeta_{\text{hh}} = 1 - \gamma_{\text{hh}}(\tau - 1) = \begin{cases} \frac{\sigma + 2 - \tau}{\sigma + 1 - (\tau - 1)/\alpha}, & \text{if } \tau \leq 2 + \sigma, \\ \frac{\sigma + 2 - \tau}{\sigma + 1}, & \text{if } \tau > 2 + \sigma. \end{cases}$$

(1.12)

The interpolation kernel allows for unified proof techniques, and shows how the possible dominant regimes (ll, lh, hh, and nn) fade into each other as the interpolation parameter $$\sigma$$ changes gradually, see Figure 1. We then state our conjecture, visualized in Figure 1, see also Table 1.

We will never use the formula for $$\zeta_{\text{hh}}$$ when it is negative: then, some other connectivity type is dominant.

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</tr>
<tr>
<td>Hyperbolic random graph [50]</td>
<td>PPP</td>
<td>(\kappa_{\text{equiv}})</td>
<td>(\varrho_{\alpha})</td>
<td>(\zeta_{\text{hh}})</td>
</tr>
<tr>
<td>Age-dependent random connection model [24]</td>
<td>PPP</td>
<td>(\kappa_{\text{prod}}, \kappa_{1,\tau-2})</td>
<td>(\varrho_{\alpha}, \varrho_{\text{thr}})</td>
<td>(\max{\zeta_{\text{hh}}, \zeta_{\text{ll}}, \zeta_{\text{nn}}})</td>
</tr>
<tr>
<td>Scale-free Gilbert graph [32]</td>
<td>PPP</td>
<td>(\kappa_{\text{equiv}}, \kappa_{1,0})</td>
<td>(\varrho_{\alpha})</td>
<td>(\zeta_{\text{nn}})</td>
</tr>
<tr>
<td>Interpolating KSRG</td>
<td>PPP</td>
<td>(\kappa_{\text{equiv}}, \kappa_{1,\sigma})</td>
<td>(\varrho_{\alpha}, \varrho_{\text{thr}})</td>
<td>(\max{\zeta_{\text{ll}}, \zeta_{\text{hh}}, \zeta_{\text{nn}}})</td>
</tr>
</tbody>
</table>

Table 1. Models belonging to the KSRG framework, their vertex sets, kernels, profiles, and the (conjectured) value of their cluster-size decay exponent $$\zeta$$. Horizontal lines separate models with different kernels.
Conjecture 1.1. We conjecture that for supercritical KSRGs in $\mathbb{R}^d$ with kernel $\kappa_{1,\sigma}$, for all $(\alpha, \tau) \in (1, \infty) \times (2, \infty)$, (1.1) and (1.2) hold with $\zeta = \max\{\zeta_l, \zeta_h, \zeta_{hh}, \zeta_{nn}\}$ up to $o(1)$-terms whenever $\zeta > 0$. When $\alpha = \infty$ and/or $\tau = \infty$, the value of $\zeta$ is obtained by taking the corresponding limit of $\zeta = \zeta(\alpha, \tau)$.

(a) Phase-diagrams of $\zeta = \zeta(\tau, \alpha)$ for models with kernels $\kappa_{\text{prod}}$ and $\kappa_{\text{pa}}$ or $\kappa_{\text{max}}$, plotted as a function of $1/(\tau - 1)$ and $1/\alpha$. The $y$-axis (i.e., $1/(\tau - 1) = 0$) also describes the phase diagram of (continuum) long-range percolation that has kernel $\kappa_{\text{triv}}$, while the models on the $x$-axis ($1/\alpha = 0$) coincide with models using a threshold profile function in (1.5). When $1/\alpha > 1$ or $1/(\tau - 1) > 1$, then $G_{\infty}$ is connected and each vertex has infinite degree almost surely \cite{31}. A white color within the square means that the model is subcritical for each value $p, \beta$ in (1.3) \cite{27}.

(b) Phase-diagrams of $\zeta = \zeta(\sigma, \tau)$ for fixed values of $\alpha$ in (1.5), plotted as a function of $1/(\tau - 1)$ on the $x$-axis and $\sigma/(\tau - 1)$ on the $y$-axis. The identity line $y = x$ corresponds to models using kernel $\kappa_{\text{prod}} \equiv \kappa_{1,1}$, the $x$-axis to models using $\kappa_{\text{max}} \equiv \kappa_{1,0}$ and the cross-diagonal $x + y = 1$ to models using $\kappa_{\text{pa}} \equiv \kappa_{1,\tau - 2}$. The origin captures models with $\kappa_{\text{triv}} \equiv \kappa_{0,0}$. Observe that $\zeta_{lh}$ (blue) is never dominant above the diagonal $y \geq x$ (equivalently, $\sigma \geq 1$), while $\zeta_{hh}$ (red) is never dominant below the cross-diagonal $x + y = 1$ (equivalently, $\sigma \leq \tau - 2$). In the quadrant $x + y \geq 1, y \leq x$ all four exponents ‘compete’ for dominance.

Figure 1. Phase diagrams of the (conjectured) cluster-size decay for kernel-based spatial random graphs. Theorem 2.2 proves the upper bound in the red regions, and the lower bounds above the $x + y \geq 1$ line on Figure 1b for all four colors simultaneously, with logarithmic correction terms on phase boundary lines.

1.4. Three related papers. The results in this paper, combined with \cite{43}, prove Conjecture 1.1 for i-KSRGs on Poisson point processes (PPP) whenever $\max\{\zeta_l, \zeta_h, \zeta_{hh}\} \geq \zeta_{nn} = \frac{d-1}{d}$ and $\sigma \leq \tau - 1$ (the
non-green regions in Figure 1. When \( \max\{\zeta_l, \zeta_h\} > \zeta_h \) (the blue and yellow regions in Figure 1), or when the maximum is non-unique (on region boundaries), we have a \( o(1) \) error in the exponent \( \zeta \), while in the open red region our bound is sharp. Regarding the speed of the lower tail of large deviations (LTLD) for the size of the giant component (as in (1.2)), we prove Conjecture 1.1 up to a \( o(1) \) error whenever \( \max\{\zeta_l, \zeta_h, \zeta_{lh}\} > 0 \) and \( \sigma \leq \tau - 1 \) (all parameters for which the model can be supercritical in \( d = 1 \)). Thus, up to \( o(1) \)-errors in some regions, these results prove Conjecture 1.1 fully for locally finite i-KSRGs on PPPs when \( d = 1 \) and \( \sigma \leq \tau - 1 \). In dimension \( d \geq 2 \), the present paper and [43] leave open (only) the regime where \( \zeta_{un} \) is maximal, but we still obtain partial results in this regime (see the following paragraphs). In [42], we prove Conjecture 1.1 fully for long-range percolation on \( \mathbb{Z}^d \) whenever the edge density is sufficiently high, i.e., also when \( \zeta_{un} \) is maximal.

Upper bounds. In this paper we prove the upper bound on the cluster-size decay and \( |C_n^{(2)}| \) for \( \zeta = \zeta_h \), which is sharp in the red regions in Figure 1 and still non-trivial whenever \( \tau < 2 + \sigma \) (above the cross-diagonal in Figure 1b), using a “backbone construction”. In [43] we show upper bounds on the speed of LTLD for the size of the giant component using a renormalization scheme driven by vertex-marks. This yields an upper bound that matches the conjectured exponent \( \zeta \) in (1.2) up to \( o(1) \) when \( \max\{\zeta_l, \zeta_h, \zeta_{lh}\} > 0 \). In [43] we use this LTLD-type result to construct a backbone (whose construction is different from the construction in this paper). Using that, we prove the upper bounds on the cluster-size decay and \( |C_n^{(2)}| \) when \( \zeta \neq \zeta_{un} \), obtaining up to \( o(1) \)-matching upper bounds for \( \zeta \in \{\zeta_l, \zeta_u\} \) in Figure 1. Only the \( h \)-regime requires \( \sigma \leq \tau - 1 \). All upper bounds for \( \zeta = \zeta_l \) extend to i-KSRGs where the vertex locations are given by \( \mathbb{Z}^d \). In [42] we show the upper bounds for long-range percolation on \( \mathbb{Z}^d \) when \( \zeta_{un} > \zeta_l \), assuming sufficiently high edge-density. We use combinatorial methods in [42], since the backbone construction fails for LRP when \( \zeta_{un} \) is dominant.

Lower bounds. This paper proves, for the whole parameter-space, the lower bound for the speed of LTLD for the size of the giant component in (1.2). LTLD is related to the lower bound on the cluster-size decay and \( |C_n^{(2)}| \), once we additionally prove that a box with appropriately restricted vertex-marks still contains a linear-sized component. Here, we prove this when \( \tau < 2 + \sigma \) using the backbone construction; the (different) backbone construction in [43] yields it for \( \tau \geq 2 + \sigma \). In [42], the high edge-density assumption ensures the existence of a giant in a box. The lower bounds do not require \( \sigma \leq \tau - 1 \), extend to i-KSRGs on \( \mathbb{Z}^d \), and do not contain \( o(1) \)-error terms.

The next section makes definitions and our findings in this paper formal.

2. Model and main results

We start by defining kernel-based random graphs, similar to the definition in [26], but specifically for the interpolating kernel \( \kappa_{1,\sigma} \) in (1.10). We denote by \( ||x - y|| \) the Euclidean distance between \( x, y \in \mathbb{R}^d \), and for \( x \in \mathbb{R}^d, s > 0 \), we denote a box of volume \( s \) in \( \mathbb{R}^d \) centered at \( x \) by

\[
\Lambda_s(x) = \Lambda(x, s) := x + [-s^{1/d}/2, s^{1/d}/2]^d
\]

If \( x = 0 \), we write \( \Lambda_s \). For a discrete set \( V \) we write \( (V)_2 := \{\{u, v\}, u \in V, v \in V, u \neq v\} \) for the list of unordered pairs in \( V \).

**Definition 2.1** (Interpolating kernel-based spatial random graph, i-KSRG). Fix \( \alpha > 1, \tau > 1, d \in \mathbb{N}, \sigma \in \mathbb{R} \). Let \( \Xi \) denote an inhomogeneous Poisson-point process on \( \mathbb{R}^d \times [1, \infty) \) with intensity measure

\[
\mu_\tau(dx \times dw) := \text{Leb} \otimes F_{W}(dw) := dx \times (\tau - 1)^{-\tau} dw
\]

For some \( p \in (0, 1], \beta > 0 \), we define the connectivity function with \( \kappa_{1,\sigma} \) from (1.10) as

\[
p\left((x_u, w_u), (x_v, w_v)\right) := \begin{cases} p \min \left\{ 1, \left( \frac{\beta_{\kappa_{1,\sigma}}(w_u, w_v)}{||x_v - x_u||^d} \right)^\alpha \right\}, & \text{if } \alpha < \infty, \\
p1 \left\{ \frac{\beta_{\kappa_{1,\sigma}}(w_u, w_v)}{||x_v - x_u||^d} \geq 1 \right\}, & \text{if } \alpha = \infty. 
\end{cases}
\]

Conditionally on \( \Xi \), the infinite random graph \( G_\infty = (V(G_\infty), E(G_\infty)) \) is given by \( V(G_\infty) = \Xi \) and for each pair \( u = (x_u, w_u), v = (x_v, w_v) \in \Xi \) the edge \( \{u, v\} \) is in \( E(G_\infty) \) with probability \( p\left((x_u, w_u), (x_v, w_v)\right) \); conditionally independently given \( \Xi \).
Write for \([a, b) \subseteq \mathbb{R}_+, \mathbb{E}[a, b) := \Xi \cap (\Lambda_n \times [a, b))\). For the finite induced subgraphs of \(\mathcal{G}_\infty\) on \(\Xi_n[a, b)\) and \(\Xi_n := \Xi_n[1, \infty)\), we respectively write
\[
\mathcal{G}_n[a, b) := (\mathcal{V}_n[a, b), \mathcal{E}_n[a, b)), \quad \mathcal{G}_n := (\mathcal{V}_n, \mathcal{E}_n). \tag{2.4}
\]
We call \(\Xi\) a realization of the vertex set. We denote the measure induced by the Palm version of \(\Xi\) where a vertex at location \(x\) is present (with unknown mark) by \(\mathbb{P}^x\).

Both the vertex set \(\mathcal{V}\) and the profile function \(\varrho\) in (2.3) can be replaced to obtain more general KSRGs. We expect that our results generalize to KSRGs with lattices as vertex sets, although some of our current proof techniques do not apply, since they make use of the independence property of PPPs. The projection of the vertex set \(\mathcal{V} = \Xi\) onto the spatial dimensions is a unit-intensity Poisson point process. The formula in (2.3) restricts to models with either a threshold-profile \((\alpha = \infty)\) or a specific long-range profile (cf. \(\varrho_n(t)\) in (1.5)), for readability. Our results readily extend to more general regularly varying profiles and fitness distributions, i.e., they allow slowly varying functions in (1.5) and (2.2), but then the exponent \(\zeta\) needs to be corrected by an additive \(o(1)\) term.

The interpolating KSRG model includes models with kernels \(\kappa_{\text{max}}, \kappa_{\text{pa}}, \kappa_{\text{prod}}\) as special cases. The parameter \(p\) governs Bernoulli percolation of the edges, and a higher \(\beta\) increases the edge density. While \(p, \beta\) do not affect \(\zeta\), they may influence supercriticality for certain values of \((\alpha, \tau, \sigma, d)\), see [10] [27] [28] on robustness of some KSRGs. See Section 1.1 above for the role of \((\alpha, \tau, \sigma, d)\).

**Main results.** We proceed to the main results of this paper. Define the *multiplicity of the maximum* of a finite set \(Z \subset \mathbb{R}\) as:
\[
m(Z) := m_Z := \sum_{\zeta \in Z} 1\{\zeta = \max(Z)\}. \tag{2.5}
\]
Recall \(\zeta_{\text{ll}}, \zeta_{\text{hh}}, \zeta_{\text{hh}}, \) and \(\zeta_{\text{mn}}\) from (1.7), (1.8), (1.12), and (1.9), respectively.

**Theorem 2.2** (Subexponential decay). Consider a supercritical \(i\)-KSRG model in Definition 2.1 with parameters \(\alpha > 1, \tau \in (2, 2 + \sigma)\), \(\sigma > 0\), and \(d \in \mathbb{N}\). Then there exists \(A > 1\) such that, for all \(k \geq 1\):

(i) For \(Z = \{\zeta_{\text{ll}}, \zeta_{\text{hh}}, \zeta_{\text{hh}}, \zeta_{\text{mn}}\}\) and \(n \in (Ak, \infty]\)
\[
\mathbb{P}^0(|C_n(0)| \geq k, 0 \notin C'_n(0)) \geq \exp\left(-A k^{\max(2)} \log^{m_Z - 1}(k)\right), \tag{2.6}
\]
(ii) if additionally \(\sigma \leq \tau - 1\), then for all \(n \in (Ak^{1 + \zeta_{\text{hh}}/\sigma}, \infty]\)
\[
\mathbb{P}^0(|C_n(0)| \geq k, 0 \notin C'_n(0)) \leq A \exp\left(-(1/A)k^{\zeta_{\text{hh}}}\right). \tag{2.7}
\]
The results also hold when \(\alpha = \infty\) by taking the appropriate limit in \(\zeta_{\text{hh}}(\alpha)\) in (1.12).

We will obtain the following statement as a corollary from Theorem 2.2.

**Corollary 2.3** (Law of large numbers for the size of the giant component). Consider a supercritical \(i\)-KSRG model in Definition 2.1 with parameters \(\alpha > 1, \tau \in (2, 2 + \sigma)\), \(\sigma \in (0, \tau - 1]\), and \(d \in \mathbb{N}\). Then,
\[
\frac{|C_n(1)|}{n} \xrightarrow{\mathbb{P}} \mathbb{P}(|C_\infty(0)| = \infty), \quad \text{as } n \to \infty. \tag{2.8}
\]

The analogue for the size of the second-largest component in \(\mathcal{G}_n\) is the following theorem.

**Theorem 2.4** (Second-largest component). Consider a supercritical \(i\)-KSRG model as in Definition 2.1 with parameters \(\alpha > 1, \tau \in (2, 2 + \sigma)\), \(\sigma \in (0, \tau - 1]\), and \(d \in \mathbb{N}\). Then the following hold:

(i) For \(Z = \{\zeta_{\text{ll}}, \zeta_{\text{hh}}, \zeta_{\text{hh}}, \zeta_{\text{mn}}\}\), there exist constants \(A, \delta, n_0 > 0\), such that for all \(n \in [1, \infty)\)
\[
\mathbb{P}\left(|C_n(2)| \geq (1/A) \left(\log(n)/((\log(n))^{m_Z - 1})\right)^{1/\max(2)}\right) \geq 1 - n^{-\delta}, \tag{2.9}
\]
(ii) If additionally \(\sigma \leq \tau - 1\), then for all \(n > 0\), there exists \(A > 0\) such that for all \(n \in [1, \infty)\)
\[
\mathbb{P}(|C_n(2)| \leq A \log^{1/\zeta_{\text{hh}}}(n)) \geq 1 - n^{-\delta}. \tag{2.10}
\]
The results also hold when \(\alpha = \infty\) by taking the appropriate limit in \(\zeta_{\text{hh}}(\alpha)\) in (1.12).

We make a few remarks about the two theorems. The assumption \(\tau \in (2, 2 + \sigma)\), (corresponding to \(x + y \geq 1\) on Figure 1) ensures that the model is supercritical for all \(p, \beta > 0\) in (2.3), see Proposition 5.14 below, implying that there exists a unique infinite component \(C_\infty(1)\). The condition \(\tau < 2 + \sigma\) also ensures \(\zeta_{\text{hh}} > 0\). In part (ii), the condition \(\sigma \leq \tau - 1\) is automatically satisfied for models with kernels \(\kappa_{1, \sigma}\) with \(\sigma \leq 1\), including \(\kappa_{\text{prod}} \equiv \kappa_{1, 1}\). because \(\tau > 2\). While we believe the restriction \(\sigma \leq \tau - 1\) is a
technical condition, it reflects the change of the tail exponent of the degree distribution at \( \sigma = \tau - 1 \): the contribution of higher-fitness vertices to the degree of a vertex is negligible compared to the total degree only when \( \sigma < \tau - 1 \). The assumption \( n > k^1+\zeta_{hh}/\alpha \) in Theorem 2.2(ii) is also a technical artifact of our proof, and can be relaxed to \( n > k^1+\varepsilon \) using results from [43]. The lower bound in (2.6) is the same as in Conjecture 1.1, here we prove it in the region \( \tau \in (2,2+\sigma) \), and in [43] when \( \tau \geq 2+\sigma \).

The next theorem shows that the stretch-exponent \( \zeta \) is also related to lower large deviations of the largest component. Contrary to the theorems above, it holds also for \( \tau \geq \sigma + 2 \).

**Theorem 2.5** (Speed of lower-tail large deviations of the giant). Consider a supercritical \( i \)-KSRG model as in Definition 2.7, with parameters \( \alpha > 1, \tau > 2, \sigma \geq 0, \) and \( d \in \mathbb{N} \). There exists a constant \( A > 0 \) such that for all \( \rho > 0 \), and \( n \) sufficiently large, with \( \mathcal{Z} = \{ \zeta_{ll}, \zeta_{hh}, \zeta_{nn} \} \),

\[
\mathbb{P}(|C_n^{(1)}| < \rho n) \geq \exp \left( - \frac{1}{A\rho} \cdot \log \log n \right),
\]

In [43], we give an almost matching upper bound for the left-hand side in (2.11) for some \( \rho > 0 \), with an error of \(-\alpha(1)\) in the exponent of \( n \). Next, we interpret our results for \( \kappa_{prod} \) models.

**Example 2.6** (Matching bounds for product-kernel models). In the open region in the \((\alpha, \tau, \sigma, d)\)-phase diagram where \( \zeta_{hh} \) is the unique maximum — corresponding to the \((hh)\)-regime — our bounds in (2.6) and (2.7), and in (2.9) and (2.10) are matching. This gives sharp bounds for scale-free percolation, geometric inhomogeneous random graphs, and hyperbolic random graphs. In these three models \( \sigma = 1 \), and whenever \( \max\{ \zeta_{ll}, \zeta_{hh}, \zeta_{nn} \} = \zeta_{hh} = (3-\tau)/(2-(\tau-1)/\alpha) \) (the red region in Figure 1), then

\[
\mathbb{P}^0(k \leq |C(0)| < \infty) = \exp(-\Theta(k^{\zeta_{hh}})),
\]

\[
\mathbb{P}(|C_n^{(1)}| < \rho n) \geq \exp \left( - \frac{1}{A\rho} \cdot \zeta_{hh} \right),
\]

\[
\mathbb{P}(A^{-1} \log^{1/\zeta_{hh}}(n) \leq |C_n^{(2)}| \leq A \log^{1/\zeta_{hh}}(n)) \geq 1 - n^{-\delta}.
\]

In [43], we present similar bounds for these \( \kappa_{prod} \)-KSRGs when \( \max(\mathcal{Z}) = \zeta_{ll} = 2 - \alpha \), which occurs when \( \alpha \in (1, \min\{\tau - 1, 1+1/d\}) \) (corresponding to the yellow region in Figure 1). For \( \kappa_{prod} \) models, \( \zeta_{hh} \) is never maximal, hence these models present only three phases in their phase diagram for \( \zeta \). For (threshold) hyperbolic random graphs (HRG), [40] proved that \( |C_n^{(2)}| = \Theta((\log n)^{2(3-\tau)}) \), which is included in Theorem 2.4 there is an isomorphism between HRGs and a 1-dimensional KSRG model with \( \sigma = 1, \tau \in (2,3), \alpha = \infty \) (see [7] or [49, Section 9]). Since for HRG we have \( \alpha = \infty \), all small components are localized in space in HRGs. However, when \( \alpha < \infty \), the long-range edges lead to de-localized small components. This makes proofs of lower bounds somewhat more and upper bounds significantly more complicated than those in [40] (at least up to our judgement).

### 2.1. Notation

For a discrete set \( \mathcal{S} \), we write \(|\mathcal{S}|\) for the size of the set. For a subset \( \mathcal{K} \subseteq \mathbb{R}^d \), let \( \text{Vol}(\mathcal{K}) \) denote the Lebesgue measure of \( \mathcal{K} \), \( \partial \mathcal{K} \) be its boundary and \( \mathcal{K}^c \) its complement. We denote the complement of an event \( \mathcal{A} \) by \( \overline{\mathcal{A}} \). Formally we define a vertex \( v \) by a pair of location and mark, i.e., \( v := (x_v, w_v) \), but we will sometimes still write \( v \in \mathcal{K} \) if \( x_v \in \mathcal{K} \). For two vertices \( u, v \), we write \( u \leftrightarrow v \) if \( u \) is connected by an edge to \( v \) in the graph \( \mathcal{G}_\infty \), and \( u \not\sim v \) otherwise. We also write \( \{u,v\} \) for the same (undirected) edge. Similarly, for a set of vertices \( \mathcal{S} \), we write \( u \leftrightarrow \mathcal{S} \) if there exists \( v \in \mathcal{S} \) such that \( u \leftrightarrow v \). We write \( X \gg Y \) if the random variable \( X \) stochastically dominates the random variable \( Y \), i.e., \( \mathbb{P}(X \geq x) \geq \mathbb{P}(Y \geq x) \) for all \( x \in \mathbb{R} \). A random graph \( \mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1) \) stochastically dominates a random graph \( \mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2) \) if there exists a coupling such that \( \mathbb{P}(\mathcal{V}_2 \subseteq \mathcal{V}_1, \mathcal{E}_2 \subseteq \mathcal{E}_1) = 1 \), and we write \( \mathcal{G}_1 \gg \mathcal{G}_2 \).

We write \( \mathcal{Q} \subseteq \mathbb{R}^d \) and \( a \leq b \)

\[
\Xi_{\mathcal{Q}}[a,b] := \Xi \cap (\mathcal{Q} \times [a,b]), \quad \Xi_{a}[a,b] := \Xi_{\Lambda_{a}}[a,b]
\]

for the restriction of the vertices with mark in \([a,b]\) and location in \( \mathcal{Q} \) and \( \Lambda_{a} \) from (2.1) respectively. We abbreviate independent and identically distributed as iid. By convention, we set \( (\alpha - 1)/\alpha := 1 \) if \( \alpha = \infty \) throughout the paper.

### 3. Methodology

#### 3.1. Upper bounds

After sketching the strategy for upper bound on the size of the second-largest component, we explain how to obtain the cluster-size decay from it, then we sketch the lower bound.
3.1.1. Second-largest component. We aim to show an upper bound of the form
\[ \Pr(|C_k| \geq k) \leq \frac{\epsilon}{k^c}, \]  
for arbitrary values of \( n \geq \text{poly}(k) \) and some constant \( c > 0 \). Such a bound yields (2.10) when one substitutes \( k = A \log^{1/\gamma_{hh}}(n) \) for a sufficiently large constant \( A = A(\delta) > 0 \). Throughout the outline we assume that \( (n/k)^{1/d} \in \mathbb{N} \). The proof consists of four revealment stages, illustrated in Figure 2.

Step 1. Building a backbone. We set \( w_{hh} := \Theta(k^{\gamma_{hh}}) \) with \( \gamma_{hh} > 0 \) from (1.11). We partition the volume-\( n \) box \( \Lambda_n \) into \( n/k \) smaller sub-boxes of volume \( k \). In this first revealment step we only reveal the location and edges between vertices in \( \Xi_n[w_{hh}, 2w_{hh}] \), obtaining the graph \( G_{n,1} := G_n[w_{hh}, 2w_{hh}] \).

We show that \( G_{n,1} \) contains a connected component \( C_{hb} \) that contains \( \Theta(k^{\gamma_{hh}}) \) many vertices in each sub-box, that we call backbone vertices. We show that this event — say \( A_{hb} \) — has probability at least 1 – \( \epsilon_{n,k} \). We do this by ordering the sub-boxes so that sub-boxes with consecutive indices share a \((d-1)\)-dimensional face, and by iteratively connecting \( \Theta(k^{\gamma_{hh}}) \) many vertices in the next subbox to the component we already built, combined with a union bound over all sub-boxes. The event \( A_{hb} \) ensures us to show that independently for all \( v \in \Xi_n[2w_{hh}, \infty) \),
\[ \Pr(v \leftrightarrow C_{hb} \mid A_{hb}, v \in \Xi_n[2w_{hh}, \infty)) \geq 1/2, \]
regardless of the location of \( v \). We call vertices in \( \Xi_n[2w_{hh}, \infty) \) connector vertices.

Step 2: Revealing low-mark vertices. We now also reveal all vertices with mark in \( [1, w_{hh}] \), and all their incident edges to \( G_{n,1} \) and towards each other, i.e., the graph \( G_{n,2} := G_{n,1}[2w_{hh}] \supseteq G_{n,1} \).

Step 3: Pre-sampling randomness to avoid merging of smaller components. To show (3.1), in the fourth revealment stage below we must avoid small-to-large merging: when the edges to/from some \( v \in \Xi_n[2w_{hh}, \infty) \) are revealed, a set of small components, each of size smaller than \( k \), could merge into a component of size at least \( k \) without connecting to the giant component. If we simply revealed \( \Xi_n[2w_{hh}, \infty) \) after Step 2, (3.2) would not be sufficient to show that small-to-large merging occurs with probability at most 1 – \( \epsilon_{n,k} \). So, we pre-sample randomness: we split \( \Xi_n[2w_{hh}, \infty) \) into two PPPs:
\[ \Xi_n[2w_{hh}, \infty) = \Xi_n^{(sure)}[2w_{hh}, \infty) \cup \Xi_n^{(unsure)}[2w_{hh}, \infty), \]
where \( \Xi_n^{(sure)}[2w_{hh}, \infty), \Xi_n^{(unsure)}[2w_{hh}, \infty) \) are independent PPPs with equal intensity: using (3.2) and helping random variables that encode the presence of edges, we pre-sample whether a connector vertex connects for sure to \( C_{hb} \) by at least one edge; forming \( \Xi_n^{(unsure)}[2w_{hh}, \infty) \) might still connect to \( C_{hb} \) since 1/2 is only a lower bound in (3.2), but we ignore that information. We crucially use the property that thinning a PPP yields two independent PPPs. The adaptation of our technique to lattices as vertex-set seems non-trivial due to this step. We reveal now \( \Xi_n^{(unsure)}[2w_{hh}, \infty) \).

Let \( G_{n,3} := G_{n,2} \) be the graph induced on the vertex set
\[ \Xi_{n,3} := \Xi_n[1, 2w_{hh}] \cup \Xi_n^{(unsure)}[2w_{hh}, \infty). \]

Step 4: Cover expansion, a volume-based argument. We now reveal \( \Xi_n^{(sure)}[2w_{hh}, \infty) \) and merge all components of size at least \( k \) with the largest component in \( G_{n,3} \) with probability at least 1 – \( \epsilon_{n,k} \). Small-to-large merging cannot happen since vertices in \( \Xi_n^{(sure)}[2w_{hh}, \infty) \) all connect to \( C_{hb} \).

We argue how to obtain (3.1).

Step 4a: Not too dense components via proper cover. For a component \( C \subseteq G_{n,3} \), the proper cover \( \mathcal{K}_n(C) \subseteq \Lambda_n \) is the union of volume-1 boxes centered at the vertices of \( C \) (the formal definition below is slightly different). Fixing a constant \( \delta > 0 \), we say that \( C \) is not too dense if
\[ \text{Vol}(\mathcal{K}_n(C)) \geq \delta |C|. \]
(3.4) Using the connectivity function \( p \) in (2.3) and \( w_{hh} = \Theta(k^{\gamma_{hh}}) \), there exists \( k_0 \) such that for any \( k \geq k_0 \) and any pair of vertices \( u \in \mathcal{V}_{n,3} \) in (3.3) and \( v \in \Xi_n^{(unsure)}[2w_{hh}, \infty) \) within the same volume-1 box,
\[ p(u, v) \geq p/2. \]
(3.5) Using this bound and that \( \Xi_n^{(sure)}[2w_{hh}, \infty) \) is a PPP, when \( |C| \geq k \), with probability at least 1 – \( \epsilon_{n,k} \), at least \( \Theta(k^{\gamma_{hh}}) \) many vertices of \( \Xi_n^{(sure)}[2w_{hh}, \infty) \) fall inside \( \mathcal{K}_n(C) \) and at least one of them connects to \( C \) by an edge. Since these vertices belong to \( \Xi_n^{(sure)}[2w_{hh}, \infty) \), they connect to \( C_{hb} \) by construction, merging \( C \) with the component containing \( C_{hb} \).

Step 4b: Too dense components via cover-expansion. We still need to handle components \( C \subseteq G_{n,3} \) with \( |C| \geq k \) that do not satisfy (3.4). These may exist (outside the component of \( C_{hb} \)) since the PPP \( \Xi_n \) contains dense areas, e.g., volume-one balls with \( \Theta(\log(n)/\log(\log(n))) \) vertices. We introduce
Figure 2. Upper bound. The y-axis represents marks, the x-axis represents space. After Steps 1 and 2 there is a component \( C^* \) containing the backbone that is connected to some small components from \( G_n[1, w_{na}) \). After Step 3, the unsure-connectors are revealed: there is small-to-large merging; some unsure-connectors connect to the backbone. After Step 4, each component of size at least \( k \) merged with the largest component via a sure-connector; unmerged small components and unsure-connectors outside \( C_n[1] \) remain, all in clusters of size at most \( k \).

a deterministic algorithm, the ‘cover-expansion algorithm’, that outputs for any (deterministic) set \( L \) of at least \( k \) vertices a set \( \mathcal{K}^{\text{exp}}(L) \subset \mathbb{R}^d \), called the expanded cover of \( L \), that satisfies (3.4) and a bound similar to (3.5) (provided that \( \sigma \leq \tau - 1 \)). In the design of the set \( \mathcal{K}^{\text{exp}}(L) \) we quantify the idea that a connector vertex can be farther away in space from a too dense subset \( L' \subset L \), while ensuring connection probability at least \( p/2 \) to \( L' \), as if \( L' \) contains a single vertex. We apply this algorithm with \( L = L \) for components of size at least \( k \) of \( G_{n,3} \) that do not satisfy (3.3) and do not contain \( C_{bb} \). The remainder of the proof is identical to Step 4a. Steps 4a, 4b, and a union bound over all components of size at least \( k \) in \( G_{n,3} \) yield (3.1).

3.1.2. Subexponential decay, upper bound. Consider \( k \) fixed. We consider the cluster-size decay (2.7) for any \( n \in [\text{poly}(k), n_k] \) with \( n_k = \exp(\Theta(k^{5/6})) \) by substituting \( n_k \) into (3.1). To extend it to larger \( n \), we first identify the lowest mark \( \overline{\pi}(n) \) such that all vertices with mark at least \( \overline{\pi}(n) \) belong to the giant component \( C_n^{(1)} \subset G_n \) with sufficiently high probability (in \( n \)). Then we embed \( \Lambda_{n_k} \) in \( \Lambda_n \) and show that

\[
\mathbb{P}^o(|C_n(0)| \geq k, 0 \notin C_n^{(1)}) \\
\leq \mathbb{P}(|C_{n_k}^{(2)}| \geq k) + \mathbb{P}(C_{n_k}^{(1)} \nsubseteq C_n^{(1)}) + \mathbb{P}^o(|C_n(0)| \geq k, 0 \notin C_n^{(1)}; |C_{n_k}(0)| < k). \tag{3.6}
\]

The first term on the right-hand side has the right error bound by (3.1). We relate the second term to the event that for some \( \bar{n} \in (n_k, n] \) there is no polynomially-sized largest component or the second-largest component is too large. The event in the third term implies that one of the at most \( k - 1 \) vertices in \( C_{n_k}(0) \) has an edge of length \( \Omega(n_k^{-1/4}/k) \), which will have probability at most \( \text{err}_{n_k,k} \), since these vertices have mark at most \( \overline{\pi}(n_k) \).

3.2. Lower bound. For the subexponential decay, we compute the probability of a specific event satisfying \( k \leq |\mathcal{C}(0)| < \infty \). We draw a ball \( B \) of volume \( \Theta(k) \) around the origin, and compute an optimally suppressed mark-profile: the PPP \( \Xi \) must fall below a \((d + 1)\)-dimensional mark-surface \( \mathcal{M} := \{(x, f(x)), x \in \mathbb{R}^d\} \), i.e., \( w_v \leq f(x_v) \) must hold for all \((x_v, w_v) \in \Xi \). We write \( \{\Xi \leq \mathcal{M}\} \) for this event. The value of \( f(x) \) is increasing in \( \|x - \partial B\| \) since high-mark vertices close to \( \partial B \) are most likely to have edges crossing \( \partial B \). \( \mathcal{M} \) is optimized so that \( \mathbb{P}(\Xi \leq \mathcal{M}) \sim \mathbb{P}(B \nleftrightarrow B' | \Xi \leq \mathcal{M}) \), where \( \{B \nleftrightarrow B'\} \) is the event that there is no edge present between vertices in \( B \) and those in its complement. Both events occur with probability \( \exp(-\Theta(k^{\max\{h, \gamma, \varepsilon, \varepsilon_{\max}\}})) \), up to logarithmic correction terms. We then find an isolated component of size at least \( k \) inside \( B \) using a technique that works when \( \tau < 2 + \sigma \). We use a boxing argument to extend this argument to the lower bound on the second-largest component of \( G_n \), similar to [40], and to obtain a lower bound on \( \mathbb{P}(|C_n^{(1)}| < \rho n) \).

3.3. Generalization of results. Most of our results extend to more general (interpolating) KSRGs than that in Definition 2.1. In particular, one can replace the spatial location of vertices to be \( \mathbb{Z}^d \) (or any other lattice), called i-KSRGs on \( \mathbb{Z}^d \). To prove this generalization, one needs to replace concentration inequalities for Poisson random variables by Chernoff bounds for Binomial random variables and replace integrals over \( \mathbb{R}^d \) by summations over \( \mathbb{Z}^d \). We believe that Theorems 2.2, 2.5 fully extend to \( \sigma > \tau - 1 \) and i-KSRGs on \( \mathbb{Z}^d \), by pre-sampling more information (similar to Step 3) already before Step 1. We decided to not include this, avoiding the extra amount of required technicalities. We highlight the
lemmas and propositions that do not immediately generalize by adding a * to their statement (e.g. Proposition 5.1).

Besides that, all results extend to i-KSRGs where the number of vertices is fixed to be $n$, where each vertex $u$ has an iid mark from distribution $F_W$ in (2.2) in Definition 2.1 and an independent uniform location in the volume-$n$ box $\Lambda_n$.

Moreover, all results (even when restricted to the graph in $\Lambda_n$) extend to the Palm-version $\mathbb{P}^x$ of $\mathbb{P}$ where the vertex set $\Xi$ is conditioned to contain a point at location $x \in \Lambda_n$ with unknown mark (which is not a priori obvious, since the model restricted to a finite box around the origin is not translation-invariant). We will omit this in our statements and proofs, and only add the superscript $x$ to our notation if we explicitly use the Palm version.

**Organization of the paper.** In Section 4 we explain the cover expansion from Step 4 in the outline of the upper bound. This technique is a novel technical contribution and is interesting in its own right. We use it also in Section 5. In Section 6 we extend it to the cluster-size decay, and show Corollary 2.3. In Section 5 we prove the upper bound on the size of the second-largest component. Then in Section 6 we extend it to the cluster-size decay, and show Corollary 2.3. In Section 7 we discuss the lower bounds, including the proof of Theorem 2.5. The first propositions in Sections 5—7 immediately imply the proofs of Theorems 2.2 and 2.4 as we verify near the end of Section 7.

## 4. The cover and its expansion

The goal of this section is to develop the cover-expansion technique in Step 4b of Section 3.1.1. The statements apply also to KSRGs on vertex sets other than a PPP. First we define a desired property notation if we explicitly use the Palm version.

### Definition 4.1 (s-expandable point-set).
Let $S \subset \mathbb{R}^d \times [1, \infty)$ be a discrete set of points, and $s > 0$. We call $S$ $s$-expandable if for all $x \in \mathbb{Z}^d$ and all $s' \in \{s + \ell/(ed^d/2^3d) : \ell \in \mathbb{N}\}$,

$$|S \cap \Lambda_{s'}(x)|/s' \leq e.$$

A discrete set $S \subset \mathbb{R}^d$ is $s$-expandable if there are no large boxes, centered at any site $z \in \mathbb{Z}^d$, with too high ratio of number of vertices in $S$ in the box vs its volume. Further, if $S$ is $s$-expandable, then any subset of $S$ is $s$-expandable; also, if $S$ is $s$-expandable, then for all $\tilde{s} \geq s$, $S$ is also $\tilde{s}$-expandable. The next proposition solves the problem of too-dense components in space, cf. (3.4).

### Proposition 4.2 (Covers and expansions for $s$-expandable sets).
Consider an i-KSRG with connection probabilities between vertices given by (2.3) with kernel $\kappa_{1,\sigma}$ and profile function with parameter $\alpha > 1$ and (arbitrary) marked vertex set $\mathcal{V} = \{(x_v, w_v)\}_{v \in \mathcal{V}}$. For a given $w > 2^d d^d/\beta$, let $s(w) > 0$ satisfy

$$s(w) := (2^d \beta w)^{1/(1-1/\alpha)}.$$  

Then, for any $s(w)$-expandable $\mathcal{L} \subseteq \mathcal{V} \cap \Lambda_n$ of vertices (for some $n > 0$), there is a set $\mathcal{K}_n(\mathcal{L}) \subseteq \Lambda_n$ with

$$\text{Vol}(\mathcal{K}_n(\mathcal{L})) \geq 1/2^{d+1} 1^{d/2} |\mathcal{L}|,$$

such that all vertices $v \in \mathcal{V} \cap \mathcal{K}_n(\mathcal{L})$ with mark $w_v \geq w$ satisfy independently of each other that

$$\mathbb{P}(v \leftrightarrow \mathcal{L} \mid \mathcal{V}) \geq p/2.$$  

We use two different constructions for the set $\mathcal{K}_n(\mathcal{L})$. If $\mathcal{L}$ is not too dense (see (3.4)), we will use a proper cover (see Definition 4.6 below). If, however, the points of $\mathcal{L}$ are densely concentrated in small areas, we will use a new (deterministic) algorithm, the cover-expansion algorithm, producing an expanded cover (see Definition 4.7 below) that still satisfies the connection probability in (4.3). This will prove Proposition 4.2. We start with some preliminaries:

### Definition 4.3 (Cells in a volume-$n$ box).
Let $\tilde{B}_z$ be a box of volume 1 centered around $z \in \mathbb{Z}^d$. For any two neighboring boxes $\tilde{B}_z, \tilde{B}_y$, allocate the shared boundary $\partial \tilde{B}_z \cap \partial \tilde{B}_y$ to precisely one of the boxes. For each $u \in \mathbb{Z}^d$ such that $u \notin \Lambda_n$ but $\tilde{B}_u \cap \Lambda_n \neq \emptyset$, let $z(u) := \arg \min \{||u - z|| : z \in \Lambda_n \cap \mathbb{Z}^d\}$, and then define for each $z \in \mathbb{Z}^d \cap \Lambda_n$ the cell of $z$ as

$$B_z := (\tilde{B}_z \cap \Lambda_n) \cup \left( \bigcup_{u \in \mathbb{Z}^d; z(u) = z} (\tilde{B}_u \cap \Lambda_n) \right).$$
In words, boxes that have their center inside $\Lambda_n$ but are not fully contained in $\Lambda_n$ are truncated, while boxes that have their centers outside $\Lambda_n$ but intersect it are merged with the closest box with center inside $\Lambda_n$. Clearly, at every point of $\Lambda_n$ at most $2^d$ cells are merged together, and only 1/2 of the radius in each coordinate can be truncated. Thus, for each cell $B_z$
\[
\sup\{\|x - y\| : x, y \in B_z\} \leq 2\sqrt{d}; \quad \text{and} \quad 2^{-d} \leq \text{Vol}(B_z) \leq 2^d. \tag{4.4}
\]

**Definition 4.4** (Notation for cells containing vertices). Let $\mathcal{L} \subset \Lambda_n$ be the set of locations of a set of vertices in any given realization $(\mathcal{V}, (w_v)_{v \in \mathcal{V}})$. Let $\{B_{z_i}\}_{i=1}^m$ be the cells with $\mathcal{L} \cap B_{z_i} \neq \emptyset$. Let $L_i := \mathcal{L} \cap B_{z_i}$, $\ell_i := |L_i|$, and $L := |\mathcal{L}| = \sum_{i=1}^m \ell_i$.

We will distinguish two cases for the arrangement of the vertices among the cells: either the number of cells is linear in the number of vertices, or there is a positive fraction of all cells that all contain ‘many’ vertices. To make this more precise, we prove the next combinatorial claim.

**Claim 4.5** (Pigeon-hole principle for cells). Let $\delta \in (0, 1)$, $\nu \geq 1$, and $\ell_1, \ldots, \ell_m \geq 1$ integers such that $\sum_{i \leq m} \ell_i = L$. If $m' < L(1 - \delta)/\nu$ then
\[
\exists I \subseteq [m'] : \forall i \in I : \ell_i \geq \nu, \quad \text{and} \quad \sum_{i \in I} \ell_i \geq \delta L. \tag{4.5}
\]

**Proof.** Assume for contradiction that $\delta, \nu, \ell_1, \ldots, \ell_m$ are such that $m' < L(1 - \delta)/\nu$ holds but (4.5) does not hold. Let $\mathcal{J} := \{j : \ell_j < \nu\} \subseteq [m']$ and let $\mathcal{J}' := [m'] \setminus \mathcal{J}$. Then $\forall i \in \mathcal{J}' : \ell_i \geq \nu$ and hence, we assumed the opposite of (4.5), it holds that $\sum_{j \in \mathcal{J}'} \ell_j < \delta L$. Since the total sum is $L$, this implies that $\sum_{j \in \mathcal{J}} \ell_j \geq (1 - \delta)L$. Moreover, since $\ell_j < \nu$ for $j \in \mathcal{J}$, it must hold that $|\mathcal{J}| \geq (1 - \delta)L/\nu$, which then gives a contradiction with the assumption in that $m' < L(1 - \delta)/\nu$. \(\square\)

We define the first possibility for the set $K_n(\mathcal{L})$, which is inspired by Claim 4.5 with $\nu = ed^d/2^{3d}$ and $\delta = 1/2$.

**Definition 4.6** (Proper cover). We say that $\mathcal{L}$ admits a proper cover if $m' \geq |\mathcal{L}|/(2ed^d/2^{3d})$ in Definition 4.4, and we define the cover of $\mathcal{L}$ as
\[
K_n^{(\text{prop})}(\mathcal{L}) := \bigcup_{i \in [m']} B_{z_i}, \quad \text{satisfying} \quad \text{Vol}(K_n^{(\text{prop})}) \geq \frac{1}{d^d/2^{3d+1}} |\mathcal{L}|. \tag{4.4}
\]

By (4.4), $\nu = ed^d/2^{3d}$, and $\delta = 1/2$, hence, we obtain the desired volume bound on the right-hand side above, establishing (4.2) for sets admitting a proper cover. Moreover, consider now $(x_u, w_u) \in (B_{z_i} \cap \mathcal{L}) \times [1, \infty)$ and $u := (x_u, w_u) \in B_{z_i} \times [w_u, \infty)$ with $B_{z_i} \subseteq K_n^{(\text{prop})}$. Then $\|x_u - x_v\| \leq 2\sqrt{d}$ by (4.4). Since we assumed $\frac{w_u}{d} \geq 2^d d/\beta$ above (4.1), using (2.3) and (1.10),
\[
p(u, v) \geq p \min\{1, (\beta K_{1, \sigma}(\nu, w, 1)/(2\sqrt{d})^{d/\alpha})\} = p \min\{1, (\beta w/2\sqrt{d})^{d/\alpha}\} \geq p. \tag{4.6}
\]
This shows (4.3) for sets admitting a proper cover. The argument for $\alpha = \infty$ is similar.

The remainder of this section focuses on sets $\mathcal{L}$ that do not admit a proper cover, i.e., the number of cells that contain vertices of $\mathcal{L}$ is too small. We define an “expanded” cover, obtained after applying a suitable volume-increasing procedure to $\bigcup_{i \in [m']} B_{z_i}$ that will be explained at the end of the section.

4.1. Cover expansion. In this section we assume that $\mathcal{L}$ does not admit a proper cover. By Claim 4.5 and re-indexing cells in Definition 4.4 without loss of generality we may assume that $I = [m'] \subseteq [m]$ satisfies (4.5) with $\nu = ed^d/2^{3d}$ and $\delta = 1/2$. We use $\Lambda(x, s)$ in (2.1) here for the box of volume $s$ centered at $x \in \mathbb{R}^d$.

**Definition 4.7** (Cover expansion). Let $\mathcal{L}$ be a set of locations of vertices that does not admit a proper cover in the sense of Definitions 4.4 and 4.6. Let $[m'] := \{j : \ell_j \geq ed^d/2^{3d}\} \subseteq [m]$ satisfying (4.5) with $\nu = ed^d/2^{3d}$ and $\delta = 1/2$. The cover expansion is defined as a subset of labels $\mathcal{J}' \subseteq [m]$ and corresponding boxes $\{B_j\}_{j \in \mathcal{J}'} \subset \mathbb{R}^d$, centered at $(z_j)_{j \in \mathcal{J}'}$, together with an allocation $\mapsto$ of the cells $B_{z_i} : i \leq m$ to these boxes, with
\[
\text{Cells}_{j} := \bigcup_{i \leq m} \{i : B_{z_i} \mapsto B_j\}, \tag{4.7}
\]
satisfying the following properties:
Proposition 4.8  (Every set has either a proper cover or a cover expansion)

Let \( B_j^{(s)} \) be a box with label \( j \) and number of vertices that are in cells allocated to \( B_j^{(s)} \), then \( \sum_{i \in \text{Cells}_j^{(s)}} \ell_i \) is proportional to \( |L_j| \). Thus, by (4.8), Observation (i) follows.

We define the expanded cover of \( L \) as

\[
K_n^{(\text{exp})}(L) := \Lambda_n \cap \left( \bigcup_{j \in \mathcal{J}^{(s)}} B_j^{(s)} \right)
\]

and call \( B_j^{(s)} \) the expanded boxes.

A few comments about Definition 4.7: (disj) and (vol) together ensure that the total volume of the expanded cover is proportional to \( |L| \). Further, (vol) ensures that \( \text{Vol}(B_j^{(s)}) \) is proportional to the number of vertices that are in cells allocated to \( B_j^{(s)} \). Finally, (near) ensures that the center \( z_i \) of each cell \( B_i \) is relatively close to the center of the box to which it is allocated. Observe that when the final box \( B_j^{(s)} \) is large, (4.9) allows for large distances between allocated initial cells and the center of \( B_j^{(s)} \).

Proposition 4.8 (Every set has either a proper cover or a cover expansion). Assume \( L \) does not admit a proper cover defined in Definition 4.6. Then there exists a cover expansion for \( L \) in the sense of Definition 4.6 and 4.7. Further, the total volume of the expanded cover is linear in \( |L| \), i.e.,

\[
\text{Vol}(K_n^{(\text{exp})}(L)) \geq \frac{1}{2d + 1} |L|.
\]

We defer the proof of existence of the desired cover expansion to the end of the section. Assuming that a cover expansion exists, we show now the linearity of its volume along with some other observations. Afterwards, we show how Proposition 4.2 follows from Proposition 4.8.

Observation 4.9 (Cover-expansion properties). Consider the cover expansion of a set \( L \) that does not admit a proper cover according to Definition 4.6.

(i) Every expanded box has volume at least 1, i.e., for all \( j \in \mathcal{J}^{(s)} \), \( \text{Vol}(B_j^{(s)}) \geq 1 \).

(ii) For any cell with \( B_z_i \rightarrow B_j^{(s)} \),

\[
\sup \left\{ \|x_u - x_v\| : x_u \in L_i, x_v \in B_j^{(s)} \right\} \leq 4\sqrt{d} \text{Vol}(B_j^{(s)})^{1/d}.
\]

(iii) For every box \( B_j^{(s)} \), there exists a box \( B_j' \) centered at \( z_j \) such that

\[
\text{Vol}(B_j') = d^{d/2} 2^{d/2} \text{Vol}(B_j^{(s)}), \quad \text{and} \quad |L \cap B_j'| \geq e \text{Vol}(B_j').
\]

(iv) If \( L \) is \( s \)-expansible, then for all \( j \in \mathcal{J}^{(s)} \)

\[
\text{Vol}(B_j^{(s)}) \leq d^{-d/2} 2^{-3d}s.
\]

(v) The total volume of a cover expansion is linear in \( |L| \), i.e., (4.11) holds.

Proof. Part (i) is a consequence of Definition 4.7, every cell with label at most \( m \) has \( \ell_i \geq d^{d/2} 2^{d} \), so by (vol), i.e., (4.8). Observation (i) follows.

For part (ii) we apply the triangle inequality: since \( x_u \in L_i \), it holds that \( x_u \in B_z_i \), and so by (4.4), \( \|x_u - z_i\| \leq 2\sqrt{d} \) and by (4.9), \( \|z_i - z_j\| \leq 2\sqrt{d} \text{Vol}(B_j^{(s)})^{1/d} \); hence \( \|x_u - z_j\| \leq 2\sqrt{d} + \sqrt{d} \text{Vol}(B_j^{(s)})^{1/d} \). Also, for any \( x_v \in B_j^{(s)} \) it holds that \( \|z_j - x_v\| \leq (\sqrt{d}/2) \text{Vol}(B_j^{(s)})^{1/d} \) by (4.8). Combining these bounds and using \( \text{Vol}(B_j^{(s)})^{1/d} \geq 1 \) yields

\[
\|x_u - x_v\| \leq 2\sqrt{d} + (3\sqrt{d}/2) \text{Vol}(B_j^{(s)})^{1/d} \leq (7\sqrt{d}/2) \text{Vol}(B_j^{(s)})^{1/d} \leq 4\sqrt{d} \text{Vol}(B_j^{(s)})^{1/d},
\]

and part (ii) is proven. For part (iii), note that part (ii) applied to \( u \in L_i \subset B_z_i \), and \( z_j \), yields that

\[
\sup_{i \in \text{Cells}_j^{(s)}} \left\{ \|x_u - z_j\| : x_u \in L_i \right\} \leq 4\sqrt{d} \text{Vol}(B_j^{(s)})^{1/d}.
\]
Consequently, the box \( B_j' \) centered at \( z_j \) of volume \( \text{Vol}(B_j') = d^{d/2}2^{3d} \text{Vol}(B_j^{(\ast)}) \) contains all \( u \) in \( \mathcal{L}_i \) with \( i \in \text{Cells}_j(\ast) \). Hence, using \((4.8)\), we obtain
\[
|\mathcal{L} \cap B_j'| \geq \sum_{i \in \text{Cells}_j(\ast)} \ell_i = ed^{d/2}2^{3d} \text{Vol}(B_j^{(\ast)}) = e \text{Vol}(B_j'),
\]
and part (iii) follows. For part (iv), by combining \((4.14)\) with Definition \((4.1)\) we see that \( \mathcal{L} \) can only be \( s \)-expandable if \( \text{Vol}(B_j') \leq s \). Rearrangement of the first part of \((4.12)\) yields \((4.13)\).

Finally, for part (v), by an argument similar to \((4.4)\), \( \text{Vol}(B_j^{(\ast)} \cap \Lambda_n) \geq 2^{-d} \text{Vol}(B_j^{(\ast)}) \) for all \( j \in \mathcal{J}(\ast) \). Since all boxes of the cover expansion are disjoint, and each cell is allocated once, \((4.8)\) and \((4.10)\) imply that
\[
\text{Vol}(\mathcal{K}_n^{(\exp)}(\mathcal{L})) = \sum_{j \in \mathcal{J}(\ast)} \text{Vol}(B_j^{(\ast)} \cap \Lambda_n) \geq 2^{-d} \sum_{j \in \mathcal{J}(\ast)} \text{Vol}(B_j^{(\ast)})
= \frac{1}{e2^{4d}d^{d/2}} \sum_{j \in \mathcal{J}(\ast)} \sum_{i \in \text{Cells}_j(\ast)} \ell_i = \frac{1}{ed^{d/2}2^{3d}} \sum_{i \leq m} \ell_i \geq \frac{1}{ed^{d/2}2^{3d+1}} \mathcal{L},
\]
where the last bound follows by the assumption in Definition \((4.7)\) that \( \ell_i \geq ed^{d/2}2^{3d} \) for \( i \leq m \), and the initial assumption that \((4.5)\) in Claim \((4.5)\) holds for \([m] \) with \( \delta = 1/2 \). \( \square \)

**Proof of Proposition \((4.12)\) assuming Proposition \((4.8)\)**: For sets \( \mathcal{L} \) that admit a proper cover, we recall the reasoning below Definition \((4.6)\) (in particular \((4.6)\)) which implies both bounds \((4.2)\) and \((4.3)\) in Proposition \((4.2)\). Let \( \mathcal{L} \) be an \( s \)-expandable set that does not admit a proper cover. Let \( \mathcal{K}_n^{(\exp)} \) be an expanded cover given by the boxes \( (B_j^{(\ast)})_{j \in \mathcal{J}(\ast)}, \mathcal{J}(\ast) \subseteq [m] \) and an allocation \( \vec{r} \) of the initial cells \( (B_{ij})_{i \in [m]} \) to these boxes. The existence of this cover expansion is guaranteed by Proposition \((4.8)\). The volume bound \((4.12)\) follows from \((4.1)\) in Proposition \((4.8)\). Hence, it only remains to verify \((4.3)\).

Let \( u = (x_u, w_u) \in \mathcal{K}_n^{(\exp)} \times [w, \infty) \). By (disj), \((4.10)\), there exists \( j \in \mathcal{J}(\ast) \) such that \( x_u \in B_j^{(\ast)} \). Recall from \((4.7)\) that \( \mathcal{J}(\ast) \) are the cells allocated to \( B_j^{(\ast)} \), and from Definition \((4.4)\) that \( \mathcal{L}_i = \mathcal{L} \cap B_{ij} \). Let now \( \mathcal{L}_j(\ast) := \bigcup_{i \in \text{Cells}_j(\ast)} \mathcal{L}_i \). Recall the formula of the connection probability from \((2.3)\).

**Case \((1)\): \( \alpha < \infty \)**. By Observation \((4.9)\)(ii) for any \( v = (x_v, w_v) \in \mathcal{L}_j^{(\ast)} \), and any \( x_u \in B_j^{(\ast)} \), using the lower bounds for the marks \( w_x \geq 1, w_u \geq w_v \), and that \( \kappa(w_u, w_v) = (w_u \vee w_v)(w_u \wedge w_v)^\sigma \) for some \( \sigma \geq 0 \), we obtain using \((2.3)\) that
\[
p(u, v) = p \min \left\{ 1, \beta^\alpha \frac{\kappa(w_u, w_v)^\alpha}{\|x_v - x_u\|^{d\alpha}} \right\} \geq p \min \left\{ 1, \frac{\beta^\alpha w_u^\alpha}{(4\sqrt{d})^{d\alpha} \text{Vol}(B_j^{(\ast)})^{1-\alpha}} \right\} =: r.
\]
By \((4.8)\), \( |\mathcal{L}_j^{(\ast)}| = \sum_{i \in \text{Cells}_j(\ast)} \ell_i = ed^{d/2}2^{3d} \text{Vol}(B_j^{(\ast)}) \). Hence, we have
\[
\mathbb{P}(\#(x_u, w_v) \in \mathcal{L}_j^{(\ast)} : (x_u, w_u) \leftrightarrow (x_v, w_v)) \leq (1 - r)^{ed^{d/2}2^{3d} \text{Vol}(B_j^{(\ast)})}
\leq \exp \left( -p ed^{d/2}2^{3d} \min \left\{ \text{Vol}(B_j^{(\ast)}), \beta^\alpha w_u^\alpha (4\sqrt{d})^{-d\alpha} \text{Vol}(B_j^{(\ast)})^{1-\alpha} \right\} \right).
\]
Since \( \alpha \geq 1 \), and \( \mathcal{L} \) is \( s \)-expandable, we can use the upper bound in \((4.13)\) on \( \text{Vol}(B_j^{(\ast)}) \) to bound the second term in the minimum on the right-hand side of the last row, and we use \( \text{Vol}(B_j^{(\ast)}) \geq 1 \) by Observation \((4.9)\)(i) to bound the first term. We obtain
\[
\mathbb{P}(\#(x_u, w_v) \in \mathcal{L}_j^{(\ast)} : (x_u, w_u) \leftrightarrow (x_v, w_v)) \leq \exp \left( -p \min \left\{ d^{d/2}2^{3d}, d^{d/2}2^{3d} \beta^\alpha w_u^\alpha (4\sqrt{d})^{-d\alpha} s\alpha(1-\alpha)(d^{-d/2}2^{-3d})^{1-\alpha} \right\} \right)
\leq \exp(-p \min \{d^{d/2}3^d, (2d^d\beta^\alpha w_u^\alpha s\alpha(1-\alpha)\}}) \leq \exp(-ep) \leq p/2,
\]
where we used in the last row that \( d^{d/2}3^d > 1 \) and also that the bound on \( w \) in \((4.1)\) ensures that the second term inside the minimum is at least \( 1 \), and that \( \exp(-ep) \leq p/2 \) for \( p \in [0, 1] \). This concludes the proposition for \( \alpha < \infty \).
Case (2): $\alpha = \infty$. Using the same bounds as for $\alpha < \infty$ on the distance, mark and volume of boxes, but now (2.3) for $\alpha = \infty$, for any $u = (x_u, w_u) \in B_j^{(r)} \times [w, \infty)$ and any $v = (x_v, w_v) \in L_j^{(r)}$ that
\[
p((x_u, w_u), (x_v, w_v)) \geq p \{ \beta w > (4\sqrt{d})^d \text{Vol}(B_j^{(r)}) \} \geq p \{ \beta w > (4\sqrt{d})^d d^{-d/2} 2^{-3d} \}
\]
are in the one-but-last step we used (4.1), finishing the proof of $\alpha = \infty$.

We observe that the case $\alpha = \infty$ does not use of the size of $L_j^{(r)}$, and only requires a single vertex in it, which is intuitive considering the threshold nature of $p$ in (2.3). It remains to prove Proposition 4.8 which is the content of the following subsection.

4.2. The algorithm producing the cover expansion. Now we give the algorithm producing the expanded cover of any discrete set $L$ without a proper cover, thence, proving the desired proposition.

Setup for the algorithm. Recall the notation from Definitions 4.4 and 4.7. Throughout, we will assume that $L$ does not admit a proper cover in Definition 4.6 and that $(B_z, \{ i \in [m] \})$ are the cells satisfying 4.5. Contrary to Definition 4.7, which allocates the initial cells $B_{z_i}$ to boxes $B_j^{(r)}$, the algorithm allocates the labels $i, j \leq m$ of the initial cells $B_{z_j}$ towards each other in discrete rounds $r \in \mathbb{N}$. We write $i \rightarrow j$ to indicate that label $i$ is allocated to label $j$ in the allocation of round $r$. We also write $\rightarrow := \{(i, j) : i \rightarrow j \}_{i \leq m}$. Cells $\rightarrow := \bigsqcup_{i \leq m} \{ i : i \rightarrow j \}; \quad J^{(r)} := \{ j : \text{Cells}^{(r)} \neq \emptyset \}.$

In each round $r \geq 0$, the boxes $\{ B_j^{(r)} \}_{j \in J^{(r)}}$, and the centers of these boxes are completely determined by $\rightarrow$ by the formula
\[
B_j^{(r)} := \Lambda \left( z_j, \frac{1}{e d^{d/2} 2^{3d}} \sum_{i \in \text{Cells}^{(r)}} \ell_i \right) \quad \text{for} \quad j \in J^{(r)};
\] (4.15)
where $\Lambda(x, s)$ is a box of volume $s$ centered at $x \in \mathbb{R}^d$ (see (2.1)). Since label $j$ corresponds to center $z_j$ across different rounds, by slightly abusing notation we also write $B_{z_j} \rightarrow B_j^{(r)}$ if and only if $i \rightarrow j$. We say that $\rightarrow$ satisfies one (or more) conditions in Definition 4.7 if $(B_j^{(r)})_{j \in J^{(r)}}$ with allocation $\rightarrow$ satisfies the condition(s)

The algorithm starts with the identity as initial allocation $0 \rightarrow$ that induces possibly overlapping boxes $B_j^{(0)}, \ldots, B_j^{(m)}$; we will show that $0 \rightarrow$ already satisfies (near) and (vol.) of Definition 4.7. In each stage the algorithm attempts to remove an overlap – a non-empty intersection – between a pair of boxes by re-allocating a few cell labels, while maintaining properties (near) and (vol.); we achieve (disj) in the last round $r^*$. The last round $r^* < \infty$ corresponds to the final output, by setting $J^{(\ast)} := J^{(r^*)} \land B_j^{(r^*}) := B_j^{(r^*)}$ and defining $B_{z_j} \rightarrow B_j^{(r^*)}$ if and only if $i \rightarrow j$.

The cover-expansion algorithm.

(input) $(B_z, \{ i \in [m] \}$ and $L_i = L \cap B_z$, satisfying 4.5 with $\nu = e d^{d/2} 2^{3d}$ and $\delta = 1/2$.

(Init.) Set $r := 0$ and allocate $j \rightarrow 0$ for all $j \leq m$.

(While) If $(B_j^{(r)})_{j \in J^{(r)}}$ in (4.15) are all pairwise disjoint, set $r^* := r$; and return $J^{(\ast)} := J^{(r^*)} \land B_j^{(r^*)} := B_j^{(r^*)}$ and $\rightarrow := \rightarrow^*$. Otherwise, let $j_1(r) \in J^{(r)}$ be the label corresponding to the largest box $B_j^{(r)}$ with an overlap with some other box in round $r$, and let $j_2(r)$ be the label of the largest box that overlaps with $B_{j_1(r)}^{(r)}$ (using an arbitrary tie-breaking rule). Define
\[
I_1^{(r)} := \text{Cells}_{j_1(r)}^{(r)} \cap \{ i : \| z_i - z_{j_1(r)} \| \leq \sqrt{d} \text{Vol}(B_{j_1(r)}^{(r)})^{1/d} \}_{i \leq m}; \quad I_2^{(r)} := \text{Cells}_{j_2(r)}^{(r)} \setminus I_1^{(r)}. \] (4.16)
Then we define $\rightarrow^{1+} \rightarrow$ by only re-allocating labels in Cells $j_2(r)$ as follows:

(i) for $i \in I_1^{(r)}$ we allocate $i \rightarrow^{1} j_1(r)$, i.e., the labels of cells that are sufficiently close to the center of $B_{j_1(r)}^{(r)}$ in order to satisfy (4.9) are re-allocated to $j_1(r)$;
(ii) for \( i \in \mathcal{I}_2^{(r)} \) we allocate \( i \overset{r \rightarrow}{\rightarrow} 1 \), i.e., the labels of cells in \( \text{Cells}_{j_2(1)}^{(r)} \) that are potentially too far away from the center of \( B_{j_1(1)}^{(r)} \) are re-allocated back to themselves;

(iii) for \( i \in [m] \setminus (\text{Cells}_{j_2}^{(r)}) \), we set \( i \overset{r \rightarrow}{\rightarrow} k \) if and only if \( i \overset{r \rightarrow}{\rightarrow} k \) (that is, \( r \overset{r \rightarrow}{\rightarrow} 1 \) agrees with \( r \overset{r \rightarrow}{\rightarrow} k \) outside labels in \( \text{Cells}_{j_2(r)}^{(r)} \)).

Increase \( r \) by one and repeat (while).

We make the immediate observation:

**Observation 4.10.** In each iteration of (while), \( \mathcal{I}_1^{(r)} \) in (4.16) is always non-empty. Moreover,

\[
\text{Vol}(B_{j_1(1)}^{(r+1)}) - \text{Vol}(B_{j_1(1)}^{(r)}) \geq 1.
\]

(4.17)

**Proof.** It can be shown inductively that \( j \overset{r \rightarrow}{\rightarrow} j \) holds for all \( j \in \mathcal{J}^{(r)} \). Since the boxes \( B_{j_1(1)}^{(r)} \) and \( B_{j_2(1)}^{(r)} \) overlap, the distance of their centers \(|z_{j_2(1)} - z_{j_1(1)}|\) is at most the diameter of \( B_{j_1(1)}^{(r)} \), which is \( \sqrt[4]{\text{Vol}(B_{j_1(1)}^{(r)})^{1/4}} \). Hence, \( j_2(1) \in \mathcal{I}_1^{(r)} \). Since each cell contains \( \ell_i \geq ed^{l/2}2^{3d} \) many vertices by the assumption in (input), we obtain by (4.15)

\[
\text{Vol}(B_{j_1(1)}^{(r+1)}) - \text{Vol}(B_{j_1(1)}^{(r)}) \geq \frac{\ell_i d^l}{ed^{l/2}2^{3d}} \geq 1.
\]

\[\square\]

**Proof of Proposition 4.8.** Once having shown that a cover expansion of \( \mathcal{L} \) exists, the bound on its volume (4.11) holds by Observation 4.9(v). So it remains to show that the algorithm produces in finitely many rounds an output satisfying all conditions of a cover expansion in Definition 4.7.

**The algorithm stops in finitely many steps.** We argue using a monotonicity argument. We say that a vector \( a = (a_1, \ldots, a_m) \in \mathbb{R}^m \) is non-increasing if \( a_i \geq a_{i+1} \) for all \( i \leq m - 1 \). We use the lexicographic ordering for non-increasing vectors \( a, b \in \mathbb{R}^m \): let \( a >_L b \) if there exists a coordinate \( j \leq m \) such that \( a_\ell = b_\ell \) for all \( \ell < j \) and \( a_j > b_j \) for \( \ell = j \).

For all \( r \in \mathbb{N} \), \( \mathcal{J}^{(r)} \subseteq [m] \), and hence, \( m^{(r)} := |\mathcal{J}^{(r)}| \leq m \). Let \( a^{(r)} \in \mathbb{R}^m \) be the non-increasing vector of the re-ordered (Vol\( (B_{j_1}^{(r)}) \)) \( j \in \mathcal{J}^{(r)} \) appended with \((m-m^{(r)})\)-many zeroes. By Observation 4.10 the entry corresponding to Vol\( (B_{j_1(1)}^{(r)}) \) in \( a^{(r)} \) increases in \( a^{(r)} \). Moreover, the entry corresponding to Vol\( (B_{j_1(1)}^{(r)}) \) “crumbles” into smaller volumes (corresponding to boxes with label in \( \mathcal{I}_2^{(r)} \) that appear in \( a^{(r)} \)).

Since by definition, \( j_1(1) \) corresponds to the largest box among \( (B_{j_1}^{(r)}) \) \( j \in \mathcal{J}^{(r)} \) that has an overlap with some other box, so also Vol\( (B_{j_2(1)}^{(r)}) \) \( \leq \text{Vol}(B_{j_1(1)}^{(r)}) \), and the allocation of labels except those in \( \text{Cells}_{j_2(1)}^{(r)} \) remains unchanged, these together imply that \( a^{(r+1)} \gg L a^{(r)} \). Finally, for any \( r \) and any \( j \in \mathcal{J}^{(r)} \), Vol\( (B_{j}^{(r)}) \) \( \leq |\mathcal{L}|/(ed^{l/2}2^{3d}) =: b \) by (4.8), implying that for all \( r \), \( (b, \ldots, b) >_L a^{(r)} \). So, \( (a^{(r)})_{r>0} \) is an increasing bounded sequence with respect to \( >_L \), with an increase of at least \( 1 \) per step by (4.17). Hence, \( (a^{(r)})_{r>0} \) converges and attains its limit after finitely many rounds, i.e., \( r^* < \infty \).

**The output corresponds to a cover expansion.** We now prove that the output \( \mathcal{J}^{(r)}, j \overset{r}{\rightarrow} \) and the corresponding boxes in (4.15) satisfy the conditions of Definition 4.7. By the stopping condition in step (while) of the algorithm, \( (B_{j}^{(r)}) j \in \mathcal{J}^{(r)} \) satisfy (disj.), and by their definition in (4.15), also (vol.).

We need to still verify (near). We show this by induction: initially, for \( (B_{j}^{(0)}) j \in \mathcal{J}^{(0)} \), (near) holds, since in (init.) all labels are allocated to themselves, so \( \text{Cells}_{j}^{(0)} = \{j\} \), and thus the left-hand side in (4.9) is 0. Assume then \( r > 0 \). We prove that (near) holds for \( r \overset{r \rightarrow}{\rightarrow} 1 \), assuming that it holds for \( r \overset{r}{\rightarrow} \). Recall from (while) that \( j_1(1) \) is the label of the largest box that has an overlap; \( j_2(1) \) is the label of the largest box overlapping with \( B_{j_1(1)}^{(r)} \); by (4.16), \( \mathcal{I}_1^{(r)} \) is the set of labels in \( \text{Cells}_{j_2(1)}^{(r)} \) re-allocated to \( j_1(1) \), and \( \mathcal{I}_2^{(r)} = \text{Cells}_{j_2(1)}^{(r)} \setminus \mathcal{I}_1^{(r)} \) is the set labels allocated in round \( r \) to \( j_2(1) \), and in round \( r + 1 \) to themselves. We distinguish between four cases for the proof of the inductive step:

- Assume \( i \notin (\text{Cells}_{j_1(1)}^{(r)} \cup \text{Cells}_{j_2(1)}^{(r)}) \) and let \( k \) be such that \( i \overset{r \rightarrow}{\rightarrow} k \). By (while) part (iii), \( \text{Cells}_{k}^{(r+1)} = \text{Cells}_{k}^{(r)} \), so by the induction hypothesis, (4.9) holds for \( i \overset{r \rightarrow}{\rightarrow} 1 \).
- Assume \( i \in \text{Cells}_{j_1(1)}^{(r)} \). Since \( B_{j_1(1)}^{(r)} \subseteq B_{j_1(1)}^{(r+1)} \) by (4.15) and (4.17), considering \( |z_i - z_{j_1(1)}| \) and the left-hand side in (4.9) stay the same, while the right-hand side increases, so the inequality required for (near) still holds.
• Assume \( i \in \mathcal{I}_1^{(r)} \subseteq \text{Cells}_{2_2(r)}^{(r)} \). The definition of \( \mathcal{I}_1^{(r)} \) in (4.16) forces that \( \|z_i - z_{j_1(r)}\| \) satisfies (4.9).

• Assume \( i \in \mathcal{I}_2^{(r)} = \text{Cells}_{2_2(r)}^{(r)} \setminus \mathcal{I}_1^{(r)} \): (4.9) holds for the same reason as for the base case, i.e., since \( i \rightarrow_{k+1} i \), \( \|z_i - z_i\| = 0 \) trivially satisfies (4.9).

Having all possible cases covered, this finishes the proof of the induction. Since \( r^* < \infty \), this finishes the proof of Proposition 4.8.

4.3. Poisson processes are expandable. We end this section by showing that a Poisson point process is typically \( s \)-expandable for \( s \) sufficiently large. Recall \( \Lambda_n = [-n^{1/d}/2, n^{1/d}/2]^d \).

Lemma 4.11 (PPPs are expandable). Let \( \Gamma \) be a Poisson point process on \( \mathbb{R}^d \) equipped with an absolutely continuous intensity measure \( \mu \) such that \( \mu(dx) \leq \text{Leb}(dx) \). Then there exists a constant \( C_{111} > 0 \) such that for any \( s > 0 \),

\[
\mathbb{P}(\Gamma \cap \Lambda_n \text{ is not } s\text{-expandable}) \leq C_{111} n \exp(-s).
\]

Proof. Using stochastic domination of point processes, without loss of generality we can assume that \( \Gamma \) has intensity measure \( \text{Leb}(dx) \). Let us define \( R(s) := \{s + \ell/(ed^{d/2}2^{3d}) : \ell \in \mathbb{N}\} \), the range of volumes of boxes that we need to consider in Definition 4.1. By a union bound over the at most \( n \) possible centers of the boxes in \( \Lambda_n \), and by translation invariance of \( \text{Leb} \), it holds that

\[
\mathbb{P}(\Gamma \cap \Lambda_n \text{ is not } s\text{-expandable}) = \mathbb{P}(\exists x \in \mathbb{Z}^d \cap \Lambda_n, \exists s' \in R(s) : |A_{s'}(x) \cap \Gamma| \geq es')
\leq n \sum_{s' \in R(s)} \mathbb{P}(|A_{s'} \cap \Gamma| \geq es').
\]

(4.18)

Since the intensity of \( \Gamma \) is equal to one, by Lemma C.1 each summand is at most \( \exp(-s') \) on the right-hand side. Hence, using that \( \sum_{\ell=0}^{\infty} f(\ell) \leq \int_{-1}^{\infty} f(x)dx \) for a monotone non-increasing function, we obtain for the summation in (4.18)

\[
\mathbb{P}(\Gamma \cap \Lambda_n \text{ is not } s\text{-expandable}) \leq n \exp(-s) \sum_{\ell=0}^{\infty} \exp\left(-\ell/(ed^{d/2}2^{3d})\right)
\leq n \exp(-s) \int_{-1}^{\infty} \exp\left(-x/(ed^{d/2}2^{3d})\right)dx = n \exp(-s)ed^{d/2}2^{3d} \exp\left(1/(ed^{d/2}2^{3d})\right).
\]

The proof of the lemma follows by setting \( C_{111} = ed^{d/2}2^{3d} \exp\left(1/(ed^{d/2}2^{3d})\right) \). \( \square \)

5. Upper bound: second-largest component

The main goal of this section is to prove the following proposition for general values of \( n \) and \( k \), which readily implies Theorem 2.4(ii), i.e., (2.10). Recall \( \rho_{hh} = 1 - \gamma_{hh}(\tau - 1) \) from (1.12). We restrict ourselves to \( \sigma \in (\tau - 2, \tau - 1] \), since \( \sigma \leq \tau - 2 \) implies that \( \rho_{hh} \leq 0 \). We recall from Section 3.3 that the superscript \( * \) below indicates that the result does not generalize to \( i \)-KSRGs on \( \mathbb{Z}^d \).

Proposition*. 5.1. Consider a supercritical \( i \)-KSRG model in Definition 2.7 and parameters \( \alpha > 1, \tau > 2 \) and \( d \in \mathbb{N} \). Assume further that also \( \sigma \in (\tau - 2, \tau - 1] \). There exists a constant \( C_{5.1} > 0 \) such that whenever \( n > k^{1+\rho_{hh}/\alpha} \) it holds that

\[
\mathbb{P}(|C_n^{(i)}| \geq k) \leq (n/q_{5.1}) \exp\left(-c_{5.1}k^{\rho_{hh}}\right).
\]

(5.1)

We follow the steps of the methodology from Section 3.1.1. The bulk of the work is to establish Steps 1 and 3, since we already developed the cover expansion of Step 4 in Section 4. We first introduce some notation. We aim to partition the box \( \Lambda_n \) into disjoint subboxes of (roughly) volume \( k \). Define

\[
n' := k[(n/k)^{1/d}]^d.
\]

(5.2)

The box \( \Lambda_n \subseteq \Lambda_n \) is the largest box inside \( \Lambda_n \) that can be partitioned into \( n'/k \) disjoint subboxes of volume exactly \( k \) (boundaries are allocated uniquely similar to Definition 4.3). Let the boxes of this partitioning of \( \Lambda_n \) be \( \mathcal{Q}_1, \ldots, \mathcal{Q}_{n'/k} \), labeled so that \( \mathcal{Q}_i \) shares a boundary (that is, a \( (d-1) \)-dimensional face) with \( \mathcal{Q}_{i+1} \) for all \( i < n'/k \). Define for each \( u = (x_u, w_u) \in \Xi_n \subset \Lambda_n \)

\[
\mathcal{Q}(u) := \arg\min_{\mathcal{Q}_i} \|x_u - Q_i\|.
\]

(5.3)
with the convention that \( \|x_u - Q_i\| = 0 \) if \( x_u \in Q_i \), and take the box with the smallest index if the minimum is non-unique. Similarly to (4.4), we observe that for any point \( u \in \Xi_n \subset A_n \)

\[
\sup_{y \in Q(u)} \|x_u - y\| \leq 2\sqrt{d}h^{1/d}.
\] (5.4)

**Step 1. Construction of the backbone.** Recall the definition of \( G_n[a, b] \) from (2.4) in Definition 2.1. We first show that, for some \( w_{hh} = w_{hh}(k) \), the graph \( G_n := G_n[w_{hh}, 2w_{hh}] \) contains a so-called backbone, a connected component \( C_{hh} \) that contains at least \( s_k = \Theta(k^{\alpha_{hh}}) \) vertices in every subbox. Using \( \alpha \) and \( \beta \) from Definition 2.1, define the constant \( C_1 \) to be the solution of the equation

\[
(p/16)\beta \alpha C_1^{-(\alpha+1)/\alpha}/(\tau-1)(2\sqrt{d})^{-\alpha d} = \log(2),
\] (5.5)

\[
\beta C_1^{-(1+\sigma)/(\tau-1)}d^{-d/2}2^{-d-2\sigma} = 1,
\] (5.6)

We set, with \( \gamma_{hh} \) from (1.11),

\[
w_{hh} := w_{hh}(k) := C_1^{1-(\tau-1)/\gamma_{hh}}, \quad s_k := (C_1/16)k^{1-\gamma_{hh}(\tau-1)} = kw_{hh}^{-(\tau-1)/16}.
\] (5.7)

To avoid cumbersome notation, we assume that \( s_k \in \mathbb{N} \). Recall the notation \( \Xi_n[a, b] \) from (2.12). Let

\[
A_{hh} := A_{hh}(n, k) := \begin{cases} G_n[a, b] \text{ contains a connected component } C_{hh}(n, k) \text{ such that } \\
\text{for all } i \leq (n'/k) : |\Xi_n[a, b], 2w_{hh}] \cap C_{hh} | \geq s_k
\end{cases}
\] (5.8)

On \( A_{hh} \), let \( C_{hh} := C_{hh}(n, k) \), the backbone, be the largest component in \( G_n[w_{hh}, 2w_{hh}] \) that satisfies the event \( A_{hh} \). In the following lemma we obtain a lower bound on the probability that there exists a backbone. Observe that \( s_k = \Theta(k^{\alpha_{hh}}) \) in (5.7), so the decay on the right-hand side in (5.9) below is of the same order as the desired decay in Proposition 5.3.

**Lemma 5.2 (Backbone construction).** Consider a supercritical i-KSRG model as in Definition 2.1 and parameters \( \alpha > 1, \tau \in (2, 2+\sigma), \sigma \geq 0, d \in \mathbb{N} \). There exist constants \( \alpha_{hh} = \Theta(k^{\alpha_{hh}/\alpha}) > 0 \) and \( k_1 \in \mathbb{N} \) such that for \( k \geq k_1 \) and all \( n \) satisfying \( n \geq k^{1+\gamma_{hh}/\alpha} \) we have

\[
\mathbb{P}(\neg A_{hh}(n, k)) \leq 3(n/k) \exp \left( -4s_k \right).
\] (5.9)

**Proof.** Towards proving (5.9), we reveal \( \Xi_n[w_{hh}, 2w_{hh}] \), i.e., only the vertex set of \( G_{n,1} \), and define

\[
A_{pol} := \{ \forall i \leq n'/k : |\Xi_n[w_{hh}, 2w_{hh}] | \geq 4s_k \}.
\] (5.10)

On \( A_{pol} \), every box contains enough vertices in \( G_{n,1} \). Recall now the edges of \( G_{n,1} \) only within each of the boxes \( (Q_i)_{i \leq n'/k} \): let \( H_i \) be the induced subgraph of \( G_{n,1} \) on \( \Xi_n[w_{hh}, 2w_{hh}] \), and define

\[
J := \min \{ i : H_i \text{ contains a connected component } C | | C | \geq s_k \}.
\] (5.11)

We write \( J = \infty \) if no such box-index exists. Then

\[
\mathbb{P}(\neg A_{hh}) \leq \mathbb{P}(\neg A_{pol}) + \mathbb{P}(J = \infty \mid A_{pol}) + \sum_{i=1}^{n'/k} \mathbb{P}(J = i \mid A_{pol}) \mathbb{P}(\neg A_{hh} \mid \{ J = i \} \cap A_{pol}).
\] (5.12)

We first bound \( \mathbb{P}(\neg A_{pol}) \) from above. The distribution of \( |\Xi_n[w_{hh}, 2w_{hh}]| \) is Poisson with mean

\[
k w_{hh}^{\tau-1}(1-2^{-\tau-1}) = 16(1-2^{-\tau})s_k \geq 8s_k \quad \text{by (2.2), (5.7) and since } \tau \geq 2.
\]

Lemma 2.1 yields

\[
\mathbb{P}(|\Xi_n[w_{hh}, 2w_{hh}]| \leq 4s_k) \leq \mathbb{P}(\text{Poi}(8s_k) < 4s_k) \leq \exp \left( -4s_k \left(1 - \log(2) \right) \right).
\]

Since \( 1 - \log(2) \geq 1/4 \), by a union bound over the at most \( n'/k \leq n/k \) subboxes we get

\[
\mathbb{P}(\neg A_{pol}) \leq (n/k) \exp(\neg s_k).
\] (5.13)

We will now show an upper bound on the summands in (5.12) that holds uniformly in \( i \). For this, we iteratively ‘construct’ a backbone. The subboxes \( Q_1, \ldots, Q_{n'/k} \) are ordered so that \( Q_i \) and \( Q_{i+1} \) share a boundary for all \( i \). On \( \{ J = i \} \), we know that \( H_i \) inside \( Q_i \) contains a connected component \( C \) with at least \( s_k \) many vertices. We now reveal edges between \( Q_i \) and \( Q_{i+1} \), and bound the probability that there are at least \( s_k \) many vertices in \( Q_{i+1} \) that are connected by an edge to \( C \); denote this set of vertices by \( \tilde{V}_{i+1} \). Next, we apply the same bound to show that at least \( s_k \) many vertices in \( Q_{i+2} \) connect by an edge to \( \tilde{V}_{i+1} \), and so on. For \( i < j \), we proceed similarly. Although \( \{ J = i \} \) implies that the induced graph \( H_{i+1} \subset Q_i \) does not contain a large enough connected component, we can, thanks to conditioning on \( A_{pol} \), still ensure that at least \( s_k \) many vertices in \( Q_{i+1} \) connect directly by an edge to the connected component \( C \) in \( Q_i \), irrespective of the vertex positions in \( Q_{i-1} \). Again, denote these
vertices by \( \hat{V}_{i-1} \). We repeat this procedure for \( \ell \in \{i-2, \ldots, 1\} \). Hence, for \( \ell \geq i \), we need to analyze the probability that a vertex in \( Q_{\ell+1} \) connects to a vertex in \( \hat{V}_i \), conditionally on \( |\hat{V}_i| \geq s_k \). Since by assumption \( \tau < 2 + \sigma \), by definition of \( \gamma_{hh} \) in (1.11), for all \( \tau < 2 + \sigma \) and \( \alpha \leq \infty \), it holds that

\[
1 - (1 + \sigma) \gamma_{hh} \geq 0, \quad \text{and} \quad 2 + \sigma - \tau > 0. \tag{5.14}
\]

Let \( c_d \) be the volume of the unit \( d \)-dimensional ball. Then let \( k_1 \) be the smallest integer so that, depending on the value of \( \alpha \), the following bounds hold (with \( C_1 = C_1(\alpha) \) from (5.5)–(5.6), respectively):

\[
(1 - p)C_1 k_{hh}^{\gamma_{hh}^1/16} \leq 1/2, \quad \text{and} \quad (C_1/4) k_{hh}^{\gamma_{hh}} \leq \exp\left( p \beta c_d 2^{-d} (C_1^{-1/(\tau - 1)} k_1)^{(2+\sigma-\tau) \gamma_{hh}^1/8} \right). \tag{5.15}
\]

The Euclidean distance between neighbors in vertices in neighboring boxes is at most \( 2\sqrt{d}/k \) (twice the diameter of a single box), and all considered vertices have mark at least \( w_{hh} \). When \( \alpha = \infty \), we use that \( \gamma_{hh} = 1/(1 + \sigma) \) (see (1.11)), and so \( w^{1+\sigma}/k = \Theta(1) \) by (5.7). We obtain using (5.6), p in (2.3), that for any \( u = (x_u, w_u) \in \Xi_{Q_{\ell+1}}[w_{hh}, 2w_{hh}] \),

\[
\mathbb{P}\left( (x_u, w_u) \leftrightarrow \hat{V}_i \mid |\hat{V}_i| \geq s_k \right) \geq 1 - \left( 1 - p \mathbb{1}\left\{ \frac{\beta \gamma_{hh}}{(2\sqrt{d})^d k} \geq 1 \right\} \right) s_k \tag{5.16}
\]

for all \( k \geq k_1 \) by the first criterion in (5.15). When \( \alpha < \infty \), using p in (2.3) for \( u \in Q_{\ell+1} \) with \( w_u \geq w_{hh} \), either the minimum is at 1 below in (5.17) (in which case the right-hand side of (5.16) remains valid) or, the minimum in p is attained at the second term below in (5.17); then we substitute \( s_k \) from (5.7),

\[
\mathbb{P}\left( (x_u, w_u) \leftrightarrow \hat{V}_i \mid |\hat{V}_i| \geq s_k \right) \geq 1 - \left( 1 - p \min \left\{ \frac{\beta \gamma_{hh}}{(2\sqrt{d})^d k} \geq 1 \right\} \right) s_k \tag{5.17}
\]

By choice of \( w_{hh} \), and \( \gamma_{hh} \) and \( C_1 \) and as defined in (5.7), (5.5), and (1.11), respectively, factors containing \( k \) cancel, and after simplification we arrive at

\[
\mathbb{P}(u \leftrightarrow \hat{V}_i \mid |\hat{V}_i| \geq s_k) \geq 1 - \exp\left( - (p/16) \beta \gamma_{hh}^1/2 \beta \gamma_{hh} \beta c_d 2^{-d} (C_1^{-1/(\tau - 1)} k_1)^{(2+\sigma-\tau) \gamma_{hh}^1/8} \right) = 1/2. \tag{5.18}
\]

Combining (5.18) with (5.10), we obtain a lower bound of \( 1/2 \) for all \( \alpha > 1 \) for any \( u \in \Xi_{Q_{\ell+1}}[w_{hh}, 2w_{hh}] \). On \( A_{poi} \) (see (5.10)) there are at least \( 4s_k \) vertices in \( \Xi_{Q_{\ell+1}}[w_{hh}, 2w_{hh}] \). Each of these vertices connects conditionally independently by an edge to vertices in \( \hat{V}_i \), with probability at least 1/2, so for all \( \ell \geq i \)

\[
\mathbb{P}(|\hat{V}_{i+1}| \geq s_k \mid |\hat{V}_i| \geq 4s_k, A_{poi}) \geq \mathbb{P}(\text{Bin}(4s_k, 1/2) \geq s_k) \geq 1 - \exp(-s_k/4),
\]

where the last bound follows by Chernoff’s bound, see e.g. [10, Theorem 2.1]. When \( J = i \) and \( \ell < i \), we can analogously bound the probability that \( |\hat{V}_i| \geq s_k \), conditionally on \( |\hat{V}_{i+1}| \geq s_k \). By a union bound over the at most \( n'/k \) subboxes (with indices both smaller as well as larger than \( i \)), \( w_{hh} \) smaller, we obtain

\[
\mathbb{P}(\neg A_{hh} \cap \{ J = i \} \cap A_{poi}) \leq (n'/k) \exp(-s_k/4) \leq (n/k) \exp(-s_k/4).
\]

The bound holds for all \( i \leq n'/k \). Using this in the summation on the right-hand side of (5.12), and using [5.13] to bound \( \mathbb{P}(\neg A_{poi}) \), [5.12] turns into

\[
\mathbb{P}(\neg A_{hh}) \leq (n/k) \exp(-s_k) + \mathbb{P}(J = \infty \mid A_{poi}) + (n/k) \exp(-s_k/4) \mathbb{P}(J \neq \infty \mid A_{poi}).
\]

It remains to bound \( \mathbb{P}(J = \infty \mid A_{poi}) \), with \( J \) from (5.11). For this we show that the graph \( H_{i+1} \) induced on \( \Xi_{Q_i}[w_{hh}, 2w_{hh}] \) stochastically dominates a soft random geometric graph above its connectivity threshold. Indeed, consider \( (x_u, w_u) \) and \( (x_v, w_v) \) in \( \Xi_{Q_i}[w_{hh}, 2w_{hh}] \). Then using (2.3), for \( \alpha = \infty \),

\[
p((x_u, w_u), (x_v, w_v)) \geq p \min\left\{ 1, \beta \gamma_{hh}^1(1+\sigma)\alpha \|x_u - x_v\|^{-\alpha} \right\} \geq p,
\]

whenever \( \|x_u - x_v\| \leq \beta^{1/d} w_{hh}^{1+\sigma}/d = r_{hh} \). The calculation for \( \alpha = \infty \) is analogous yielding the same radius \( r_{hh} \) and same lower bound \( p \). Hence, \( H_{i+1} \succeq RG_1 \) where in \( RG_1 \) is a soft random geometric graph: conditioned on \( A_{poi} \), there are say \( N_i \geq 4s_k \) many vertices in \( \Xi_{Q_i}[w_{hh}, 2w_{hh}] \), and two vertices are connected by an edge with probability \( p \) whenever they are within distance \( r_{hh} = \beta^{1/d} w_{hh}^{1+\sigma}/d \) of each other. Writing \( B_r \) for the Euclidean ball of radius \( r \) around the origin and for some dimension
dependent constant \( c_d \), the expected degree of a vertex at any location in \( R \) is thus, conditional on \( N_i \):

\[
\mathbb{E}[\deg_{RG_i}(v) \mid N_i] \geq pN_i \frac{\text{Vol}(B_{\rho})}{k} = N_i \cdot \frac{pc_d \beta 2^{-d} \cdot w_{hh}^{1+\sigma}}{k}.
\]

(5.20)

We apply Equation (2.6) below Theorem 2.1] to \( R \), and combine it with (5.19), this yields the statement of the lemma in (5.9) with

\[
\mathbb{P}(\mathcal{H}_{\ell, i} \text{ not connected } \mid N_i) \leq \mathbb{P}(\mathcal{R}(\mathcal{G}) \text{ not connected } \mid N_i) \leq N_i \exp (-\mathbb{E}[\deg_{RG_i}(v) \mid N_i]).
\]

On the event \( A_{\text{poi}}, N_i \geq 4k \) holds for all \( i \). Hence, by (5.20), uniformly for all \( i \leq n' / k \),

\[
\mathbb{P}(\mathcal{H}_{\ell, i} \text{ not connected } \mid A_{\text{poi}}) \leq \sup_{N_i \geq 4k} N_i \exp (-\mathbb{E}[\deg_{RG_i}(v) \mid N_i])
\]

\[
\leq 4k \exp (-4sk \cdot \frac{pc_d \beta 2^{-d} \cdot w_{hh}^{1+\sigma}}{k})
\]

\[
\leq (C_1/4)^k \mathbb{E}[\deg_{\text{poi}}] \exp (-\frac{pc_d \beta 2^{d} / \alpha}{k^{1+\sigma}}),
\]

where we used \( s_k \) from (5.7) to obtain the last row. Using the second criterion in (5.15), for all \( k \geq k_1 \),

\[
\mathbb{P}(\mathcal{H}_{\ell, i} \text{ not connected } \mid A_{\text{poi}}) \leq \exp(-pc_d \beta 2^{-d} w_{hh}^{2+\sigma} / k) =: p_1.
\]

By (5.14), the exponent of \( w_{hh} \) is positive. Since the induced subgraphs \( (\mathcal{H}_{\ell, i})_{\leq n' / k} \) are independent (being contained in disjoint boxes \( Q_i \)), and the number of boxes is \( n' / k = [(n/k)^{1/d}] d \geq n/(2k) \), using the definition of \( \mathbb{P}(J = \infty \mid A_{\text{poi}}) \leq \mathbb{P}(\forall i \leq n' / k : \mathcal{H}_{\ell, i} \text{ not connected } \mid A_{\text{poi}}) \leq p_1^{n/(2k)} \)

\[
\leq \exp (-pc_d \beta 2^{-d} w_{hh}^{2+\sigma} / k)
\]

\[
\leq \exp (-pc_d \beta 2^{-d} w_{hh}^{2+\sigma} / k).
\]

Using (1.11) and (1.12), it is elementary to check that \( \gamma_{hh}(2 + \sigma - \tau) = \zeta_{hh}(1 - 1/\alpha) \). Hence, whenever \( n' / k > k_{1+\sigma}^\alpha / \alpha \), the exponent of \( k \) on the right-hand side is (at least) \( \zeta_{hh} \). Since we assumed \( n \geq k_{1+\sigma}^\alpha / \alpha \), this is indeed the case. Furthermore, since \( s_k = (C_1/16)^k \) by (5.7), with \( c_2 := (pc_d \beta 2^{-d}) C_1^{-2+\sigma-\tau}(1 - 1)^{-1} \), using \( C_1 \) from (5.5), we obtain that \( \mathbb{P}(J = \infty \mid A_{\text{poi}}) \leq \exp(-c_2 s_k) \). Combined with (5.19), this yields the statement of the lemma in (5.9) with \( c_{\text{cd}} := \min\{c_2, 1/4\} \). \( \square \)

We will end Step 1 with a claim that shows (3.2), using notation for the construction of the graph \( \mathcal{G}_n \) that facilitates later steps that we introduce first. We recall the definition of i-KSRG from Definition 2.1. Given the vertex set \( \Xi \), it is standard practice to use independent uniform random variables to facilitate couplings with the edge-set. This definition here is more general and allows for other helping random variables as well, leading to different distributions on graphs. This will be useful later.

**Definition 5.3** (Graph encoding). Let \( \xi \subset \mathbb{R}^d \times [1, \infty) \) be a discrete set and assume that \( \Psi_\xi = \{ \varphi_{u,v} : \varphi_{u,v} \in [0,1], \{u,v\} \in (\xi) \} \) is a collection of random variables given \( \xi \). For a given connectivity function \( p : (\mathbb{R}^d \times [0,1])^2 \rightarrow [0,1] \), we call \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) the (sub)graph encoded by \( (\xi, \Psi_\xi, p) \). If \( \mathcal{V} = \{u,v\} \) and for all \( \{u,v\} \in (\xi) \), with \( u = (x_u, w_u), v = (x_v, w_v) \),

\[
\{\{u,v\} \in (\xi) \} \iff \{\varphi_{u,v} \leq p((x_u, w_u), (x_v, w_v))\}.
\]

(5.21)

Given \( \Xi \) in (2.2), and \( p \) from (2.3), let \( \mathcal{G}_\xi \) be a collection of independent \( \text{Unif}[0,1] \) random variables given \( \Xi \). \( \mathcal{G}_\infty \) in Definition 2.1 is then the graph encoded by \( (\Xi, \Psi_\Xi, p) \). Writing \( \Psi_n[a,b] := \{\varphi_{u,v} \in \Psi_\Xi : \{u,v\} \in (\Xi_n[a,b]) \} \) and \( \Psi_{n,\Xi} := \Psi_n[1, \infty), \mathcal{G}_n \) in (2.4) is then the graph encoded by \( (\Xi_n, \Psi_{n,\Xi}, p) \).

An immediate corollary is the following:

**Corollary 5.4.** Assume \( \hat{\mathcal{G}}, \hat{\mathcal{G}} \) are two random graphs, encoded respectively by \( (\Xi, \hat{\Psi}, p) \), and \( (\Xi, \hat{\Psi}, p) \) for respective point processes \( \Xi, \hat{\Xi} \) on \( \mathbb{R}^d \times [1, \infty) \) using the same connectivity function \( p \). If \( (\Xi, \Psi) \) and \( (\hat{\Xi}, \hat{\Psi}) \) have the same law then the encoded graphs \( \hat{\mathcal{G}} \) and \( \hat{\mathcal{G}} \) also have the same law.

The collection of (conditionally) independent uniform variables \( \Psi_n = \{\varphi_{u,v} : \{u,v\} \in \Xi_n\} \) and the connectivity function \( p \) determine the presence of edges in \( \mathcal{G}_n \). By (5.21), if for some \( r > 0 \) it holds that \( \varphi_{u,v} \leq r \leq p(u,v) \), then \( \{u \leftrightarrow v\} \). Writing \( \mathcal{Q}(u) \) for the box containing or closest to \( u \in \Xi_n \) (see (5.3), let \( v_1(1), v_2(2), \ldots, v_n(s_k) \), denote the vertices in \( \mathcal{Q}(u) \cap C_{\text{pois}} \) in decreasing order with respect to their marks. Let

\[
\mathcal{S}(u) := \{v_1(1), \ldots, v_n(s_k)\}.
\]

(5.22)
Claim 5.5 (Connections to the backbone). Consider an $i$-KSRG satisfying the conditions of Proposition 5.1. Fix $n$ and $k$ and assume $G_{n,1}$ satisfies the event $A_{bb}(n,k)$. Let $\Psi_n = \{ \varphi_{u,v} : u,v \in (\Xi_n) \}$ be a collection of iid $\text{Unif}[0, 1]$ random variables and $r_k := 1 - 2^{-1/s_k}$. Then, for all $u \in \Xi_n[2w_{hh}(k), \infty)$ and $v \in S(u)$, $p(u, v) \geq r_k$ and

$$P(\forall v \in S(u) : \varphi_{u,v} > r_k | G_{n,1}, A_{bb}) = P(\exists v \in S(u) : \varphi_{u,v} \leq r_k | G_{n,1}, A_{bb}) = 1/2.$$  

(5.23)

Proof. On the event $A_{bb}, C_{bb} \subseteq G_{n,1}$ satisfies (5.8) and in particular $S(u)$ in (5.22) is well-defined and has size $s_k$. Since $\{ \varphi_{u,v} \}$ is a collection of iid $\text{Unif}[0, 1]$ random variables, (cf. Definition 5.3), one must set $r_k := 1 - 2^{-1/s_k}$ for (5.23) to hold. Hence, it only remains to show $p(u, v) \geq r_k$ in the statement.

With $Q(u)$ and $S(u)$ from (5.3) and (5.22), respectively, by (5.19), every $u \in \Xi_n[2w_{hh}, \infty)$ is at distance at most $2\nu dk$ from any vertex in $v \in S(u)$. Since $w_u \geq 2w_{hh} \geq w_{hh}$, and $|S(u)| = s_k$, the computations (5.16) - (5.18) word-by-word carry through with $\hat{V}$ replaced by $S(u)$, obtaining

$$P(u \leftrightarrow S(u) | G_{n,1}, \Xi_n, A_{bb}) = 1 - \prod_{v \in S(u)} (1 - p(u, v)) \geq 1 - (1 - z_k)^{s_k} \geq 1/2,$$

with $z_k$ either equaling $p$ in the right-hand side of (5.16) or the appropriate expression in the right-hand side of (5.17), that bound individually each $p(u, v)$ from below. Following now the calculations towards (5.18) ensures that in both cases $z_k \geq 1 - 2^{-1/s_k}$. The assumptions on $k \geq k_1$ in (5.15) are also needed for this (for instance that $p \geq r_k$).

Step 2. Revealing low-mark vertices. Having established that $G_{n,1}$ contains a backbone with sufficiently high probability, we define $G_{n,2} := G_{n,1}[1, 2w_{hh}] \supseteq G_{n,1}$.

Step 3. Presampling the vertices connecting to the backbone. We make Step 3 of Section 3.1.1 precise now. Step 3 ensures that during Step 4 below no small-to-large merging occurs when revealing the connector vertices of $\Xi_n[2w_{hh}, \infty)$. That is, components of size smaller than $k$ do not merge into a larger component via edges to a vertex $v \in \Xi_n[2w_{hh}, \infty)$ that is not connected to the backbone $C_{bb} (C_{bb} \subseteq G_{n,1})$ is contained in the giant component of $G_n$). So, we partially pre-sample some randomness that encodes the presence of some edges.

For a pair $n, k$, we now present the alternative graph-encoding $\tilde{G}_n$ of KSRGs (cf. Definitions 2.1 and 5.3) and verify that $\tilde{G}_n$ and $G_n$ in Definition 2.1 have the same law. The difference between the encoding in Definition 5.3 and the construction of $G_n$ is that in the latter the edge-variables $\varphi_{u,v}$ are no longer independent $\text{Unif}[0, 1]$ random variables, but are sampled from a suitable (conditional) joint distribution, whenever $u \in \Xi_n[2w_{hh}(k), \infty)$ and $v \in S(u)$ from (5.22). Recall $r_k = 1 - 2^{-1/s_k}$ from Claim 5.5 with $w_{hh}(k) := w_{hh}$ and $s_k$ defined in (5.7).

Definition 5.6 (Alternative graph construction). Fix $n$ and $k$. Let $G_{n,2} = G_{n,1}[1, 2w_{hh}(k)]$ from Definition 2.1 be the graph encoded by $(\Xi_n[1, 2w_{hh}], \Psi_n[1, 2w_{hh}], p)$. Let $\hat{\Xi}_n^{(\text{unsure})}[2w_{hh}, \infty)$ and $\hat{\Xi}_n^{(\text{sure})}[2w_{hh}, \infty)$ be two independent Poisson point processes on $\Lambda_n \times [2w_{hh}, \infty)$ with intensity $(1/2)\text{Leb} \otimes F_W(\text{d}w)$, with $F_W$ as in (2.2), and let

$$\hat{\Xi}_n[2w_{hh}, \infty) := \hat{\Xi}_n^{(\text{unsure})}[2w_{hh}, \infty) \cup \hat{\Xi}_n^{(\text{sure})}[2w_{hh}, \infty).$$

(5.24)

Let $\Sigma_n := \{ U_{u,v} : u \in \hat{\Xi}_n[2w_{hh}, \infty), v \in \Xi_n[1, 2w_{hh}] \cup \hat{\Xi}_n[2w_{hh}, \infty) \}$ be a collection of iid $\text{Unif}[0, 1]$ random variables (conditionally on these PPPs).

(i) If $G_{n,1} = G_n[2w_{hh}, 2w_{hh}] \subseteq G_{n,2}$ does not satisfy the event $A_{bb}$ in (5.8), then set $\hat{\Psi}_n := \Psi_n[1, 2w_{hh}] \cup \Sigma_n$ in Definition 5.3 to construct $\tilde{G}_n \supseteq G_{n,2}$ on $\Xi_n[1, 2w_{hh}] \cup \hat{\Xi}_n[2w_{hh}, \infty)$, i.e.,

$$\tilde{G}_n := (\Xi_n[1, 2w_{hh}] \cup \hat{\Xi}_n[2w_{hh}, \infty), \Psi_n[1, 2w_{hh}] \cup \Sigma_n, p).$$

(ii) If $G_{n,1} \subseteq G_{n,2}$ satisfies the event $A_{bb}$, then we construct $\tilde{G}_n \supseteq G_{n,2}$ conditionally on $G_{n,2}$ as follows. For each $u \in \hat{\Xi}_n[2w_{hh}, \infty)$ in (5.24), the set of vertices $S(u) \subseteq \Xi_n[1, 2w_{hh}]$ is a deterministic function of $G_{n,1} \subseteq G_{n,2}$, given by (5.22). Let

$$\hat{\Psi}_{n}^{(\text{unsure})}(u) := \{ U_{u,v} : u \in \hat{\Xi}_n^{(\text{unsure})}[2w_{hh}, \infty), v \in \hat{\Xi}_n^{(\text{unsure})}[2w_{hh}, \infty) \cup \Xi_n[1, 2w_{hh}] \setminus S(u) \},$$

$$\hat{\Psi}_{n}^{(\text{sure})}(u) := \{ U_{u,v} : u \in \hat{\Xi}_n^{(\text{sure})}[2w_{hh}, \infty), v \in \hat{\Xi}_n^{(\text{sure})}[2w_{hh}, \infty) \cup \Xi_n[1, 2w_{hh}] \setminus S(u) \},$$

$$\hat{\Psi}_{n}^{(\text{id,both})}(u) := \{ U_{u,v} : u \in \hat{\Xi}_n^{(\text{sure})}[2w_{hh}, \infty), v \in \hat{\Xi}_n^{(\text{unsure})}[2w_{hh}, \infty) \}$$

(5.25)
be disjoint subsets of $\Sigma_n$, and write $\hat{\Psi}^{(\text{ind})} := \hat{\Psi}^{(\text{ind, unsure})} \cup \hat{\Psi}^{(\text{ind, sure})} \cup \hat{\Psi}^{(\text{ind, both})}$ for the union. Conditionally on $\hat{\Xi}_n^{(\text{unsure})}[2w_{\text{hh}}, \infty)$, $\hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty)$ and $\mathcal{G}_n[1, 2w_{\text{hh}})$, define also the collections of random variables

$$\hat{\Psi}_n^{(\text{cond, unsure})} := \{ \hat{\varphi}_{u,v} : u \in \hat{\Xi}_n^{(\text{unsure})}[2w_{\text{hh}}, \infty), v \in S(u) \},$$

$$\hat{\Psi}_n^{(\text{cond, sure})} := \{ \hat{\varphi}_{u,v} : u \in \hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty), v \in S(u) \},$$

so that for different vertices $u_1, u_2 \in \hat{\Xi}_n[2w_{\text{hh}}, \infty)$, the collections $\{ \hat{\varphi}_{u_1,v} \}_v \subseteq S(u_1)$ and $\{ \hat{\varphi}_{u_2,v} \}_v \subseteq S(u_2)$ are independent. The joint distribution of $\{ \hat{\varphi}_{u,v} \}_{v \in S(u)}$ for a single $u \in \hat{\Xi}_n^{(\text{unsure})}[2w_{\text{hh}}, \infty)$ is as follows: for any sequence $(z_{u,v})_{v \in S(u)} \in [0,1]^{|S(u)|}$ of length $s_k$, and with $r_k = 1 - 2^{-1/s_k}$,

$$P\left( \forall v \in S(u) : \hat{\varphi}_{u,v} \leq z_{u,v} \mid u \in \hat{\Xi}_n^{(\text{unsure})}[2w_{\text{hh}}, \infty) \right):= P\left( \forall v \in S(u) : U_{u,v} \leq z_{u,v} \mid \forall v \in S(u) : U_{u,v} > r_k \right).$$

Similarly we define the joint distribution of $\{ \hat{\varphi}_{u,v} \}_{v \in S(u)}$ for a single $u \in \hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty)$ as follows: for any sequence $(z_{u,v})_{v \in S(u)} \in [0,1]^{|S(u)|}$ of length $s_k$,

$$P\left( \forall v \in S(u) : \hat{\varphi}_{u,v} \leq z_{u,v} \mid u \in \hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty) \right):= P\left( \forall v \in S(u) : U_{u,v} \leq z_{u,v} \mid \exists v \in S(u) : U_{u,v} \leq r_k \right).$$

We define $\hat{\mathcal{G}}_n$ as the graph encoded by $(\hat{\Xi}_n, \hat{\Psi}_n, p)$, where

$$\hat{\Xi}_n := \Xi_n[1, 2w_{\text{hh}}) \cup \hat{\Xi}_n^{(\text{unsure})}[2w_{\text{hh}}, \infty) \cup \hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty),$$

$$\hat{\Psi}_n := \Psi_n[1, 2w_{\text{hh}}) \cup \hat{\Psi}_n^{(\text{ind})} \cup \hat{\Psi}_n^{(\text{cond, unsure})} \cup \hat{\Psi}_n^{(\text{cond, sure})}.$$ 

An immediate corollary is the following statement.

**Corollary 5.7.** Consider an $i$-KSRG $\hat{\mathcal{G}}_n$ following Definition 5.6 for some $n, k$. On the event $\mathcal{A}_{hh}(n,k)$, every vertex in $\hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty)$ is connected by an edge to $\mathcal{C}_{hh}(n,k)$ in $\hat{\mathcal{G}}_n$.

**Proof.** The conditioning in (5.29) guarantees that for each $u \in \hat{\Xi}_n^{(\text{sure})}[2w_{\text{hh}}, \infty)$ at least one $\hat{\varphi}_{u,v} \leq r_k$ occurs among the edge-variables $\{ \hat{\varphi}_{u,v} : v \in S(u) \}$, where $S(u) \subseteq \mathcal{C}_{hh}$ (cf. (5.22)). Then since $\hat{\varphi}_{u,v} \leq r_k \leq p(u,v)$ holds by Claim 5.5, this ensures that $\{ u, v \}$ is in the edge set of $\hat{\mathcal{G}}_n$ by the graph-encoding in Definition 5.3.

**Proposition 5.8.** Fix a connectivity function $p$. The law of the random graph $\hat{\mathcal{G}}_n$ formed by Definition 5.6 is identical to the law of the random graph $\hat{\mathcal{G}}_n$ formed by Definition 5.3.

**Proof.** By Corollary 5.4 it is sufficient to show that $(\hat{\Xi}_n, \hat{\Psi}_n)$ defined in (5.30) has the same distribution as $(\Xi_n, \Psi_n)$ from Definition 5.6. By standard properties of PPPs, $\Xi_n$ can be written as two independent PPPs defined on the same space, each having half the intensity measure of $\Xi_n$. Thus, by the construction of $\Xi_n[1, 2w_{\text{hh}})$ in Definition 5.6 it holds that the PPPs $\Xi_n$ and $\hat{\Xi}_n$ have the same law. Hence, we may couple the vertex sets so that a.s. $\Xi_n = \hat{\Xi}_n$, and only need to show that the collection of variables encoding the edges, $\Psi_n$ and $\hat{\Psi}_n$, share the same law, conditioned under the given vertex set realization (say) $\Xi_n$. By (5.30) in Definition 5.6 the graph $\hat{\mathcal{G}}_{n,2}$ spanned on $\Xi_n[1, 2w_{\text{hh}}) \subseteq \hat{\Xi}_n$ is determined by $\Psi_n[1, 2w_{\text{hh}}) = \{ \varphi_{u,v} : u, v \in \Xi_n[1, 2w_{\text{hh}}) \}$ in Definition 5.3. Thus $\hat{\mathcal{G}}_{n,2}$ has the same distribution both in Definition 5.3 and in Definition 5.6.

(i) If now $\Psi_n[1, 2w_{\text{hh}})$ is such that the graph $\mathcal{G}_{n,2}$ does not satisfy the event $\mathcal{A}_{hh}$, by (i) of Definition 5.6 the statement holds since both $\{ \varphi_{u,v} \}$ and $\{ U_{u,v} \}$ are iid uniforms whenever $u \in \Xi_n[2w_{\text{hh}}, \infty)$, i.e., $\Psi_n \setminus \Psi_n[1, 2w_{\text{hh}}) = \Sigma_n$ and $\Psi_n \setminus \Psi_n[1, 2w_{\text{hh}})$ have the same distribution. 

(ii) If $\Psi_n[1, 2w_{\text{hh}})$ is such that the graph $\mathcal{G}_{n,2}$ does satisfy the event $\mathcal{A}_{hh}$, then we work conditionally on a realization of the graph $\hat{\mathcal{G}}_{n,2} = (\Xi_n[1, 2w_{\text{hh}}), \Psi_n[1, 2w_{\text{hh}}), p)$, and also on the coupled realization of the PPPs $\Xi_n[2w_{\text{hh}}, \infty)$ $\Xi_n[2w_{\text{hh}}, \infty)$. Let us define the conditional probability measure (of the edges) under the coupling by

$$P^*(\cdot) := P(\cdot | \mathcal{G}_{n,2}, \Xi_n[2w_{\text{hh}}, \infty)) = P(\cdot | \mathcal{G}_{n,2}, \text{unlabeled } \Xi_n[2w_{\text{hh}}, \infty)),$$

where in the conditioning we do not reveal to which sub-PPP $\Xi_n^{(\text{sure})}[2w_{\text{hh}}, \infty)$ or $\Xi_n^{(\text{unsure})}[2w_{\text{hh}}, \infty)$ it belongs to. Using $\hat{\Psi}_n$ from (5.30) and $\hat{\Psi}_n^{(\text{ind})} \subseteq \Sigma_n$ from (5.25) (containing independent copies $U_{u,v}$ of Unif[0, 1] random variables, like $\Psi$ in Definition 5.3), we see that variables
in $\Psi_n \setminus \Psi_n[1, 2w_{hh})$ and $\hat{\Psi}_n \setminus \Psi_n[1, 2w_{hh})$ also share the same (joint) law of iid $\text{Unif}[0, 1]$ whenever $u$ and $v$ are such $u, v \in \hat{\Xi}_n[2w_{hh}, \infty)$ and that $v \notin S(u)$. Moreover, in \cite{5.26, 5.27}, the collections $\{\hat{\varphi}_{u,v} \mid v \in S(u)\}$ are independent across $u$ for different vertices $u \in \hat{\Xi}_n[2w_{hh}, \infty)$. So for $\hat{\mathcal{G}}_n \triangleq \mathcal{G}_n$ it is left to show that for any fixed $u \in \hat{\Xi}_n[2w_{hh}, \infty) = \Xi_n[2w_{hh}, \infty)$, under the measure $\mathbb{P}^*$, 
\begin{equation}
\{ \varphi_{u,v} \in S(u) \} \triangleq \{ \hat{\varphi}_{u,v} \in S(u) \}. \tag{5.32}
\end{equation}

We first analyze the distribution of the left-hand side, i.e., $\varphi_{u,v}$ being iid from Definition 5.3. Let $(z_{u,v})_{v \in S(u)} \in [0, 1]^{S(u)}$ be any sequence of length $s_k$. By Claim 5.5 and the law of total probability
\begin{equation}
\mathbb{P}^*(\forall v \in S(u) : \varphi_{u,v} \leq z_{u,v}) = (1/2)\mathbb{P}^*(\forall v \in S(u) : \varphi_{u,v} \leq z_{u,v} \mid \forall v \in S(u) : \varphi_{u,v} > r_k) + (1/2)\mathbb{P}^*(\forall v \in S(u) : \varphi_{u,v} \leq z_{u,v} \mid \exists v \in S(u) : \varphi_{u,v} \leq r_k). \tag{5.33}
\end{equation}

We now analyze the right-hand side in (5.32). By the construction in (5.24), $\hat{\Xi}_n[2w_{hh}, \infty)$ is the union of two iid sub-PPPs. Under $\mathbb{P}^*$ in (5.31) we did not reveal to which sub-PPP vertices belong to. Hence, for each $u \in \hat{\Xi}_n[2w_{hh}, \infty)$, independently of each other
\begin{equation}
\mathbb{P}^*(u \in \hat{\Xi}_n^{(unsure)}[2w_{hh}, \infty) \mid u \in \hat{\Xi}_n[2w_{hh}, \infty)) = \mathbb{P}^*(u \in \hat{\Xi}_n^{(sure)}[2w_{hh}, \infty) \mid u \in \hat{\Xi}_n[2w_{hh}, \infty)) = 1/2.
\end{equation}

Thus, by the law of total probability, and using the distributions of $(\varphi_{u,v})_{v \in S(u)}$ given by (5.28) and (5.29),
\begin{equation}
\mathbb{P}^*(\forall v \in S(u) : \varphi_{u,v} \leq z_{u,v}) = (1/2)\mathbb{P}^*(\forall v \in S(u) : \varphi_{u,v} \leq z_{u,v} \mid u \in \hat{\Xi}_n^{(unsure)}[2w_{hh}, \infty)) + (1/2)\mathbb{P}^*(\forall v \in S(u) : \varphi_{u,v} \leq z_{u,v} \mid u \in \hat{\Xi}_n^{(sure)}[2w_{hh}, \infty)) \tag{5.34}
\end{equation}

Note that $\{U_{u,v}\}_{u,v}$ and $\{\varphi_{u,v}\}_{u,v}$ are both sets of independent $\text{Unif}[0, 1]$ random variables by Definitions 5.6 and 5.3 respectively. Hence, (5.32) follows by combining (5.33) and (5.34).

For the remainder of this section, we assume that we construct $\mathcal{G}_n$ following Definition 5.6 and write
\begin{equation}
\Xi_n[2w_{hh}, \infty) := \Xi_n^{(unsure)}[2w_{hh}, \infty) \cup \Xi_n^{(sure)}[2w_{hh}, \infty)
\end{equation}
as the union of two independent PPPs of equal intensity, such that if $\mathcal{G}_{n,2} = \mathcal{G}[1, 2w_{hh})$ satisfies $\mathcal{A}_{hh}$ in (5.8), every vertex in $\Xi_n^{(sure)}[2w_{hh}, \infty)$ connects by an edge to $\mathcal{C}_{hh}$, by Corollary 5.7.

To finish Step 3, on the event $\mathcal{A}_{hh}$, we define $\mathcal{G}_{n,3} := (\mathcal{V}_{n,3}, \Psi_{n,3})$, with
\begin{equation}
\mathcal{V}_{n,3} := \Xi_n[1, 2w_{hh}) \cup \Xi_n^{(sure)}[2w_{hh}, \infty), \quad \Psi_{n,3} := \Psi_n[1, 2w_{hh}) \cup \hat{\Psi}_{n}^{(unsure)} \cup \hat{\Psi}_{n}^{(cond,unsure)}, \tag{5.35}
\end{equation}
i.e., the graph spanned on $\mathcal{V}_{n,3}$. We call the vertices in $\Xi_n^{(sure)}[2w_{hh}, \infty)$ sure-connector vertices. If the event $\mathcal{A}_{hh}$ does not hold then we say that the construction failed and we leave $\mathcal{G}_{n,3}$ undefined.

**Step 4. Cover expansion.** In this step, we ensure that all components of size at least $k$ of $\mathcal{G}_{n,3}$ merge with the giant component of $\mathcal{G}_n$ via edges towards sure-connector vertices, with error probability $\mathsf{err}_{n,k}$ from (3.1). The next lemma proves this using the cover-expansion technique of Section 4. The notion of expandability is from Definition 4.1 and $s_k$ is from (5.7). Let for some $q_{n,k} > 0$
\begin{equation}
\mathcal{A}_{\exp}(q_{n,k}) := \mathcal{A}_{\exp}(n, k, q_{n,k}) \triangleq \{ \mathcal{V}_{n,3} \text{ is } (q_{n,k})\text{-expandable} \}. \tag{5.36}
\end{equation}

**Lemma** 5.9 (Cover-expansion and $\sigma \leq \tau - 1$). Consider an $i$-KSRG satisfying the conditions of Proposition 5.7. For any $q_{n,k} > 0$
\begin{equation}
\mathbb{P}(\neg \mathcal{A}_{\exp}(q_{n,k})) \leq \mathcal{C}_{n,k} \exp(-q_{n,k}^p). \tag{5.37}
\end{equation}

Moreover, if $\sigma \leq \tau - 1$, there exist $k_2$, $\mathcal{C}_{n,k_2} > 0$, such that conditionally on any realization of $\mathcal{G}_{n,3}$ satisfying $\mathcal{A}_{hh} \land \mathcal{A}_{\exp}(q_{n,k})$, for all $k \geq k_2$ and any connected component $\mathcal{C}$ of $\mathcal{G}_{n,3}$ with $|\mathcal{C}| \geq k$,
\begin{equation}
\mathbb{P}(\mathcal{C} \neq \Xi_n^{(sure)}[2w_{hh}, \infty) \mid \mathcal{G}_{n,3}, \mathcal{A}_{hh} \land \mathcal{A}_{\exp}(q_{n,k})) \leq \exp(-q_{n,k}^p). \tag{5.38}
\end{equation}

**Proof.** The statement (5.37) follows directly for any $q_{n,k} > 0$ from its definition in (5.36) and Lemma 4.11. In Proposition 4.2, for a given mark $w$, the function $s(w)$ in (4.1) describes the necessary “expandability parameter”, such that all vertices with mark at least $w$ in $\mathcal{K}_n(\mathcal{C})$ connect to any $s(w)$-expandable set $\mathcal{L}$ of vertices with probability at least $p/2$. We shall take $w := 2w_{hh}$, the lowest
Since $V_{n,3}$ is $(c_3 n^{k_{33}})$-expansible, it is also $s(2w_{wh})$-expansible whenever

$$s(2w_{wh}) \geq c_3^{1/\alpha} n^{k_{33}}. \tag{5.39}$$

We compute the left-hand side using the value of $w_{wh}$ from (5.7) and the function $s(w)$ from (4.1):

$$s(2w_{wh}) = (2^{d+1} \beta C_1^{-1/(\tau-1)})^{1/(1-1/\alpha)} n^{\gamma_{wh} / (1-1/\alpha)}.$$  

By (5.7) and (1.12), it holds that $s_k = \Theta(k^{2+\gamma_{wh}(\tau-1)}) = \Theta(k^{\gamma_{wh}})$. Using (1.12), it is elementary to verify that $\gamma_{wh} / (1-1/\alpha) \geq \zeta_{wh}$ if and only if $\sigma \leq \tau - 1$. Thus, if $\sigma \leq \tau - 1$, (5.39) holds for some $c_3^{1/\alpha} > 0$, whenever $k > k_2$ for some sufficiently large $k_2 \geq 1$, that ensures the condition $w > 2^d d^{d/2} / \beta$. Hence, Proposition 4.1 holds for any set $\mathcal{C} \subseteq \mathcal{V}_{n,3}$ satisfying (4.2), and guarantees the existence of a set $\mathcal{K}_n(\mathcal{C})$ connected to $\mathcal{C}$ as

$$\mathcal{H}_C := \{ v \in \mathcal{K}_n(\mathcal{C}) \cap \Xi_{n}(2w_{wh}, \infty) : v \leftrightarrow \mathcal{C} \}. \tag{5.40}$$

Since $\Xi_{n}(2w_{wh}, \infty)$ is a Poisson process, the number of its points in $\mathcal{K}_n(\mathcal{C})$ follows a Poisson distribution. Since each of these points connects by an edge independently to $\mathcal{C}$ with probability at least $p/2$ by (4.3), and an independent thinning of a PPP is another PPP; we obtain using the intensity measure in Definition 5.6 by the volume bound (4.2) on $\mathcal{K}_n(\mathcal{C})$ for $|\mathcal{C}| \geq k$ and $c := (p/4)^2 (2^{d+1})^{2(\tau-1)/d} / e > 0$

$$\mathbb{P}(|\mathcal{H}_C| = 0 \mid \mathcal{G}_{n,3}, \mathcal{A}_{hh} \cap \mathcal{A}_{exp}(c_{33})) \leq \mathbb{P}(\text{Poi}\{p/2\} \cdot (1/2) \cdot \text{Vol}(\mathcal{K}_n(\mathcal{C})) \cdot (2w_{wh})^{-(\tau-1)}) = 0) \leq \exp\left(- \frac{c \cdot k w_{wh}^{-(\tau-1)}}{16 c \cdot s_k}\right),$$

where we used $s_k$ from (5.7) in the last step. Since $\{|\mathcal{H}_C| > 0\}$ in (5.40) implies $\{\mathcal{C} \leftrightarrow \Xi_{n}(2w_{wh}, \infty)\}$, this finishes the proof of (5.38) with $c_{33} := 16(p/4)^2 (2^{d+1})^{2(\tau-1)/d} / e$. □

Combining everything: preventing too large components.

Proof of Proposition 5.1. Assume that $k \geq \max\{k_1, k_2\}$ defined in Lemmas 5.2 and 5.9, respectively, and that $n > k^{1+\zeta_{wh}/\alpha}$, i.e., $n$ satisfies the bound in Lemma 5.2. We construct $\mathcal{G}_n \supseteq \mathcal{G}_{n,3}$ following Definition 5.6, where $\mathcal{G}_{n,3}$ from (5.35) is the subgraph of $\mathcal{G}_n$ induced on $\mathcal{V}_{n,3} = \Xi_{n}(2w_{wh}, \infty)$. The events $\mathcal{A}_{hb}$ in (5.8) and $\mathcal{A}_{exp}(c_{33}) := \mathcal{A}_{exp}$ in (5.36) are measurable with respect to $\mathcal{G}_{n,3}$. Hence, by the law of total probability (taking expectation over realizations of $\mathcal{G}_{n,3}$), we obtain

$$\mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k) \leq \mathbb{E}[\mathcal{A}_{hb} \cap \mathcal{A}_{exp}] \mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k \mid \mathcal{G}_{n,3}, \mathcal{A}_{exp} \cap \mathcal{A}_{hb}) + \mathbb{P}(\neg \mathcal{A}_{hb}) + \mathbb{P}(\neg \mathcal{A}_{exp}). \tag{5.41}$$

The notion of revealed vertices after Step 3 are $\mathcal{V}_{n,3} \subseteq \Xi_{n}(2w_{wh}, \infty)$, and by Corollary 5.7 each vertex in $\Xi_{n}(2w_{wh}, \infty)$ connects by an edge to $\mathcal{C}_{hb}$. Thus each component $\mathcal{C} \notin \mathcal{C}_{hb}$ of $\mathcal{G}_{n,3}$ either remains the same in $\mathcal{G}_n$ or it merges with the component containing $\mathcal{C}_{hb}$ via a vertex in $\mathcal{V} \setminus \mathcal{V}_{n,3}$. Hence, conditionally on $\mathcal{A}_{hb}$ and $\mathcal{G}_{n,3}$,

$$\{\mathcal{C}_{n}^{(2)} \geq k\} \subseteq \{ \exists \text{ a component } \mathcal{C} \text{ of } \mathcal{G}_{n,3} \text{ with } |\mathcal{C}| \geq k : \mathcal{C} \notin \Xi_{n}(2w_{wh}, \infty) \}. \tag{5.42}$$

By a union bound over the at most $\mathcal{V}_{n,3} / k$ components of size at least $k$, (5.38) of Lemma 5.9 yields

$$\mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k \mid \mathcal{G}_{n,3}, \mathcal{A}_{exp} \cap \mathcal{A}_{hb}) \leq \mathbb{P}(\mathcal{V}_{n,3} / k) \mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k) \mathbb{P}(\mathcal{V}_{n,3} / k) \mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k).$$

Substituting this bound into (5.41), and using Lemmas 5.2 and 5.9 to bound the last two terms yields

$$\mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k) \leq \mathbb{E}[\mathcal{V}_{n,3} / k] \mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k) + (3n/k) \mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k) + (3n/k) \mathbb{P}(\mathcal{C}_{n}^{(2)} \geq k).$$

Since $\mathcal{V}_{n,3} \subseteq \Xi_{n}$ by construction, $\mathbb{E}[\Xi_{n}] = n$ by (2.2), and $s_k = \Theta(k^{\gamma_{wh}})$ by (5.7), this finishes the proof of Proposition 5.1 for $k \geq \max\{k_1, k_2\}$ and $n > k^{1+\zeta_{wh}/\alpha}$. For $k < \max\{k_1, k_2\}$, (5.1) is trivially satisfied for $c_{33}^{1/\alpha} > 0$ sufficiently small. □
The backbone: intermediate results. We state two corollaries of the proof of Proposition 5.1 and two propositions based on the backbone constructions for later use. We defer the detailed proofs to the appendix and only give a sketch here. We start with a corollary of the proof of Proposition 5.1.

Corollary 5.10 (Backbone becoming part of the giant). Consider an i-KSRG satisfying the conditions of Proposition 5.7. Then conditionally on the graph $G_{n,2} = G_n[1, 2w_{hh}]$ satisfying $A_{bb}(n,k)$ in (5.8),
\[
\mathbb{P}(C_{bb}(n, k) \notin C_n^{(1)} | G_{n,2}, A_{bb}(n,k)) \leq n/(\Theta(n^{1/\delta})) \exp (-n^{1/\delta}) .
\] (5.43)

Proof sketch. By definition of the backbone in (5.8), the backbone contains at least $k$ vertices. The proof of Proposition 5.1 merges each component of size at least $k$ with the backbone, with error probability given on the right-hand side of (5.43). Hence, the component containing the backbone is the only remaining component of size at least $k$.

The next corollary follows from Lemma 5.2. It is not sharp but it yields a useful temporary estimate.

Corollary 5.11 (Lower bound on largest component). Consider a supercritical interpolating KSRG model as in Definition 2.7 with parameters $\alpha > 1$, $\tau \in (2, 2 + \sigma)$, $\sigma \geq 0$, $d \in \mathbb{N}$. For each $\delta > 0$, there exists a constant $A > 0$ such that for all $n$ sufficiently large
\[
\mathbb{P}(|C_n^{(1)}| \leq n(A \log(n))^{-1/\delta_h}) \leq n^{-\delta}.
\]

Proof. If $A_{bb}(n,k_n)$ holds for some $k_n$, then the largest component $C_n^{(1)}$ must have at least the size of the backbone $C_{bb}(n,k_n)$. Setting $k = k_n = (A \log(n))^{1/\delta_h}$ in (5.9), the backbone exists with probability at least $1 - n^{-\delta}$ for $A = A(\delta)$ sufficiently large, since $s_k = s_{k_n} = (C_1/16)A \log(n)$ by (5.7). Then its size is at least $(n'/k)s_k \geq (n'/k) = \Theta(nk_n^{-1/\delta_h})$ by definition of $n'$ in (5.2), finishing the proof.

The next proposition shows that all vertices with a sufficiently high mark (simultaneously) belong to the largest component $C_n^{(1)}$ with polynomially small error probability.

Proposition 5.12 (Controlling marks of non-giant vertices). Consider an i-KSRG in the setting of Theorem 2.3(ii). For all $\delta > 0$, there exists $M_\delta > 0$ such that for $\overline{w}(n,\delta) = (M_\delta \log(n))^{(1-\sigma_{hh})/\delta_h}$
\[
\mathbb{P}(\exists u \in \Xi_n[\overline{w}(n,\delta), \infty) : u \notin C_n^{(1)}) \leq n^{-\delta}.
\] (5.44)

Proof sketch. We give the detailed proof in the Appendix on page 42 and here a sketch. We consider $k$ as a free parameter, so using Lemma 5.2 with $k = k_n = \Theta((\log(n))^{1/\delta_h})$, a backbone $C_{bb}(n,k_n)$ exists and satisfies $C_{bb}(n,k_n) \subseteq C_n^{(1)}$, with probability at least $1 - n^{-\delta}$ by Corollary 5.10 and Lemma 5.2 (the same calculation as the proof of Corollary 5.11). The choice of $\overline{w}(n,\delta)$ is so that a vertex $u$ with mark $w_u \geq \overline{w}(n,\delta)$ connects to each backbone-vertex in its own subbox with probability at least $p$ in (2.3). For $u$ to not be contained in $C_n^{(1)}$, these $s_{k_n}$ many edges must be all absent. Then we apply concentration inequalities to obtain the result.

Remark 5.13. Combined with the proof of the lower bound of Theorem 2.4 below in Section 7, one may show that Proposition 5.12 is sharp up to a constant factor, i.e., there exist constants $\delta, m_w > 0$ such that for all $n$ sufficiently large
\[
\mathbb{P}(\exists v \in \Xi_n[(m_w \log(n))^{(1-\sigma_{hh})/\delta_h}, \infty) : v \notin C_n^{(1)}) = 1 - n^{-\delta}.
\]

We state a proposition that 0 is with constant probability in a linear-sized component in the induced subgraph $G_{n,2} = G_n[1, 2w_{hh}(k)]$ with $k = k_n = \text{poly}(n)$ and $w_{hh}(k)$ from (5.7). Let $C_{n,2}(0)$ be the component containing 0 in $G_{n,2} \subseteq G_n$, by setting $C_{n,2}(0)$ to be the empty set if $w_0 \geq 2w_{hh}(k)$.

Proposition 5.14 (Existence of a large component). Consider a supercritical i-KSRG model as in Definition 2.7 with parameters $\alpha > 1$, $\tau \in (2, 2 + \sigma)$, $\sigma \geq 0$, $d \in \mathbb{N}$. There exist constants $\rho, m > 0$ such that for all $n$ sufficiently large, when $k = k_n = (m \log(n))^{1/\delta_h}$,
\[
\mathbb{P}^0(|C_{n,2}(0)| \geq pm) \geq \rho, \quad \text{and} \quad \mathbb{P}^0(|C_{\infty}(0)| = \infty) \geq \rho.
\] (5.45)

Proof sketch. For the first inequality in (5.45), we build a connected backbone on vertices with mark in $[w_{hh}(k_n), 2w_{hh}(k_n)]$ using Lemma 5.2. Then we use a second-moment method to show that the origin and linearly many other vertices are connected to this backbone via paths along which the vertex marks are increasing. The second statement follows similarly, forming an infinite path along which the marks are increasing. The detailed proof can be found in Appendix A.1.
6. Upper bound: subexponential decay

In this section we prove Theorem 2.2(ii). We carry out the plan in Section 3.1.2 in detail. Instead of arguing directly for KSRGs with parameters described in Theorem 2.2(ii), we derive general conditions that ensure that a bound on the size of the second-largest component as in Proposition 5.1 readily implies subexponential decay. Recall from Definition 2.1 that \( \mathbb{P}^x \) denotes the conditional measure that \( V \) contains a vertex at location \( x \in \Lambda_n \), that has an unknown mark from distribution \( F_W \).

**Proposition 6.1** (Prerequisites for subexponential decay). Consider a supercritical \( i \)-KSRG model as in Definition 2.1, with parameters \( \alpha > 1, \tau > 2 \) and \( d \in \mathbb{N} \). Assume that there exist \( \zeta, \eta, c, c', \eta, M_c > 0 \), and a function \( n_0(k) = O(k^{1+c'}) \), such that for all \( k \) sufficiently large constant and whenever \( n \in [n_0(k), \infty) \), with \( \overline{w}(n,c) := M_c \log^n(n) \) it holds that for any \( x \in \Lambda_n \)

\[
\mathbb{P}^x (|C_n^{(2)}| \geq k) \leq n^{c'} \exp \left( -ck\zeta \right), \tag{6.1}
\]

\[
\mathbb{P}^x (|C_n^{(1)}| \leq n^c) \leq n^{-1-c}, \tag{6.2}
\]

\[
\mathbb{P}^x \left( \exists v \in \Xi_n(\overline{w}(n,c), \infty) : v \notin C_n^{(1)} \right) \leq n^{-c}. \tag{6.3}
\]

Then there exists a constant \( A > 0 \) such that for all \( k \) sufficiently large constant and \( n \) satisfying \( n \in [n_0(k), \infty) \),

\[
\mathbb{P}^0 (|C_n(0)| \geq k, 0 \notin C_n^{(1)}) \leq A \exp \left( -\left( 1/A \right) k^2 \right), \tag{6.4}
\]

and

\[
\frac{|C_n^{(1)}|}{n} \xrightarrow{\mathbb{P}} \mathbb{P} \left( |C_\infty(0)| = \infty \right), \quad \text{as } n \to \infty. \tag{6.5}
\]

Observe that (6.4) does not follow from a naive application of (6.1), since we sharpened the polynomial prefactor on the right-hand side of (6.1) to a universal constant \( A \) in (6.4), and \( n = \infty \) is also allowed in (6.4). The inequalities (6.1)–(6.3) are satisfied when \( \tau \in (2 + \sigma) \) and \( \sigma \leq \tau - 1 \) by Propositions 5.1 and 5.12 and Corollary 5.11 (we leave it to the reader to verify that the results hold also for the Palm-version \( \mathbb{P}^x \) of \( \mathbb{P} \)). Thus, Theorem 2.2(ii) follows immediately after we prove Proposition 6.1. We state and prove an intermediate claim that we need for Proposition 6.1. We write \( C_n^k \) for the largest component in the graph induced on vertices in \( Q \subseteq \mathbb{R}^d \).

**Claim 6.2** (Leaving the giant). Consider a supercritical \( i \)-KSRG model as in Definition 2.1, under the same setting as Proposition 6.1. Assume that \( \zeta, \eta, c, c', \eta, M_c > 0 \), and \( n \) is sufficiently large and let \( N \in [n, \infty) \). Then there exists \( \delta > 0 \) such that for any box \( Q_n \subseteq \Lambda_N \) with \( \text{Vol}(Q_n) = n \), any \( x \in Q_n \), \( n \) sufficiently large, and all \( N \geq n \), it holds for \( u := (x, w_u) \) that

\[
\mathbb{P}^x \left( u \in C_n^{(1)} \cap u \notin C_N^{(1)} \right) \leq n^{-\delta}. \tag{6.6}
\]

**Proof.** We will first prove the following bound that holds generally for a sequence of increasing graphs \( G_n \subseteq G_{n+1} \subseteq \ldots \), whose largest and second-largest components we denote by \( C_n^{(1)} \) and \( C_n^{(2)} \), respectively. Let \( (k_n)_{n \geq 0} \) and \( (K_n)_{n \geq 0} \) be two non-negative sequences such that \( k_n < K_n \) for all \( n \geq 0 \). Then, for all \( 0 < n < N \leq \infty \),

\[
\mathbb{P} (u \in C_n^{(1)} \cap u \notin C_N^{(1)}) \leq \sum_{n=\tilde{n}}^{N} \left( \mathbb{P}(|C_N^{(1)}| < K_{\tilde{n}}) + \mathbb{P}(|C_N^{(2)}| > k_{\tilde{n}}) \right). \tag{6.7}
\]

We verify the bound using an inductive argument. We define for \( \tilde{n} \geq n \) the events

\[
\mathcal{A}(\tilde{n}) := \{|C_{\tilde{n}}^{(1)}| \geq K_{\tilde{n}} \} \cap \{|C_{\tilde{n}+1}^{(1)}| \leq k_{\tilde{n}+1} \}.
\]

Since by assumption \( k_{\tilde{n}+1} < K_{\tilde{n}} \), the event \( \mathcal{A}(\tilde{n}) \) ensures that \( |C_{\tilde{n}}^{(1)}| \) is already larger than \( |C_{\tilde{n}+1}^{(1)}| \), hence \( \mathcal{A}(\tilde{n}) \) implies that \( C_{\tilde{n}}^{(1)} \subseteq C_{\tilde{n}+1}^{(1)} \). Iteratively applying this argument yields that \( \cap_{\tilde{n} \in [n,N]} \mathcal{A}(\tilde{n}) \) implies that \( \{C_n^{(1)} \subseteq C_N^{(1)} \} \). We combine this with the observation that given that \( u \in V_n \), it holds that \( \{u \in C_n^{(1)} \cap u \notin C_N^{(1)} \} \subseteq \{C_n^{(1)} \notin C_N^{(1)} \} \). This yields

\[
\mathbb{P}(u \in C_n^{(1)} \cap u \notin C_N^{(1)}) \leq \mathbb{P}(C_n^{(1)} \notin C_N^{(1)}) \leq \mathbb{P}(\{C_n^{(1)} \notin C_N^{(1)} \} \cap \bigcap_{\tilde{n}=n}^{N-1} \mathcal{A}(\tilde{n}) \} + \sum_{\tilde{n}=n}^{N-1} \mathbb{P}(\neg \mathcal{A}(\tilde{n})) \tag{6.8}
\]

\[
\leq 0 + \sum_{\tilde{n}=n}^{N-1} \left( \mathbb{P}(|C_{\tilde{n}}^{(1)}| < K_{\tilde{n}}) + \mathbb{P}(|C_{\tilde{n}}^{(2)}| > k_{\tilde{n}}) \right),
\]

as claimed.
showing (6.11). We move on to (6.6) for which we have to define the increasing sequence of graphs. Consider any sequence of boxes \((Q_n)_{n\geq 2}\) such that \(Q_0 \subseteq Q_{n+1} \subseteq \cdots \subseteq Q_N := \Lambda_{\infty}\) and Vol\((Q_n) = \tilde{n}\), and let \(G_n\) denote the induced subgraph of \(\mathcal{G}\) on \(Q_n\) for \(\tilde{n} \in [n, N]\), that is conditioned to contain a vertex at location \(x \in Q_n\). Using the assumed lower bound on \(|C_{\tilde{n}}^{(1)}|\) in (6.2), we set \(K_{\tilde{n}} := \tilde{n}^{c}\), and using the assumed upper bound on \(|C_n^{(2)}|\) in (6.1), there exists a large constant \(A\) so that whenever \(n\) is sufficiently large, and \(\tilde{n} \geq n\), by setting \(k_{\tilde{n}} := (A \log(\tilde{n}))^{1/k_{\tilde{n}}n}\), it holds that
\[
\mathbb{P}^x(\{|C_{\tilde{n}}^{(1)}| < K_{\tilde{n}}\} \leq \tilde{n}^{-c-1}, \quad \mathbb{P}^x(\{|C_{\tilde{n}}^{(2)}| > K_{\tilde{n}}\} \leq \tilde{n}^{-c-1}.
\]
Clearly \(k_{n+1} < K_{\tilde{n}}\) for \(\tilde{n} \geq n\), so that substituting the bounds into (6.8) and summing over \(\tilde{n} \geq n\) yields the assertion (6.6) for any \(\delta < c\) and \(n\) sufficiently large.

We continue to prove Proposition 6.1 starting with some notation. For some \(c \in (0, \min\{\alpha - 1, \tau - 2\})\), and using \(c, c', \eta, \zeta\), and \(w(n, c)\) from the statement of Proposition 6.1
\[
N_k := \exp(k^c(2c')) , \quad n_k := N_k \cup \{N_k, c\} = M_c(k^c)^n, \quad t_k := n_k^{1/d}(2k).
\]
Note that \(N_k, n_k = \exp(\Theta(k^c))\). For \(n < N_k\), the statement (6.4) follows directly from (6.1), since when \(n = N_k\), the right-hand side of (6.1) becomes \(\exp(-k^c/2c)\), so we may set any \(A\) such that \(1/A < c^2/2\) in (6.4). So we may assume in the remainder of the section that \(n > N_k\). We write for \(C_{\Lambda(x,n)}(u)\) the component of vertex \(u := (x, w_u) \in \mathcal{V}\) in the graph \(\mathcal{G}_{\infty}\) restricted to \(\Lambda(x, n)\). Define for \(x \in \mathbb{R}^d\) the two events
\[
A_{\text{low-edge}}(x, n_k, N_k, n_k, \mathbb{W}) := \left\{ C_{\Lambda(x,n)}(u) < k, \exists v_1, v_2 \in \Xi_{\Lambda(x,n)}[1, \mathbb{W}] \text{ such that } v_1 \in C_{\Lambda(x,n)}(u), \|x_{v_1} - x_{v_2}\| \geq t_k, \text{ and } v_1 \leftrightarrow v_2 \right\},
\]
(6.10)
\[
A_{\text{long-edge}}(x, n_k, N_k, n_k, \mathbb{W}) := \left\{ \exists v_1 \in \Xi_{\Lambda(x,n)}[1, \mathbb{W}], \exists v_2 \in \Xi \setminus \Xi_{\Lambda(x,n)}, v_1 \leftrightarrow v_2 \right\}.
\]
(6.11)
The next lemma relates the probability of the event \(|C_n(u)| \geq k, u \notin C_n^{(1)}\) to the events \(A_{\text{low-edge}}\) and \(A_{\text{long-edge}}\) used in the assumption of Proposition 6.1

**Lemma 6.3 (Extending the box-sizes).** Consider a supercritical i-KSRG model as in Definition 2.1 under the same setting as Proposition 6.1. Assume that (6.1)–(6.3) hold. Then there exists \(A' > 0\) such that for all \(n\) with \(n \in [N_k, \infty]\), whenever \(u := (x, w_u) \in \mathcal{V}\) is a vertex with \(\|x - \partial \Lambda_n\| \geq N_k^{1/2}/2\),
\[
\mathbb{P}^x(|C_n(u)| \geq k, u \notin C_n^{(1)}) \leq A' \exp\left(- (1/A') k^c \right) + \mathbb{P}^x(A_{\text{low-edge}}(0, n_k, N_k, \mathbb{W})) + \mathbb{P}^x(A_{\text{long-edge}}(0, n_k, N_k, \mathbb{W})).
\]
(6.12)

**Proof.** Let \(\tilde{n} \geq 1\), and denote by \(C_{\Lambda(x,n)}^{(1)}\) the largest connected component in the induced subgraph of \(\mathcal{G}_{\infty}\) inside the box \(\Lambda(x, \tilde{n}) \subseteq \Lambda_{\tilde{n}}\). For a vertex \(u := (x, w_u) \in \mathcal{V}\) define
\[
A_{\text{leave-giant}}(x, \tilde{n}) := \{ u \in C_{\Lambda(x,n)}^{(1)}, u \notin C_n^{(1)} \},
\]
(6.13)
\[
A_{\text{mark-giant}}(x, N_k, n_k, \mathbb{W}) := \{ \forall v \in \Xi_{\Lambda(x,n)}[1, \mathbb{W}], v \notin C_{\Lambda(x,n)}^{(1)} \},
\]
(6.14)
The first events relates to (6.6) in Claim 6.2 while the second one to (6.3) of Proposition 6.1. The values of \(n_k < N_k \leq n\) from (6.9) and that we assumed \(\|x - \partial \Lambda_n\| \geq N_k^{1/2}/2\) ensure that \(\Lambda(x, n_k) \subseteq \Lambda(x, N_k) \subseteq \Lambda_{n_k}\). Then we bound
\[
\left\{ |C_n(u)| \geq k, u \notin C_n^{(1)} \right\} \subseteq \left\{ |C_n(u)| \geq k, u \notin C_n^{(1)}, u \notin C_{\Lambda(x,n)}^{(1)} \right\} \cup \left\{ u \in C_{\Lambda(x,n)}^{(1)}, u \notin C_n^{(1)} \right\}
\]
\[
\cup \left\{ |C_n(u)| \geq k, u \notin C_n^{(1)}, u \notin C_{\Lambda(x,n)}^{(1)} \right\} \subseteq \left\{ |C_{\Lambda(x,n)}^{(1)}| \geq k \right\} \cup A_{\text{leave-giant}}(x, n_k)
\]
\[
\cup \left\{ |C_n(u)| \geq k, u \notin C_n^{(1)}, u \notin C_{\Lambda(x,n)}^{(1)} \right\} \subseteq \left\{ |C_{\Lambda(x,n)}^{(1)}| \geq k \right\} \cup A_{\text{leave-giant}}(x, n_k)
\]
(6.15)
Applying probabilities on both sides we obtain the inequality stated in (3.6) for \(x = 0\). We introduce a shorthand notation for the third event on the right-hand side of (6.15), i.e.,
\[
A_{\text{goal}} := \left\{ |C_n(u)| \geq k, u \notin C_n^{(1)}, u \notin C_{\Lambda(x,n)}^{(1)} \right\}
\]
(6.15)
Define the auxiliary events

\[ A_{\text{becomes-large}} := \{|C_{\Lambda(x,n_k)}(u)| < k, |C_n(u)| \geq k\}, \]
\[ A_{\text{outof-giant}}(n_k, n) := \{u \notin C_{n}^{(1)}, u \notin C_{\Lambda(x,n_k)}^{(1)}\}, \]

and observe that \( A_{\text{goal}} = A_{\text{becomes-large}} \cap A_{\text{outof-giant}}(n_k, n) \). In order to bound \( \mathbb{P}(A_{\text{goal}}) \), we distinguish whether \( u \) enters the giant at the intermediate box of size \( N_k \in (n_k, n) \) or not:

\[ A_{\text{goal}} \subseteq \{u \in C_{\Lambda(x,n_k)}^{(1)}, u \notin C_{n}^{(1)}\} \cup \{|C_n(u)| \geq k, u \notin C_{\Lambda(x,n_k)}^{(1)}, |C_{\Lambda(x,n_k)}(u)| < k\} = A_{\text{leave-giant}}(x, N_k) \cup (A_{\text{becomes-large}} \cap A_{\text{outof-giant}}(n_k, N_k)), \]

(6.17)

with \( A_{\text{leave-giant}}(x, \tilde{n}) \) defined in (6.14). We observe that \( A_{\text{becomes-large}} \subseteq A_{\text{edge}} \). By the pigeon-hole principle, if all edges adjacent to all vertices in \( C_{\Lambda(x,n_k)}(u) \) were shorter than \( t_k \), the furthest point that could be reached from \( x \) with at most \( k - 1 \) edges has Euclidean norm at most \((k - 1)t_k < n_k^{1/2}/2\), and thus its location would be inside \( \Lambda(x, n_k) \), contradicting the definition of \( A_{\text{becomes-large}} \) in (6.16). Returning to (6.17), we obtain that

\[ A_{\text{goal}} \subseteq (A_{\text{leave-giant}}(x, N_k) \cup (A_{\text{edge}} \cap A_{\text{outof-giant}}(n_k, N_k))) \]

\[ \subseteq A_{\text{leave-giant}}(x, N_k) \cup A_{\text{edge}} \cap \{u \notin C_{\Lambda(x,n_k)}^{(1)}\} \].

In order to bound the existence of long edges, we put restrictions on the marks: we distinguish whether all vertices in \( \Xi_{\Lambda(x,N_k)} \setminus C_{\Lambda(x,n_k)}^{(1)} \) have mark at most \( \overline{\omega}_{N_k} \) — this is the event \( A_{\text{mark-giant}}(x, N_k, \overline{\omega}_{N_k}) \) in (6.14) — or not. We obtain

\[ A_{\text{goal}} \subseteq (A_{\text{leave-giant}}(x, N_k) \cup (\neg A_{\text{mark-giant}}(x, N_k, \overline{\omega}_{N_k})) \cup (A_{\text{edge}} \cap \{u \notin C_{\Lambda(x,n_k)}^{(1)}\} \cap A_{\text{mark-giant}}(x, N_k, \overline{\omega}_{N_k})). \]

(6.19)

The intersection with \( \{u \notin C_{\Lambda(x,n_k)}^{(1)}\} \cap A_{\text{mark-giant}}(x, N_k, \overline{\omega}_{N_k}) \) in the last event ensures that all vertices in the cluster of \( u \) with location in \( \Lambda(x, N_k) \supseteq \Lambda(x, n_k) \) have mark at most \( \overline{\omega}_{N_k} \). We make another case distinction, with respect to the locations of the vertices the edge \( e \geq t_k \) of length at least \( t_k \) exists on the event \( A_{\text{edge}} \) in (6.18). Namely, \( e \geq t_k \) either has both endpoints in \( \Lambda_{n_k} \) or it has one endpoint inside \( \Lambda_{n_k} \) and the other one outside \( \Lambda_{N_k} \). For the first event, we obtain the event \( A_{\text{low-edge}}(x, n_k, N_k, \overline{\omega}_{N_k}) \) and for the latter \( A_{\text{long-edge}}(x, n_k, N_k, \overline{\omega}_{N_k}) \), respectively (defined in (6.10)–(6.11)). Hence,

\[ A_{\text{edge}} \cap \{u \notin C_{\Lambda(x,n_k)}^{(1)}\} \cap A_{\text{mark-giant}}(x, N_k, \overline{\omega}_{N_k}) \subseteq A_{\text{low-edge}}(x, n_k, N_k, \overline{\omega}_{N_k}) \cup A_{\text{long-edge}}(x, n_k, N_k, \overline{\omega}_{N_k}). \]

Using this in (6.19), then substituting (6.19) back into (6.15), and then taking probabilities yields

\[ \mathbb{P}^x(|C_n(u)| \geq k, u \notin C_{n}^{(1)}) \leq \mathbb{P}^x(|C_{\Lambda(x,n_k)}^{(2)}| \geq k) + \mathbb{P}^x(\neg A_{\text{mark-giant}}(x, N_k, \overline{\omega}_{N_k})) + \sum_{\tilde{n} \in (n_k, N_k)} \mathbb{P}^x(A_{\text{leave-giant}}(x, \tilde{n})) \]

\[ + \mathbb{P}^x(A_{\text{low-edge}}(x, n_k, N_k, \overline{\omega}_{N_k})) + \mathbb{P}^x(A_{\text{long-edge}}(x, n_k, N_k, \overline{\omega}_{N_k})). \]

The event \( A_{\text{leave-giant}}(x, \tilde{n}) = \{u \in C_{\Lambda(x,\tilde{n})}, u \notin C_{\Lambda(x,n_k)}^{(1)}\} \) considers the graph in the box \( \Lambda_n \) (which is centered at the origin) and therefore does not necessarily have the same probability for all \( x \in \Lambda_n \). The four other events consider the graph in boxes centered at \( x \). Hence, we translate those events (and the Palm measure \( \mathbb{P}^x \)) by \(-x\) to obtain

\[ \mathbb{P}^x(|C_n(u)| \geq k, u \notin C_{n}^{(1)}) \leq \mathbb{P}^x(|C_{\Lambda(x,n_k)}^{(2)}| \geq k) + \mathbb{P}^x(\neg A_{\text{mark-giant}}(0, N_k, \overline{\omega}_{N_k})) + \sum_{\tilde{n} \in (n_k, N_k)} \mathbb{P}^x(u \in C_{\Lambda(x,\tilde{n})}^{(1)}, u \notin C_{n}^{(1)}) \]

\[ + \mathbb{P}^x(A_{\text{low-edge}}(0, n_k, N_k, \overline{\omega}_{N_k})) + \mathbb{P}^x(A_{\text{long-edge}}(0, n_k, N_k, \overline{\omega}_{N_k})). \]
The first two terms can be bounded by substituting the definitions of \( n_k, N_k = \exp \left( \Theta(k^c) \right) \) in (6.9) into the assumed bounds on the probabilities in Proposition 6.3. The terms in the sum are bounded from above by \( n_k^\delta = \exp \left( - \Theta(k^c) \right) \) by Claim 6.2. This finishes the proof of (6.12). \( \square \)

We move on to bounding \( \mathbb{P}^0(A_{\text{low-edge}}) \) on the right-hand side of (6.12) in Proposition 6.3 with \( A_{\text{low-edge}} \) from (6.10). To do so, we need an auxiliary claim that controls the probability that for every point in \( \Xi_{N_k}[1, \overline{w}_{N_k}] \) there are not "too many" points at distance at least \( t_k \), with \( t_k = n_k^{1/d}/2k \). We define first for \( i \geq 1 \), and \( u = (u, w_u) \in \Xi_{n_k}[0, \overline{w}] \) the annuli

\[
\mathcal{R}_i(u) := \left( A(u, (2t_k)^d) \setminus \Lambda(x_u, (2^{i-1}t_k)^d) \right) \times [1, \infty).
\]

(6.20)

With the measure \( \mu_\tau \) from (2.2), we then define the bad events

\[
\mathcal{A}_{\text{dense}} := \left\{ \exists i \geq 1, u \in \Xi_{n_k}[1, \overline{w}_{N_k}] : |\Xi_{N_k} \cap \mathcal{R}_i(u)| > 2 \cdot \mu_\tau(\mathcal{R}_i(u)) \right\}.
\]

(6.21)

In the following auxiliary claim we give an upper bound on \( \mathbb{P}(\mathcal{A}_{\text{dense}}) \). Its proof is standard, based on Palm theory and Chernoff bounds, see page 46 of Appendix B.

Claim 6.4. For all \( c, \delta > 0 \), there exists \( n_0 \) such that for all \( n_k \geq \max\{n_0, k^{d+\delta}\} \), \( \mathbb{P}^0(\mathcal{A}_{\text{dense}}) \leq n_k^{-c} \).

We can now analyze \( \mathbb{P}^0(\mathcal{A}_{\text{low-edge}}) \) in Lemma 6.3. The next claim holds for the specific choices of \( n_k, N_k, \overline{w}_{N_k} \) and \( t_k \) in (6.9) for all \( k \) sufficiently large, but extends to more general settings, e.g. KSRGs defined on any vertex set that satisfies \( \neg \mathcal{A}_{\text{dense}} \) in (6.21).

Claim 6.5 (No low-mark edge from a small component). Consider an \( i \)-KSRG that satisfies the conditions in Proposition 6.7. For any \( \varepsilon \in (0, \min\{\alpha - 1, \tau - 2\}) \) in (6.9) there exists a constant \( A' > 0 \), such that for sufficiently large \( A' \) such that for sufficiently large \( \alpha < \infty \)

\[
\mathbb{P}^0(\mathcal{A}_{\text{low-edge}}(0, n_k, N_k, \overline{w}) \mid \neg \mathcal{A}_{\text{dense}}) \leq \begin{cases} A' \exp(-(1/A')k^c), & \text{if } \alpha < \infty, \\ 0, & \text{if } \alpha = \infty. \end{cases}
\]

(6.22)

Proof of Claim 6.5. Assume first \( \alpha = \infty \). The event \( \mathcal{A}_{\text{low-edge}}(0, n_k, N_k, \overline{w}_{N_k}) \) is by definition in (6.10) restricted to vertices of mark at most \( \overline{w}_{N_k} = \overline{w}(N_k, c) \) in (6.9). We write \( t_k \) in (6.9) as \( t_k = \exp((\varepsilon/d)k^c)/(2k) \), which is larger than \( \beta \overline{w}_{N_k}^{1+\delta} = \beta M_1^{c+\sigma}((c/(2c'))k^c)^{(\sigma+1)} \) for \( k \) sufficiently large. Hence, the indicator in \( p(u, v) \) is then 0 by (2.3), so a connection between \( u, v \) can not occur.

Assume \( \alpha < \infty \). To obtain an upper bound on the left-hand side of (6.22), we condition on the full realization \( \Xi \) satisfying the event \( \neg \mathcal{A}_{\text{dense}} \) containing 0:

\[
\mathbb{P}^0(\mathcal{A}_{\text{low-edge}}(0, n_k, N_k, \overline{w}) \mid \neg \mathcal{A}_{\text{dense}}) = \mathbb{E}^0\left[ \mathbb{P}^0(\mathcal{A}_{\text{low-edge}}(0, n_k, N_k, \overline{w}) \mid \Xi, \neg \mathcal{A}_{\text{dense}}) \right].
\]

(6.23)

Let us denote the subgraph of \( G_{n_k} \) with edges of length at most \( t_k = n_k^{1/d}/(2k) \) by \( G_{n_k}(\leq t_k) \) and write \( C_{n_k}(0, \leq t_k) \) for the component in this graph containing the origin. Clearly,

\[
\left\{ |C_{n_k}(0)| < k, \exists v_1, v_2 \in \Xi_{N_k}[1, \overline{w}_{N_k}] \text{ s.t.} \right\} \subseteq \left\{ |C_{n_k}(0, \leq t_k)| < k, \exists v_1, v_2 \in \Xi_{N_k}[1, \overline{w}_{N_k}] \text{ s.t.} \right\} \subseteq \left\{ v_1, v_2 \in \Xi_{n_k}[0, \leq t_k, \|x_{v_1} - x_{v_2}\| \geq t_k, \text{ and } v_1 \leftrightarrow v_2 \right\},
\]

where the left-hand side is the definition of \( \mathcal{A}_{\text{low-edge}} \) in (6.10). Conditionally on \( \Xi \), all edges of length at least \( t_k \) are present independently of edges shorter than \( t_k \). We obtain by a union bound over all vertices in \( \Xi_{n_k}[1, \overline{w}_{N_k}] \leq \Xi \),

\[
\mathbb{P}^0(\neg \mathcal{A}_{\text{low-edge}}(0, n_k, N_k, \overline{w}_{N_k}) \mid \Xi, \neg \mathcal{A}_{\text{dense}}) \leq \sum_{v_1 \in \Xi_{n_k}[1, \overline{w}_{N_k}]} \mathbb{P}^0\left( v_1 \in C_{n_k}(0, \leq t_k), |C_{n_k}(0, \leq t_k)| < k \mid \Xi, \neg \mathcal{A}_{\text{dense}}, \sum_{v_2 \in \Xi_{N_k}[1, \overline{w}_{N_k}]: \|x_{v_1} - x_{v_2}\| \geq t_k} p(v_1, v_2) \right).
\]

(6.24)

Using the definitions of \( p \) in (2.3), \( \kappa, \sigma \) from (1.10), the upper bound on \( |\mathcal{R}_i(u)| \) in \( \mathcal{A}_{\text{dense}} \) in (6.21), mark bounds \( w_{v_1}, w_{v_2} \leq \overline{w}_{N_k} \), and the distance bound \( \|x_{v_1} - x_{v_2}\| \geq 2^{i-1}t_k \) when \( x_{v_2} \in \mathcal{R}_i(u) \) in
The following bound holds uniformly for all \( v_1 \in \Xi_{N_k}[1, w_k] \):

\[
T(v_1) \leq \sum_{i \geq 1} \sum_{v_2 \in \Xi_{N_k}[1, w_k]} p(v_1, v_2) \leq \sum_{i \geq 1} \sum_{v_2 \in \Xi_{N_k}[1, w_k]} p^\alpha \kappa_1^{\alpha}(\overline{w}_{N_k}, \overline{w}_{N_k}) 2^{-(i-1)ad} t_k^{-(i-1)ad} \\
\leq 2 \sum_{i \geq 1} t_k^{2d - 2(i-1)d} p^\alpha \kappa_1^{\alpha}(\overline{w}_{N_k}, \overline{w}_{N_k}) 2^{-(i-1)ad} t_k^{-(i-1)ad} \\
= 2(2^d - 1)p^\alpha \kappa_1^{\alpha}(\overline{w}_{N_k}, \overline{w}_{N_k}) 2^{-(i-1)(i-1)d}.
\]

(6.25)

Since \( \alpha > 1 \) by assumption in Theorem 2.2, the sum on the right-hand side is finite. This gives a bound on \( T(v_1) \) in (6.24) that does not depend on \( v_1 \). Hence, returning to (6.24),

\[
\sum_{v_1 \in \Xi_{N_k}[1, w_{N_k}]} p^0(v_1 \in C_{N_k}(0, \leq t_k), |C_{N_k}(0, \leq t_k)| < k \mid \Xi, \neg A_{\text{dense}})
= \mathbb{P}^0(1_{|C_{N_k}(0, \leq t_k)| < k}) \sum_{v_1 \in \Xi_{N_k}[1, w_{N_k}]} 1_{\{v_1 \in C_{N_k}(0, \leq t_k)\}} \mid \Xi, \neg A_{\text{dense}} < k,
\]

since on realizations of the graph satisfying \( |C_{N_k}(0, \leq t_k)| < k \), the sum that follows is at most \( k - 1 \). Substituting this with (6.25) into (6.24) yields for (6.23), that for some constant \( C > 0 \)

\[
\mathbb{P}^0(A_{\text{low-edge}}(0, n_k, N_k, \overline{w}_{N_k}) \mid \neg A_{\text{dense}}) \leq C k \overline{w}_{N_k}^{\alpha(\sigma + 1)} t_k^{(1-\alpha)d}.
\]

Substituting on the right-hand side the choices of \( t_k = n_k^{1/d}/(2k) \), and \( n_k = \exp(\varepsilon(c/2)k^2) \) from (6.9), yield (6.22) for any \( \varepsilon > 0 \) (using \( \overline{w}_{N_k} \) is polynomial in \( k \)).

The last claim bounds the last term in Lemma 6.3. Recall \( A_{\text{long-edge}} \) from (6.11).

**Claim 6.6** (No long edge from a small component). Consider a supercritical \( i \)-KSRG model in Definition 2.1 with parameters \( \alpha > 1, \tau > 2 \) and \( d \in \mathbb{N} \). Assume \( N \geq n \geq 1 \), and \( \overline{w} \geq 1 \) such that

\[
(N^{1/d} - n^{1/d})/2 \geq \max \{ \sqrt{\alpha n^{1/d}}, (\beta w^{1+\sigma})^{1/d}, N^{1/d}/4 \}.
\]

(6.26)

There exists a constant \( C > 0 \) such that

\[
\mathbb{P}^0(\exists v_1 \in \Xi[n, 1, w], \exists v_2 \in \Xi \setminus \Xi_N : v_1 \leftrightarrow v_2) \leq C n \overline{w}^{-\alpha + 2} N^{-\min\{\alpha - 1, \tau - 2\}}(1 + \log(N)).
\]

(6.27)

In particular, for \( n_k, N_k, \overline{w}_{N_k} \) as in (6.9), if \( \varepsilon = \varepsilon(M, n) \in (0, \min\{\alpha - 1, \tau - 2\}) \) is sufficiently small, then for \( k \) sufficiently large, with \( A_{\text{long-edge}} \) defined in (6.11),

\[
\mathbb{P}^0(A_{\text{long-edge}}(0, n_k, N_k, \overline{w}_{N_k})) \leq \exp(-\varepsilon k^{60h}).
\]

(6.28)

**Proof.** We defer the proof of (6.27) (based on a first-moment method) to Appendix B on page 47. The bound (6.28) follows directly from (6.27) by substituting \( n_k, N_k \) and \( \overline{w}_{N_k} \) from (6.9) to (6.27), then using that \( \overline{w}_{N_k} \) and \( \log(N_k) \) are polynomial in \( k \) and of much smaller order than \( n_k \) and \( N_k \).}

Having bounded all terms on the right-hand side in (6.12), we finish the section:

**Proof of Proposition 6.1.** For \( n \leq N_k \), using that \( n_k = \exp((c/2)k^2) \), Proposition 6.1 follows directly from (6.1), since

\[
\mathbb{P}^0(|C_n(0)| \geq k, 0 \notin C^{(1)}_n) \leq \mathbb{P}^0(|C^{(2)}_n| \geq k) \leq N_k \exp(-ck^2) = \exp(-(c/2)k^2).
\]

We now consider \( n > N_k \). Recall the values of \( n_k, \overline{w}_{N_k}, \) and \( t_k \) from (6.9). Lemma 6.3 and Claims 6.4-6.6 directly imply (6.4) in Proposition 6.1. We will now prove the law of large numbers (6.5). In [33] it is shown that finite \( i \)-KSRGs \( G_n = (V_n, \mathcal{E}_n) \) rooted at a vertex at the origin (see Definition 2.1) converge locally to their infinite rooted version \( (G_{\infty}, 0) \) as \( n \to \infty \). We refer to [33] and its references for an introduction to local limits. We use the concept of local limits as a black box and verify a necessary and sufficient condition for the law of large numbers for the size of the giant component for graphs that have a local limit by Van der Hofstad [33, Theorem 2.2]. We state the condition: let
$(G_n)_{n \geq 1}$ be a sequence of graphs with $|V_n| = n$ that converges locally in probability to $(G_{\infty}, \emptyset)$ (here $\emptyset$ denotes the root of the graph). Assume that
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E} \left[ \sum_{u,v \in V_n} \mathbb{1}_{\{|C_n(u)| \geq k, |C_n(v)| \geq k, C(u) \neq C(v)\}} \right] = 0, \tag{6.29}
\]
then, as $n \to \infty$,
\[
\frac{|C_n(u)|}{n} \xrightarrow{\mathcal{P}} \mathcal{P}^0 \left( |C_{\infty}(\emptyset)| = \infty \right).
\]
To verify condition (6.29), we have to consider a model satisfying $|V_n| = n$. Therefore, we consider i-KSRGs on $G_n$ that are conditioned to have $|V_n| = n$. All our previous results extend to this model, as remarked in Section 3.3. We analyze the indicator random variables in (6.29) for a fixed pair $u, v \in V_n$.

We distinguish whether one of them is part of the largest component to obtain
\[
\{|C_n(u)| \geq k, |C_n(v)| \geq k, C_n(u) \neq C_n(v)\} \subseteq \{|C_n(u)| \geq k, u \notin C_n^{(i)} \} \cup \{|C_n(v)| \geq k, v \notin C_n^{(i)} \}.
\]
Using linearity of expectation over the $n^2$ vertex pairs in the sum of (6.29), and that all $n$ vertices are identically distributed, we obtain that
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E} \left[ \sum_{u,v \in V_n} \mathbb{1}_{\{|C_n(u)| \geq k, |C_n(v)| \geq k, C(u) \neq C(v)\}} \right] \leq \lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P}(|C_n(U)| \geq k, U \notin C_n^{(i)}),
\]
where $U$ is a vertex with a uniform location $X \in \Lambda_n$ and mark $w_u$ following distribution $F_W$. We condition on the location $X = x$, to obtain that
\[
\mathbb{P}(|C_n(U)| \geq k, U \notin C_n^{(i)}) \leq \frac{1}{n} \int_{x \in \Lambda_n: \|x - \partial \Lambda_n\| \geq N_{k,2}^{1/d}/2} \mathbb{P}_x \left( |C_n(u)| \geq k, u \notin C_n^{(i)} \right) \mathbb{P}(\|X - \partial \Lambda_n\| < N_{k,2}^{1/d}/2).
\]
The second term tends to zero as $n \to \infty$, while the first term is of order $\exp \left( -\Omega(k^2) \right)$ by Lemma 6.3 and Claims 6.4,6.6 that apply for all $n \geq N_k$ and $x$ such that $\|x - \partial \Lambda_n\| \geq N_{k,2}^{1/d}/2$. Thus, the first term tends to zero as $k \to \infty$. This finishes the proof of the condition (6.29). The law of large numbers follows. \hfill \square

7. LOWER BOUNDS

In this section we prove Theorem 2.2). We first show that certain prerequisite inequalities imply the lower bound on the cluster-size decay, and afterwards we verify these inequalities for $\tau \in (2,2+\sigma)$. Eventually, we prove Theorem 2.3. We define for some constant $\eta > 0$ the function and set
\[
w_\eta(\ell) := (\log(\ell))^{\eta}, \quad \Lambda_{\ell,\eta} := \Lambda_{\ell} \times [1, w_\eta(\ell)), \quad C_{\ell,\eta}(0) := \text{the component containing (0,} w_\eta(\ell)), \tag{7.1}
\]
i.e., $C_{\ell,\eta}(0)$ is the component of $(0, w_\eta(\ell))$ in the graph induced on vertices in $\Lambda_{\ell,\eta}$ and let $C_{\ell,\eta}(0) := \emptyset$ if $w_\eta \geq w_\eta(\ell)$. We recall $m(Z)$, the multiplicity of the maximum of $Z$, from (2.5), and the exponents $\zeta_{ll}, \zeta_{lh}, \zeta_{hl}, \zeta_{nn}$ from (1.7), (1.8), (1.12), and (1.9). Define for some small constant $\varepsilon > 0$ to be defined later, and $Z = \{\zeta_{ll}, \zeta_{lh}, \zeta_{hl}, \zeta_{nn}\} \cup \{\varepsilon\}$
\[
\kappa_{\varepsilon,n} := \begin{cases} \left( \frac{\varepsilon \log(n)/\log(\log(n))}{m_Z-1} \right)^{1/m_Z}, & \text{if max}(Z) > 0, \\ \exp \left( \frac{\varepsilon \log(n)}{1/(m_Z-1)} \right), & \text{if max}(Z) = 0, \varepsilon > 0, \end{cases} \quad m_Z > 1. \tag{7.2}
\]
Note that $\zeta_{nn} = \frac{d-1}{\alpha}$, which is equal to 0 only if $d = 1$, and positive otherwise. Thus, only in dimension $d = 1$ can have $\kappa_{n,\varepsilon}$ different from $\Theta(\text{polylog}(n))$. More precisely, for $d = 1$, $\kappa_{n,\varepsilon} = \Theta(\text{poly}(n))$ if only one out of $\{\zeta_{ll}, \zeta_{lh}, \zeta_{hl}\}$ is zero, and the others are negative; and it is stretched exponential in the logarithm if at least two elements out of $\{\zeta_{ll}, \zeta_{lh}, \zeta_{hl}\}$ are zero, and none of them is positive. If the three values $\zeta_{ll}, \zeta_{lh}, \zeta_{hl}$ are all negative, then the model is always subcritical when $d = 1$ as shown by Gracar, Lüchtrath, and Mönch 27.

By an "i-KSRG on $\mathbb{Z}^d$" we mean the model mentioned in Section 3.3, i.e., when $\Xi$ is replaced by vertices in $\mathbb{Z}^d$ having iid marks from distribution $F_W$ in (2.2) in Definition 2.1.

**Proposition 7.1** (Lower bound holds when linear-sized giant on truncated weights exists). Consider a supercritical i-KSRG model in Definition 2.1 with parameters $\alpha \in (1, \infty], \tau \in (2, \infty], \sigma \geq 0, d \in \mathbb{N}$,
or a supercritical i-KSRG on \( \mathbb{Z}^d \) with additionally \( \min\{p, p^{\alpha}\} < 1 \) in (2.3). Assume that there exist constants \( \eta, \rho > 0 \) such that for all \( \ell \) sufficiently large, and with \( C_{\ell, \eta}(0) \) from (7.1),
\[
\mathbb{P}^x(|C_{\ell, \eta}(0)| \geq \rho^\ell) \geq \rho. \tag{7.3}
\]
Then there exists \( A > 0 \) such that for all \( n \in [Ak, \infty] \), with \( \mathcal{Z} = \{ \zeta_{il}, \zeta_{ih}, \zeta_{hh}, \zeta_{nn} \} \),
\[
\mathbb{P}^x(|C_n(0)| \geq k, 0 \notin C^{(i)}_n) \geq \exp \left(-Ak^{\max(\mathcal{Z})} \log^{m-1}(k)\right). \tag{7.4}
\]
Moreover, there exists \( \delta, \varepsilon > 0 \), such that for all \( n \) sufficiently large, with \( k_{n, \varepsilon} \) from (7.2),
\[
\mathbb{P}(|C_n^{(2)}| > k_{n, \varepsilon}) \geq 1 - n^{-\delta}. \tag{7.5}
\]

By Proposition 5.14, the condition (7.3) is satisfied when \( \tau \in (2, 2 + \sigma) \), implying Theorem 2.2. In 43, we show that (7.3) holds whenever \( \max\{\zeta_{il}, \zeta_{ih}, \zeta_{hh}\} > 0 \) and the model is supercritical. We prove Proposition 7.1 by formalizing the reasoning at the beginning of Section 1.2. First, we assume that the vertex set is given by the PPP \( \Xi \) in Definition 2.1. Then we explain how to extend to i-KSRGs on \( \mathbb{Z}^d \).

7.1. Finding a localized component. The goal of this section is to prove Proposition 7.1. First we aim to bound the probability of \( \{C_n(0) \geq k, 0 \notin C^{(i)}_n\} \) in (7.4) from below. To do so, we write it as the intersection of “almost independent” events, for which we introduce some notation now:

**Two large components.** We write \( \Lambda(x, s) = \Lambda_s(x) \) for a box of volume \( s \) centered at \( x \), see (2.1). We define for a constant \( M_{in} > 0 \) the radius and Euclidean ball
\[
r_k := (kM_{in})^{1/d}, \quad B_{kM_{in}} := \{x \in \mathbb{R}^d : \|x\| \leq r_k\}, \tag{7.6}
\]
where \( M_{in} > 1 / \rho \) is implicitly defined given \( \rho \) from Proposition 7.1 such that \( \|\partial \Lambda_{k/\rho} - \partial B_{kM_{in}}\| = r_k / 2 \), implying also that \( \Lambda_{k/\rho} \subseteq B_{kM_{in}} \). We would like to constrain \( C_n(0) \) to the ball \( B_{kM_{in}} \), and we would like to find a component outside \( B_{kM_{in}} \) that is larger than \( |C_n(0)| \). We ‘construct’ these two components, with the constant \( M_{out} := 2^{d+2}M_{in} \) on vertices in the (hyper)rectangles (see also (7.1))
\[
\mathcal{R}_{in} := \Lambda_{k/\rho, \eta} = \Lambda(0, k/\rho) \times \left[1, w_\eta(k/\rho)\right), \quad \mathcal{R}_{out} := \Lambda(x_{out}, kM_{out}/\rho) \times \left[1, w_\eta(kM_{out}/\rho)\right), \quad \Lambda_{in} := \Lambda(0, k/\rho), \quad \Lambda_{out} := \Lambda(x_{out}, kM_{out}/\rho), \tag{7.7}
\]
where \( x_{out} := (x_{out}^1, 0, \ldots, 0) \in \mathbb{R}^d \) is defined implicitly given \( \rho \) such that \( \|\partial \Lambda(x_{out}, kM_{out}/\rho) - \partial B_{kM_{out}}\| := r_k / 2 \). We assume that \( n \) is sufficiently large so that \( \Lambda_{in} \cup \Lambda_{out} \subseteq \Lambda_{n} \). Let \( C_{in}(0) \) be the component of \( (0, w_0) \) in the subgraph of \( \mathcal{G}_n \) induced on vertices in \( \mathcal{R}_{in} \), where \( C_{in}(0) = \emptyset \) if \( w_0 \geq w_\eta(k/\rho) \). Since \( \mathcal{R}_{in} \subseteq B_{kM_{in}} \), it is immediate that \( C_{in}(0) \subseteq \Xi_{B_{kM_{in}}} \). Let \( C^{(i)}_{out} \) be the largest component in the subgraph of \( \mathcal{G}_n \) induced on vertices in \( \mathcal{R}_{out} \). Define the events
\[
\mathcal{A}^{(\text{in giant})} := \{ |C_{in}(0)| \geq k \}, \quad \mathcal{A}^{(\text{out giant})} := \{ |C^{(i)}_{out}| \geq kM_{out} \}, \quad \mathcal{A}^{(\text{components})} := \mathcal{A}^{(\text{in giant})} \cap \mathcal{A}^{(\text{out giant})}. \tag{7.8}
\]

![Figure 3](image-url)
Isolation. On \( A^{(k)}_{\text{components}} \), \( C_n(0) \supseteq C_{in}(0) \) could still connect to the giant/infinite component. To prevent this, we will ban edges that cross the boundary of \( B_{kMn} \) as follows. In Section \( 3.2 \) we informally described an “optimally-suppressed mark-profile”. We first define a suppressed mark-profile, then optimize its shape. Let \( \gamma \in (0, 1/(\sigma + 1)] \) be a constant. We define, for \( C_\beta := (2\beta)^{1/d} \), and for \( x \in \mathbb{R}^d \) with \( ||x - \partial B_{kMn}|| = ||x|| - r_k || =: z \), the \( \gamma \)-suppressed profile by

\[
\begin{align*}
f_\gamma(z) := & \begin{cases} 
1 & \text{if } z \leq C_\beta, \\
(z/C_\beta)^d & \text{if } z \in (C_\beta, r_k), \\
(z/C_\beta)^d(r_k/C_\beta)^{-d(1-\gamma)} & \text{if } z > r_k, 
\end{cases} \\
M_\gamma := & \{ (x, f_\gamma(||x|| - r_k) ) : x \in \mathbb{R}^d \}. 
\end{align*}
\]

(7.9)

We say that \( v \) is below, on, or above \( M_\gamma \), if \( v \) is at most, equal to, or strictly larger than \( f_\gamma(||x|| - r_k) \), respectively. We say \( \Xi \supseteq M_\gamma \) if all points in \( \Xi \) are below \( M_\gamma \), see Figure \( 3 \). We split the PPP \( \Xi \) into four independent PPPs, whether points fall below or above \( M_\gamma \), and inside or outside \( B_{kMn} \):

\[
\Xi_{\leq M_\gamma} := \{ (x, u, w) \in \Xi : u \in B_{kMn}, u \leq f_\gamma(||x|| - r_k) \}, \\
\Xi_{\geq M_\gamma} := \{ (x, v, w) \in \Xi : v \notin B_{kMn}, v \leq f_\gamma(||x|| - r_k) \}, \\
\Xi_{\text{in}} := \{ (x, u, w) \in \Xi : u \in B_{kMn}, u > f_\gamma(||x|| - r_k) \}, \\
\Xi_{\text{out}} := \{ (x, v, w) \in \Xi : v \notin B_{kMn}, v > f_\gamma(||x|| - r_k) \}. 
\]

(7.11)

For \( A, B \subseteq \Xi \) we denote by \( |E(A, B)| \) the number of edges between vertices in \( A \) and \( B \). Define

\[
\{ \Xi \leq M_\gamma \} = \{ [\Xi_{\leq M_\gamma} \cup \Xi_{\geq M_\gamma}, | = 0 \}, \quad A^{(k)}_{\text{noedge}}(\gamma) := \{ |E(\Xi_{\leq M_\gamma}, \Xi_{\geq M_\gamma})| = 0 \}. 
\]

(7.12)

Under \( \{ \Xi \leq M_\gamma \} \cap A^{(k)}_{\text{noedge}}(\gamma) \), the vertices in \( B_{kMn} \) are not connected to the unique infinite component when \( n = \infty \) and are isolated from the rest of \( \mathcal{G}_n \) when \( n < \infty \). We comment on the profile function \( f_\gamma \) in \( 7.9 \). The event \( \{ \Xi \leq M_\gamma \} \) demands no vertices within distance \( C_\beta \) from \( \partial B_{kMn} \), since \( f_\gamma(||x|| - r_k) = 1 \) for \( ||x - \partial B_{kMn}|| \leq C_\beta \), and vertex marks are above \( 1 \) a.s. The function \( f_\gamma \) is continuous and increasing in \( z \); the closer a point \( x \) is to the boundary of \( B_{kMn} \), the stronger the restriction on its mark in \( \Xi \leq M_\gamma \). This is natural since vertices with higher mark close to \( \partial B_{kMn} \) are more likely to have an edge crossing this boundary, which we want to prevent. While the event \( \{ \Xi \leq M_\gamma \} \) becomes less likely when \( \gamma \) is small, \( A^{(k)}_{\text{noedge}}(\gamma) \) becomes more likely. This naturally leads to an optimization problem, that we set up next. Let \( Y = (0, \alpha - (\tau - 1), (\sigma + 1)\alpha - 2(\tau - 1)) \), and define

\[
\gamma_{\text{an}} := \begin{cases} 
\max_{0 < \gamma \leq 1/(\sigma + 1)} \{ \gamma : 1 - \gamma(\tau - 1) \geq 2 - \alpha + \gamma \max(Y) \}, & \text{if } \alpha < \infty, \\
1/(\sigma + 1), & \text{if } \alpha = \infty. 
\end{cases} \\
\zeta_{\text{an}} := 1 - \gamma_{\text{an}}(\tau - 1), 
\]

(7.13)

The inequality \( (\ast) \) holds for \( \gamma = 0 \) since \( \alpha > 1 \) by assumption, therefore \( \gamma_{\text{an}} > 0 \) and is well-defined.

We call \( f_* := f_\gamma_{\text{an}} \) the optimally-suppressed mark-profile (and show below that it is indeed “optimal”) and write \( M_* := M_{\gamma_{\text{an}}} \). We define the isolation event using the events in \( 7.12 \), see also Figure \( 3 \)

\[
A^{(k)}_{\text{isolation}} := \{ \Xi \leq M_* \} \cap A^{(k)}_{\text{noedge}}(\gamma_{\text{an}}). 
\]

(7.14)

\section*{Ensuring almost independence.}

The events \( A^{(k)}_{\text{isolation}} \) and \( \{ |C_{in}(0)| \geq k \} \) in \( 7.8 \) are correlated, since \( \{ |C_{in}(0)| \geq k \} \) may push up the number of high-mark vertices in \( \mathcal{R}_{in} \), making \( A^{(k)}_{\text{isolation}} \) less likely. To overcome the dependence, we introduce two auxiliary events that ensure regularity of \( \Xi \) in the hyperrectangles \( \mathcal{R}_{in}, \mathcal{R}_{out} \) from \( 7.7 \). Using \( w_\eta \) from \( 7.1 \), define, using \( c_m := 1/\rho, c_{out} := M_{out}/\rho \):

\[
j_{in}^* := \min \{ j \in \mathbb{N} : w_\eta(kc_{in})/2^j < 1 \}, \quad j_{out}^* := \min \{ j \in \mathbb{N} : w_\eta(kc_{out})/2^j < 1 \}. 
\]

(7.15)

and writing \( \text{loc} \in \{ \text{in}, \text{out} \} \), define

\[
I_j := [w_\eta(kc_{\text{loc}})/2^j, w_\eta(kc_{\text{loc}})/2^{j-1}], \quad 1 \leq j < j_{\text{out}}^*. 
\]

(7.16)
Using $A_{in}, A_{out}$ in \((\ref{eqn:components})\), the intensity measure $\mu_x$ of $\Xi$ in \((\ref{def:mu})\), and $\Xi_{loc}(I_j^{\infty})$ for the vertices in $\Xi \cap (\Lambda_{loc} \times I_j^{\infty})$, consider the following events (see also Figure \(3\) for $loc \in \{in, out\}$:

\[
\begin{align}
A^{(k,loc)}_{regular}(\eta) &:= \{ \forall j \leq j_{loc} : |\Xi_{loc}(I_j^{\infty})| \leq 2\mu_x(\Lambda_{loc} \times I_j^{\infty}) \}, \\
A^{(k)}_{regular}(\eta) &:= A^{(k,\in)}_{regular}(\eta) \cap A^{(k,\out)}_{regular}(\eta). 
\end{align}
\]  
\tag{7.17}

Finally, fix a realization of the induced subgraphs $G_{R_{in}} \cup G_{R_{out}} = (\Xi_{R_{in}}, \mathcal{E}(G_{R_{in}})) \cup (\Xi_{R_{out}}, \mathcal{E}(G_{R_{out}}))$ so that the vertex set $\Xi_{R_{in}} \cup \Xi_{R_{out}}$ satisfies the event $A^{(k,\text{regular})}(\eta)$ for some $\eta > 0$, and define the conditional probability measure and expectation by

\[
\bar{P}_{\text{io}}(\cdot) := P(\cdot | G_{R_{in}} \cup G_{R_{out}}, A^{(k,\text{regular})}(\eta)), \quad \bar{E}_{\text{io}}[\cdot] := \bar{E}_{\text{io}}[\cdot | G_{R_{in}} \cup G_{R_{out}}, A^{(k,\text{regular})}(\eta)]
\]  
\tag{7.18}

In the conditioning we reveal both the vertex and edge sets in the disjoint boxes $R_{in}, R_{out}$. The event $A^{(k,\text{regular})}(\eta)$ checks the number of vertices in hyperrectangles inside $R_{in}, R_{out}$, hence $A^{(k,\text{regular})}(\eta)$ is measurable with respect to $G_{R_{in}}, G_{R_{out}}$ and in principle can be left out of the conditioning in \((\ref{eqn:iso})\).

We state a lemma that will imply Proposition \((\ref{thm:7.1})\).

**Lemma 7.2** (Lower bound for isolation). Consider a supercritical $i$-KSRG model in Definition \((\ref{def:i})\) with parameters $\alpha \in (1, \infty)$, $\tau \in (2, \infty)$, $\sigma \geq 0$, and $d \in \mathbb{N}$, or a supercritical $i$-KSRG on $\mathbb{Z}^d$ with additionally $p < 1$ in \((\ref{def:deg})\). For any constant $\eta > 0$ in \((\ref{def:eta})\), there exists $A > 0$, such that for any realization of $G_{R_{in}} \cup G_{R_{out}}$ that satisfies $A^{(k,\text{regular})}(\eta)$, with $A = \{\xi, \zeta, \chi, \nu, \zeta, \eta\}$, with $\bar{P}_{\text{io}}$ from \((\ref{eqn:iso})\)

\[
\bar{P}_{\text{io}}(A^{(\text{iso})}) \geq \exp \left( - Ar^{d \max(2)} \log^3 m - 1(r_k) \right).
\]  
\tag{7.19}

The same bound holds for the Palm-version $\bar{P}_{\text{io}}^0$ of $\bar{P}_{\text{io}}$.

We now show that Proposition \((\ref{thm:7.1})\), follows directly from Lemma \((\ref{lem:7.2})\).

**Proof of Proposition \((\ref{thm:7.1})\), assuming Lemma \((\ref{lem:7.2})\).** We first show \((\ref{prop:7.4})\). Recall $A^{(k,\text{components}} \cap A^{(k,\text{isolation})}$ from \((\ref{def:components-isolation})\), and $A^{(k,\text{regular})}$ from \((\ref{def:components-regular})\), respectively. The intersection of these events implies $|C_n(0)| \geq k, n \notin C^{(1)}_n$, since $|C_n(0)| \geq |C_n(0)| \geq k$, and $A^{(k,\text{isolation})}$ ensures that $C_n(0)$ is fully contained in $B_{kM_n}$. Hence, $|C_n(0)| \leq |B_{kM_n}| \leq kM_{out}/2$ and $|C_n| \geq kM_{out}$, because that $C_n(0)$ is not the largest component of $\mathcal{G}_n$. So, by the law of total probability

\[
P^0\big( |C_n(0)| \geq k, 0 \notin C^{(1)}_n \big) 
\geq P^0\big( A^{(k,\text{components}} \cap A^{(k,\text{isolation})} \cap A^{(k,\text{isolation})} \cap A^{(k,\text{regular})} (\eta) \big) 
\geq P^0\big( A^{(k,\text{components}} \cap A^{(k,\text{isolation})} \cap A^{(k,\text{isolation})} \cap A^{(k,\text{regular})} (\eta) \big) - P^0\big( (\neg A^{(k,\text{isolation})} \big)
\geq P^0\big( A^{(k,\text{components}} \cap A^{(k,\text{isolation})} \cap A^{(k,\text{isolation})} \cap A^{(k,\text{regular})} (\eta) \big) - P^0\big( (\neg A^{(k,\text{isolation})} \big).
\]  
\tag{7.20}

Recall $A^{(k,\text{isolation})} = \{ |B_{kM_n} \leq kM_{out}/2 \}$ from \((\ref{def:components-isolation})\), and $M_{out} = 2^{d+2}M_n$ above \((\ref{def:components-isolation})\). The box with side-length $2r_k = 2(kM_n)^{1/d}$ (by definition in \((\ref{def:components-isolation})\)) centered at the origin is the smallest box that contains $B_{kM_n}$. Using the intensity measure $\mu_x$ from \((\ref{def:mu})\), and writing $B_{kM_n}^\leq := \{ (x, \nu) \in \mathbb{R}^{d+1} : x \in B_{kM_n} ; (x, \nu) \leq \mu_x \}$, we have

\[
\mu_x(B_{kM_n}^\leq) \leq 2^d kM_n = 2^{d+2}kM_n/4 = kM_{out}/4.
\]

By a standard concentration inequality for Poisson random variables (see Lemma \(\text{C.1}\) for $x = 2$), there exist $c', c > 0$ such that, since $r_k = \Theta(k^{1/d})$,

\[
P^0\big( (\neg A^{(k,\text{isolation})} \big) \leq \exp(-c'r_k^d) = \exp(-ck).
\]  
\tag{7.21}

Returning to \((\ref{eqn:components-regular})\), the event $A^{(k,\text{regular})}(\eta) = A^{(k,\text{isolation})}(\eta) \cap A^{(k,\text{isolation})}(\eta)$, defined in \((\ref{eqn:components-regular})\), holds with probability tending to 1 as $k \to \infty$, again by concentration inequalities for Poisson random variables (see Lemma \(\text{C.1}\) for $x = 2$). Hence,

\[
P^0\big( A^{(k,\text{components}} \cap A^{(k,\text{regular})} (\eta) \big) \geq P^0\big( A^{(k,\text{components}} \cap A^{(k,\text{regular})} (\eta) \big) - P^0\big( (\neg A^{(k,\text{isolation})} \big)
\geq P^0\big( A^{(k,\text{components}} \cap A^{(k,\text{regular})} (\eta) \big) - o_k(1).
\]  
\tag{7.22}

We recall from \((\ref{def:components-isolation})\) that $A^{(k,\text{components}} = \{ |C_n(0)| \geq k \} \cap \{ |C_n^{(1)}| \geq kM_{out} \}$. Translate the hyperrectangle $R_{out}$ in \((\ref{def:components-isolation})\) containing $C_n^{(1)}$ to the origin of $\mathbb{R}^d$ (using $w_x(\eta)$) in \((\ref{def:components-isolation})\):

\[
R_{out}' := \Lambda(0, kM_{out}/\rho) \times [1, w_\eta(kM_{out}/\rho)] = \Lambda(kM_{out}/\rho, \eta).
\]  
\tag{7.23}
and write $C_{\text{out}}^{(1)}$, $C_{\text{out}}(0)$ for the largest component and for the component containing $(0, w_0)$ in the subgraph of $\mathcal{G}_n$ induced by vertices in $\mathcal{R}_{\text{out}}'$. As before, we ignore the conditioning $(0, w_0) \in \Xi$ in Definition [7] in our computations. By the translation invariance of the annealed measure and that the events $\{C_{\text{in}}(0) \geq k\}$ and $\{|C_{\text{out}}^{(1)}| \geq M_{\text{out}}k\}$ are independent because they are induced subgraphs of the disjoint hyperrectangles $\mathcal{R}_{\text{in}}$ and $\mathcal{R}_{\text{out}}$ in (7.7).

$$P^0(A_{\text{components}}^{(1)}) = P^0(|C_{\text{in}}(0)| \geq k)P^0(|C_{\text{out}}^{(1)}| \geq M_{\text{out}}k) \geq P^0(|C_{\text{in}}(0)| \geq k)P^0(|C_{\text{out}}(0)| \geq M_{\text{out}}k).$$

The bound $P^0(|C_{\ell, n}(0)| \geq \rho^\ell) \geq \rho$ in (7.3) in Proposition 7.1 holds for all $\ell$ sufficiently large. In particular, since $C_{\text{in}}(0) = C_{k/\rho, n}(0)$ by definition of $\mathcal{R}_{\text{in}}$ in (7.7), and $C_{\text{out}}(0) = C_{kM_{\text{out}}/\rho, n}(0)$ by definition of $\mathcal{R}_{\text{out}}$ in (7.23), we obtain for $k$ sufficiently large

$$P^0(A_{\text{components}}^{(1)}) \geq P^0(|C_{k/\rho, n}(0)| \geq k)P^0(|C_{kM_{\text{out}}/\rho, n}(0)| \geq M_{\text{out}}k) \geq \rho^2,$$  

(7.24)

implying that $P^0(A_{\text{components}}^{(1)} \cap A_{\text{regular}}(\eta)) \geq \rho^2 - o_k(1) > 3\rho^2/4$ in (7.22). Since the event $A_{\text{components}}^{(1)} \cap A_{\text{regular}}(\eta)$ is measurable with respect to the $\sigma$-algebra generated by the subgraph $\mathcal{G}_{\mathcal{R}_{\text{in}}} \cup \mathcal{G}_{\mathcal{R}_{\text{out}}}$, we take expectation over all possible realizations of the latter, and recalling the measure $\tilde{P}_{io}$ from (7.18),

$$P^0(A_{\text{components}}^{(1)} \cap A_{\text{regular}}(\eta)) = P^0[\tilde{P}_{io}(A_{\text{components}}^{(1)} \cap A_{\text{regular}}(\eta))] = P^0[\tilde{P}_{io}(A_{\text{isolation}}^{(k)}|A_{\text{components}}^{(1)} \cap A_{\text{regular}}(\eta))].$$

Apply now Lemma 7.2 on the right-hand side, and substitute the bound $3\rho^2/4$ below (7.24) into (7.22) and then in turn into (7.20) and (7.21), to obtain for $k$ sufficiently large

$$P^0(A_{\text{components}}^{(1)} \cap A_{\text{isolation}}^{(k)} \cap A_{\text{in small}}^{(k)} \cap A_{\text{regular}}(\eta)) \geq (\rho^2/2) \exp \left(-A_r^{d\max(\Xi)}z_{m-1}(r_k)\right) \geq (\rho^2/2) \exp \left(-A_r^{d\max(\Xi)}z_{m-1}(1/k)\right).$$  

(7.25)

We obtained the second row by substituting $r_k = (kM_{\text{in}})^{1/d}$ in (7.6) and setting $A' := A_r^{d\max(\Xi)}/2$ that also compensates for the constants from the log-correction term. Observe that $\max(\Xi) < 1$, so that the right-hand side in (7.21) is of smaller order than the right-hand side in (7.19). By (7.20) this finishes the proof of (7.4). We turn to the proof of (7.5).

Lower bound on second-largest component. We generalize an argument from [10]. We have to bound $P(|C_{\text{in}}(2)| \leq k_{n, \varepsilon})$ from above for a suitably chosen $\varepsilon$ in the definition of $k_{n, \varepsilon}$ in (7.2). To do so, we fix $\theta > 0$ to be specified later, and assume for simplicity that $n^{(1-\theta)/d} \in \mathbb{N}$. Then we partition $\Lambda_n$ into $m_n := n^{1-\theta}$ many subboxes $\Lambda_n^1, \ldots, \Lambda_n^m$, centered respectively at $x_1, \ldots, x_m$, each of volume $\tilde{n} := n^\theta$. By disjointness, the induced subgraphs $\mathcal{G}_{\Lambda_n}^1, \ldots, \mathcal{G}_{\Lambda_n}^m$ in these boxes are independent realizations of $\mathcal{G}_{\Lambda_n}$, translated to $x_1, \ldots, x_m$. We write $\mathcal{G}_{\Lambda_n}^1, \ldots, \mathcal{G}_{\Lambda_n}^m$ for the induced subgraphs, translated back to the origin; $\Xi_{\Lambda_n}^{(m_n)}$ for the vertex set in $\mathcal{G}_{\Lambda_n}^1$ that is below $\mathcal{M}_r$ and inside $\mathcal{B}_{kM_{\text{in}}}$ after the translation, see (7.11), and write $\Xi_{\Lambda_n}^{(m_n)}$ for the same vertex set before the translation. For the translated subgraphs $(\mathcal{G}_{\Lambda_n}^i)^{\prime \prime}_{\Lambda_n}$, we define for $k = k_{n, \varepsilon}$ the same events as in (7.8), (7.14), (7.17),

$$A_{\text{good}}^{(i)} := A_{\text{components}}^{(k_{n, \varepsilon}, i)} \cap A_{\text{isolation}}^{(k_{n, \varepsilon}, i)} \cap A_{\text{in small}}^{(k_{n, \varepsilon}, i)} \cap A_{\text{regular}}(\eta),$$

where now in the definition of these events we replace $C_{\Xi}(0)$ with the component containing the point of $\Xi$ closest to the origin $0 \in \mathbb{R}^d$ for $\square \in \{\in, \tilde{n}\}$. We also assume that $\tilde{n} = n^\theta$ is sufficiently large compared to $k_{n, \varepsilon}$ in (7.2) so that the spatial projection of the box $\mathcal{R}_{\text{out}}$ still fits within $\Lambda_n$. This can be ensured even if $k_{n, \varepsilon} = \Theta(n^{\theta})$ is maximal in (7.2) by choosing $\varepsilon < \theta$. If $A_{\text{good}}^{(i)}$ holds for some $i \leq m_n$, then the induced graph $\mathcal{G}_{\Lambda_n}^i$ in subbox $\Lambda_n^i$ contains a component $C_{\Xi}^{(i)}$ in $\Xi_{\Lambda_n}^{(n)}$ (which we call a 'candidate' second-largest component of $\mathcal{G}_{\Lambda_n}$) with size at least $k_{n, \varepsilon}$ that is not the largest component in its own box, and all vertices in $\Lambda_n^i$ are below $\mathcal{M}_r(x_i)$, i.e., $\mathcal{M}_r$ shifted to $x_i$. If additionally for all $i$ there is no edge between $\Xi_{\Lambda_n}^{(m_n)}$ to a vertex outside $\Lambda_n^i$, then the component $C_{\Xi}^{(i)}$ is not the giant and is larger than $k_{n, \varepsilon}$. Taking complements we obtain that

$$\{C_{\Xi}^{(i)} \leq k_{n, \varepsilon}\} \subseteq \{\exists i \leq m_n : \Xi_{\Lambda_n}^{(m_n)} \leftrightarrow \Xi_{\Lambda_n} \setminus \Xi_{\Lambda_n}^{(i)} \cup \bigcup_{i \leq m_n} (-A_{\text{good}}^{(i)})\}. $$

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By translation invariance, a union bound, and the independence of \((G_n^{(i)})_{i \leq m_n}\),
\[
\mathbb{P}(C_n^{(2)} \leq k_{n, \varepsilon}) \leq m_n \mathbb{P}(\Xi_{<, M_\varepsilon} \leftrightarrow \Xi_n \setminus \Xi_{\varepsilon}) + (1 - \mathbb{P}(A^{(1)}_{\text{good}})) m_n =: T_1 + T_2.
\] (7.26)

By the definitions in (7.11) and (7.6), each \(u \in \Xi_{<, M_\varepsilon}\) has \(\|x_u\| \leq r_k\) with \(k = k_{n, \varepsilon}\), and mark \(w_u = f_s(r_{2k_{n, \varepsilon}})\), \((f_s = f_{\gamma_{nn}})\) is from below (7.13), hence \(\Xi_{<, M_\varepsilon} \subseteq \Xi(2r_{2k_{n, \varepsilon}}) \cup \{1, f_s(r_{2k_{n, \varepsilon}})\} \subseteq \hat{\xi}_n\), whenever \((2r_{2k_{n, \varepsilon}})^d = k_{n, \varepsilon} M_{in} \varepsilon^d < n^\vartheta\), which holds whenever \(\varepsilon < \vartheta\) by (7.2). Hence we can bound \(T_1\) as
\[
T_1 \leq m_n \mathbb{P}(\Xi(2r_{2k_{n, \varepsilon}}) \cup \{1, f_s(r_{2k_{n, \varepsilon}})\} \leftrightarrow \Xi_n \setminus \Xi_{\varepsilon}).
\]

We can directly apply Claim 6.6 on the right-hand side, i.e., setting there \(N := \hat{n} = n^\vartheta\) and \(n := (2r_{2k_{n, \varepsilon}})^d = k_{n, \varepsilon} M_{in} \varepsilon^d\) by (7.6) and \(w := f_s(r_{2k_{n, \varepsilon}})\). The profile \(f_s = f_{\gamma_{nn}}\) is defined below (7.13), using (7.9) with exponent \(\gamma_{nn}\) and \(r_k = (M_{in} k)^{1/d}\) in (7.6), and finally \(k_{n, \varepsilon}\) from (7.2) we obtain

\[
\mathbb{P}(\Xi(2r_{2k_{n, \varepsilon}}) \cup \{1, f_s(r_{2k_{n, \varepsilon}})\} \leftrightarrow \Xi_n \setminus \Xi_{\varepsilon})
\]

Condition (6.26) holds since \(N = n^\vartheta \geq k_{n, \varepsilon} \gg w\). Then Claim 6.6 yields for some \(C > 0\)
\[
T_1 \leq n^{1-\vartheta} C_{6.6} f_s(r_{2k_{n, \varepsilon}}) \max(2r_{2k_{n, \varepsilon}})^d n^{-\vartheta \min(\alpha - 1, \tau - 2)} (1 + \frac{1}{\alpha - \tau - 1} \log(n^\vartheta))
\]
\[
\leq C \log(n) \cdot k_{n, \varepsilon}^{1+\vartheta - \alpha - 1} n^{-\vartheta \min(\alpha, \tau - 1)}.
\]

Since \(k_{n, \varepsilon} \in (7.2)\) is at most \(n^\varepsilon\), as long as \(1 - \vartheta \min(\alpha, \tau - 1) < 0\), we can choose \(\varepsilon > 0\) in (7.2), sufficiently small such that for any \(\delta \in (0, \vartheta \min(\alpha, \tau - 1) - 1)\), for all \(n\) sufficiently large,
\[
T_1 \leq n^{-\delta}.
\] (7.27)

We turn to bound \(T_2\) in (7.26) using \((1 - x)^{m_n} \leq \exp(-m_n x)\), where we apply (7.25) on \(x = \mathbb{P}(A^{(1)}_{\text{good}})\) to obtain a lower bound on the exponent
\[
m_n \mathbb{P}(A^{(1)}_{\text{good}}) \geq (\rho^2/2) n^{1-\vartheta} \exp\left(-A^{\text{max}(Z)}_{<, n, \varepsilon} \log n^{-1}(k_{n, \varepsilon})\right)
\]
\[
= (\rho^2/2) \exp\left((1- \vartheta) \log(n) - A^{\text{max}(Z)}_{<, n, \varepsilon} \log n^{-1}(k_{n, \varepsilon})\right).
\]

In order to show \(T_2 \leq n^{-\delta}\) in (7.26), it is much stronger to show that with \(m(Z) = 1 = m'\)
\[
\forall \varepsilon' > 0, \text{ there exists } \varepsilon_1 > 0 \text{ such that for all } \varepsilon < \varepsilon_1: A^{\text{max}(Z)}_{<, n, \varepsilon} \log m'(k_{n, \varepsilon}) < \varepsilon' \log(n).
\] (7.28)

We recall the definition of \(k_{n, \varepsilon}\) in (7.2). While \(k_{n, \varepsilon}\) may be as large as \(n^\varepsilon\), only when \(\max(Z) = 0\), so effectively the expression in (7.28) is still small. We formally check:

Case 1. \(\max(Z) > 0\). We substitute \(k_{n, \varepsilon}\) in the first row of (7.2) to (7.28)
\[
A^{\text{max}(Z)}_{<, n, \varepsilon} \log m'(k_{n, \varepsilon}) = A(\varepsilon \log(n)/\log log(n))m' \log m'(\varepsilon \log(n)/\log log(n))^{1/\max(Z)}.
\]

The last factor is at most \((\max(Z))^{-m'}(\log(n))^{m'}\), and (7.28) follows whenever \(\varepsilon < \varepsilon'(\max(Z))^{m'}/A'\).

Case 2. \(\max(Z) = 0\). We substitute \(k_{n, \varepsilon}\) in the second row of (7.2) to (7.28)
\[
A^{\text{max}(Z)}_{<, n, \varepsilon} \log m'(k_{n, \varepsilon}) = A' \left(\log \left(\exp(\varepsilon \log(n))^{1/m'}\right)\right)^{m'} = A' \log(n),
\]
and (7.28) again follows. Choose now any \(\vartheta < 1/\min(\alpha, \tau - 1)\), and then combine (7.27) with \(T_2 \leq n^{-\delta}\) to bound (7.26). This finishes the proof of (7.5) and hence Proposition 7.1 subject to Lemma 7.2. \(\square\)

7.2. Isolation. We aim to prove Lemma 7.2. We suppress the superscripts \((k)\) of events, and leave it to the reader to verify that the results extend to the Palm-version \(\hat{\mathbb{P}}^0_{\text{io}}\) of \(\hat{\mathbb{P}}_{\text{io}}\), which is defined in (7.18). We work towards bounding the event in (7.19) via a few lemmas/claims, also related to the back-of-the-envelope reasoning of Section 1.2. Recall \(f_\gamma, M_\gamma\) from (7.9), (7.10), the PPPs in (7.11), and \(\zeta_{nn} = \frac{d-1}{\delta}\) from (1.9).

Lemma 7.3 (Vertices above \(M_\gamma\)). Consider a supercritical \(i\)-KSRG under the conditions of Lemma 7.2. For each \(\gamma \in (0, 1/(\alpha + 1)]\), there exists a constant \(C_{\text{iso}} > 0\) such that
\[
\hat{\mathbb{E}}_{\text{io}}(\Xi_{> M_\gamma} \cup \Xi_{> M_\gamma}^\text{out}) \leq \begin{cases} C_{\text{iso}} d_{\text{out}}^{(1-\gamma)(\tau-1)}, & \text{if } 1 - \gamma(\tau - 1) > \zeta_{nn}, \\ C_{\text{iso}} d_{\text{out}}^{\gamma(\tau-1)} \log(r_k), & \text{if } 1 - \gamma(\tau - 1) = \zeta_{nn}, \\ C_{\text{iso}} d_{\text{out}}^{\gamma(\tau-1)}, & \text{if } 1 - \gamma(\tau - 1) < \zeta_{nn}. \end{cases}
\] (7.29)
Recall the multiplicity $m$ of the maximum from \[2.5\].

**Lemma 7.4** (Edges between vertices below $M_υ$). Consider a supercritical $i$-KSRG under the conditions of Lemma 7.2 with $α < ∞$. Let $Y = \{0, α - (τ - 1), (σ + α - 2)(τ - 1)\}$. For each $γ ∈ (0, 1/(σ + 1)]$ there exists a constant $C(υ)(γ > 0)$ such that for any realization of $Ξ_{R_{in}} \cup Ξ_{R_{out}}$ that satisfies $A_{reg}(γ)$ in (7.17) for some $η > 0$,

\[\mathbb{P}_{io}[E(Ξ_{M_υ} \cup Ξ_{M_υ})] \leq \begin{cases} C(υ)k^{d(2 - α + γ)}\log^m(γ)^{-1}(r_k), & \text{if } 2 - α + γ > 0, \\
C(υ)\log^m(γ)(r_k), & \text{if } 2 - α + γ = 0, \\
C(υ)k^{d(2α)} & \text{if } 2 - α + γ < 0, 
\end{cases}\] (7.30)

**Proof sketch of Lemmas 7.3 and 7.4.** We defer the lengthy integrals to the appendix on page 48 but we give some intuition. The expectation $\mathbb{E}_{io}$ in (7.18) is conditional on $Ξ \cap (R_{in}∪R_{out})$ where $R_{in}, R_{out}$ in (7.7) are hyperrectangles below $M_υ$ for all $η$, since $w_η(ℓ)$ is poly-logarithmic in $ℓ$ in (7.1). Hence $Ξ_{M_υ}$ is independent of the conditioning in $\mathbb{E}_{io}$, so

\[\mathbb{E}_{io}[E(Ξ_{M_υ} \cup Ξ_{M_υ})] = E[E(Ξ_{M_υ} \cup Ξ_{M_υ})] = μ_τ(d+1)[R_{M_υ} \cup R_{M_υ}],\] (7.31)

with the intensity $μ_τ$ in (2.2), and where we denote $R_{M_υ}$ the points of $R_{M_υ}$ above the $d$-dimensional surface $M_υ$, see (7.10). Define the hyperrectangle $R^d := [-2r_k, 2r_k]^d \times [r_k^n, ∞]$ and $A_β := \{x ∈ R^d, ||x|| ≤ [r_k^n - Cβ, r_k + Cβ] \times [1, ∞), an annulus in R^d times all mark-coordinates. Then by definition of $f_τ$ in (7.9), $R^d \cap A_β \subseteq R_{M_υ}$ and $μ_τ(R^d \cap A_β) = \Theta(r_k^{d - (γ(τ - 1) + 1)})$. Careful integration shows that the right-hand side of (7.31) is the same order, unless $1 - γ(τ - 1) = (d - 1)/d = 0$, then we get an extra log($r_k$) factor.

For Lemma 7.3, we use independence of $Ξ$ in disjoint sets: inside and outside of $R_{in}, R_{out}$. When counting edges with at least one point in $R_{in} \cup R_{out}$, we use that $Ξ$ in these sets is regular, i.e., $\mathcal{A}_{reg}$ holds, and outside these sets we integrate using the intensity $μ_τ$ in (7.29). We explain now how the exponents of $r_k$ in (7.30) arise: the expected number of edges between vertices of constant mark within constant distance of $∂B_{k,M_{in}}$ is $Θ(r_k^{d - 1})$, which yields lower bounds for all cases in (7.30).

Let $0 ≤ γ_1 ≤ γ_2 ≤ γ$ such that $σγ_1 + γ_2 ≤ σ + 1$. Using $μ_τ$ in (2.2), the expected number of pairs $(u, v) ∈ Ξ_{M_υ} \times Ξ_{M_υ}$ within distance $Θ(r_k)$ from $∂B_{k,M_{in}}$ and marks $u_υ, w_υ = Θ(k^{γ_1}), w_υ = Θ(k^{γ_2})$ is

\[E[Edge(γ_1, γ_2)] = Θ(r_k^{d(1 - γ_1(τ - 1))} \cdot r_k^{d(1 - γ_2(τ - 1))}) = Θ(r_k^{d(2γ_1(σ + 1))}).\] (7.32)

The typical Euclidean distance between such vertices is $Θ(r_k)$. Therefore, by the connection probability $p$ in (2.3), a pair of such vertices is connected with probability roughly $Θ(r_k^{dα(σγ_1 + γ_2 - 1)})$, and hence there are

\[E[Edges(γ_1, γ_2)] = E[Edge(γ_1, γ_2)] \cdot Θ(r_k^{dα(σγ_1 + γ_2 - 1)}) = Θ(r_k^{d(2 - α + γ(σα - (τ - 1)) + 2α(α - (τ - 1)))}).\] (7.32)

such edges in expectation. The proof below on page 49 reveals that the expectation of $E(Ξ_{M_υ} \cup Ξ_{M_υ})$ can be bounded by maximizing (7.32) with respect to $γ_1, γ_2$, i.e., edges between vertices of mark $Θ(k^{γ_1})$ and $Θ(k^{γ_2})$, with the optimal pair $(γ_1, γ_2)$ dominate the whole expectation. Depending on $(σ, α, τ)$, the optimal $(γ_1, γ_2)$ is either

- (ii) $(0, 0)$, pairs of low-mark vertices, which, when substituted in (7.32) yields $0 \in Y$ in (7.30);
- (li) $(0, γ)$, pairs of one low-mark vertex and one high-mark vertex, which, when substituted in (7.32) yields $α - (τ - 1) \in Y$ in (7.30);
- (hh) $(γ, γ)$, pairs of high-mark vertices, which, when substituted in (7.32) yields $σ + 1 = 2(τ - 1) \in Y$ in (7.30);

or the convex combinations of these if the maximum in $Y$ is non-unique, leading to the poly-logarithmic correction terms in (7.30). These three options of $(γ_1, γ_2)$, combined with the $Θ(r_k^{d})$ many low-mark edges close to the boundary explain what we call the dominant types of connectivity in Section 1.2. We remark that without the assumption that $γ ≤ 1/(σ + 1)$, the optimal solution in (hh) would change to $(γ \wedge 1/(σ + 1), γ \wedge 1/(σ + 1))$. □

Lemmas 7.3 and 7.4 hold for any $γ ≤ 1/(σ + 1)$. Now we show that setting $γ = γ_{nn}$ from (7.13) minimizes the sum of (7.29) (increasing in $γ$) and (7.30) (decreasing in $γ$).
Claim 7.5 (The optimally-suppressed mark-profile). Consider a supercritical \( i\)-KSRG under the conditions of Lemma 7.2. Let \( Z = \{\zeta_{i1}, \zeta_{ih}, \zeta_{hh}, \zeta_{mh}\} \). There exists a constant \( c_{\text{opt}} > 0 \) such that when \( \alpha < \infty \)

\[
\widetilde{P}_io(\{Z \leq M, X \leq M, X \leq Z\}) \leq \beta \max \left( \frac{d}{\log d} \right) \log_{M \alpha} \left( \frac{r_k}{r_k - \max(2, \log_{M \alpha})} \right),
\]

and when \( \alpha = \infty \)

\[
\widetilde{P}_io(\{Z \leq M, X \leq M, X \leq Z\}) \leq \beta \max \left( \frac{d}{\log d} \right) \log_{M \alpha} \left( \frac{r_k}{r_k - \max(2, \log_{M \alpha})} \right).
\]

We postpone the proof, based on elementary rearrangements of the formulas of \( \zeta_{i1}, \zeta_{ih}, \zeta_{hh}, \) and \( \zeta_{mh} \), to the appendix on page 52. We proceed with a claim towards bounding the event \( A^{\text{noedge}}_{\text{out}} \) in (7.12), its proof explaining the restriction \( \gamma \leq 1/(\sigma + 1) \) in \( \gamma_{\text{un}} \) in (7.13). We recall \( p(u, v) \) defined in (2.3).

Claim 7.6 (Cross-edge probability bounds). Consider a supercritical \( i\)-KSRG under the conditions of Lemma 7.2, and assume \( \gamma \leq 1/(\sigma + 1) \). For any \( (u, v) \in \Xi_{\leq M} \times \Xi_{\leq M} \)

\[
p(u, v) \leq \begin{cases} 2^{-\alpha}, & \text{if } \alpha < \infty, \\ 0, & \text{if } \alpha = \infty. \end{cases}
\]

Proof. We show that whenever \( u, v \) are below \( M \), and on different sides of \( \partial B_{kM} \), then \( \beta_{k1}(u, v)/\|x_u - x_v\|^d \leq 1/2 \). By definition of \( p \) in (2.3), this directly implies (7.35). To see this bound, for \( \sigma \geq 0 \) the connection probability \( p \) is increasing in the marks. Therefore, without loss of generality we will assume that \( u \in B_{kM} \) and \( v \notin B_{kM} \) fall exactly on \( M \), and that \( k \) is large enough that \( r_k \geq C_\beta \).

Since \( f_\gamma \) in (7.9) switches formula at \( r_k \) outside \( \partial B_{kM} \), we distinguish two cases.

Case 1. Assume \( \|v\| \leq r_k \). Since \( f_\gamma \) in (7.9) and the explanation below (7.12) both \( u, v \) are at least \( C_\beta \) distance from \( \partial B_{kM} \). Thus, for any \( (u, v) \), there exist \( t \geq 2C_\beta \), \( \nu \in (0, 1) \) such that \( \|x_u - \partial B_{kM}\| = r_k - \|x_v\| = (1 - \nu)t \), and \( \|x_v - \partial B_{kM}\| = \|x_v\| = r_k = \|x_v\| \). By triangle inequality \( \|x_u - x_v\| \leq t \). Hence, \( w_o = C_\beta^{-d(\nu)}(\nu)^d \), \( w_v = C_\beta^{-d(\nu)}((1 - \nu)t)^d \) by assuming \( u, v \) being on \( M \), and \( f_\gamma \) in (7.9).

Using \( K_{1, \nu} \), we have

\[
\beta_{k1}(u, v) \leq \beta_{k1}(u, v) \leq \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} \leq \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} \leq \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d}.
\]

The right-hand side is non-increasing in \( t \) whenever \( \gamma \leq 1/(\sigma + 1) \). Since \( t \geq 2C_\beta \geq C_\beta \), and \( C_\beta = (2\beta)^{1/d} \)

\[
\beta_{k1}(u, v) \leq \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} \leq \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} = \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} = \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} = \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} = \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d} = \beta_{C_\beta^{-d(\nu)}(\nu)^d \min((1 - \nu)t, \nu t)^d}.
\]

where to obtain the second inequality we used that \( f_\gamma(\|x_v\|) \leq r_k \), and to obtain the second term the power of \( \|x_v\| - r_k \) canceled. The bound 1/2 follows because \( r_k \geq C_\beta \), \( \gamma \leq 1/(\sigma + 1) \) and \( C_\beta = (2\beta)^{1/d} \).

We are ready to prove Lemma 7.2.

Proof of Lemma 7.2 We will obtain a lower bound on \( \tilde{P}_io(\{\Xi \leq M \}) \). Then we maximize this bound by setting \( \gamma = \gamma_{\text{un}} \), yielding the event \( A^{\text{noedge}}_{\text{out}} \) in (7.14) and the bound in Lemma 7.2. The events \( \{\Xi \leq M \} \) and \( A^{\text{noedge}}_{\text{out}} \) (defined in (7.12)) are independent of each other under \( \tilde{P}_io \) in (7.18), since having no points above \( M \) is independent of the conditioning in \( \tilde{P}_io \) (since each point in \( R_{\text{in}} \cup R_{\text{out}} \) is below \( M \) if \( k \) is sufficiently large), and \( A^{\text{noedge}}_{\text{out}} \) only depends on points of \( \Xi \) below \( M \).

Hence

\[
\tilde{P}_io(\{\Xi \leq M \}) = \tilde{P}_io(\{\Xi \leq M \}) \tilde{P}_io(\{\Xi \leq M \}).
\]
We analyze the two probabilities separately. For the first term, by the above independence,
\[ \tilde{P}_{io}(\Xi \leq \mathcal{M}_\gamma) = \tilde{P}_{io}(|\Xi_{in}^\gamma \cup \Xi_{out}^\gamma| = 0) = \exp \left( - \tilde{E}_{io} \left[ |\Xi_{in}^\gamma \cup \Xi_{out}^\gamma| \right] \right), \] (7.37)
for which we will use Lemma 7.3 shortly. We now turn to the second factor in (7.36). By definition of \( A_{\text{noedge}} \) in (7.12), and using the conditional independence of edges,
\[ \tilde{P}_{io}(A_{\text{noedge}}^{(k)}(\gamma)) = \tilde{E}_{io} \left[ \prod_{u \in \Xi_{in}^\gamma, v \in \Xi_{out}^\gamma} (1 - p(u, v)) \right]. \] (7.38)

We distinguish now the two cases \( \alpha = \infty \) and \( \alpha < \infty \).

**Case \( \alpha = \infty \).** By Claim 7.6 since \( \gamma \leq 1/(\sigma + 1) \) by assumption, \( p(u, v) = 0 \) for each factor, hence \( \tilde{E}_{io}(A_{\text{noedge}}^{(k)}(\gamma)) = 1 \). Since the right-hand side of (7.37) is increasing in \( \gamma \), we set \( \gamma = 1/(\sigma + 1) = \gamma_{\text{nn}} \) when \( \alpha = \infty \) by (7.13). Combining then (7.36) with (7.37) and Claim 7.5 it follows that
\[ \tilde{P}_{io}(\{\Xi \leq \mathcal{M}_\gamma\} \cap A_{\text{noedge}}^{(k)}(\gamma_{\text{nn}})) \geq \exp \left( - C_{\text{max}}^{d \max(Z)} k \log m z - 1(r_k) \right), \]
finishing the proof of the lemma for \( \alpha = \infty \).

**Case \( \alpha < \infty \).** By Claim 7.6 for all \((u, v) \in \Xi_{in}^\gamma \times \Xi_{out}^\gamma\) it holds that \( 1 - p(u, v) \geq 1 - 2^{-\alpha} \) since \( \gamma \leq 1/(\sigma + 1) \) by assumption. Hence, there exists a constant \( c > 0 \) such that for all such \((u, v)\) we have \( 1 - p(u, v) \geq \exp(-c \cdot p(u, v)) \). Using this in (7.38) and that \( s \mapsto \exp(-s) \) is a convex function, Jensen’s inequality gives a lower bound in terms of the expected number of edges between vertices below \( \mathcal{M}_\gamma \), i.e.,
\[ \tilde{P}_{io}(A_{\text{noedge}}^{(k)}(\gamma)) \geq \tilde{E}_{io} \left[ \exp \left( - c \sum_{u \in \Xi_{in}^\gamma, v \in \Xi_{out}^\gamma} p(u, v) \right) \right] \geq \exp \left( - c \tilde{E}_{io} \left[ \sum_{u \in \Xi_{in}^\gamma, v \in \Xi_{out}^\gamma} p(u, v) \right] \right) = \exp \left( - c \tilde{E}_{io} \left[ E(\Xi_{in}^\gamma, \Xi_{out}^\gamma) \right] \right). \] (7.39)

Together with (7.36) and (7.37), we obtain that
\[ \tilde{P}_{io}(\{\Xi \leq \mathcal{M}_\gamma\} \cap A_{\text{noedge}}^{(k)}(\gamma)) \geq \exp \left( - \left( \tilde{E}_{io}[|\Xi_{in}^\gamma \cup \Xi_{out}^\gamma|] + c \tilde{E}_{io}[E(\Xi_{in}^\gamma, \Xi_{out}^\gamma)] \right) \right). \] (7.40)

By Lemmas 7.3 and 7.4, the first expectation is decreasing, while the second one is increasing in \( \gamma \) on the right-hand side. By (7.33) in Lemma 7.5, when we set \( \gamma = \gamma_{\text{nn}} \) in (7.13), we obtain that (7.40) turns into (7.19), finishing the proof of Lemma 7.2 when \( \mathcal{V} = \Xi \). \( \square \)

**Proof of Lemma 7.2 for i-KSRGs on \( \mathbb{Z}^d \).** We explain how to adjust the proof to i-KSRGs on \( \mathbb{Z}^d \) using the assumption \( \min\{p, p^\beta\} < 1 \) in Lemma 7.2. Since the vertex set \( \mathcal{V} = \mathbb{Z}^d \), the event \( \Xi \leq \mathcal{M}_\gamma \), as defined in (7.10) would never hold, since \( \mathbb{Z}^d \) does have points within distance \( C_\beta \) from \( \partial B_{k \min} \) (see \( f_\gamma \) in (7.9) and the reasoning below (7.12)). Hence, we must adjust the definition of \( f_\gamma \) within distance \( C_\beta \) of \( \partial B_{k \min} \) to be any constant \( c > 1 \). Then the upper bound \( 2^{-\alpha} \) on \( p(u, v) \) in Claim 7.6 can be replaced by \( p < 1 \) (the edge-retention probability for vertices at distance 1 in (2.3)), that only affects constant prefactors in (7.39) and (7.40).

The proofs of the preliminary Lemmas 7.3 and 7.4 remain valid by replacing concentration bounds for Poisson random variables by concentration bounds for Binomial random variables, and replacing integrals over \( \mathbb{R}^d \) by summations over \( \mathbb{Z}^d \). The proof of Claim 7.5 remains verbatim valid. \( \square \)

**Proofs of Theorems 2.2, 2.4, and Corollary 2.3.** Proposition 7.1 yields the lower bounds for both theorems, since its condition (7.3) on having a large enough component on restricted marks occurs with positive probability by Proposition 5.14 under the assumption \( \tau < 2 + \sigma \). The assumption \( \tau > 2 \) is necessary to have a locally finite graph. Propositions 5.1 and 6.1 imply the upper bounds (i.e., part (ii)) in Theorems 2.4 and Corollary 2.3, respectively. The conditions in Proposition 6.1 are satisfied by Propositions 5.1, 5.12 and Corollary 5.11 that all hold for the Palm-version \( \mathbb{P}^\ast \) of \( \mathbb{P} \) as well. The condition \( \tau < 2 + \sigma \) is needed for Claim 5.5 which yields that a constant proportion of high-mark vertices, i.e., vertices having mark \( \Omega(k^{\tau_{\text{th}}}) \), is connected by an edge to another high-mark vertex. This allows to construct a backbone of high-mark vertices (Lemma 5.2), and to merge components of size at least \( k \) with the backbone via a high-mark vertex in (5.42). The additional condition \( \sigma \leq \tau - 1 \) in
Theorems 2.2 and 2.4 is needed in the cover-expansion step (Lemma 5.9) in the proof of Proposition 5.1.

It remains to prove Theorem 2.5 whose proof is based on Lemma 7.2.

Proof of Theorem 2.5 For \( \rho \geq 1 \) the statement is trivial. There exists a constant \( C > 0 \) such that for any \( \rho \in (0,1) \) and \( n \geq 1 \) a box of volume \( n \) is contained in the union of \([C/\rho]\) (partially overlapping) balls of volume \( n\rho/2 \). Fix \( \rho \in (0,1) \), and write \( \Xi(\rho) \) for the vertices in the \( i \)-th ball of such a cover of balls of volume \( n\rho/2 \). Recall that \( |E(A,B)| \) denotes the number of edges between the sets \( A, B \). Then

\[
\{ |C_n(\rho)| < \rho n \} \supseteq \bigcap_{i \leq [C/\rho]} \{ |E(\Xi(\rho) \setminus \Xi(\rho))| = 0 \} \cap \{ |\Xi(\rho)| < \rho n \} \tag{7.41}
\]

Indeed, on the event on the right-hand side, each connected component of \( G_\rho \) is fully contained in some ball (or the intersection of some balls) with at most \( \rho n \) vertices. We apply an FKG inequality to bound the probability of intersection from below.

We give a (natural) definition of increasing events, using the collection \( \Psi = \{ \varphi_{u,v} : u, v \in \Xi \} \) that encodes the presence of edges using a set of uniform random variables. We say that a function \( f(\Xi, \Psi) \) defined on the marked vertex set \( \Xi \) and edge-variable set \( \Psi \) is increasing if it is non-decreasing in \( \Xi \) with respect to set inclusion (formally, if \( \Xi' \supseteq \Xi, \Psi_{\Xi'} \supseteq \Psi_{\Xi}, \) then \( f(\Xi', \Psi_{\Xi'}) \geq f(\Xi, \Psi_{\Xi}) \) holds), as well as coordinate-wise non-increasing with respect to the edge variables (formally, if \( \Psi' \) satisfies \( \varphi'_{u,v} \leq \varphi_{u,v} \) for all \( u, v \in (\Xi(\rho)) \), then \( f(\Xi, \Psi') \geq f(\Xi, \Psi) \) holds). Intuitively this means that more vertices and edges increase the value of \( f \). Similarly to [13], we obtain that for two increasing functions \( f_1, f_2 \),

\[
\mathbb{E}[f_1(\Xi, \Psi_{\Xi}) \cdot f_2(\Xi, \Psi_{\Xi})] = \mathbb{E}[\mathbb{E}[f_1(\Xi, \Psi_{\Xi}) \cdot f_2(\Xi, \Psi_{\Xi}) | \Xi]] \geq \mathbb{E}[\mathbb{E}[f_1(\Xi, \Psi_{\Xi}) | \Xi] \cdot \mathbb{E}[f_2(\Xi, \Psi_{\Xi}) | \Xi]] \geq \mathbb{E}[f_1(\Xi, \Psi_{\Xi})] \cdot \mathbb{E}[f_2(\Xi, \Psi_{\Xi})],
\]

by applying FKG to the random graph conditioned to have \( \Xi \) as its vertex set for the first inequality, and then FKG for point processes for the second inequality [20, Theorem 20.4]. We say that an event \( A \) is decreasing iff the function \( -1_A \) is increasing. It follows that for decreasing events \( A, A' \)

\[
\mathbb{P}(A \cap A') = \mathbb{E}[(-1_A(\Xi, \Psi_{\Xi})) \cdot (-1_A(\Xi, \Psi_{\Xi}))] \geq \mathbb{E}[\mathbb{1}_A(\Xi, \Psi_{\Xi})] \cdot \mathbb{E}[\mathbb{1}_{A'}(\Xi, \Psi_{\Xi})] = \mathbb{P}(A) \cdot \mathbb{P}(A'). \tag{7.42}
\]

Observe that the events on the right-hand side in (7.41) are all decreasing (adding vertices/edges make the events less likely to occur) so that (7.42) applies. Hence,

\[
\mathbb{P}(\{|C_n(\rho)| < \rho n\}) \supseteq \prod_{i \leq [C/\rho]} \mathbb{P}(|E(\Xi(\rho) \setminus \Xi(\rho))| = 0) \cdot \mathbb{P}(|\Xi(\rho)| < \rho n), \tag{7.43}
\]

Since each ball has volume \( n\rho/2 \), the event \( \{|\Xi(\rho)| < \rho n\} \) holds with probability at least \( 1/2 \) by concentration inequalities for Poisson random variables (Lemma C.1 for \( x = 2 \)). To bound \( \mathbb{P}(|E(\Xi(\rho), \Xi(\rho)| = 0) \), we consider the optimally-suppressed mark profile translated to the center of the \( i \)-th ball, with \( kM_{\text{in}} \) replaced by \( n\rho/2 \). We restrict \( \Xi \) to be below the mark profile, and to have no edges between \( \Xi(\rho) \) and \( \Xi(\rho) \). We apply Lemma 7.2 integrate over all realisations of \( G_{\text{in}}, G_{\text{out}} \) satisfying \( \mathcal{A}_{\text{regular}} \), and use that the event \( \mathcal{A}_{\text{regular}} \) in the conditioning in \( \mathbb{P}_{\text{io}} \) in (7.18) holds with high probability by Poisson concentration (Lemma C.1 for \( x = 2 \)), see the argument below (7.21). We obtain that for all \( i \leq [C/\rho] \),

\[
\mathbb{P}(|E(\Xi(\rho) \setminus \Xi(\rho))| = 0) \cdot \mathbb{P}(|\Xi(\rho)| < \rho n) \supseteq \exp \left( -\Theta(n_{\text{max}}(\rho) \log n_{\text{max}}^{-1}(n)) \right)/2,
\]

which yields the desired statement in (2.11) when taking the product over \([C/\rho]\) balls in (7.43). \( \square \)

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APPENDIX A. PROOFS BASED ON BACKBONE CONSTRUCTION

We present the proofs of the propositions at the end of Section [5].

Proof of Proposition 5.12: We will first derive a bound on \( \mathbb{P}(\exists v \in \Xi_n[\overline{w}, \infty) : v \not\in C^{(1)}_n) \) for arbitrary \( \overline{w} \geq 1 \). We make use of the backbone construction from Section [7] we will show that vertices with mark at least \( \overline{w} \) are likely to connect to the backbone \( \mathcal{C}_{bb} \), which will be a subset of the giant component. Observe that the event in (5.4) allows us to choose the size of the boxes when we build the backbone, i.e., the value of \( k \) is not yet defined with respect to \( \overline{w} \). We define \( k = k(\overline{w}) \) implicitly by \( \overline{w} = A_1 k^{1 - \sigma_{hh}} \), where \( A_1 \) is a large enough constant to be determined later. We aim to show that for some \( A_2 > 0 \),

\[
\mathbb{P}(\neg A_{\text{mark-giant}}(n, \overline{w})) := \mathbb{P}(\exists v \in \Xi_n[A_1 k^{1 - \sigma_{hh}}, \infty) : v \not\in C^{(1)}_n) \leq n \exp(- A_2 k^{\sigma_{hh}}). \tag{A.1}
\]

If this bound holds, then substituting \( \overline{w} = \overline{w}_n = (M_w \log(n))^{(1-\sigma_{hh})/\sigma_{hh}} \) yields

\[
k(\overline{w}_n) = A_1^{-1/(1 - \sigma_{hh})} \frac{1}{\overline{w}_n^{1/(1 - \sigma_{hh})}} = A_1^{-1/(1 - \sigma_{hh})} (M_w \log(n))^{1/\sigma_{hh}},
\]

which, when substituted back to (A.1) yields (5.44) if \( M_w \) is chosen sufficiently large.

Recall \( A_{bb}(n, k) \) and \( \mathcal{C}_{bb}(n, k) \) from (5.8). Distinguishing two cases depending on whether \( A_{bb}(n, k) \) holds for \( \mathcal{G}_{n,2} = \mathcal{G}_n[1, 2w_{hh}] \) or not (with \( w_{hh}(k) \) in (5.7)), by Lemma 5.2

\[
\mathbb{P}(\neg A_{\text{mark-giant}}(n, \overline{w})) \leq \mathbb{P}(\neg A_{bb}) + \mathbb{E}[\mathbb{1}_{\{A_{bb}\}} \mathbb{P}(\neg A_{\text{mark-giant}}(n, \overline{w}) | \mathcal{G}_{n,2}, A_{bb})] \leq 3n \exp(- A_2 k^{\sigma_{hh}}) + \mathbb{E}[\mathbb{1}_{\{A_{bb}\}} \mathbb{P}(\neg A_{\text{mark-giant}}(n, \overline{w}) | \mathcal{G}_{n,2}, A_{bb})]. \tag{A.2}
\]

On the event \( A_{bb} \), there is a backbone \( \mathcal{C}_{bb} \). This backbone is either not part of the giant component, or if it is, then a vertex with mark at least \( \overline{w} \) outside the giant has no connection to any of the vertices in the backbone. Hence, conditionally on the event \( A_{bb} \),

\[
\neg A_{\text{mark-giant}}(n, \overline{w}) \subseteq \{ \mathcal{C}_{bb} \not\subseteq C^{(1)}_n \} \cup \{ \exists v \in \Xi_n[\overline{w}, \infty) : v \not\leftrightarrow \mathcal{C}_{bb}, \mathcal{C}_{bb} \subseteq C^{(1)}_n \}.
\]

By a union bound and Corollary 5.10, this implies that

\[
\mathbb{P}(\neg A_{\text{mark-giant}}(n, \overline{w}) | \mathcal{G}_{n,2}, A_{bb}) \leq (n/2)^{\mathcal{A}_{bb}(n, 2w_{hh})} + \mathbb{P}(\exists v \in \Xi_n[\overline{w}, \infty) : v \not\leftrightarrow \mathcal{C}_{bb} | \mathcal{G}_{n,2}, A_{bb}) \tag{A.3}
\]

Recall that \( \mathcal{G}_{n,2} \) is the graph spanned on vertices with mark in \([1, 2w_{hh}]\), see Definition 5.6. With \( C_1 \) from (5.5–5.6), we may assume \( A_1 \geq 2C_1^{-1/(\tau-1)} \). Since \( w_{hh} = C_1^{-1/(\tau-1)} k^{\gamma_{hh}} \), defined in (5.7), and since \( 1 - (1 + \sigma) \gamma_{hh} \geq 0 \) (see (5.14)), this implies that

\[
\overline{w} = A_1 k^{1-\sigma_{hh}} \geq 2w_{hh} = 2C_1^{-1/(\tau-1)} k^{\gamma_{hh}}.
\]

Hence, vertices of mark at least \( \overline{w} \) are part of \( \Xi_n[2w_{hh}, \infty) \) and are not revealed in \( \mathcal{G}_{n,2} \). Conditioning on the number of vertices \( |\Xi_n[\overline{w}, \infty)| \), the location of each vertex is independent and uniform in \( \Lambda_n \). Taking a union bound over these vertices in \( \Xi_n[\overline{w}, \infty) \), yields

\[
\mathbb{P}(\exists v \in \Xi_n[\overline{w}, \infty) : u \not\leftrightarrow \mathcal{C}_{bb} | \mathcal{G}_{n,2}, A_{bb}) \leq \mathbb{E}[|\Xi_n[\overline{w}, \infty)|] \cdot \mathbb{P}(v \not\leftrightarrow \mathcal{C}_{bb} | \mathcal{G}_{n,2}, A_{bb}, v \in \Xi_n[\overline{w}, \infty)) \leq n \mathbb{P}(v \not\leftrightarrow \mathcal{C}_{bb} | \mathcal{G}_{n,2}, A_{bb}, v \in \Xi_n[\overline{w}, \infty)) \tag{A.4}
\]

Let \( Q(v) \) be as in (5.3). Conditionally on \( A_{bb} \), \( Q(v) \) contains at least \( s_k = \Theta(k^{\gamma_{hh}}) \) vertices in \( \mathcal{C}_{bb} \) with mark in \([w_{hh}, 2w_{hh})\), where \( w_{hh} \) is defined in (5.7), yielding the set of vertices \( S(v) \) in (5.22). We use the distance bound in (5.4), and \( p, \kappa_1, \sigma \) defined in (2.3), and (1.10), respectively, and the value \( \overline{w} \) in (A.4), to obtain that for any \( v \in \Xi_n[\overline{w}, \infty) \) and \( u \in S(v) \), when \( \alpha < \infty \),

\[
p(u, v) \geq p \min \left\{ 1, (\beta \kappa_1, \sigma(w_{hh}, A_1 k^{1-\sigma_{hh}})(2\sqrt{d})^{-d}k^{-\alpha}) \right\} = p \min \left\{ 1, (\beta C_1^{-\sigma/(\tau-1)} k^{\gamma_{hh}} A_1 k^{1-\sigma_{hh}}(2\sqrt{d})^{-d}k^{-\alpha}) \right\} = p,
\]

whenever \( A_1 \geq (2\sqrt{d})^d C_1^{-\sigma/(\tau-1)} / \beta \), since the exponent of \( k \) in the second term of the minimum is 0. The same bound holds when \( \alpha = \infty \). Since \( v \) connects by an edge to each of the \( s_k = \Theta(k^{\gamma_{hh}}) \) many backbone vertices in \( S(u) \) with probability at least \( p \), conditionally independently of each other, we bound (A.4) by

\[
\mathbb{P}(\exists v \in \Xi_n[\overline{w}, \infty) : v \not\leftrightarrow \mathcal{C}_{bb} | \mathcal{G}_{n,2}, A_{bb}) \leq n(1 - p)^{s_k}.
\]
Since \( s_k = \Theta(k^{\zeta_{bh}}) \) in (5.7), combining this with (A.2) and (A.3) yields (A.1) for \( A_2 \) sufficiently small. As argued below (A.1), this yields (5.44).


This section presents the proof of the first statement of Proposition 5.14. At the end of the section, we comment how the proof can be adjusted to obtain the second statement considering the infinite model. Throughout the proof, we will consider the Palm version of \( P \), conditioning \( \Xi \) to contain a vertex at location 0. We will leave it out in the notation. We will show using a second-moment method that linearly many vertices connect to the backbone \( C_{bh}(n, k) \) for a properly chosen \( k \). First we introduce some notation. Recall \( C_1 \) from (5.5)-(5.6). We implicitly define \( k = k_n \) as the solution of the equation

\[
(C_1/16)^{\zeta_{bh}} := (2/|\Xi_1| \log(n)), \tag{A.5}
\]

yielding \( m = (2/|\Xi_1|)^{1/\zeta_{bh}} \) and the mark-truncation value in the definition of \( \mathcal{G}_{n,2} \) at \( 2w_{bh}(k_n) = 2C_1^{-1/(\tau-1)}k_n^{\zeta_{bh}} \) from (5.7), with \( w_{bh}(k_n) = (M_w \log(n))^{\gamma_{bh}/\zeta_{bh}} \) for some constant \( M_w \). We reveal the realization of the graph \( \mathcal{G}_{n,1} = \mathcal{G}_n[w_{bh}(k_n), 2w_{bh}(k_n)) \) (defined above (5.5)), conditioned to satisfy the event \( \mathcal{A}_{bh}(n, k_n) \) in (5.8). Recalling the intensity measure of the Poisson vertex set \( \mu_r \) from (2.2), for any constant \( m_w \geq 1 \) we define the event

\[
\mathcal{A}'_{reg} := \{ |\Xi_n[m_w, 2m_w]/\mu_r(\Lambda_n \times [m_w, 2m_w])| \in [1/2, 2] \}, \tag{A.6}
\]

that is, that the PPP in \( \Lambda_n \) is regular in the sense that the number of constant-mark vertices is roughly as expected. Writing \( w_0 \) for the mark of 0, we define the conditional probability measure

\[
\tilde{P}_{bh}(\cdot) := P(\cdot | \mathcal{A}'_{reg}, \mathcal{A}_{bh}(n, k_n), \mathcal{G}_{n,1}, w_0 \in [m_w, 2m_w]), \tag{A.7}
\]

with corresponding expectation \( \tilde{E}_{bh} \). We state a lemma that implies Proposition 5.14.

**Lemma A.1.** (Constructing a component). Consider a supercritical \( \iota \)-KSRG under the same conditions as Proposition 5.14. Take \( k_n \) as in (A.5). For any \( m_w \geq 1 \), for all sufficiently large \( n \),

\[
P(\mathcal{A}'_{reg}, \mathcal{A}_{bh}(n, k_n), w_0 \in [m_w, 2m_w]) \geq m_w^{\rho/(\tau-1)/2}. \tag{A.8}
\]

Moreover, there exist constants \( m_w \geq 1, \rho > 0 \), such that for all sufficiently large \( n \)

\[
\tilde{P}_{bh}(\mathcal{C}_{n,2}(0)) \geq \rho. \tag{A.9}
\]

To prove the lemma (in particular the second statement), we need to define auxiliary notation and an auxiliary claim: we define for \( u := (x_u, w_u) \in \Xi_n[m_w, 2m_w] \) the event

\[
\{ u \xrightarrow{\pi} C_{bh} \} := \left\{ \exists \text{ a path in } \mathcal{G}_{n,2} \text{ from } u \text{ to some vertex in } C_{bh}(n, k_n), \text{ with all vertices (except } u \text{) having mark in } \mathbb{R}^+ \setminus [m_w, 2m_w] \right\}. \tag{A.10}
\]

Recall \( \Lambda(0, s) \) from (2.1) the box centered at 0 of volume \( s \), and that \( u = (x_u, w_u) \) denotes the location and mark of a vertex in \( \Xi \). The next claim states that if \( u \) and 0 are both vertices with mark in \([m_w, 2m_w] \), falling into different subboxes \( Q(0) \neq Q(u) \) (5.3) with respect to the boxing defined by \( k_n \) above (5.3), then the event that both 0 and \( u \) connect to the backbone \( C_{bh} \), happens with constant probability.

**Claim A.2.** (Paths to the backbone). Consider a supercritical \( \iota \)-KSRG under the conditions of Proposition 5.14. There exist positive constants \( m_w, \epsilon_{A.2} > 0 \) such that for all \( u \in \Xi_n[m_w, 2m_w] \) with \( x_u \notin \Lambda(0, C_{A.2} k_n) \), and \( n \) sufficiently large

\[
\tilde{P}_{bh}(\{ 0 \xrightarrow{\pi} C_{bh}(n, k_n) \} \cap \{ u \xrightarrow{\pi} C_{bh}(n, k_n) \} | u \in \Xi_n[m_w, 2m_w], x_u \notin \Lambda(0, C_{A.2} k_n)) \geq \epsilon_{A.2}
\]

**Proof.** We will build what we call "mark-increasing paths". Recall \( k_n = (m \log(n))^{1/\zeta_{bh}} \) in Prop. 5.14 and that \( w_{bh}(k_n) = (M_w \log(n))^{\gamma_{bh}/\zeta_{bh}} \) computed below (A.1). Define

\[
j_s := \max\{ j : 2^j m_w < w_{bh}(k_n) \}, \tag{A.11}
\]

and define for \( 0 \leq j \leq j_s \) and \( x \in \Lambda_n \) the following boxes and disjoint mark intervals

\[
Q_j(x) := \Lambda(x, 2^{-\sigma} d^{-d/2}(2^j m_w)^{\sigma+1}) \cap \Lambda_n, \quad I_j := [2^j m_w, 2^{j+1} m_w], \tag{A.12}
\]
and write $Q_{j,x+1} := Q(x)$, i.e., the volume-$k_n$ subbox containing $x$ in the partitioning of $\Lambda_n$ for the backbone construction given in [5.3], and define also $I_{j,x+1} := [w_{hh}(k_n), 2w_{hh}(k_n))$. Even with the truncation by $\Lambda_n$ in (A.12), the volume bound

$$\beta2^{-\sigma-d-d/2}(2^j m_w)^{\sigma+1} \leq \text{Vol}(Q_j(x)) \leq \beta2^{-\sigma-d-d/2}(2^j m_w)^{\sigma+1}$$

(A.13)

holds for all $x \in \Lambda_n$ and $j \leq j_x$ similarly to (A.4). By (2.2), the number of vertices of $\subset$ in $Q_j(x) \times I_j$ has Poisson distribution with mean

$$\mu_r(Q_j(x) \times I_j) \geq 2^{-1}\beta(2^j m_w)^{-(\tau-1)}2^{-\sigma-d-d/2}(2^j m_w)^{\sigma+1},$$

(A.14)

where we used that $1 - 2^{-\sigma-1} \geq 1 - \tau > 2$. Moreover, (A.11) implies $2^j m_w \leq w_{hh}(k_n)$ for all $j \leq j_x$ and substituting this in (A.12) with $w_{hh}(k_n) = C_1^{-1/(\tau-1)}k_n^{\gamma_{hh}}$ from (5.7) yields

$$Q_j(x) \subseteq \Lambda(x, \beta2^{-\sigma-d-d/2}(w_{hh}(n))^{\sigma+1}) = \Lambda(x, \beta2^{-\sigma-d-d/2}C_1^{-1/(\tau-1)}k_n^{\gamma_{hh}(\sigma+1)})$$

$$\subseteq \Lambda(x, \beta2^{-\sigma-d-d/2}(\sigma+1)/(\tau-1)k_n^{\gamma_{hh}(\sigma+1)}) =: Q^*(x),$$

where the last inclusion follows from $\gamma_{hh} \leq 1/(-\sigma + 1)$ by (5.14). Let $\text{diam}(Q^*(x)) := Ck_n^{1/d}$ denote the diameter of $Q^*(x)$, and let $Q^\circ := \Lambda(0, 2^j I_C(k_n))$ denote the box centered at 0, such that $\text{diam}(Q^\circ) = 2\text{diam}(Q^\circ)$. For any $x_u \notin Q^\circ$, and all pairs $j, j' \leq j_x$, it holds that

$$Q_j(x_u) \cap Q_{j'}(0) = \emptyset,$$

(A.15)

and thus the PPPs restricted to $Q_j(x_u) \times I_j$ are independent of each other and of $Q_{j'}(0) \times I_j$ for all $j, j' \leq j_x$. On the conditional measure $F_{bb}$ defined in (A.7), we fixed (revealed) the realization of $\Xi_n[w_{hh}(k_n), 2w_{hh}(k_n))$. Edges among $\Xi_n[w_{hh}(k_n), 2w_{hh}(k_n))$ and $\Xi \cap (Q_j(x_u) \times I_j)$ are thus also present conditionally independently. We define for $u = (x_u, w_u) =: u_0$ the event of having a "mark-increasing path" (a subevent of $\{u \overset{\pi}{\rightarrow} C_{bb}\}$ defined in (A.10)):

$$\{u \sim C_{bb}\} := \{\exists(u_1, \ldots, u_{j+1}), u_{j+1} \in C_{bb} : \forall j \in [j_x+1] : u_j \in Q_j(x_u) \times I_j, u_{j+1} \dashv u_j\}.$$  (A.16)

on which there is a path from $u$ to the backbone, where the $j$th vertex on the path is in $Q_j(x_u) \times I_j$ (that are disjoint across $j$). The mark of $u_{j+1}$ is in the right range by definition $I_{j,x+1}$ of below (A.12).

By this disjointness and (A.15), the events $\{0 \sim C_{bb}\}$ and $\{u \sim C_{bb}\}$ are independent conditionally on $\Xi_n[w_{hh}(k_n), 2w_{hh}(k_n)).$

To bound $F_{bb}(u \sim C_{bb})$ from below, we greedily ‘construct’ a path from $u = u_0$ to the backbone. By assumption $u_0 \in [m_w, 2m_w)$ hence $u_0 \in Q_0(x_u) \times I_0$. We first bound the probability that $u_0$ connects by an edge to a vertex $u_1 \in Q_1(x_u) \times I_1$. Then, if there is such a connection, we choose $u_1$ to be an arbitrary vertex connected to $u_0$ and give a uniform lower bound (over the possible $u_1$) on the probability that it connects by an edge to a vertex $u_2 \in Q_2(x_u) \times I_2$. We continue this process until we reach $u_{j_x}$ that has mark just smaller than the minimal mark of vertices in the backbone, by definition of $j_x$ in (7.15). Then we find a connection from $u_{j_x}$ to the backbone. We now bound the probability that two vertices $u_{j_x-1}$ and $u_j$ are connected by an edge.

By construction, $w_{u_{j_x-1}} \geq 2^{j_x-1}m_w$, $w_{u_{j_x}} \geq 2^j m_w.$ Hence, with the kernel $\kappa_1, \sigma$ from (1.10), and volume bound (A.13), we obtain

$$\beta \kappa_1, \sigma(w_{u_{j_x-1}}, w_{u_{j_x}}) = \beta w^{\sigma}_{\kappa_1, \sigma} w_{u_{j_x}} \geq \beta m_{\kappa_1, \sigma}^{\sigma+1} = \beta 2^{-\sigma-d/2}(2^j m_w)^{\sigma+1} \text{Vol}(Q_j(x_u)).$$

Further, their distance $\|x_{u_{j_x-1}} - x_{u_{j_x}}\| \leq d^{d/2} \text{Vol}(Q_j(x_u)) \leq \beta 2^{-\sigma-d/2}(2^j m_w)^{\sigma+1}$ by (A.12), hence

$$\text{p}(u_{j_x-1}, u_{j_x}) \geq \text{p} \min \left\{ \frac{1}{1 \leq (\beta \kappa_1, \sigma(w_{u_{j_x-1}}, w_{u_{j_x}}))^{\alpha}} \right\} = \text{p}.$$
by (2.2) and (A.14). For \( j = j_* + 1 \) we use that the vertices in the backbone in \( Q_{j_*+1} \times I_{j_*+1} \) are exactly those in \( Q(u) \times [w_{hh}, 2w_{hh}] \), that we denoted by \( S(u) \) in (5.22):

\[
\tilde{\mathbb{P}}_{bb}(\{ u \ni C_{bb} \}) \leq \tilde{\mathbb{P}}_{bb}(u, \not\ni S(u)) + \sum_{j=1}^{j_*} \mathbb{P}(\text{Poi}(p\beta2^{-\alpha-d-1}d^{-d/2}(2^j m_w)^{\alpha+2-\tau}) = 0) \\
\leq \tilde{\mathbb{P}}_{bb}(u, \not\ni S(u)) + \sum_{j=1}^{\infty} \exp \left( -p\beta2^{-\alpha-d-1}d^{-d/2}(2^j m_w)^{\alpha+2-\tau} \right). \quad (A.17)
\]

Since \( \tau < 2 + \sigma \) by assumption, the sum is finite and decreasing in \( m_w \). By definition of \( j_* \) and \( I_{j_*} \), we recall that \( Q(x_u) \) is the subbox of volume \( k_n \) in the partitioning for the backbone containing \( x_u \) (5.3), that contains at least \( s_{k_n} \) backbone vertices of mark at least \( w_{hh}(k_n) \), both defined in (5.7). When \( \alpha < \infty \), we follow the computations in (5.17) (that are also valid under the conditional measure \( \tilde{\mathbb{P}}_{bb} \) from (A.7), ensuring that \( C_{bb}(n, k_n) \) exists and \( |S(u)| \geq s_{k_n} \) by \( A_{bb} \) in (5.8), and use that \( u, j_* \) is at distance at most \( 2\sqrt{d}k_n^{1/d} \) from any vertex in \( S(u) \), to obtain

\[
\tilde{\mathbb{P}}_{bb}(u, \not\ni S(u)) \geq 1 \left( 1 - p \min\{ 1, \beta\alpha d^{-\alpha d/2 - 2(2\sigma + d)\alpha w_{hh}^{(1+\sigma)\alpha} k_n^{-\alpha} \} \} \right)_{s_{k_n}} \\
\geq 1 \left( 1 - \exp \left( p \min\{ s_{k_n}, \beta\alpha d^{-\alpha d/2 - 2(2\sigma + d)\alpha w_{hh}^{(1+\sigma)\alpha} k_n^{-\alpha} s_{k_n} \} \} \right) \right) \quad (A.18)
\]

Using that \( s_{k_n} = k_n w_{hh}^{-(\tau-1)} / 16, w_{hh} = C_1^{1/(\tau-1)} k_n w_{hh}, \) and \( \gamma_{hh} = (\alpha - 1)/(\sigma + 1) - (\tau - 1) \), we have

\[
w_{hh}^{(1+\sigma)\alpha} k_n^{-\alpha} s_{k_n} \geq 2^{-4} w_{hh}^{(1+\sigma)\alpha - (\tau - 1)} k_n^{-\alpha} = 2^{-4} C_1^{1 - (1+\sigma)\alpha/(\tau - 1)}.
\]

Using this bound on the right-hand side in (A.18), this yields combined with (A.17) that there exists a constant \( q > 0 \) such that if \( m_w \) is sufficiently large \( \tilde{\mathbb{P}}_{bb}(u \ni C_{bb}) \geq q \) establishing Claim \( A_2 \) when \( \alpha < \infty \), by the reasoning about independence below (A.16).

When \( \alpha = \infty \), the choice of \( C_1 \) in (5.6) ensures that for any vertex \( u_{bb} \in C_{bb} \cap Q(u) \) that

\[
p(u_{bb}, u_{bb}) \geq p \min\{ \beta \gamma_{kk}(w_{hh} / 4, w_{hh}), \}
\]

establishing Lemma \( A_2 \) for \( \alpha = \infty \) when combined with (A.17) for \( m_w \) sufficiently large. \qed

We are now ready to prove Lemma \( A.1 \).

**Proof of Lemma \( A.1 \)** We first show (A.8). By a union bound, concentration inequalities for Poisson random variables (Lemma \( C.1 \) for \( x \in \{ 1/2, 2 \} \)) and \( F_W (\mu_0) = (\tau - 1)w_0^{-\tau} dw \) in Definition 2.1 it follows that \( \mathbb{P}(\neg A_{reg}^\prime) = \exp(-\Theta(m_w^{-(\tau - 1)})) = o(1) \), and thus

\[
\mathbb{P}(\neg (A_{reg}^\prime \cap A_{bb} \cap \{ w_0 \in [m_w, 2m_w] \})) \\
\leq \mathbb{P}(\neg A_{bb}) + \mathbb{P}(w_0 \notin [m_w, 2m_w] \cap A_{bb}^\prime) + o(1) \leq \mathbb{P}(\neg A_{bb}) + 1 - (1 - 2^{-\tau - 1})m_w^{-(\tau - 1)} + o(1).
\]

By the choice of \( k_n \) in (A.5), the first term tends to zero by Lemma 5.2 as \( n \) tends to infinity. Since \( 1 - 2^{-\tau - 1} > 1/2 \) for \( \tau > 2 \), for \( n \) sufficiently large (depending also on the constant \( m_w \) ) it follows that

\[
\mathbb{P}(\neg (A_{reg}^\prime \cap A_{bb} \cap \{ w_0 \in [m_w, 2m_w] \})) \leq 1 - m_w^{-(\tau - 1)} (1 - 2^{-\tau - 1}) + o(1) \leq 1 - m_w^{-(\tau - 1)}/2,
\]

and (A.8) follows. We proceed to (A.9). Conditionally on the realization of \( G_{n,1} \) satisfying \( A_{bb} \) (present in the conditioning of \( \tilde{\mathbb{P}}_{bb} \) in (A.7)), we define the following set and random variable:

\[
\mathcal{U} := \{ u \in \Xi_n [m_w, 2m_w] : u \ni C_{bb} \}, \quad \mathcal{X} := \sum_{u \in \Xi_n [m_w, 2m_w]} 1_{\{ u \ni C_{bb} \}}, \quad (A.19)
\]

with \( \ni \) from (A.10). The measure \( \tilde{\mathbb{P}}_{bb} \) is a conditional measure where \( A_{reg}^\prime \) (defined in (A.6)) holds and so \( |\Xi_n [m_w, 2m_w] | \leq 2\mu_r (A_n \times [m_w, 2m_w]) \). Using \( \mu_r \) in (2.2), we obtain that deterministically under \( \tilde{\mathbb{P}}_{bb} \):

\[
X^2 \leq 4 (\mu_r (A_n \times [m_w, 2m_w]^2) \leq 4 (m_w^{-(\tau - 1)n})^2.
\]
When $0 \in \mathcal{U}$ holds, then $|\mathcal{C}_n(0)| \geq |\mathcal{U}| \geq X$, and so we apply Paley-Zygmund’s inequality to $X$ under the measure $\tilde{\mathbb{P}}_{bb}$, which yields for $\rho' := \tilde{\mathbb{E}}_{bb}[X]/(2n)$ that
\[
\tilde{\mathbb{P}}_{bb}(|\mathcal{C}_n(0)| \geq \rho'n) \geq \tilde{\mathbb{P}}_{bb}(|\mathcal{U}| \geq \rho'n, 0 \in \mathcal{U}) \geq \tilde{\mathbb{P}}_{bb}(X \geq \tilde{\mathbb{E}}_{bb}[X]/2) \\
\geq (1/4) \frac{\tilde{\mathbb{E}}_{bb}[X]^2}{\tilde{\mathbb{P}}_{bb}[X]^2} \geq \frac{\tilde{\mathbb{E}}_{bb}[X]^2}{16n^2w^{-2(\tau-1)}}. \tag{A.20}
\]
We now bound the numerator on the right-hand side from below. Conditionally on $|\Xi_n[m_w, 2m_w]|$, the vertices have a uniform location in $\Lambda_n$, so
\[
\tilde{\mathbb{P}}_{bb}(x_u \notin \Lambda(0, C_{A.2}k_n) | u \in \Xi_n[m_w, 2m_w], |\Xi_n[m_w, 2m_w]|) = (n - C_{A.2}k_n)/n \geq 1/2,
\]
where the last inequality follows from assuming that $n$ is sufficiently large (recall $k_n = \Theta(\log(n))/\zeta$) by (A.5). The conditioning on $A_{\text{reg}}$ implies that $|\Xi_n[m_w, 2m_w]| \geq \mu_\tau(\Lambda_n \times [m_w, 2m_w])/2$. Using linearity of expectation of $X$ in (A.19), and the tower rule, (by first conditioning on $|\Xi_n[m_w, 2m_w]|$) we obtain for $n$ sufficiently large
\[
\tilde{\mathbb{E}}_{bb}[X] \geq (\mu_\tau(\Lambda_n \times [m_w, 2m_w])/2) \\
\cdot \tilde{\mathbb{E}}_{bb}\left[ \tilde{\mathbb{E}}_{bb}(x_u \notin \Lambda(0, C_{A.2}k_n) | u \in \Xi_n[m_w, 2m_w], |\Xi_n[m_w, 2m_w]|) \right] \\
\cdot \tilde{\mathbb{E}}_{bb}\left( \left( 0 \rightarrow C_{bb} \right) \cap \{ x_u \rightarrow C_{bb} \} | u \in \Xi_n[m_w, 2m_w], x_u \notin \Lambda(0, C_{A.2}k_n) \right) \\
\geq nm_nw^{-(\tau-1)}(1 - 2^{-2(\tau-1)})/2 \cdot (1/2) \cdot A.2 \geq n^2m_n^{(\tau-2)}2^{-3},
\tag{A.21}
\]
where the second bound follows if $m_n$ is chosen as in Claim A.2, and from the definition of $\mu_\tau$ in (2.2); the last bound holds since $2^{-2(\tau-1)} \leq 1/2$ for $\tau > 2$. Substituting the last bound (A.21) into the numerator on the right-hand side to (A.20), we have obtained that
\[
\tilde{\mathbb{P}}_{bb}(|\mathcal{C}_n(0)| \geq \rho'n) \geq 2^{-10}q_{A.2}^{-2}
\]
holds with $\rho' = \tilde{\mathbb{E}}_{bb}[X]/(2n) \geq q_{A.2}^{-2}m_n^{(\tau-1)}2^{-4}$, which yields the statement of Lemma A.1 for $\mu_{A.1} = \min\{q_{A.2}2^{(\tau-2)4}, 2^{-10}q_{A.2}^{-2}\}$. \hfill \Box

Proof of Proposition 5.14. We start with the first inequality in (5.45). Using $\tilde{\mathbb{P}}_{bb}$ in (A.7), we observe that the bound in Lemma A.1 holds uniformly over all realizations of $G_{n,1}$ satisfying $A_{bb}$. Hence, by first taking expectation over these possible realizations, we obtain that
\[
\mathbb{P}(\mathcal{C}_n(0) | A_{\text{reg}}, A_{bb}(n, k_n), w_0 \in [m_w, 2m_w]) \geq \mu_{A.1}
\]
also holds. The statement now follows with $\rho := (A.1)m_n^{(\tau-1)/2}$ by the law of total probability combining (A.8) and (A.9) of Lemma A.1. The second inequality in (5.45) for $n = \infty$ follows from the same construction as the greedy path in the proof of Claim 5.5 below (A.16), making the greedy path infinitely long. We leave it to the reader to fill in the details. \hfill \Box

Appendix B. Proofs using first-moment method

We start with the proof of Claim 6.4.

Proof of Claim 6.4. By Palm theory and symmetry of the PPP $\mu_\tau$ on $\mathbb{R}^d$, using that we conditioned on $0 \in \Xi$ in the Palm version, we consider $\mathcal{R}_i(0)$ in (6.20) for some $i \geq 1$. We estimate the mean from below using $t_k = n_k^{1/d}/(2k)$
\[
\lambda_i := \mu_\tau(\mathcal{R}_i(0)) = (2^it_k)d - (2^{i-1}t_k)d = (2^{i-1}t_k)^d(2^d - 1) \geq 2^{d(i-1)}(2k)^d/dn_k \geq 2^{d(i-2)}n_k^\delta.
\]
Hence, since $|\Xi_{N_k} \cap \mathcal{R}_i(0)|$ is distributed as $\text{Poi}(\lambda_i)$ with $\lambda_i > 2^{d(i-2)}n_k^\delta$, by Chernoff bounds for Poisson random variables (see Lemma C.1 applied with $x = 2$, using $1 + 2\log(2) - 2 \geq 1/4$),
\[
\mathbb{P}( |\Xi_{N_k} \cap \mathcal{R}_i(0)| > 2 \cdot \mu_\tau(\mathcal{R}_i(0)) ) \leq \exp(-\lambda_i(1 + 2\log(2) - 2)) \leq \exp\left(-2^{d(i-2)}n_k^\delta,\right)
\]
and the statement follows for every \( c > 0 \), i.e.,
\[
\mathbb{P}^0(\mathcal{A}_{\text{dense}}) = \mathbb{P}^0(\exists u \in \Xi_n[1, \overline{w}], i \geq 1 : |\Xi_N_i \cap \mathcal{R}_i(x_u)| > 2 \cdot \mu_{\tau}(\mathcal{R}_i(x_u)))
\leq \mathbb{P}(\exists u \in \Xi_n[1, \overline{w}]) \cdot \mathbb{P}^0(\exists i \geq 1 : |\Xi_N_i \cap \mathcal{R}_i(0)| > 2 \cdot \mu_{\tau}(\mathcal{R}_i(0)))
\leq n_k \sum_{i \geq 1} \exp \left( -2^{d(i-2)-2} n_k^\delta \right) \leq 2n_k \exp \left( -2^{-d-2} n_k^\delta \right) = o(n_k^{-c}).
\]
\[
\square
\]

Now we prove Claim 6.6 used for the upper bound of subexponential decay.

**Proof of Claim 6.6** For compact sets \( K_1, K_2 \subseteq \mathbb{R}^d \), we define \( |K_1 - K_2| := \min\{||x - y||, x \in K_1, y \in K_2\} \). We define
\[
\tilde{t}_k := \|\partial \Lambda_N - \partial \Lambda_n\| = (N^{1/d} - n^{1/d})/2.
\]
The definition of \( \mathcal{A}_{\text{long-edge}} \) in (6.11) implies that
\[
\mathcal{A}_{\text{long-edge}}(0, n, N, \overline{w}) \subseteq \{ \exists u \in \Xi_n[1, \overline{w}], v \in \Xi : \|x_u - x_v\| \geq \tilde{t}_k, u \leftrightarrow v \},
\]
so that after conditioning on \( \Xi_n[1, \overline{w}] \) it follows by a union bound that
\[
\mathbb{P}^0(\mathcal{A}_{\text{long-edge}}(0, n, N, \overline{w})) \leq \mathbb{E}^0 \left[ \sum_{u \in \Xi_n[1, \overline{w}]} \mathbb{P}(\exists v \in \Xi : \|x_u - x_v\| \geq \tilde{t}_k, u \leftrightarrow v | u \in \Xi_n[1, \overline{w}]) \right]
\leq \mathbb{E}^0 \left[ \sum_{u \in \Xi_n[1, \overline{w}]} q(u) \right] \tag{B.1}
\]

**Assume \( \alpha < \infty \).** Since the diameter of \( \Lambda_n \) is \( \sqrt{dn^{1/d}} \), the lower bound on \( N \) in the statement of Claim 6.6 implies that \( \|x_u - x_v\| \leq \tilde{t}_k \) for all \( x_u, x_v \in \Lambda_n \). Hence, \( v \notin \Xi_n \) whenever \( \|x_u - x_v\| > \tilde{t}_k \). This implies by Markov’s bound, using the connection probability in (2.3), and that \( w_u \leq \overline{w} \) and the intensity \( \mu_{\tau} \) and the boundary, the intensity invariance of the intensity of \( \Xi \) in (2.2), that for all \( u \in \Xi_n \),
\[
q(u) \leq \mathbb{E}^0 \left[ \sum_{v \in \Xi : \|x_u - x_v\| > \tilde{t}_k} p \min \left\{ 1, \left( \frac{\beta \kappa_{1, \sigma}(\overline{w}, w_u)}{\|x_u - x_v\|^\alpha} \right)^\tau \right\} \right]
\]
\[
= p(\tau - 1) \int_{w_v=1}^{\infty} \int_{x_u : \|x_u - x_v\| > \tilde{t}_k} \min \left\{ 1, \left( \frac{\beta \kappa_{1, \sigma}(\overline{w}, w_u)}{\|x_u - x_v\|^\alpha} \right)^\tau \right\} w_v^{-\tau} dw_v dx_v \tag{B.2}
\]
\[
= p(\tau - 1) \int_{w_v=1}^{\infty} w_v^{-\tau} \int_{x_u : \|x_u - x_v\| > \tilde{t}_k, \|x_u - x_v\|^d \leq \beta \kappa_{1, \sigma}(\overline{w}, w_v)} du_v dx_v
\]
\[
+ p(\tau - 1) \int_{w_v=1}^{\infty} (\beta \kappa_{1, \sigma}(\overline{w}, w_v))^{\alpha} w_v^{-\tau} \int_{x_u : \|x_u\| > \tilde{t}_k, \|x_u\|^d \geq \beta \kappa_{1, \sigma}(\overline{w}, w_v)} dx_v =: T_1 + T_2, \tag{B.3}
\]
where we cut the integral into two based on the value on the minimum. We analyze the two integrals separately. Analyzing \( T_1 \), the integration with respect to \( x_v \) gives the Lebesgue measure of the set \( \{ x_v : \tilde{t}_k^d \leq \|x_v\|^d \leq \beta \kappa_{1, \sigma}(\overline{w}, w_v) \} \), which is nonzero only if this set is nonempty, and then can be bounded from above by \( c_d \beta \kappa_{1, \sigma}(\overline{w}, w_v) \) for some constant depending only on \( d \). So we obtain
\[
T_1 \leq c_d p(\tau - 1) \int_{w_v=1}^{\infty} \mathbb{1}(\tilde{t}_k^d \leq \beta \kappa_{1, \sigma}(\overline{w}, w_v)) \kappa_{1, \sigma}(\overline{w}, w_v) w_v^{-\tau} dw_v
\]
\[
= c_d p(\tau - 1) \left( \int_{w_v=1}^{\overline{w}} \mathbb{1}(\tilde{t}_k^d \leq \beta \overline{w} w_v^d) \overline{w} w_v^{-\tau} dw_v + \int_{w_v=\overline{w}}^{\infty} \mathbb{1}(\tilde{t}_k^d \leq \beta \overline{w}^d w_v) \overline{w}^d w_v^{-\tau} dw_v \right),
\]
where in the last step we used the definition of \( \kappa_{1, \sigma} \) in (1.10) and cut the integration into two based on the minimum of \( \overline{w} \) and \( w_v \). Since we assumed \( t_k \geq \sqrt{\beta w_v} \) in the statement of the lemma, and \( w_v^\sigma \leq \overline{w}^\sigma \), the indicator in the first integral is 0. Moving the indicator in the second integral into the integration boundary yields
\[
T_1 \leq c_d p(\beta \overline{w}^\sigma) (\tilde{t}_k^d / (\beta \overline{w}^\sigma))^{-(\tau - 2)} = c_d p(\beta^{\tau - 1} \overline{w}^\sigma (\tau - 1) \tilde{t}_k^{-d(\tau - 2)}). \tag{B.4}
\]
We turn to $T_2$ in (B.3). For some $d$-dependent constant $c_d$, using again $\kappa_{1,\sigma}$ in (1.10)

$$T_2 \leq pc_d'(\tau - 1) \int_{w_v=1}^{\infty} (\beta \kappa_{1,\sigma}(w_v))^{\alpha}w_v^{-\tau} \max\{\tilde{d}_k, \beta \kappa_{1,\sigma}(w_v)\}^{1-\alpha}dw_v$$

$$= pc_d'(\tau - 1) \int_{w_v=1}^{\infty} (\beta \kappa_{1,\sigma}(w_v))^{\alpha}w_v^{-\tau} \max\{\tilde{d}_k, \beta \kappa_{1,\sigma}(w_v)\}^{1-\alpha}dw_v$$

$$+ pc_d'(\tau - 1) \int_{w_v=1}^{\infty} (\beta \kappa_{1,\sigma}(w_v))^{\alpha}w_v^{-\tau} \max\{\tilde{d}_k, \beta \kappa_{1,\sigma}(w_v)\}^{1-\alpha}dw_v.$$  \hspace{1cm} (B.5)

For the first integral, we bound $w_v^{\alpha} \leq w_v^\sigma$, and observe that by the assumption $\tilde{d}_k \geq \beta \kappa_{\sigma+1}$ in the statement of the lemma, the maximum is always attained at $\tilde{d}_k$. Hence,

$$\int_{w_v=1}^{\infty} (\beta \kappa_{1,\sigma}(w_v))^{\alpha}w_v^{-\tau} \max\{\tilde{d}_k, \beta \kappa_{1,\sigma}(w_v)\}^{1-\alpha}dw_v \leq pc_d'(\beta \kappa_{\sigma+1})\tilde{d}_k^{-(\alpha-1)d}.$$ \hspace{1cm} (B.7)

We split the integral in (B.6) according to where the maximum is attained, i.e.,

$$\int_{w_v=1}^{\infty} (\beta \kappa_{1,\sigma}(w_v))^{\alpha}w_v^{-\tau} \max\{\tilde{d}_k, \beta \kappa_{1,\sigma}(w_v)\}^{1-\alpha}dw_v$$

$$= pc_d'(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)) - pc_d'(\beta \kappa_{\sigma}(\tilde{d}(\tau-1))) \int_{w_v=1}^{\infty} (\beta \kappa_{1,\sigma}(w_v))^{\alpha}w_v^{-\tau} \max\{\tilde{d}_k, \beta \kappa_{1,\sigma}(w_v)\}^{1-\alpha}dw_v.$$

where the second step follows from the assumption that $\tilde{d}_k \geq \beta \kappa_{\sigma+1}$ in the statement of the lemma.

For the remaining integral on the right-hand side of (B.8), say $T_22$, we consider three cases, i.e., for some $C > 0$

$$T_{22} \leq \begin{cases} C\tilde{d}_k^{-(\alpha-1)}d(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)))^{\alpha-\tau} & \text{if } \alpha > \tau - 1, \\ C\tilde{d}_k^{-(\alpha-1)}d(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)))^{\alpha} & \text{if } \alpha = \tau - 1, \\ C\tilde{d}_k^{-(\alpha-1)}d(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)))^{\alpha} & \text{if } \alpha < \tau - 1, \end{cases}$$

Elementary rewriting of the first and third case yields

$$T_{22} \leq \begin{cases} C\tilde{d}_k^{-(\alpha-1)}d(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)))^{\alpha} & \text{if } \alpha > \tau - 1, \\ C\tilde{d}_k^{-(\alpha-1)}d(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)))^{\alpha} & \text{if } \alpha = \tau - 1, \\ C\tilde{d}_k^{-(\alpha-1)}d(\beta \kappa_{\sigma}(\tilde{d}(\tau-1)))^{\alpha} & \text{if } \alpha < \tau - 1. \end{cases}$$

Combining this bound with (B.8), then (B.7) and (B.4), gives in (B.3) and (B.1) that for some $C > 0, b > 0$

$$\mathbb{P}^\alpha(A_{\text{long-edge}}(0, n, \overline{w})) \leq C\overline{w}^\alpha \tilde{d}_k^{-(\alpha-1)\min\{\alpha-1, \tau-2\}}(1 + 1_{\alpha=\tau-1})(\log(t_k))E[\Xi_n[1, \overline{w}]]$$

$$\leq C\overline{w}^\alpha N^{-\alpha\min\{\alpha-1, \tau-2\}}(1 + 1_{\alpha=\tau-1})\log(n),$$

where the last bound follows by the assumed bound in (6.26), since $\tilde{d}_k = (N^{1/d} - n^{1/d})/2$, and the intensity of $\Xi_n[1, \overline{w}]$ in (2.2). This finishes the proof of (6.27) when $\alpha < \infty$.

It remains to show the bound for the case $\alpha = \infty$. In this case, the same calculations hold, with only $T_1$ in (B.3) present, since the connection probability is 0 when the minimum in (B.2) is not at 1.

We proceed with the proofs of two lemmas for the lower bound.

**Proof of Lemma 7.3.** We recall from (7.31) that it is sufficient to bound $\mu_\tau(R_{d+1,M}) = E[\Xi_{\text{in}}[M]] + E[\Xi_{\text{out}}[M]]$. We introduce some notation: for two functions $g(k), h(k)$, we write $g \lesssim h$ if $g = O(h)$.
Since $f_\gamma(x)$ is symmetric around the boundary of $\partial B_{kM_\gamma}$ (see its definition in (7.9)), it is easy to see that $\mathbb{E}[\xi_{\gg M_\gamma}^\text{in}] \leq \mathbb{E}[\xi_{\gg M_\gamma}^\text{out}]$. Since $\zeta_{\text{out}} = (d - 1)/d$ by (1.9), for (7.29) it is sufficient to show that
\[
\mathbb{E}[\xi_{\gg M_\gamma}^\text{out}] \lesssim r^{d(1-(\gamma-1))} + (1 + 1_{1 - (\gamma - 1) = 1 - 1/d}) \log(r)r^{d - 1}.
\]  
(7.9)

Using the intensity measure of $\Xi$ in (2.2), switching to polar coordinates in the first $d$ directions, and integrating with respect to the mark-coordinate, we obtain using the exact form of $f_\gamma$ in (7.9)
\[
\mathbb{E}[\xi_{\gg M_\gamma}^\text{out}] \lesssim \int_0^\infty (z + r_k)^{d-1} \int_0^\infty (\tau - 1)w^{-\tau} dw dz
\]
\[
\lesssim \int_0^{C_\beta} (z + r_k)^{d-1} dz + \int_{z = C_\beta}^{r_k} (z + r_k)^{d-1} z^{-d\gamma (\gamma - 1)} dz
\]
\[
+ \int_{z = r_k}^{\infty} (z + r_k)^{d-1} (z^2 - d\gamma (\gamma - 1))^{-\gamma (\gamma - 1)} dz =: I_1 + I_2 + I_3.
\]  
(10.10)

The integration length of $I_1$ is a constant, so $I_1 \lesssim r_k^{d-1}$. For $I_2$ we apply the binomial theorem, i.e.,
\[
I_2 \lesssim \sum_{j = 0}^{d - 1} r_k^j \int_{C_\beta}^{r_k} z^{(1 - \gamma (\gamma - 1))d - 1 - j} dz.
\]  
(11.11)

Analyzing the summands separately, we obtain for $j \leq d - 1$
\[
r_k^j \int_{C_\beta}^{r_k} z^{(1 - \gamma (\gamma - 1))d - 1 - j} dz \lesssim \begin{cases} r_k^{d(1-(\gamma-1))} & \text{if } d(1 - \gamma (\gamma - 1)) > j, \\
\log(r_k)r_k^j & \text{if } d(1 - \gamma (\gamma - 1)) = j, \\
r_k^j & \text{if } d(1 - \gamma (\gamma - 1)) < j.
\end{cases}
\]

Using these bounds, which are increasing for $j \in [d - 1]$, in (B.11), we obtain
\[
I_2 \lesssim (1 + 1_{1 - (\gamma - 1) = 1 - 1/d}) \log(r_k)r_k^{d - 1} + r_k^{d(1-(\gamma-1))}.
\]  
(12.12)

It remains to bound $I_3$ in (B.10). Using that $\tau > 2$ by assumption, and $z + r_k \leq 2z$,
\[
I_3 \lesssim r_k^{d(1-(\gamma-1))} \int_{z = r_k}^{\infty} z^{-d(\tau - 2) - 1} dz \lesssim r_k^{d(1-(\gamma-1)) - (\gamma - 2)} = r_k^{d(1-(\gamma-1))}.
\]

Together with the bound on $I_1$ below the definitions of $I_1, I_2, I_3$ in (B.11), and on $I_2$ in (B.12), this proves (B.9) and also finishes the proof of (7.29). \hfill \Box

**Proof of Lemma 7.2** We split the expected number of edges depending on the locations of the endpoints of the vertices.
\[
\mathbb{E}_{\text{io}} \left[ \mathbb{E} \left[ \xi_{\gg M_\gamma}^\text{in} \mid \xi_{\gg M_\gamma}^\text{out} \right] \right] = \mathbb{E}_{\text{io}} \left[ \mathbb{E} \left[ \xi_{\gg M_\gamma}^\text{in} \mid \xi_{\leq M_\gamma}^\text{in} \cap \xi_{\leq M_\gamma}^\text{out} \cap \xi_{\leq M_\gamma}^\text{io} \cap \xi_{\leq M_\gamma}^\text{out} \right] \right]
\]
\[
+ \mathbb{E}_{\text{io}} \left[ \mathbb{E} \left[ \xi_{\gg M_\gamma}^\text{in} \mid \xi_{\leq M_\gamma}^\text{in} \cap \xi_{\leq M_\gamma}^\text{out} \right] \right] + \mathbb{E}_{\text{io}} \left[ \mathbb{E} \left[ \xi_{\leq M_\gamma}^\text{in} \cap \xi_{\leq M_\gamma}^\text{out} \right] \right] + \mathbb{E}_{\text{io}} \left[ \mathbb{E} \left[ \xi_{\leq M_\gamma}^\text{in} \cap \xi_{\leq M_\gamma}^\text{out} \right] \right] \]  
(13.13)

We analyze the first term on the right-hand side and at the end we sketch how the bounds could be adapted for the other three terms. Since $A_{\text{regular}}(\eta)$ is measurable with respect to $\xi_{\leq M_\gamma}^\text{in} \cup \xi_{\leq M_\gamma}^\text{out}$, it can be left out of the conditioning. Further, points of $\Xi$ in disjoint sets are independently present, hence
\[
\mathbb{E} \left[ \mathbb{E} \left[ \xi_{\gg M_\gamma}^\text{in} \mid \xi_{\leq M_\gamma}^\text{in} \cap \xi_{\leq M_\gamma}^\text{out} \cap A_{\text{regular}}(\eta) \right] \right] = \mathbb{E} \left[ \mathbb{E} \left[ \xi_{\leq M_\gamma}^\text{in} \mid \xi_{\leq M_\gamma}^\text{in} \cap \xi_{\leq M_\gamma}^\text{out} \right] \right] \]  
(14.14)

We use the notation $q \lesssim h$ if $q = O(h)$. We integrate over the locations and marks of the vertices in $\xi_{\leq M_\gamma}^\text{in} \cup \xi_{\leq M_\gamma}^\text{out}$ by writing $z(x) = \|x - \partial B_{kM_\gamma}\|$, and upper bound the connectivity function $p$ in (2.3) to obtain
\[
\mathbb{E} \left[ \mathbb{E} \left[ \xi_{\gg M_\gamma}^\text{in} \mid \xi_{\gg M_\gamma}^\text{out} \cap M \right] \right] \]
\[
\lesssim \int_{x_u : 1 \leq u \leq r_k} \int_{x_v : 1 \leq v = 1} f_\gamma(z(x_u)) \int_{w_u = 1} f_\gamma(z(x_v)) \int_{w_v = 1} \frac{(\kappa_{1,\sigma}(w_u, w_v))^\alpha}{\|x_u - x_v\|^\alpha} w_u^{-\tau} w_v^{-\tau} dw_u dw_v dx_u dx_v.
\]  
(15.15)
We analyze the double integral over the marks. Define

\[ g_1(z_1, z_2) := \int_{w_1 = 1} \int_{w_2 = 1} f_\gamma(z_1) f_\gamma(z_2) \kappa_{1, \sigma}(w_1, w_2) w_1^{-\tau} w_2^{-\tau} \, dw_1 dw_2, \]

Using the definition and symmetry of \( \kappa_{1, \sigma} \) in [1.10], we reparametrize by \( w \leq \tilde{w} \) and using that \( f_\gamma \) is increasing

\[ g_1(z_1, z_2) \lesssim \int_{w = 1} \int_{\tilde{w} = w} f_\gamma(z_1 \wedge z_2) f_\gamma(z_1 \vee z_2) \kappa_{1, \sigma}(w, \tilde{w})(w\tilde{w})^{-\tau} \, w \, dw \tilde{w} = \int_{w = 1} \int_{\tilde{w} = w} \kappa_{1, \sigma}(w, \tilde{w})(w\tilde{w})^{-\tau} \, w \, dw \tilde{w}. \]

The definition \( f_\gamma \) in (7.9) undergoes a change at \( z = r_k \). When integrating, we have nine cases depending on whether the exponents are below, equal to, or above \(-1\) each. This yields for \( z_2 \leq \min\{z_1, r_k\} \) (so \( f_\gamma(z_2) = 1 \wedge (z_2/C_\beta)^{d}\)) that \( g_1(z_1, z_2) \lesssim g(z_1, z_2) \), with

\[ g(z_1, z_2) := \begin{cases} f_\gamma(z_1)^{\alpha-(\tau-1)} z_2^{\gamma d(\alpha-(\tau-1))}, & \text{if } \alpha > \tau - 1, \sigma \alpha > \tau - 1, \\ f_\gamma(z_1)^{\alpha-(\tau-1)} \log(z_2), & \text{if } \alpha > \tau - 1, \sigma \alpha = \tau - 1, \\ f_\gamma(z_1)^{\alpha-(\tau-1)}, & \text{if } \alpha > \tau - 1, \sigma \alpha < \tau - 1, \\ (1 + \log(f_\gamma(z_1)/f_\gamma(z_2))) z_2^{\gamma d((\sigma+1)\alpha-2(\tau-1))}, & \text{if } \alpha = \tau - 1, \sigma \alpha > \tau - 1, \\ \log(f_\gamma(z_1)) \log(z_2), & \text{if } \alpha = \tau - 1, \sigma \alpha = \tau - 1, \\ \log(f_\gamma(z_1)), & \text{if } \alpha = \tau - 1, \sigma \alpha < \tau - 1, \\ z_2^{\gamma d((\sigma+1)\alpha-2(\tau-1))}, & \text{if } \alpha < \tau - 1, (\sigma + 1)\alpha > 2(\tau - 1), \\ \log(z_2), & \text{if } \alpha < \tau - 1, (\sigma + 1)\alpha = 2(\tau - 1), \\ 1, & \text{if } \alpha < \tau - 1, (\sigma + 1)\alpha < 2(\tau - 1). \end{cases} \]  

(B.16)

Returning to (B.15), we use \( g(z_1, z_2) \) to bound the inner two integrals from above. We take a case-distinction on whether the vertex \( u \) (inside) or \( v \) (outside) is closer to the boundary \( \partial B_{kM_{\text{in}}} \). Then we obtain (since \( f_\gamma(x) = 1 \) when \( z(x) \leq C_\beta \) by (7.9)),

\[ \mathbb{E}[\mathcal{E}(\Xi_{\leq M_\gamma}, \Xi_{\leq M_\gamma}^\text{out})] \lesssim \int_{x, u: ||x_u|| \leq r_k - C_\beta} \int_{x, v: ||x_v|| > 2r_k} \|x_u - x_v\|^{-\alpha d} g(z(x_u), z(x_v)) \, dx_u \, dx_v \\
+ \int_{x, u: ||x_u|| \leq r_k - C_\beta} \int_{x, v: ||z(x_v)|| \leq z(x_u)} \|x_u - x_v\|^{-\alpha d} g(z(x_u), z(x_v)) \, dx_u \, dx_v \\
+ \int_{x, v: r_k \leq ||x_v|| \leq r_k - C_\beta} \int_{x, w: ||z(x_w)|| \leq z(x_v)} \|x_u - x_v\|^{-\alpha d} g(z(x_u), z(x_v)) \, dx_u \, dx_v \\
=: I_1 + I_2a + I_2b =: I_1 + I_2. \]  

(B.17)

To evaluate the integrals we change variables. For \( I_1 \) we use \( z(x_u) \leq r_k \), and \( g \) is increasing in both its arguments, and that \( \|x_u - x_v\| \geq \|x_v\| - r_k \geq z(x_v) := t \) and use polar coordinates in the second row below. Then \( \|x_v\| = t + r_k \). Thus, since there are \( \Theta((t + r_k)^{d-1}) \) points outside at distance \( t \) from \( \partial B_{kM_{\text{in}}} \),

\[ I_1 \lesssim \int_{x, u: ||x_u|| \leq r_k - C_\beta} \int_{x, v: ||x_v|| > 2r_k} \|x_v\| - r_k \|^{-\alpha d} g(r_k, z(x_v)) \, dx_u \, dx_v \\
\lesssim r_k^d \int_{r_k}^{(t + r_k)^{d-1}} t^{-\alpha d} g(r_k, t) \, dt \lesssim r_k^d \int_{t \geq r_k} t^{-d(\alpha-1)-1} g(r_k, t) \, dt. \]  

(B.18)

Before substituting the definition of \( g \) into the bound, we recall that \( f_\gamma(t) = C_\beta^{-\alpha d} t^{d}(r_k)^{-d(1-\gamma)} \) for \( t > r_k \) by (7.9). The following elementary integration inequalities will be helpful soon:

\[ r_k^d \int_{t \geq r_k} t^{-d(\alpha-1)-1} f_\gamma(t)^{\alpha-(\tau-1)} \, dt \lesssim r_k^d \int_{t \geq r_k} t^{-d(\tau-2)-1} \, dt \lesssim r_k^{-d(2-\alpha+\gamma(\alpha-(\tau-1)))}, \]  

(B.19)

\[ r_k^d \int_{t \geq r_k} t^{-d(2-\alpha)} \, dt \lesssim r_k^d(2-\alpha). \]  

(B.20)
Substituting the definition of $g$ in (B.16) into (B.18), since $r_k < t$, we must set $z_2 = r_k$ in $g$ in (B.16). We obtain then by elementary integration on (B.18) and the bounds in (B.19)-(B.20) that:

$$I_1 \lesssim \begin{cases} r_k^{d(2-\alpha+y)((\sigma+1)\alpha-2(\tau-1))}, & \text{if } \alpha > \tau - 1, \sigma \alpha > \tau - 1, \\ r_k^{d(2-\alpha+y(\alpha-(\tau-1)))} \log(r_k), & \text{if } \alpha > \tau - 1, \sigma \alpha = \tau - 1, \\ r_k^{d(2-\alpha+y((\sigma+1)\alpha-2(\tau-1)))}, & \text{if } \alpha = \tau - 1, \sigma \alpha > \tau - 1, \\ r_k^{d(2-\alpha+y(\alpha-(\tau-1)))}, & \text{if } \alpha > \tau - 1, \sigma \alpha < \tau - 1, \\ r_k^{d(2-\alpha+y((\sigma+1)\alpha-2(\tau-1)))}, & \text{if } \alpha = \tau - 1, \sigma \alpha < \tau - 1, \\ r_k^{d(2-\alpha+y(\alpha-(\tau-1)))}, & \text{if } \alpha < \tau - 1, (\sigma + 1)\alpha > 2(\tau - 1), \\ r_k^{d(2-\alpha)} \log(r_k), & \text{if } \alpha < \tau - 1, (\sigma + 1)\alpha = 2(\tau - 1), \\ r_k^{d(2-\alpha)} \log^2(r_k), & \text{if } \alpha = \tau - 1, \sigma \alpha = \tau - 1, \\ r_k^{d(2-\alpha)} \log(r_k), & \text{if } \alpha < \tau - 1, (\sigma + 1)\alpha < 2(\tau - 1). \\ \end{cases} \quad (B.21)$$

We turn to $I_{2a}$ in (B.17), handling the case when the outside vertex $v$ is closer to the boundary $\partial B_{kM_u}$ than $u$, implying $z(x_v) \leq z(x_u)$. We reparametrise this integral based on the distance $z_u$ from $\partial B_{kM_u}$ of the inside vertex $u$. Indeed, when $z_u \in [\beta, r_k]$ then $\|x_u\| = r_k - z_u$. Since $v$ is closer, we must also have that $z_v = \|x_v\| - r_k \in [C\beta, z_u]$, and hence $t := \|x_u - x_v\| \in [z_u + C\beta, 2r_k]$. Hence,

$$I_{2a} \lesssim \int_{z_u = C\beta}^{r_k} \int_{x_u \mid r_k \|x_u\| = z_u}^{2r_k} \int_{z_u = C\beta}^{z_u} \int_{x: x = x_v \|z(x_v) = z_v\|, \|x_u - x_v\| = t}^{z_u} t^{-d\alpha} g(z_u, z_v) dx_v d\mu_x dt dx_u dz_u. \quad (B.22)$$

The integrand does not depend on $x_v$ anymore, hence the most inside integral, over $x_v$, can be bounded from above by maximizing the Lebesgue measure of where $x_v$ may fall: $x_v$ has distance $t$ from $x_u$ and distance $z_v$ from the boundary. Some geometry shows that $x_v$ is then on the intersection of two spheres with radii $t$ and $r_k + z_v$, respectively, with Lebesgue measure then at most $\Theta(t^{d-2})$. We can also integrate over all the potential locations $x_u$, giving a factor $\Theta((r_k - z_u)^{d-1})$, so we obtain

$$I_{2a} \lesssim \int_{z_u = C\beta}^{r_k} (r_k - z_u)^{d-1} \int_{z_u = C\beta}^{z_u} t^{-d\alpha} g(z_u, z_v) dz_v dt dz_u \lesssim \int_{z_u = C\beta}^{r_k} (r_k - z_u)^{d-1} z_u^{d(\alpha - 1) - 1} \int_{z_u = C\beta}^{z_u} g(z_u, z_v) dz_u dz_v, \quad (B.22)$$

where we integrated over $t$ to obtain the second row. Treating $I_{2b}$ in (B.17) is very similar, but now we reparametrise the integral based on the distance $z_v$ of $v$ from the boundary and the distance $t = \|x_u - x_v\|$. We obtain

$$I_{2b} \lesssim \int_{z_v = C\beta}^{r_k} \int_{x_v \mid \|x_v\| - r_k = z_v}^{r_k} \int_{z_v = C\beta}^{z_v} \int_{x: x = x_v \|z(x_v) = z_v\|, \|x_u - x_v\| = t}^{z_v} t^{-d\alpha} g(z_u, z_v) dx_u d\mu_x dt dx_u dz_v \lesssim \int_{z_v = C\beta}^{r_k} (r_k + z_v)^{d-1} z_v^{d(\alpha - 1) - 1} \int_{z_u = C\beta}^{z_v} g(z_u, z_v) dz_u dz_v. \quad (B.22)$$

This bound dominates the bound on $I_{2a}$ in (B.22). Applying the binomial theorem on $(r_k + z_v)^{d-1}$, we obtain

$$I_2 = I_{2a} + I_{2b} \lesssim \sum_{j=0}^{d-1} \int_{z_u = C\beta}^{r_k} z_u^{d(2-\alpha) - 2 - j} \int_{z_u = C\beta}^{z_v} g(z_u, z_v) dz_u dz_v. \quad (B.23)$$
We evaluate the inner integral using the definition of \( g \) in (B.16), and since \( z_u \leq z_v \) we set \( z_1 = z_v \), \( z_2 = z_u \) in (B.16), and \( f_\gamma(z) = (z/C_\gamma)^{\gamma - d} \), and obtain the nine cases:

\[
\int_{z_u = C_\gamma}^{z_v} g(z_u, z_v) dz_u \lesssim \begin{cases} 
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1) - 1), & \text{if } \alpha > \gamma - 1, \sigma \alpha > \gamma - 1, \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)) + 1, & \text{if } \alpha > \gamma - 1, \sigma \alpha = \gamma - 1, \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)), & \text{if } \alpha > \gamma - 1, \sigma \alpha < \gamma - 1, \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)), & \text{if } \alpha = \gamma - 1, \sigma \alpha > \gamma - 1, \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)), & \text{if } \alpha = \gamma - 1, \sigma \alpha < \gamma - 1, \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)), & \text{if } \alpha < \gamma - 1, (\sigma + 1)\alpha > 2(\gamma - 1), \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)), & \text{if } \alpha < \gamma - 1, (\sigma + 1)\alpha = 2(\gamma - 1), \\
  z_v^d((\sigma + 1)\alpha - 2(\gamma - 1)), & \text{if } \alpha < \gamma - 1, (\sigma + 1)\alpha < 2(\gamma - 1). 
\end{cases}
\]

(B.24)

For \( \mathcal{Y} = \{0, \alpha - (\tau - 1), (\sigma + 1)\alpha - 2(\gamma - 1)\} \), recall that \( m(\mathcal{Y}) \) counts the multiplicity of the maximum in \( \mathcal{Y} \). Substituting (B.24) into (B.23), the nine cases can be summarized as obtaining the integrand of \( \int_{z_u}^{d - 2 + \alpha} \gamma \log m(\mathcal{Y}) - 1(z_u) \), so that following the similar reasoning as from (B.11) to (B.12),

\[
I_2 \lesssim \begin{cases} 
  r_k^d(2 - \alpha + \gamma \max(\mathcal{Y})) \log m(\mathcal{Y}) - 1(r_k), & \text{if } d(2 - \alpha + \gamma \max(\mathcal{Y})) > d - 1, \\
  r_k^d(2 - \alpha + \gamma \max(\mathcal{Y})) \log m(\mathcal{Y}) - 1(r_k), & \text{if } d(2 - \alpha + \gamma \max(\mathcal{Y})) = d - 1, \\
  r_k^d, & \text{if } d(2 - \alpha + \gamma \max(\mathcal{Y})) < d - 1,
\end{cases}
\]

where the second bound follows from similar reasoning as in (B.11) leading to (B.12). The presence of a \((d - 1)\) term and the additional log-factors ensure that the bound on \( I_2 \) dominates the bound on \( I_1 \) in (B.21). Recalling that \( I_1 + I_2 \) dominates the expected number of edges below the \( \gamma \)-suppressed profile from (B.17), this yields by (B.14) and \( \xi_{\text{sn}} = (d - 1)/d \) that

\[
\mathbb{E} \left[ |\mathcal{E}(\Xi_{\leq M_z \setminus R_{\text{in}}}, \Xi_{\leq M_z \setminus R_{\text{out}}})| \right] \lesssim r_k^d(2 - \alpha + \gamma \max(\mathcal{Y})) \log m(\mathcal{Y}) - 1(r_k), \quad \text{if } d - \alpha + \gamma \max(\mathcal{Y}) > \xi_{\text{sn}},
\]

\[
\lesssim r_k^d(2 - \alpha + \gamma \max(\mathcal{Y})) \log m(\mathcal{Y}) - 1(r_k), \quad \text{if } d - \alpha + \gamma \max(\mathcal{Y}) = \xi_{\text{sn}},
\]

\[
\lesssim r_k^d, \quad \text{if } d - \alpha + \gamma \max(\mathcal{Y}) < \xi_{\text{sn}}.
\]

To obtain bounds on the other three expectations in (B.13), one can replace the integrals over the marks in (B.15) by summing over the mark intervals \( I_2 \) defined in (7.15), the upper bounds on the number of vertices using the definitions of \( \mathcal{A}_{\text{regular}}(\eta) \) and \( \mathcal{A}_{\text{regular}}^*(\eta) \) in (7.17) ensure that the total number of points in each interval only differs from its expectation by a constant factor. Then one can use an upper bound on the mark of each vertex in \( I^\text{loc}_j \) in (7.16), given by \( I^\text{loc}_j = \{w_\eta(k)^/2^j, w_\eta(k)^/2^j-1\} \), and thus also the mark is at most a factor two larger than the mark of a typical vertex in \( I^\text{loc}_j \). Lastly, the distance between vertices in \( R_{\text{in}} \) and outside \( B_{k,M_{\text{in}}} \) (but within distance \( r_k \) of \( B_{k,M_{\text{in}}} \)) can be bounded from below by \( r_k/2 \) by Lemma A.1 and analogously we can bound the distance between vertices in \( R_{\text{out}} \) and vertices inside \( B_{k,M_{\text{in}}} \). We leave it to the reader to fill in the details.

\[\square\]

APPENDIX C. AUXILIARY PROOFS

It remains to prove Claim 7.5.

**Proof of Claim 7.5** We first assume \( \alpha < \infty \). We set \( \gamma := \gamma_{\text{sn}} \) defined in (7.13) in Lemmas 7.3 and 7.4 and compare the exponents of \( r_k^d \) on the right-hand sides of (7.29) and (7.30), respectively. We will show that the three cases there can be ‘merged’ by taking the exponent of \( r_k^d = \Theta(k) \) to be \( \max(\mathcal{Z}) \), and the exponent of the log-factors in (7.29) and (7.30) is then at most \( m_{\mathcal{Z}} - 1 \). We distinguish whether \((*)\) in (7.13) holds with equality or inequality. Let \( \mathcal{Y} = \{0, \alpha - (\tau - 1), (\sigma + 1)\alpha - 2(\gamma - 1)\} \) as defined above (7.13) and \( \mathcal{Z} = \{\xi_{\text{ll}}, \xi_{\text{hh}}, \xi_{\text{hh}}, \xi_{\text{sn}}\} \) as usual.

**Case 1A. Assume** \( 1 - \gamma_{\text{sn}}(\tau - 1) > 2 - \alpha + \gamma_{\text{sn}} \max(\mathcal{Y}), \) and \( \alpha < \infty \). This, by (7.13) implies that \( \gamma_{\text{sn}} = 1/(\sigma + 1) \). Considering that \( \mathcal{Y} = \{0, \alpha - (\tau - 1), (\sigma + 1)\alpha - 2(\gamma - 1)\} \), the assumed inequality
of Case 1A turns then into the following three inequalities after elementary rearrangements:

\[(\sigma + 1)(\alpha - 1) > \tau - 1, \quad \alpha < (\sigma + 1)(\alpha - 1), \quad (\sigma + 1)\alpha - (\tau - 1) < (\sigma + 1)(\alpha - 1). \tag{C.1}\]

Solving the last inequality yields \(\sigma < \tau - 2\), which readily implies \(\zeta_{hh} < 0\) by its definition in (1.12).

The second inequality implies that \(\alpha > 1 + 1/\sigma > 1 + 1/(\tau - 2)\), which is equivalent to \(\zeta_{hh} < 0\). It is left to show that \(\zeta_l = 2 - \alpha < 0\). The first inequality, and \((\sigma + 1)/(\tau - 1) < 1\) (coming from the third inequality), yield

\[\frac{1}{\alpha - 1} < \frac{\sigma + 1}{\tau - 1} < 1,\]

which is equivalent to \(\alpha > 2\). Summarizing, we obtained that under the assumption of Case 1A, \(\max\{\zeta_l, \zeta_h, \zeta_{hh}\} < 0 \leq \zeta_{nn}\), hence \(\max(Z) = \zeta_{nn}\) and \(m(Z) = 1\). We also check that in this case the exponents of \(r_k^d\) in (7.29) and (7.30) are respectively at most \(\zeta_{nn} = (d - 1)/d \geq 0\). This is true since

\[2 - \alpha + \gamma_{-\alpha} \max(Y) < 1 - \gamma_{-\alpha} (\tau - 1) = 1 - \frac{\tau - 1}{\sigma + 1} < 0,\]

where the first inequality follows by the assumption of Case 1A, and the second one by the last inequality in (C.1). Hence, under Case 1A, the third case holds in (7.29) and (7.30), the exponent

\[\gamma = \frac{\alpha - 1}{\max(Y) + \tau - 1} = \min \left\{ \frac{\alpha - 1}{\tau - 1}, \frac{\alpha - 1}{\alpha}, \frac{\alpha - 1}{(\sigma + 1)\alpha - (\tau - 1)} \right\}. \tag{C.2}\]

Since \(\gamma_{-\alpha} \leq 1/(\sigma + 1)\) by (7.13), it is elementary to verify that the third term may be minimal only when \(\tau \leq 2 + \sigma\).

Assume \(\tau \leq 2 + \sigma\). Using (C.2), (1.7), (1.8), (1.12), it is elementary to verify that

\[\zeta_{-\alpha} = 1 - \min \left\{ \frac{\alpha - 1}{\tau - 1}, \frac{\alpha - 1}{\alpha}, \frac{\alpha - 1}{(\sigma + 1)\alpha - (\tau - 1)} \right\} \approx \max(Z_{\zeta_{}, \zeta_{gh}, \zeta_{hh}}). \tag{C.3}\]

It immediately follows that \(\max\{\zeta_{-\alpha}, \zeta_{-\alpha}\} = \max(Z)\). Hence, the exponents of \(r_k^d\) in (7.29) and (7.30) are equal to \(\max(Z)\), which is what we aimed for in (7.33). We now treat the exponent of the logarithm in (7.30) and in (7.29). Recall that \(m\) in (2.5) denotes the multiplicity of the maximum of a set. From (C.2) and (C.3) it follows that \(m(Y) = m(\{\zeta_l, \zeta_h, \zeta_{hh}\})\). Hence, using that \(\zeta_{nn} = \max(\{\zeta_l, \zeta_h, \zeta_{hh}\})\),

\[m(Z) = m(\{\zeta_l, \zeta_h, \zeta_{hh}, \zeta_{nn}\}) = m(Y) 1_{\{\zeta_{-\alpha} > \zeta_{nn}\}} + m(Y) 1_{\{\zeta_{-\alpha} = \zeta_{nn}\}} + 1_{\{\zeta_{-\alpha} < \zeta_{nn}\}}. \tag{C.4}\]

The relations between \(\zeta_{nn}\) and \(1 - \gamma_{-\alpha} (\tau - 1)\) on the one hand, and \(\zeta_{nn}\) and \(2 - \alpha + \gamma_{-\alpha} \max(Y)\) on the other hand, are the same by our equality assumption in Case 1B. This confirms that the log-factor in (7.29) is at most the log-factor in (7.30), showing therefore the first inequality of (7.33), and that the log-factor in (7.30) equals what we aimed for in the second inequality of (7.33), thereby finishing Case 1B when \(\tau \leq 2 + \sigma\).

Assume \(\tau > 2 + \sigma\). The minimum in (C.2) is never attained at the third term (and equivalently the maximum in \(Y\) is never attained at \((\sigma + 1)\alpha - 2(\tau - 1)\)). Similarly to (C.3), we obtain that \(\zeta_{-\alpha} = \max(\{\zeta_l, \zeta_h\})\). Moreover, \(\zeta_{hh} < 0 \leq \zeta_{nn}\) by the formula for \(\zeta_{hh}\) in (1.12), yielding \(\max(Z) = \max(\{\zeta_l, \zeta_h, \zeta_{nn}\})\). Hence, the exponents of \(r_k^d\) in (7.29) and (7.30) equal to \(\max(Z)\), which is what we aimed for in (7.33). For the log-factor we argue similarly as in (C.4) and below, with the additional restriction that the maximum in \(Y\) is never attained at \((\sigma + 1)\alpha - 2(\tau - 1)\), and the maximum in \(Z\) is never attained at \(\zeta_{hh}\).

Case 2. Assume \(\alpha = \infty\). Since \(\zeta_l = 2 - \alpha = -\infty, \zeta_h = -(\tau - 2) < 0\ (\tau > 2)\) by assumption in Lemma 7.2, and \(\zeta_{nn} = (d - 1)/d \geq 0\), we obtain \(\max(\{\zeta_l, \zeta_h, \zeta_{hh}, \zeta_{nn}\}) = \max(\{\zeta_{hh}, \zeta_{nn}\}) = \max(\{\zeta_{hh}, \zeta_{nn}\})\) by the formula of \(\zeta_{hh}\) in (1.12). Since \(\gamma_{hh} = \gamma_{-\alpha}\) when \(\alpha = \infty\), it follows that the exponents of \(r_k^d\) and \(\log(r_k^d)\) in (7.29) and (7.34) coincide. This finishes the proof.

Lastly, we state a Poisson concentration bound (without proof) that we often rely on in the paper.

**Lemma C.1** (Poisson bound [57]). For \(x > 1\),

\[P(\text{Poi}(\lambda) \geq x\lambda) \leq \exp(-\lambda(1 + x\log(x) - x)),\]
and for $x < 1$, 

$$
\mathbb{P}(\text{Poi}(\lambda) \leq x\lambda) \leq \exp(-\lambda(1 - x - x \log(1/x))).
$$

**REFERENCES**


