Controller synthesis for L2 behaviors using rational kernel representations

Citation for published version (APA):

DOI:
10.1109/CDC.2008.4739069

Document status and date:
Published: 01/01/2008

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.
Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.
• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

Take down policy
If you believe that this document breaches copyright please contact us at: openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 23. Oct. 2023
Controller synthesis for $L_2$ behaviors using rational kernel representations

Mark Mutsaers and Siep Weiland

Abstract—This paper considers the controller synthesis problem for the class of linear time-invariant $L_2$ behaviors. We introduce classes of LTI $L_2$ systems whose behavior can be represented as the kernel of a rational operator. Given a plant and a controlled system in this class, an algorithm is developed that produces a rational kernel representation of a controller that, when interconnected with the plant, realizes the controlled system. This result generalizes similar synthesis algorithms in the behavioral framework for infinitely smooth behaviors that allow representations as kernels of polynomial differential operators.

I. INTRODUCTION

The analysis of system interconnections is at the heart of many problems in modeling, simulation and control. Indeed, when focusing on control, the controller synthesis question amounts to finding a dynamical system (a controller) that, after interconnection with a given plant, results in a controlled system that is supposed to perform a certain task in a more desirable manner than the plant. Usually the control synthesis problem is formulated as a feedback optimization problem in which the plant and controller interact through a number of distinguished channels that have been divided in input- and output variables.

The behavioral theory of dynamical systems has been advocated as a conceptual framework in which especially interconnection structures of dynamical system can be studied in an input-output independent setting. The behavioral approach to systems theory is advocated as a unifying framework for systems of diverse nature. The behavioral viewpoint for dynamical system has been developed in many papers during the past four decades with a focus on polynomial representations.

Within the behavioral framework, the interconnection of dynamical systems involves the question when a given dynamical system $\Sigma_K$ can be implemented (or realized) as the interconnection of a dynamical system $\Sigma_P$, that is supposed to be given, and a second dynamical system $\Sigma_C$, that is supposed to be designed. With the interpretation that $\Sigma_P$ and $\Sigma_K$ denote the plant- and (desired) controlled system, this question is therefore equivalent to a synthesis question for the controller $\Sigma_C$.

Within the behavioral framework this question received a very complete and elegant answer for the class of linear time-invariant systems that admit representations in terms of polynomial difference or polynomial differential operators [5], [6]. A rather complete theory has been developed for such representations that covers, among other things, $H_\infty$, LQ and $H_2$ optimal control.

It is the purpose of this paper to reconsider the controller synthesis question for specific classes of linear and time-invariant $L_2$ systems that admit representations in terms of rational functions. In doing so, we depart from the setting proposed in [11] of considering infinitely smooth trajectories as solutions of “rational” differential equations. Instead, we view rational functions in $H_\infty$ as multiplicative operators on $L_2$ functions and define $L_2$ systems through the kernel of such operator. In this way, rational functions naturally define dynamical systems in the frequency domain and offer distinct algebraic advantages over polynomial kernel representations.

The paper is organized as follows. Section II contains the formulation of the main problem that is discussed in this paper. In Section III some notational remarks about spaces and operators are introduced. Sections IV and V contain the introduction of $L_2$ behaviors, the interconnection problem and a novel controller synthesis algorithm. An example using this synthesis algorithm is given in Section VI. In the last section of this paper, the results of this paper are discussed and some recommendations for further research on $L_2$ systems are given.

II. PROBLEM FORMULATION

Following the behavioral formalism, a dynamical system [1] is described by a triple:

$$\Sigma = (T, \mathbb{W}, \mathbb{B}),$$

where $T \subseteq \mathbb{R}$ or $T \subseteq \mathbb{C}$ is the time- or frequency-axis, $\mathbb{W}$ is the variable signal space, which typically contains inputs and outputs and will be taken to be a finite dimensional vector space throughout, and $\mathbb{B} \subseteq \mathbb{W}^2$ is the behavior, that is defined in more explicit terms in Section IV.

Using (1) it is possible to describe plants, controllers and desired controlled systems (as $\Sigma_P$, $\Sigma_C$ and $\Sigma_K$ respectively).

Fig. 1 illustrates the interconnection $\Sigma_K$ of two systems $\Sigma_P = (T, \mathbb{W}, \mathbb{P})$ and $\Sigma_C = (T, \mathbb{W}, \mathbb{C})$. It is defined as $\Sigma_K = (T, \mathbb{W}, \mathbb{P} \cap \mathbb{K})$ and motivated by the idea that the heart of the analysis of system interconnections is at the heart of many problems in modeling, simulation and control. Indeed, when focusing on control, the controller synthesis question amounts to finding a dynamical system (a controller) that, after interconnection with a given plant, results in a controlled system that is supposed to perform a certain task in a more desirable manner than the plant. Usually the control synthesis problem is formulated as a feedback optimization problem in which the plant and controller interact through a number of distinguished channels that have been divided in input- and output variables.

The behavioral theory of dynamical systems has been advocated as a conceptual framework in which especially interconnection structures of dynamical system can be studied in an input-output independent setting. The behavioral approach to systems theory is advocated as a unifying framework for systems of diverse nature. The behavioral viewpoint for dynamical system has been developed in many papers during the past four decades with a focus on polynomial representations.

Within the behavioral framework this question received a very complete and elegant answer for the class of linear time-invariant systems that admit representations in terms of polynomial difference or polynomial differential operators [5], [6]. A rather complete theory has been developed for such representations that covers, among other things, $H_\infty$, LQ and $H_2$ optimal control.

It is the purpose of this paper to reconsider the controller synthesis question for specific classes of linear and time-invariant $L_2$ systems that admit representations in terms of rational functions. In doing so, we depart from the setting proposed in [11] of considering infinitely smooth trajectories as solutions of “rational” differential equations. Instead, we view rational functions in $H_\infty$ as multiplicative operators on $L_2$ functions and define $L_2$ systems through the kernel of such operator. In this way, rational functions naturally define dynamical systems in the frequency domain and offer distinct algebraic advantages over polynomial kernel representations.

The paper is organized as follows. Section II contains the formulation of the main problem that is discussed in this paper. In Section III some notational remarks about spaces and operators are introduced. Sections IV and V contain the introduction of $L_2$ behaviors, the interconnection problem and a novel controller synthesis algorithm. An example using this synthesis algorithm is given in Section VI. In the last section of this paper, the results of this paper are discussed and some recommendations for further research on $L_2$ systems are given.

II. PROBLEM FORMULATION

Following the behavioral formalism, a dynamical system [1] is described by a triple:

$$\Sigma = (T, \mathbb{W}, \mathbb{B}),$$

where $T \subseteq \mathbb{R}$ or $T \subseteq \mathbb{C}$ is the time- or frequency-axis, $\mathbb{W}$ is the variable signal space, which typically contains inputs and outputs and will be taken to be a finite dimensional vector space throughout, and $\mathbb{B} \subseteq \mathbb{W}^2$ is the behavior, that is defined in more explicit terms in Section IV.

Using (1) it is possible to describe plants, controllers and desired controlled systems (as $\Sigma_P$, $\Sigma_C$ and $\Sigma_K$ respectively).

Fig. 1 illustrates the interconnection $\Sigma_K$ of two systems $\Sigma_P = (T, \mathbb{W}, \mathbb{P})$ and $\Sigma_C = (T, \mathbb{W}, \mathbb{C})$. It is defined as $\Sigma_K = (T, \mathbb{W}, \mathbb{P} \cap \mathbb{K})$ and motivated by the idea that the
behavior of the interconnection satisfies the laws of both $\Sigma P$ and $\Sigma C$.
Fig. 1(a) gives an illustration of the problem treated in this paper, namely given a plant $\Sigma P$ and a desired controlled system $\Sigma C$, construct, if it exists, a controller $\Sigma C$ that after interconnection with the plant results in the desired controlled system.
We address this problem for very specific classes of $L_2$ systems. More specifically, we address the problems of existence, (non-) uniqueness of controllers, together with the problem to parametrize all controllers that establish a desired controlled system after interconnection.
As mentioned in the introduction, earlier research, using infinitely smooth trajectories, has been carried out for this problem [6], [9], [10]. This paper contributes to the controller synthesis question by considering various $L_2$ systems, represented through rational operators.

III. NOTATION

A. Hardy spaces

Hardy spaces are denoted by $H_p^+$ and $H_p^-$, where $p = 1, 2, \ldots, \infty$, and defined by:

$$H_p^+ := \{ f : C^+ \rightarrow C^q \mid \| f \|_{H_p^+} < \infty \},$$

$$H_p^- := \{ f : C^- \rightarrow C^q \mid \| f \|_{H_p^-} < \infty \},$$

where $C^+ := \text{Re}\{s\} > 0$ and $C^- := \text{Re}\{s\} < 0$, with $s = \sigma + j\omega$. So, functions in $H_p^+$ are analytic in $C^+ \cup \{\infty\}$ and functions in $H_p^-$ are analytic in $C^- \cup \{-\infty\}$. The $H_p^+$ spaces are the classical Hardy spaces [4].

The norms of functions in $H_p^+$ and $H_p^-$ are defined as:

$$\| f \|_{H_p^+} := \begin{cases} \lim_{\sigma \searrow 0} \left( \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^p d\omega \right)^{\frac{1}{p}} & 0 < p < \infty, \\ \sup_{\omega \in \mathbb{R}} |f(\sigma + j\omega)|, & p = \infty, \end{cases}$$

and

$$\| f \|_{H_p^-} := \begin{cases} \lim_{\sigma \searrow 0} \left( \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^p d\omega \right)^{\frac{1}{p}} & 0 < p < \infty, \\ \sup_{\omega \in \mathbb{R}} |f(\sigma + j\omega)|, & p = \infty. \end{cases}$$

It is remarked that the tangential limits $\sigma \rightarrow 0$ in the above expressions exist, which makes the Hardy spaces well defined normed spaces, cf. [4].

B. Rational functions and Units

The prefixes $R$ and $U$ denote, respectively, rational functions and units in the Hardy spaces $H_p^+$ and $H_p^-$ as in

$$R H_p^+ := \{ f \in H_p^+ \mid f \text{ is rational} \},$$

$$R H_p^- := \{ f \in H_p^- \mid f \text{ is rational} \},$$

$$U H_1^+ := \{ U \in R H_1^+ \mid U^{-1} \in R H_1^- \},$$

$$U H_1^- := \{ U \in R H_1^- \mid U^{-1} \in R H_1^- \}.$$

Note that units are necessarily square rational matrices.

C. Laplace transformation

The Laplace transform $L : L_2(\mathbb{R}, \mathbb{R}^q) \rightarrow L_2(\mathbb{C}, C^q)$ defines an isometry between the $L_2$ Hilbert space and the inner product space $L_2$:

$$L := H_2^+ \oplus H_2^- = \{ f : C \rightarrow C^q \mid \| f \|_2 < \infty \},$$

which inherits the following norm:

$$\| f \|_2^2 = \int_{-\infty}^{\infty} f(j\omega)^H f(j\omega) d\omega,$$

and the inner product on complex valued functions:

$$(f, g) = \int_{-\infty}^{\infty} f(j\omega)^H g(j\omega) d\omega.$$

Any element $w \in L_2$ can be uniquely decomposed as

$$w = w_+ + w_-,$$

where

$$w_+ := \Pi_+ w, \text{ with } \Pi_+ : L_2 \rightarrow H_2^+, \quad w_- := \Pi_- w, \text{ with } \Pi_- : L_2 \rightarrow H_2^-.$$

Here, $\Pi_+$ and $\Pi_-$ denote the canonical projections from $L_2$ onto $H_2^+$ and $H_2^-$, respectively.

D. Mappings in $R H_\infty^+$ and $R H_\infty^-$

Elements of $R H_\infty^+$ and $R H_\infty^-$ (also known as stable- and anti-stable functions of $R H_{\text{stable}}$ and $R H_{\text{anti-stable}}$) define operators in the following manner. Let $\Theta \in R H_\infty^+$ and define $\Theta : L_2 \rightarrow L_2$ by:

$$(\Theta w)(s) := \Theta(s) w(s), \quad \text{where } w \in L_2,$$

which is the usual “multiplication” or Laurent operator in the frequency domain [4]. Similarly, let $\Psi \in R H_\infty^-$ and define $\Psi : L_2 \rightarrow L_2$ by:

$$(\Psi w)(s) := \Psi(s) w(s), \quad \text{where } w \in L_2.$$

When restricted to the domains $H_\infty^+$ or $H_\infty^-$, these operators define functions as:

Lemma 3.1: Let $\Theta \in R H_\infty^+$, with possible domains $L_2, H_\infty^+$ and $H_\infty^-$. Then

$$\Theta : L_2 \rightarrow L_2, \quad \Theta : H_\infty^+ \rightarrow H_\infty^+, \quad \Theta : H_\infty^- \rightarrow L_2.$$

Similarly, let $\Psi \in R H_\infty^-$, with possible domains $L_2, H_\infty^+$ and $H_\infty^-$. Then

$$\Psi : L_2 \rightarrow L_2, \quad \Psi : H_\infty^+ \rightarrow L_2, \quad \Psi : H_\infty^- \rightarrow H_\infty^-.$$

The proof of this lemma and more details about Hardy spaces can be found in [4].

1 A function is analytic if it is complex differentiable.
First, behaviors associated with mappings \( \mathcal{P} \) from the space of rational stable Hardy functions are discussed. For any \( P \in \mathcal{RH}_\infty^+ \), the following three dynamical systems are defined:

\[
\Sigma_P := (C, C^0, \mathcal{P}(P)), \\
\Sigma_{P,+} := (C, C^0, \mathcal{P}_+(P)), \\
\Sigma_{P,-} := (C, C^0, \mathcal{P}_-(P)),
\]

where

\[
\mathcal{P}(P) := \{ w \in \mathcal{L}_2 \mid Pw = 0 \} = \ker \mathcal{P} \subset \mathcal{L}_2, \\
\mathcal{P}_+(P) := \{ w \in \mathcal{H}_2^+ \mid Pw = 0 \} = \ker \mathcal{P}_+ \subset \mathcal{H}_2^+, \\
\mathcal{P}_-(P) := \{ w \in \mathcal{H}_2^- \mid Pw \in \mathcal{H}_2^- \} = \ker \mathcal{P}_- \subset \mathcal{H}_2^-.
\]

Here, \( \mathcal{P}_- \) is the canonical projection that is introduced before. For these sets we have the following properties:

**Lemma 4.1:** For \( P \in \mathcal{RH}_\infty^+ \), the behaviors \( \mathcal{P}(P) \), \( \mathcal{P}_+(P) \) and \( \mathcal{P}_-(P) \) are linear and right-shift invariant subsets of \( \mathcal{L}_2, \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \), respectively. A system \( \Sigma \) with either of these behaviors is called an \( \mathcal{L}_2 \) right-shift invariant system.

**Definition 4.1:** The classes of all linear and right-shift invariant systems in \( \mathcal{L}_2, \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \) that admit representations as the kernel of a rational element \( P \in \mathcal{RH}_\infty^+ \) are denoted by \( \mathcal{M}_+, \mathcal{M}_- \) and \( \mathcal{M}_0 \).

Similarly, for any \( \hat{P} \in \mathcal{RH}_\infty^- \), the following three dynamical systems are introduced as:

\[
\Sigma_{\hat{P}} := (C, C^0, \mathcal{P}(\hat{P})), \\
\Sigma_{\hat{P},+} := (C, C^0, \mathcal{P}_+(\hat{P})), \\
\Sigma_{\hat{P},-} := (C, C^0, \mathcal{P}_-(\hat{P})),
\]

where the behaviors are given by:

\[
\mathcal{P}(\hat{P}) := \{ w \in \mathcal{L}_2 \mid \hat{P}w = 0 \} = \ker \hat{P} \subset \mathcal{L}_2, \\
\mathcal{P}_+(\hat{P}) := \{ w \in \mathcal{H}_2^+ \mid \hat{P}w \in \mathcal{H}_2^+ \} = \ker \Pi_+ \hat{P} \subset \mathcal{H}_2^+, \\
\mathcal{P}_-(\hat{P}) := \{ w \in \mathcal{H}_2^- \mid \hat{P}w = 0 \} = \ker \hat{P} \subset \mathcal{H}_2^-.
\]

As introduced before, \( \Pi_+ : \mathcal{L}_2 \to \mathcal{H}_2^+ \) is the canonical projection.

**Lemma 4.2:** For \( \hat{P} \in \mathcal{RH}_\infty^- \) the behaviors \( \mathcal{P}(\hat{P}) \), \( \mathcal{P}_+(\hat{P}) \) and \( \mathcal{P}_-(\hat{P}) \) are linear and left-shift invariant subsets of \( \mathcal{L}_2, \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \), respectively. A system \( \Sigma \) with either of these behaviors is called an \( \mathcal{L}_2 \) left-shift invariant system.

**Definition 4.2:** The classes of all linear and left-shift invariant systems in \( \mathcal{L}_2, \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \) that admit representations as the kernel of a rational element \( \hat{P} \in \mathcal{RH}_\infty^- \) are denoted by \( \mathcal{L}_-, \mathcal{L}_+ \) and \( \mathcal{L}_0 \).

Now dynamical systems (1) can be described using \( \mathcal{L}_2 \) behaviors, some properties, using rational elements, are introduced:
Theorem 4.1: Let \( P, K \in \mathcal{RH}^+_{\infty} \) and let \( \mathcal{P}(\pm) = \mathcal{P}(\pm)(P) \) and \( \mathcal{K}(\pm) = \mathcal{K}(\pm)(K) \) be as defined in (2). Then the following statements are equivalent:

i. \( \mathcal{K} \subset \mathcal{P} \),
ii. \( \mathcal{K}_+ \subset \mathcal{P}_+ \),
iii. \( \mathcal{K}_- \subset \mathcal{P}_- \),
iv. \( \exists F \in \mathcal{RH}^+_{\infty} \) such that \( P = FK \).

Moreover, \( \mathcal{K} = \mathcal{P} \iff \mathcal{K}_+ = \mathcal{P}_+ \iff \mathcal{K}_- = \mathcal{P}_- \iff \exists U \in \mathcal{UH}^-_{\infty} \) such that \( P = UK \).

The proof of this theorem can be found in the Appendix. Also anti-stable mappings can be used in the representations, which yields the following theorem:

Theorem 4.2: Let \( \hat{P}, \hat{K} \in \mathcal{RH}^+_{\infty} \) and let \( \mathcal{P}(\pm) = \mathcal{P}(\pm)(\hat{P}) \) and \( \mathcal{K}(\pm) = \mathcal{K}(\pm)(\hat{K}) \) as in (3). Then the following statements are equivalent:

i. \( \mathcal{K} \subset \mathcal{P} \),
ii. \( \mathcal{K}_+ \subset \mathcal{P}_+ \),
iii. \( \mathcal{K}_- \subset \mathcal{P}_- \),
iv. \( \exists \hat{F} \in \mathcal{RH}^+_{\infty} \) such that \( \hat{P} = \hat{F} \hat{K} \).

Moreover, \( \mathcal{K} = \mathcal{P} \iff \mathcal{K}_+ = \mathcal{P}_+ \iff \mathcal{K}_- = \mathcal{P}_- \iff \exists \hat{U} \in \mathcal{UH}^-_{\infty} \) such that \( \hat{P} = \hat{U} \hat{K} \).

The proof of this theorem is similar to the one of Theorem 4.1 and therefore is not included in this paper.

V. CONTROLLER SYNTHESIS

A. Full Interconnection problem

For each of the above classes of \( \mathcal{L}_2 \) systems, the synthesis problem defined in Section II can now be formally stated as follows:

Problem 5.1: Given two linear left-shift invariant systems \( \Sigma_P \) and \( \Sigma_K \) in the class \( \mathcal{L}_0 \) (or \( \mathcal{L}_+ \) or \( \mathcal{L}_- \)),

i. Verify whether there exists \( \Sigma_C \in \mathcal{L}_0 \) \( (\mathcal{L}_+ \) or \( \mathcal{L}_-) \) such that \( \Sigma_C \cap \Sigma_P = \mathcal{K} \). Any such system is said to implement \( \mathcal{K} \) for \( \Sigma_P \) by full interconnection through \( w \) (Fig. 1(a)).

ii. If such controller exists, find a representation \( C_0 \in \mathcal{RH}^+_{\infty} \) of \( \Sigma_C = (T, W, C) \) in the sense that \( C = \ker C_0 \) (or \( C = \ker \Pi_+ C_0 \) or \( C = \ker C_0 \)).

iii. Characterize the set \( \mathcal{C}_{\text{par}} \) of all \( C \in \mathcal{RH}^+_{\infty} \) for which \( \Sigma_C = (T, W, \ker C) \) implements \( \mathcal{K} \) for \( \Sigma_P \).

A similar problem formulation applies for the model classes \( \mathcal{M}_0 \) \( (\mathcal{M}_+ \) or \( \mathcal{M}_-) \).

Our synthesis algorithm is inspired by the polynomial analog that has been treated in [5], [6] and leads to explicit rational representations of behaviors \( \mathcal{C} \) that implement \( \mathcal{K} \) for \( \Sigma_P \).

Theorem 5.1: Given the systems \( \Sigma_P = (T, W, P) \) and \( \Sigma_K = (T, \mathcal{W}, \mathcal{K}) \) in the class \( \mathcal{L}_0 \) \( (\mathcal{M}_0) \),

i. There exists a controller \( \Sigma_C = (T, \mathcal{W}, C) \in \mathcal{L}_0 \) \( (\mathcal{M}_0) \) that implements \( \mathcal{K} \) for \( \Sigma_P \) by full interconnection if and only if \( \mathcal{K} \subset \mathcal{P} \).

ii. Whenever one of the equivalent conditions of item i holds, the set \( \mathcal{C}_{\text{par}} \) of all possible kernel representations of controllers that implement \( \mathcal{K} \) for \( \Sigma_P \) by full interconnection is given in Step 5 of Algorithm 1 below.

The proof of Theorem 5.1 is inspired by the polynomial analog in [5] and [6] and is given in the next subsection.

B. Algorithm

The following algorithm results in the explicit construction of all controllers \( \Sigma_C \) that solve Problem 5.1 for the class \( \mathcal{L}_2 \) of \( \mathcal{L}_2 \) systems. A similar algorithm applies for the solution of Problem 5.1 for the model classes \( \mathcal{L}_0 \) \( (\mathcal{M}_0) \).

Algorithm 1: Given \( P, K \in \mathcal{RH}^+_{\infty} \) that define the systems \( \Sigma_P \) and \( \Sigma_K \) as in (3).

Aim: Find all \( C \in \mathcal{RH}^+_{\infty} \) that define the system \( \Sigma_C = (T, \mathcal{W}, C) \in \mathcal{L}_0 \) \( (\mathcal{M}_0) \) with \( C = \ker C \) such that \( C \) implements \( \mathcal{K} \) for \( \Sigma_P \) in the sense that \( \Sigma_P \cap \Sigma_C = \mathcal{K} \) by full interconnection.

Step 1: Verify whether \( \mathcal{K} \subset \mathcal{P} \). Equivalently, verify whether there exists a mapping \( F \in \mathcal{RH}^+_{\infty} \) such that \( P = FK \). If not, the algorithm ends and no controller exists that implements \( \mathcal{K} \) for \( \Sigma_P \).

Step 2: Determine a unit \( U \in \mathcal{UH}^-_{\infty} \) which brings \( F \) into column reduced form: \( \tilde{F} = FU = [F_1, 0] \), where \( F_1 \in \mathcal{RH}^+_{\infty} \) is square and of full rank.

Step 3: Extend the matrix \( \tilde{F} \) with \( \tilde{W} = [0, I] \) such that

\[
\tilde{X} = \begin{bmatrix} \tilde{F} \\ \tilde{W} \end{bmatrix} = [F_1, 0, I],
\]

belongs to \( \mathcal{UH}^-_{\infty} \). Factorize \( \tilde{W} = WU \) with \( W = \tilde{W} U^{-1} \).

Step 4: Set \( \Sigma_C = (T, \mathcal{W}, C) \), where \( C = \ker C_0 \) and \( C_0 = WK \in \mathcal{RH}^+_{\infty} \). The controller \( \Sigma_C \) then belongs to \( \mathcal{L}_0 \) and implements \( \mathcal{K} \) for \( \Sigma_P \).

Step 5: Set \( \mathcal{C}_{\text{par}} = \{ Q_1 P + Q_2 W K \mid Q_1, Q_2 \in \mathcal{RH}^+_{\infty}, Q_2 \text{ full rank} \} \).

Then \( \mathcal{C}_{\text{par}} \) is a parametrization of all controllers \( \Sigma_C = (T, \mathcal{W}, C) \) that implement \( \mathcal{K} \) for \( \Sigma_P \) by ranging over all kernel representations \( \mathcal{C} = \ker C \) with \( C \in \mathcal{C}_{\text{par}} \).

Proof: Proof of Theorem 5.1:

i. \((\Rightarrow)\): This is trivial.

\((\Leftarrow)\): If \( \mathcal{K} \subset \mathcal{P} \), then there exists a \( F \) as in Theorem 4.1 or 4.2. In the controlled behavior \( \mathcal{K} \), the restrictions of the plant as well as the restrictions applied by the controller have to be satisfied:

\[
\mathcal{K} = \ker \begin{bmatrix} F \\ W \end{bmatrix} = \ker K = \ker (\Gamma K),
\]

where \( \Gamma = \text{col}(F, W) \) with \( W \) unknown. The extended matrix \( \hat{X} \) in step 3 is a multiplication of a unit \( U \) with \( \Lambda \), so \( \Lambda \) has to be a unit. Therefore:

\[
\mathcal{K} = \ker \begin{bmatrix} F \\ W \end{bmatrix} = \ker \begin{bmatrix} F \end{bmatrix} \mathcal{K},
\]

so, \( \mathcal{C} = WK \) which results in \( \mathcal{C} = \ker C = \ker \{ WK \} \).

ii. One can apply a multiplication with a unit \( Q \in \mathcal{UH}^-_{\infty} \):

\[
\mathcal{K} = \ker \begin{bmatrix} F_1 \\ 0 \\ Q_2 \end{bmatrix} \begin{bmatrix} P \\ W \end{bmatrix} = \ker \{ Q_1 P + Q_2 WK \}.
\]
where \( Q_1, Q_2 \in \mathcal{RH}_\infty \) and \( Q_2 \) is full rank. Then all possible rational functions \( C \) can be parametrized by \( \mathbb{C}_{\text{par}} \) as in step 5.

VI. EXAMPLE: QUADRATIC COST

In this example, the plant behavior \( P \) of an unstable plant \( \Sigma_P \) is described by the state-space realization:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0, \\
y(t) &= x(t),
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \).

The desired controlled behavior \( \Sigma_K \) consists of all pairs \( (u, y) \in L_2(\mathbb{R}_+, \mathbb{R}^{n \times n}) \) that minimize the cost function:

\[
J(x_0, x(t), u(t)) = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt
\]

subject to the system equations (4) of the plant model \( \Sigma_P \). Here, \( 0 \leq Q \in \mathbb{R}^{n \times n}, 0 < R \in \mathbb{R}^{m \times m} \).

As discussed in [3], [8], this controlled behavior can be written as a dynamical system \( \Sigma_K \) with the state-space realization:

\[
\begin{align*}
\dot{x}(t) &= (A - BR^{-1}B^T S)x(t), & x(0) &= x_0, \\
y(t) &= -R^{-1}B^T Sx(t), \\
u(t) &= x(t),
\end{align*}
\]

where \( S \) is a solution of an Algebraic Ricatti Equation. The numerical values used for those matrices are the following:

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 4 & -15 \\ -15 & 76 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}^+, \quad \alpha \neq \beta.
\]

The dynamical systems are specified by trajectories in the time domain, but we are interested in \( L_2 \) behaviors using rational kernel representations as system representations. Because the controlled system is autonomous, the left-shift invariance property is required, which restricts us to use anti-stable mappings for \( P, K \) and also \( C \). This results in:

\[
P(s) = -1 - (sI - A)^{-1}B \in \mathcal{RH}_\infty^0, \\
K(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty, \\
\gamma(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty,
\]

where \( u(s) = \gamma(s)u(s) \). Due to the requirement, the anti-stable “poles” \( \alpha \) and \( \beta \) are introduced. Of course, no “pole-zero” cancellation should occur when \( \alpha \) and \( \beta \) are chosen. Using those representations, the full interconnection algorithm can be applied to the problem:

Step 1: The first step in the full interconnection algorithm is to verify whether \( K \subset P \), which should be the case. Equivalently, we need to verify whether there exists a \( F(s) \in \mathcal{RH}_\infty \) such that \( P(s) = F(s)K(s) \):

\[
F(s) = \Gamma(s)A(s) \in \mathcal{RH}_\infty, \quad \Gamma(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty, \quad \Lambda(s) = (sI - A)^{-1}B(sI - B)^{-1}.
\]

Step 2: The next step is to column reduce \( F(s) \). This can be done using algorithms as in [2], which results in:

\[
\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0, \quad \begin{bmatrix} -\frac{1}{2} & -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0,
\]

and

\[
\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0.
\]

Step 3-4: Then, as discussed in Step 4 of Algorithm 1, a possible controller behavior \( \mathcal{C} = \ker C_0 \) is expressed as:

\[
C_0(s) = W(s)K(s) = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -1 & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0.
\]

As mentioned before, the behavioral framework does not require a separation of the variable \( w \) into inputs and outputs. This can be seen in the result above, because the controller restricts the outputs of the plant in the first row when a separation in the variable space is made. So, mathematically the interconnection with this controller results in the desired controlled behavior, but this representation is not directly implementable for a real system, because outputs of a plant can’t always be used as inputs. For a practical reason, we consider the parametrization of all controllers that implement \( \Sigma_K \) in the next step.

Step 5: Another controller can be found using the matrices \( Q_1(s) \) and \( Q_2(s) \) as defined in Algorithm 1. When these matrices are chosen to be:

\[
Q_1 = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0 \quad \text{and} \quad Q_2 = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0,
\]

the resulting controller is equal to:

\[
C(s) = Q_1(s)P(s) + Q_2(s)C_1(s) = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \in \mathcal{RH}_\infty^0.
\]

In fact, this controller allows a feedback implementation as it is equivalent to the general LQR, namely:

\[
u(t) = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} x(t).
\]

Note: The values in the Ricatti solution \( S \) and the values in the estimated rational expressions are rounded to integers for simplification.

VII. CONCLUSIONS AND RECOMMENDATIONS

We considered the problem of controller synthesis for specific classes of \( L_2 \) functions. Operators in the classes \( \mathcal{RH}_\infty^0 \) of stable rational functions and \( \mathcal{RH}_\infty^0 \) of anti-stable rational functions define linear right-shift invariant \( L_2 \) behaviors and linear left-shift invariant \( L_2 \) behaviors by considering their kernel spaces. Given two \( L_2 \) systems \( \Sigma_P \) and \( \Sigma_K \) we solve the question to synthesize a third \( L_2 \) system \( \Sigma_C \) that realizes
Σ_K in the sense that the full interconnection of Σ_P and Σ_C satisfies K = P ∩ C. Necessary and sufficient condition for the existence of an L_2 system Σ_C is the inclusion K ⊂ P. An explicit controller synthesis algorithm for the class of all controllers that implement an L_2 controlled system for an L_2 plant has been derived. An example is given to demonstrate the algorithm for the construction of a rational representation of C.

This paper only introduced the case when the plant behavior P and controller behavior C are fully interconnected, which is not always the case. Therefore, some further research has to be done for the “partial interconnection” case using those classes of rational functions. Studies already started for infinite smooth continuous behaviors in [6]. In this case, disturbances like noise can be taken into account, which may yield in robust control problems.

APPENDIX

PROOF OF THEOREM 4.1

Proof:

(iv ⇒ {i,ii,iii}): 

• iv ⇒ i:
  K, P ⊂ L_2, so w ∈ L_2: If P = FK and take a w ∈ K. Then, v = Kw = 0, so also Pu = FKw = Fv = 0. This implies that P(s)w(s) = 0, so w ∈ P, and K ⊂ P.

• iv ⇒ ii:
  K, P ⊂ H_2^+, so w ∈ H_2^+:
  This proof is identical to the case when K, P ⊂ L_2.

• iv ⇒ iii:
  K, P ⊂ H_2^−, so w ∈ H_2^−:
  Again, if P = FK and w ∈ K, one can say that v = Kw ∈ H_2^− and hence Pu = FKw = Fv. Now, F ∈ RH_2^∞, so Fv ∈ H_2^−. So, Pw ∈ H_2^+ which implies using (2) that w ∈ P, which is equal to K ⊂ P.

(iv ⇐ {i,ii,iii}): 

• iv ⇐ i:
  K, P ⊂ L_2, so w ∈ L_2: Using the definition of K, one can write:
  \[ K = \{ w ∈ L_2 | \langle Kw, v \rangle_{L_2} = 0 \ \forall v ∈ L_2 \} = \{ w ∈ L_2 | \langle w, K^* v \rangle_{L_2} = 0 \ \forall v ∈ L_2 \} = (\text{im } K^*)^\perp, \]
  where K^* : L_2 → L_2 is the dual- or adjoint operator in RH_2^∞, defined by K^*(s) = K^T(s^{-1}). Something similar can be applied to the plant behavior. So, K ⊂ P implies that P^⊥ ⊂ K^⊥ and using the previous definition of K, this results in
  \[ (\text{im } P^*) \subseteq (\text{im } K^*), \]
  where the bar denotes the closure in L_2.
  For rational operators the latter implies that:
  \[ (\text{im } P^*) \subseteq (\text{im } K^*), \]
  because in that case the images are closed.

Then we can say that for some e_i^2, P*e_i ∈ im K*, so there exists a v_i such that:
  \[ P*e_i = K^*v_i. \]

This can be extended to a set of v_i’s, such that:
  \[ P^* = K^*X \text{ with } X = (v_1, \ldots, v_p) ∈ RH_2^∞ \subset RH_2^∞. \]

Then, we can rewrite this to P = X*K, where F is equal to the dual operator X* ∈ RH_2^∞.

• iv ⇐ ii:
  K, P ⊂ H_2^+, so w ∈ H_2^+:
  This proof is similar to the one in the previous item, except that now the H_2^+ inner product is used. However, H_2^+ inherited this inner product from L_2.

• iv ⇐ iii:
  K, P ⊂ H_2^−, so w ∈ H_2^−: Now, K can be written as:
  \[ K = \{ w ∈ H_2^− | \langle Kw, v \rangle_{H_2^−} = 0 \ \forall v ∈ H_2^− \} = \{ w ∈ H_2^− | \langle w, K^* v \rangle_{H_2^−} = 0 \ \forall v ∈ H_2^− \} = (\text{im } K^*\Pi^*_{i=1}^p)^\perp, \]
  where K^* and Π_{i=1}^p are adjoint operators. This can also be done for the plant behavior P. As in item (iv ⇐ i), P^⊥ ⊂ K^⊥, so:
  \[ (\text{im } P^*\Pi^*_{i=1}^p) \subset (\text{im } K^*\Pi^*_{i=1}^p). \]

Equality condition:

Using the previous items, one can say that P = K if and only if P = U_1K and K = U_2P with both U_1 and U_2 in RH_2^∞. Moreover, if U_1 and U_2 satisfy these conditions, then P = U_1U_2K and K = U_2U_1P. If P and K are full rank, we find that U_1 = U_2^{-1}, which completes the proof.

REFERENCES


