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Multidimensional Hosting Capacity Region in Dutch MV Grid Congestion Management

Sicheng Gong, Graduate Student Member, IEEE, Vladimír Čuk, and J. F. G. Cobben

Abstract—The hosting capacity determines the grid ability to host integrated installations, where mutual-limitations among points of connection (POCs) are important and formulate a combinational feasible region. Intended to assess such interdependence for defining the region, this paper proposes a multidimensional hosting capacity region derivation scheme in a radial grid. As this scheme can be designed conservative, from a risk-averse viewpoint, it gains significant acceptance in the industrial sector. This multidimensional region not only exploits grid power delivery potential, but also aids in making grid congestion management decisions. A typical 10.5kV Dutch grid case has been tested accordingly, and corresponding results have confirmed the conservative property preservation of the region. The case study demonstrates that compared to the original hosting capacity concept, the region hypervolume gain ratio remains higher than 1.94. With proper measure selection, the estimated region occupation ratio can keep up to 92.50%. The congestion management decision computation time can also be reduced by 58.0% compared to that using the monolithic model. The effectiveness of the proposed scheme and its advantages in grid congestion management have been successfully confirmed.

Index Terms—hosting capacity, power quality, feasible region, multidimensional geometry.

I. INTRODUCTION

ELECTRIC vehicle (EV) is promising for transport decarbonization to promote Dutch mobility energy transition [1], [2]. Facing rapid EV ownership growth and current EV subsidy policy in the Netherlands, supported by Dutch distribution system operators (DSOs) and local municipalities, a large-scale MV ultra-fast EV charging infrastructure system has been put on the agenda [3]–[5]. Under this Dutch charging infrastructure vision, massive power delivery in MV grids is predictable, leading to grid congestion [6]. The DSO is principally responsible to handle such problem by capacity expansion or local energy storage installation [7].

Ahead of taking above strategies to host such charging infrastructure system, its grid impacts should be analyzed. Hosting capacity is a concept originally quantifying the grid ability to host distribution energy resources, while now also used to define the acceptable integration capacity for various installations in practices [8]–[10]. The evolved concept aids in evaluating POC-wise grid impacts of those new installations. Towards a certain POC, the determined hosting capacity is treated as its own installation capacity threshold, eventually benefiting grid congestion mitigation. Moreover, the capacity threshold is correlated to connection fees in practices [11]. If a POC expects a higher hosting capacity, which is commonly determined by the DSO, it should pay a higher connection fee accordingly. Nevertheless, nowadays new installations need to wait on connection due to limited hosting capacity. They own strong willingness to pay, while in some cases they might need to wait for years until there is enough expansions for them [12]. Considering above technical and financial incentives, rational hosting capacity determination is expected for Dutch DSOs.

Several factors should be accounted during hosting capacity decision, including POC voltage harmonic distortions, POC voltage level, cable line current, etc [13]. In the context of disruptive power electronics technology for harmonic emission reduction, harmonic problems are neglected in this paper, and the focus is on restrictions related to voltage level and line current. [14], [15]. Moreover, with grid modelling technology developing, there is rising possibility to analyze hosting capacity using predetermined network parameters [16]. Referring to a certain grid model, the DSO will test a quantity of scenarios by taking all maximal or minimal loads/generations, as there is no coordination between those connections.

It is unfortunate that such determination scheme would naturally cause unexpected redundancy, as each involved POC is hypothesized to keep working in full load mode, which is rare in practices. The higher margin exploration is expected if installations can coordinate to utilize such redundancy. References [17]–[19] have explored the overall hosting capacity limit of the grid for various installations. For testing scenario reduction to save computation source, corresponding probabilistic methods have also been developed, thereby eliminating the need of repeated scenario testing [20]–[22]. However, all calculations above are implemented from a network-wide viewpoint, where detailed cooperative operation schemes among various POCs are missing.

From the perspective of a single POC, two paths can be taken to leverage redundancy: one involves temporal correlation and the other includes exploration of the multidimensional feasible region. The first approach seeks to incorporate time-dependent hosting capacity, enabling dynamic hosting capacity allocation among POCs [23]–[25]. Accordingly, Dutch DSOs are contemplating a specific hosting capacity determination scheme that varies over time [26]. However, this dynamic capacity determination scheme demands the amalgamation of all POCs’ operation profiles, requiring extensive scenario testing and potentially raising issues of user data privacy. Moreover, the responsibility of implementing capacity allocation still

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falls on the DSO, thus causing equality concerns as well. In contrast, the approach involving a multidimensional feasible region imposes combinational constraints on POCs, not only allowing full grid power delivery potential exploitation, but also eliminating the need for user profile integration [27].

Intuitively, The DSO can use power flow tools to conduct as many scenario tests as possible in a brute-force manner, eventually establishing the envelope of those possible points. In practices, there might not be a need to know all possibilities, as the DSO would only investigate the scenarios asked by the customer. Despite such workload reduction, the testing scenario number will still grow exponentially as more POCs are involved, effectively demanding greater computational capabilities. Aiming for direct region derivation, reference [28] searches for a feasible neighbourhood named "security region", ultimately calculating the final region through iterative union of regions. However, this relies on a linearized grid model. Towards the non-convex grid model, individual security region is also derivable although there lacks discussions on union of these regions [29]. For a monolithic region, the "hosting capacity region" approach was proposed and verified effective in [27], with the principle of region derivation to be further clarified in Section II-C. However, its application is restricted to two dimensions (2D). An extension to multidimensional hosting capacity region is needed.

This paper seeks to tackle this challenge by devising dedicated multidimensional hosting capacity region algorithms. By breaking the dimension limit, it allows for the involvement of more POCs. With the intention of leveraging model convexity for effective polytope exploration, a conservative model approximation scheme is developed, which theoretically guarantees the feasibility of all points in the derived region. This enables DSOs to accept such concept from a risk-averse perspective. During grid congestion management, this multidimensional hosting capacity region can further provide advantages by speeding up the resolution of corresponding optimal decision problems. This paper offers three primary contributions:

- **By grid model modification and geometric computation, a novel scheme has been developed to derive a conservative multidimensional hosting capacity region.**
- **To mitigate the Matthew Effect in assessing the multidimensional hosting capacity region, various facet selection measures have been proposed and validated.**
- **Through applying the hosting capacity region to optimal energy storage deployment problems, its potential to accelerate the solving process has been verified.**

The remaining part of this paper is organized as follows. Section II explains the grid model, together with responsive multidimensional hosting capacity region derivation principles. Then corresponding concrete derivation algorithms are designed and analyzed in Section III. Algorithm validation and concept application are implemented in Section IV and a conclusion is provided in Section V.

**II. Hosting Capacity Region in Radial Grids**

"Hosting capacity region" is a concept naturally evolving from "hosting capacity". In this section, it is further illustrated about how this evolution can help grid power delivery potential exploitation. Moreover, in advance of answering how to derive the multidimensional hosting capacity region, the region derivation principle would be also discussed.

### A. Concept Evolution

Based on original "hosting capacity" concept, the DSO would set independent installation capacity limits over specific POCs [8]. Those limitations are concise to both DSO and installation owners. Meanwhile, considering compliance to extreme scenarios when all POCs are in heavy-load mode, part of flexibility would be sacrificed, for instance temporary single-POC hosting capacity expansion when remaining POCs are in light-load mode. Intuitively, the original concept evolves into "hosting capacity region", formulating a combinational feasible region for involved POCs. The multidimensional hosting capacity region encompasses all operational possibilities that are acceptable for the radial grid. Through coordinated operation among those POCs, this region opens up the potential for some POCs to temporarily exchange more power when other POCs are not heavily loaded. Additionally, when addressing grid congestion management problems, the multidimensional hosting capacity region can serve as a compact alternative to the original grid model, eventually accelerating the solving process.

For deeper explanation, a simple testing case is provided in Fig. 1, with cable and substation parameters given in Table I. The hosting capacity region of solar panel and charging facility is targeted using ergodic algorithm, which is plotted in Fig. 2 [27]. In this figure, gradual color is used to show different loading ratios of the cable section between charging point as low loading ratio of 0.163+j0.083Ω/km and heavy loading ratio of 0.163+j0.398Ω/km. Meanwhile, considering compliance to extreme scenarios when all POCs are in heavy-load mode, part of flexibility would be sacrificed, for instance temporary single-POC hosting capacity expansion when remaining POCs are in light-load mode. Intuitively, the original concept evolves into "hosting capacity region", formulating a combinational feasible region for involved POCs. The multidimensional hosting capacity region encompasses all operational possibilities that are acceptable for the radial grid. Through coordinated operation among those POCs, this region opens up the potential for some POCs to temporarily exchange more power when other POCs are not heavily loaded. Additionally, when addressing grid congestion management problems, the multidimensional hosting capacity region can serve as a compact alternative to the original grid model, eventually accelerating the solving process.

![Fig. 1: 3-Node testing case schematic](image)

**TABLE I: 3-Node grid parameters**

<table>
<thead>
<tr>
<th>Type</th>
<th>Cable (240A)</th>
<th>Transformer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impedance (70°C, 50Hz)</td>
<td>0.163+j0.083Ω/km</td>
<td>0.0067+j0.398Ω/km</td>
</tr>
<tr>
<td>Current / Capacity Limit</td>
<td>360A</td>
<td>36MVA</td>
</tr>
</tbody>
</table>

If still using independent capacity limits, dashed lines for a magenta rectangular feasible region would be set as shown in Fig. 2. It leads to smaller operation space for those instal-
lem, from a conservative perspective, the cable impedance is sensitive to working temperature. Intended to avoid the generation of a 4km cable is far lower than 1kvar [31]. An approximate reactive power generator can be integrated to all variables in (1) is given in Table II. Regarding cable ground capacitance can be even ignored. The notation of given in (1), DistFlow model is adopted to mathematically describe the radial grid as proposed in [30]. The notation of this section would discuss such possibility of acceleration and compactness.

B. DistFlow Model

Since radial distribution grids are widespread deployed in LV and MV electrical systems, from the perspective of DSO, the grid topology can be set to be radial. Accordingly, as given in (1), DistFlow model is adopted to mathematically describe the radial grid as proposed in [30]. The notation of all variables in (1) is given in Table II. Regarding cable ground capacitance neglecting in (1), due to constrained voltage level, an approximate reactive power generator can be integrated to terminal POCs. In Dutch MV grids, the approximate reactive generation of a 4km cable is far lower than 1kvar [31]. Therefore, compare to the Mvar-scale hosting capacity region, the cable ground capacitance can be even ignored.

\[
P_{ij} = \sum_{k:(j,k) \in E} P_{jk} + r_{ij}l_{ij} - P_j \quad (1a)
\]

\[
Q_{ij} = \sum_{k:(j,k) \in E} Q_{jk} + x_{ij}l_{ij} - Q_j \quad (1b)
\]

\[
v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)l_{ij} \quad (1c)
\]

\[
l_{ij} = \frac{P_{ij}^2 + Q_{ij}^2}{v_i} \quad (1d)
\]

\[
P_{i}^{\text{min}} \leq P_i \leq P_{i}^{\text{max}} \quad (1e)
\]

\[
Q_{i}^{\text{min}} \leq Q_i \leq Q_{i}^{\text{max}} \quad (1f)
\]

\[
v_{i}^{\text{min}} \leq v_i \leq v_{i}^{\text{max}} \quad (1g)
\]

\[
l_{ij}^{\text{min}} \leq l_{ij} \leq l_{ij}^{\text{max}} \quad (1h)
\]

1) Looking for convexification: Cable parameters can be sensitive to working temperature. Intended to avoid the temperature dependence which leads to a non-polynomial problem, from a conservative perspective, the cable impedance is assumed constant with working temperature being fixed to 70°C [32]. This will lead to higher voltage level deviation, equivalently setting stricter voltage level constraints from a risk-averse perspective. Moreover, since POC load profiles are not being used during multidimensional hosting capacity region derivation, for uninvolved POCs, their integration power would be set constant. In Dutch hosting capacity calculation practices, all existing installations are set to full/zero power mode for extreme scenario testing, which is in analogy to this presumption. Even with changes in those uninvolved installations, the derived region remains valid following the appropriate geometric transformation. In summary, in advance of discussing region derivation principle, there are three presumptions made in this paper as listed below, without harming the universal applicability of the concept in Dutch practices.

1) The distribution grid is radial;
2) The cable operates in 70°C, with constant impedance;
3) Uninvolved POCs are in constant power mode.

C. Region derivation principle

As explained in [27], there exists a path to derive multidimensional hosting capacity region through model approximation, especially considering relatively low values of \( l_{ij} \)-related items in (1a)-(1c).

1) Model convexification: Here we propose an approximate model as illustrated in (2), which is denoted as Model-I. It is derived by ignoring \( l_{ij} \)-related items in (1a)-(1c). Since Model-I is convex, the combinational feasible region of \( P_{ij}, Q_{ij} \) must be convex.

\[
P_{ij} = \sum_{k:(j,k) \in E} P_{jk} - P_j \quad (2a)
\]

\[
Q_{ij} = \sum_{k:(j,k) \in E} Q_{jk} - Q_j \quad (2b)
\]

\[
v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) \quad (2c)
\]

\[
\frac{P_{ij}^2 + Q_{ij}^2}{v_i} \quad (2d)
\]

\[
v_{ij}^{\text{min}} \leq v_j \leq v_{ij}^{\text{max}} \quad (2e)
\]

Model-I: (2)

Convexity contributes to quick geometrical processing and compact data structure when shaping the region. With convexity guaranteed, as shown in Fig. 3, when we figure out two boundary points \( A \) and \( B \), all points located in the connection line between \( A \) and \( B \) are feasible. Likewise, with another boundary point \( C \), a triangle with \( A, B, C \) being
vertices can be defined as a feasible region, which is covered by stripes in Fig. 3. Afterwards, through importing extra boundary point \( D \), a quadrangle feasible region colored by grey can be derived. With more boundary points, the whole feasible region can grow into a polygon. This polygon can be numerically represented by a series of linear constraints as illustrated in Fig. 3. Unlike monolithic raw point processing in classic convex hull algorithm, the proposed polygon derivation method is tailor-made for hosting capacity region problems, taking advantage of heuristic boundary point selection \([33]\). In a 2D case, only a sequence of boundary points is enough to represent such polygon, where each two adjacent points can generate a linear constraint accordingly.

From a multidimensional view, those linear constraints would be upgraded into hyperplanes, and edges into facets. The polytope also evolves into a polytope. When expanding the polytope, we can select a certain facet at first. Using a 3-dimensional (3D) case for illustration, as shown in Fig. 4, there is a known polyhedron feasible region with points \( G, A, B, C \) being vertices. Through choosing another interior point \( D \) in face \( ABC \), vector \( OD \) would be stretched until reaching the boundary. Let this critical boundary point be marked as \( E \).

Lemma 1. In Model-II, \( f(C_{ij}) \rightarrow C_j \), \( f \) is a bijection.

Proof. See Appendix A-1.

Corollary 1. In Model-II, \( \exists f, f(C_{ij}) \rightarrow C_j, f \) is a bijection.

Let the subscript \( i \) or \( l \) distinguishes different \( C_{ij} \) and \( C_j \) in Model-I and Model-II. Model-III is generated by reconsidering \( l_{ij} \)-related items in (1c). The subscript \( l \) denotes respective variables derived from Model-III.

Model-III: \( (2a)-(2b), (1c)-(1d), (1g)-(1h) \).

The proof of Lemma 1 naturally leads to Corollary 2.

Corollary 2. In Model-III, \( \exists f, f(C_{ij}) \rightarrow C_j, f \) is a bijection.

As Model-III is more close to DistFlow model, first we connect Model-I,II to Model-III, thus claiming Proposition 1.

Proposition 1. \( \overline{C_j} \subset C_j^{III} \), where \( \overline{C_j} = C_j^I \cap C_j^{II} \).

Proof. See Appendix A-2.

According to Proposition 1, we can intersect feasible regions of Model-I,III. The intersection is guaranteed a subset of feasible region of Model-III as illustrated in Fig. 5. The necessity of Model-III lies in its function as an intermediary during the derivation process. Working towards the feasible region of Model-III is a vital step in beginning geometric operations on derived regions, thereby shrinking the gap between Model-I,II and the exact model.

Intended to further proceed towards DistFlow model, Model-IV is given by ignoring (1e)-(1f) in DistFlow Model. Let the subscript \( IV \) denote respective variables derived from Model-IV. Based on Definition 1.2, we can further connect Model-I to Model-IV by Proposition 2 through iterative shifting and intersection.

Model-IV: \( (1a)-(1d), (1g)-(1h) \).

Definition 1. In \( C_j \oplus r_{ij}^{max} \), if \( P_j \) is a dimension of \( C_j \), \( \oplus \) is an operator which moves \( C_j \) along the dimension \( P_j \) with \( r_{ij}^{max} \). Otherwise, \( C_j \oplus x_{ij}^{max} \leftarrow C_j \).

Definition 2. In \( C_j \oplus x_{ij}^{max} \), if \( Q_j \) is a dimension of \( C_j \), \( \oplus \) is an operator which moves \( C_j \) along the dimension \( Q_j \) with \( x_{ij}^{max} \). Otherwise, \( C_j \oplus x_{ij}^{max} \leftarrow C_j \).

Proposition 2. \( \bigcap_{(i,j) \in E} \{ R_j \} \subset C_j^{IV} \), where \( R_j = C_j^{III} \cap (C_j^{III} \oplus r_{ij}^{max}) \cap (C_j^{III} \oplus x_{ij}^{max}) \).

Proof. See Appendix A-3.
By combining Proposition 1,2, as stated in Remark 1, it is possible to approximately derive the feasible region in Model-IV, based on convex Model-I and Model-II.

**Remark 1.** $\mathcal{R} \subseteq \mathcal{C}_{ij}^{IV}$, where 
\[
\mathcal{R} = \bigcap_{(i,j) \in E} \{\mathcal{R}_{ij}\}, \quad \mathcal{R}_{ij} = \overline{\mathcal{C}_j} \cap (\overline{\mathcal{C}_j} \oplus r_{ij} l_{ij}^{max}) \cap (\overline{\mathcal{C}_j} \oplus x_{ij} l_{ij}^{max})
\]

*Proof.* See Appendix A-4.

Compared to the final expected feasible region $\mathcal{C}_{ij}^{IV}$ in DistFlow model, we can directly apply (1e)-(1f) to $\mathcal{C}_{ij}^{IV}$. Therefore, after add constraints (1e)-(1f) to $\mathcal{R}$, referring to Remark 1, the final derived region must be a subset of $\mathcal{C}_{ij}^{IV}$. Therefore, Method-II is proposed to derive the hosting capacity region from a risk-averse perspective, indicating that all points in such region are theoretically guaranteed feasible for DistFlow model.

**Method-II:** Compute the intersection between feasible regions of Model-I and Model-II, thus deriving a feasible region subset of Model-III. Through further intersecting this polytope and its shifted ones, a feasible region subset of Model-IV is derived. After recycling (1e)-(1f), a subset of DistFlow model feasible region is achieved eventually.

### III. ALGORITHM DESIGN

With hosting capacity region derivation principle explained in Section II-C, it allows us to look into details during concrete algorithm design.

#### A. General hosting capacity region derivation

Algorithm 1 summarizes the region derivation progress in Method-II, eventually realizing convexification compensation in Method-I. Since the polytope feasible region generation in Model-I and Model-II is independent, Algorithm 1 allows distributive computation for acceleration.

#### B. Heuristic polytope generation

In polytope region generation part at Algorithm 1, Bisection method is a classic numerical method to figure out critical boundary points [34]. In a 3D case, as shown in Fig. 4, vector $\overrightarrow{OD}$ and $\gamma \overrightarrow{OD}$ can be selected as two initial points in Bisection, where $\gamma$ is a big enough positive number. Through iterative bisection testing, $\gamma_0$ can be calculated so that $\gamma_0 \overrightarrow{OD}$ leads to a critical point $E$.

Based on Bisection ideology, as explained in Section II-C, benefiting from model convexity, Algorithm 2 is proposed to quickly generate the polytope feasible region. This heuristic algorithm works in multi-dimension for both Model-I and Model-II, eventually getting rid of 2D limit in [27]. In the initial stage of Algorithm 2, matrix $\mathbf{U}$ should owns a rank of $n$, where $\mathbf{U} = [v_1; v_2; \ldots; v_{n+1}]$, ensuring an initial $n$-polytope with nonzero hypervolume.

#### Algorithm 1: Hosting capacity region derivation

**Data:** Grid graph $E$, region dimension.

**Result:** multidimensional hosting capacity region $\mathcal{R}$. Initialize Model-I and Model-II;

Generate polytope feasible region about $P_j$ and $Q_j$ in Model-I, and denote it as $\mathcal{C}_{ij}^I$;

Generate polytope feasible region about $P_j$ and $Q_j$ in Model-II, and denote it as $\mathcal{C}_{ij}^I$;

$\mathcal{R} \leftarrow \mathcal{R} \cap \mathcal{R}_{ij}$;

**end**

Add constraints (1e)-(1f) to $\mathcal{R}$;

#### Algorithm 2: Polytope generation in $\mathbb{R}^n$

**Data:** Vector basis set $\mathbf{V} = \{v_1; v_2; \ldots; v_{n+1}\}$.

**Result:** Facet set $\mathbf{F}$ to represent the polytope.

Set stopping criteria $\epsilon$ and $\xi$; Initialize dictionary $\mathbf{D}_n \leftarrow \emptyset$, $\mathbf{F} \leftarrow \emptyset$;

**for** $(i,j) \in E$ **do**

Run **Bisection** to find boundary point $\gamma_{ij} v_j$;

Add a pair $\{v_j : \gamma_{ij}\}$ to $\mathbf{D}_n$;

**end**

**for** $v_j \in \mathbf{V}$ **do**

**V** ← $\mathbf{V} \setminus v_j$;

Define facet $\mathbf{f}_j$ from $\mathbf{V}$, add $\mathbf{f}_j$ to $\mathbf{F}$;

**end**

**repeat**

Set a new vector $v_0$ based on selected facet $\mathbf{f}_0$;

Run **Bisection** to find boundary point $\gamma_0 v_0$;

Add a pair $\{v_0 : \gamma_0\}$ to $\mathbf{D}_n$, add $v_0$ to $\mathbf{V}$;

Remove $\mathbf{f}_0$ from $\mathbf{F}$;

**for** $\mathbf{f}_j$ **do**

if $\mathbf{f}_j$ intersects with $\mathbf{f}_0$ then

Initialize set $\Delta \mathbf{V}$ by all vertices of $\mathbf{f}_j$ except the intersection vector, add $\gamma_0 v_0$ to $\Delta \mathbf{V}$;

Define facet $\mathbf{f}_j$ from $\Delta \mathbf{V}$, add $\mathbf{f}_j$ to $\mathbf{F}$;

**end**

until $\text{size}(\mathbf{V}) \geq \xi$;

$\mathbf{H} \leftarrow \emptyset$, add all keys $\mathbf{h}_j$ in $\mathbf{D}_n$ to $\mathbf{H}$;

#### C. Efficient facet selection

There exists a flexible step about new vector generation in Algorithm 2. In this paper, we treat such flexibility as a facet selection problem, where the centroid of selected
facet will be adopted to generate a new vector. Efficient facet selection is expected to take better advantage of such flexibility. The centroid is straightforward to compute in high-dimension scenarios. Through direct facet selection, we can avoid additional computations needed to determine which facet should be removed. From a 3D view in Fig. 4, we need extra measures to rationalize selecting face \(ABC\) among all four faces in this pyramid.

If randomly choosing next-round face for polyhedron growth, since these three new faces share the same possibility with other faces, the next-round vector is more likely to cross original \(ABC\). It is equivalent to choose another interior point neighboring \(D\) in \(ABC\), tendentially making less contributions to polyhedron growth in that iteration. Accumulatively, more and more new faces are likely to be descendents of \(ABC\), thus influencing the polyhedron generation algorithm performance. This phenomenon is a variant of Matthew effect, which can be qualified by the gap ratio \(G\) calculated as (4) [35],

\[
G = \frac{\max_{i,j \in \mathcal{O}} (D_i - D_j)}{\sum_{i \in \mathcal{O}} D_i} \times 100\% \quad (4)
\]

where \(\mathcal{O}\) denotes the set of original ancestor facets formulating the initial polytope. \(D_i\) is the equivalent facet descendant number of the \(i\)th ancestor facet.

As illustrated in Fig. 6, in 3D scenarios with random facet selection, by running the code with 1000 various random seeds, the average gap ratio will raise up smoothly until hitting a threshold of 58.87\%, although each single case still owns an uneven ratio curve with uncertain thresholds. This high expectation of \(G\) corroborates the risk of Matthew Effect during random facet selection. In the remainder of this section, aiming for Matthew effect mitigation, several measures are proposed and discussed to help efficient facet selection.

For a known \(n\)-polytope \(C\), let \(\mathcal{F}\) denote its facet, which is an \((n - 1)\)-dimensional polytope. \(C(\cdot)\) is a new \(n\)-polytope, as a convex-hull of all input facets or points. \(V(\cdot)\) denotes the polytope hypervolume. \(g(\cdot)\) is the facet generation order, which is equal to 0 for first \(n+1\) facets formulating the starting polytope. \(\Delta R\) is the incremented polytope hyper volume after facet alternation. Intuitively, we expect to keep a high \(\Delta R\) value at each iteration, and greedy ideology can be adopted, which directly takes \(\Delta R\) as the measure during facet selection.

Meanwhile, due to its ergodic nature in each round, this greedy algorithm may be questioned in the aspect of computation time expense. With the hosting capacity region dimension number growing, the needed vertex number will increase exponentially. Therefore, more alternative measures can be considered for acceleration during each iteration of facet selection, which has been listed in Table III.

### TABLE III: Measure table for facet selection

<table>
<thead>
<tr>
<th>Measure</th>
<th>Meaning in 3D case</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta V(\mathcal{F}))</td>
<td>Polyhedron volume increment</td>
<td>Maximal</td>
</tr>
<tr>
<td>(V(\mathcal{F}))</td>
<td>Face area</td>
<td>Maximal</td>
</tr>
<tr>
<td>(V(C(\mathcal{F}, \Theta)))</td>
<td>Origin-apex triangular pyramid volume</td>
<td>Maximal</td>
</tr>
<tr>
<td>(g(\mathcal{F}))</td>
<td>Face generation order</td>
<td>Minimal</td>
</tr>
</tbody>
</table>

Intuitively, in 3D cases, a higher \(V(\mathcal{F})\) implies a larger basis of the new generated pyramid, and a higher \(V(C(\mathcal{F}, \Theta))\) suggests a pyramid basis far from the origin point. Both results contribute empirically to faster polytope growth. \(g(\mathcal{F})\) seeks to directly mitigate the Matthew Effect without any geometric considerations on current polytope status. In summary, compared to \(\Delta R\), benefiting from reduced computation complexity, these alternative measures are more empirical and own weaker causal relationship with efficient polytope growth.

### IV. ALGORITHM VALIDATION AND APPLICATION

Intended to verify the effectiveness of proposed algorithms in Section III, several numerical tests are implemented. Referring to the derived hosting capacity region, we further discuss its advantage in congestion management practices, which has been confirmed by extra numerical tests as well. The testing case is illustrated in Fig. 7, which is based on 10.5kV MV network case provided by a dutch DSO, Alliander [36]. All POCs are already connected to constant-power loads, whose respective nominal current is listed in Fig. 7 with the power factor being 0.98. More grid parameters are listed in Table IV.

### TABLE IV: Power cable and transformer parameters

<table>
<thead>
<tr>
<th>Object (max current)</th>
<th>Resistance</th>
<th>Reactance (50Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>240 AL Cable (412A)</td>
<td>126mΩ/km</td>
<td>116mΩ/km</td>
</tr>
<tr>
<td>150 AL Cable (312A)</td>
<td>320mΩ/km</td>
<td>188mΩ/km</td>
</tr>
<tr>
<td>95 AL Cable (224A)</td>
<td>641mΩ/km</td>
<td>204mΩ/km</td>
</tr>
<tr>
<td>36MVA 50kV/10.5kV Transformer</td>
<td>0.0022p.u.</td>
<td>0.065p.u.</td>
</tr>
</tbody>
</table>

Bus 2,4,7 are considered for extra EV charging integration in this case. Considering emerging bidirectional charging facilities, respective POC injection power can be either positive or negative [37]. In this section, we select injected active power from these three POCs as dimensions, eventually targeting a...
3D hosting capacity region. Pandapower toolkit is adopted for power flow calculation to check the feasibility of points [38]. All codes were run on a computer with an Intel i7-9750H processor running at 2.60 GHz using 15.8 GB of RAM, running Windows 10 Enterprise version.

A. Algorithm validation

Concerning the derived multidimensional hosting capacity region, it should be inspected in two aspects, including its conservative property and degree. The former one checks if all points in the derived region are feasible for the grid, eventually answering whether the DSO can trust the results. The latter one estimates how much feasible space is sacrificed in such derivation progress, thus evaluating our algorithm approximation performance.

Considering unknown exact hosting capacity region, point estimation method can be adopted. After generating uniformly distributed random points in an excessively big polytope, we check each point feasibility in both derived region \( \mathcal{R} \) and DistFlow model. Accordingly, indices \( P_{RD} \) and \( P_R \) are proposed to realize inspection in a quantitative way. When \( P_{RD} \) is equal to 1, the conservative property can be confirmed. Moreover, combined with a higher \( P_R \), which can be taken as an estimated region occupation ratio, we claim smaller hosting capacity region loss, indicating better approximation performance.

\[
P_{RD} = \frac{N_{RD}}{N_R} \times 100\%, \quad P_R = \frac{N_R}{N_D} \times 100\% \tag{5}
\]

where \( N_R \) is the number of points inside \( \mathcal{R} \), \( N_D \) denotes how many points are verified feasible in DistFlow model. \( N_{RD} \) is the number of points that satisfy both conditions above.

To figure out a minimal orthotope encapsulating the derived hosting capacity region, which is a rectangular cuboid in 3D scenario, we expand it along each axis by 2MW and create a significantly larger polytope as depicted by the grey box in Fig. 8a. The total random testing point number is set to 10000.

Moreover, as illustrated in Fig. 8b, following the original hosting capacity concept, it leads to a inscribed orthotope region equivalently. In this section, we set it to be the maximum-hypervolume orthotope in the convex hull of all feasible points, which is denoted as \( \hat{\mathcal{R}} \) and able to be solved through conic optimization as stated in [39]. Since the real feasible region may be non-convex, \( V(\mathcal{R}) \) is an over-estimator of its maximal interior orthotope volume. Intended to quantify the advantage of the evolved concept, \( V(\mathcal{R}) \) and \( V(\hat{\mathcal{R}}) \) are compared as (6), where \( \alpha \) is the estimated region hypervolume gain ratio when adopting such evolved concept.

\[
\alpha = \frac{V(\hat{\mathcal{R}})}{V(\mathcal{R})} \tag{6}
\]

1) Under various methods: Prior to employing Algorithm 2 to use Method-II as a springboard to approximate DistFlow Model, we need to confirm its advantage compared to directly adopting Method-I. Moreover, different measures are adopted sequentially to inspect their impacts on algorithm performance. After setting the generated vertex number to 500, respective testing results are summarized in Table V. With Method-I, \( P_{RD} \) may surpass 100\% as Method-I provides an approximation region without conservative property guarantees. Due to the limitation of iterative algorithm in power flow computation, \( P_{RD} \) based on Method-II cannot be exactly equivalent to 100\%. Therefore, we list relevant extreme values of voltage or current level among those computer-claimed infeasible points, checking whether respective constraint violations are acceptable. As given in Table V, \( P_{RD} \) under Method-II keeps higher than that under Method-I, denoting the advantage of proposed Algorithm 1. Moreover, \( P_{RD} \) under Method-II keeps close to 100\% and extreme cable load ratio of respective infeasible points is only up to 102.03\%. In practices, we can tighten these grid constraints by scaling down current level upper bounds before applying Method-II, thus ensuring rigorous compliance to original constraints.

Simultaneously, according to Remark 1, a conservative hosting capacity region will be derived directly through Method-II. To confirm the validity of Remark 1, points deemed

---

**TABLE V: Testing results during algorithm validation**

<table>
<thead>
<tr>
<th>Method</th>
<th>Measure</th>
<th>( P_{RD} / P_R ) (%)</th>
<th>( V(\mathcal{R}) ) (MW(^3))</th>
<th>( \alpha ) max( {\hat{v}_j} ) (%)</th>
<th>max/min( {v_j} ) (p.u.)</th>
<th>Time expense (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method-I</td>
<td>Random</td>
<td>98.53 / 89.57</td>
<td>540.87</td>
<td>2.06</td>
<td>102.87</td>
<td>1.014 / 0.956</td>
</tr>
<tr>
<td>Method-I</td>
<td>( \Delta V(\mathcal{R}) )</td>
<td>97.14 / 100.68</td>
<td>602.97</td>
<td>2.68</td>
<td>102.88</td>
<td>1.029 / 0.954</td>
</tr>
<tr>
<td>Method-I</td>
<td>( V(\mathcal{F}(\mathcal{R})) )</td>
<td>97.43 / 95.72</td>
<td>577.53</td>
<td>2.57</td>
<td>102.90</td>
<td>1.051 / 0.953</td>
</tr>
<tr>
<td>Method-I</td>
<td>( g(\mathcal{F}) )</td>
<td>97.54 / 97.88</td>
<td>590.54</td>
<td>2.63</td>
<td>102.88</td>
<td>1.034 / 0.953</td>
</tr>
<tr>
<td>Method-I</td>
<td>( g(\mathcal{F}) )</td>
<td>97.96 / 97.04</td>
<td>584.53</td>
<td>2.60</td>
<td>102.90</td>
<td>1.027 / 0.954</td>
</tr>
<tr>
<td>Method-II</td>
<td>Random</td>
<td>99.56 / 70.31</td>
<td>426.63</td>
<td>1.65</td>
<td>101.25</td>
<td>1.006 / 0.957</td>
</tr>
<tr>
<td>Method-II</td>
<td>( \Delta V(\mathcal{R}) )</td>
<td>98.66 / 97.86</td>
<td>592.45</td>
<td>2.64</td>
<td>101.79</td>
<td>1.019 / 0.953</td>
</tr>
<tr>
<td>Method-II</td>
<td>( V(\mathcal{F}(\mathcal{R})) )</td>
<td>98.89 / 91.44</td>
<td>552.61</td>
<td>2.46</td>
<td>101.33</td>
<td>1.032 / 0.956</td>
</tr>
<tr>
<td>Method-II</td>
<td>( g(\mathcal{F}) )</td>
<td>98.92 / 94.63</td>
<td>574.02</td>
<td>2.56</td>
<td>101.62</td>
<td>1.021 / 0.954</td>
</tr>
<tr>
<td>Method-II</td>
<td>( g(\mathcal{F}) )</td>
<td>99.06 / 93.24</td>
<td>562.70</td>
<td>2.51</td>
<td>102.03</td>
<td>1.028 / 0.953</td>
</tr>
</tbody>
</table>

Fig. 8: 3D hosting capacity region from various concepts

(a) Evolved concept  (b) Original concept
infeasible by Pandapower need to be further scrutinized, as the inability of iterative algorithms to compute a feasible solution for an operation point doesn’t necessarily negate its feasibility. Therefore, we built an optimization model with same constraints and an artificial objective function, and input it into the Gurobi solver. Of the 26 suspect points based on measure $g(\mathcal{F})$, 22 points are confirmed to be feasible. Given the limitations of the non-convex solver in Gurobi, among all 10000 testing points, hardly can 4 left ill-posed points harm the conclusion on point feasibility. Therefore, the conservative region property under Method-II has been confirmed.

Among all measures, the greedy measure $\Delta V(R)$ owns the best performance with the price of high computation time expenses. Accordingly, $P_R$ is up to 97.86% while owning the maximal $\ell_{ij}^{ab}$ constrain lower than 101.79%. Among three alternative measures, their respective computation time is reduced and similar performance is achieved. Measure $g(\mathcal{F})$ owns the highest speed, where 78.69% computation time can be saved compared to that of $\Delta V(R)$. Measure $V(\mathcal{F})$ holds the performance bottom line with its $P_R$ being 91.44%. Nevertheless, compared to the original hosting capacity concept, it still provides a high region hypervolume gain ratio up to 2.46, indicating this measure is still worth practical consideration for DSOs.

2) Under various measures: After the region conservative property under Method-II is verified above, we mainly focus on efficient facet selection inspection. Through monitoring polytope hypervolume growing with vertex number raising, we can analyze the impacts of various measures on algorithm performance. As given in Fig. 9, $\Delta V(R)$ keeps contributing to quicker polytope hypervolume growth, indicating more efficient facet selection. Combining with time expense data in Table V, we can adopt $V(C(\mathcal{F}, \Theta))$ with similar algorithm performance achieved, thus saving over half computation time than $\Delta V(R)$. Unfortunately, $V(\mathcal{F})$-based algorithm still suffers Matthew Effect when generating first 50 vertices in this case, although its subsequent efficient facet selection quickly makes up the hypervolume gap.

3) Under various grid parameters: In above study case, voltage level constraints are not a limiting factor when generating the hosting capacity region. Intended to make those constraints work during region derivation, we consider sensitivity to cable lengths through increasing all of them by 2 times. Intended to examine the derived hosting capacity region variation, based on measure $\Delta V(R)$, we plot the original one by grey and the updated one by light red in Fig. 10a. The grey and light red lines inside are respective polytope edges. Respective three-view drawings are provided in Fig. 10b-10d. With cable length increasing, a reduction in the region can be observed as outlined grey parts in Fig. 10b-10c. In particular, there is noticeable cross-section area loss as marked by black-line polygons in Fig. 10b, 10c, where the Bus 7 dimension keeps involved. It indicates that the grid congestion is more sensitive to Bus 7 installation capacity during grid cable length scaling.

Simultaneously, through inspecting testing results provided in Table VI, high $P_{RD}$ and weak constraint violations are observed, which verifies the algorithm effectiveness in this case. Although the region loss increases indicated by dropping $P_D$, it keeps acceptable especially when adopting $\Delta V(R)$.

4) Under various involved POCs: Intended to further verify the effectiveness of proposed algorithms, consistent in a 3D view, we move to explore the hosting capacity region for Bus 3, 5, 7. Unlike Bus 2, 4, Bus 3, 5 own stronger coupling relationship with Bus 7, as they are regarded as its parent bus in load streams. Here we still focus on Model-IV to check the conservative property and degree of the derived region. The testing results are provided in Table VII. High $P_{RD}$ and $P_R$ have verified the algorithm effectiveness in this case.

In summary, the proposed hosting capacity region derivation scheme based on Method-II have tested, whose conservative property and low region loss are verified accordingly. Compare to $\Delta V(R)$, three alternative measures contributes less to efficient facet selection, while their advantage of computation acceleration can benefit for future higher-dimensional hosting capacity region derivation.
TABLE VI: Testing results under various parameters

<table>
<thead>
<tr>
<th>Measure</th>
<th>( P_{RD} / P_R ) (%)</th>
<th>( V(\mathcal{R}) ) (MW)</th>
<th>( \alpha )</th>
<th>max{( \hat{P}_{ij} )} (% )</th>
<th>max/min{( v_j )} (p.u.)</th>
<th>Time expense (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta V(\mathcal{R}) )</td>
<td>98.12 / 92.50</td>
<td>508.88</td>
<td>2.17</td>
<td>104.69</td>
<td>1.087 / 0.904</td>
<td>1032.58</td>
</tr>
<tr>
<td>( V(\mathcal{F}) )</td>
<td>98.31 / 84.16</td>
<td>456.00</td>
<td>1.94</td>
<td>103.63</td>
<td>1.077 / 0.911</td>
<td>279.68</td>
</tr>
<tr>
<td>( V(C(\mathcal{F}, \mathcal{O})) )</td>
<td>98.39 / 87.71</td>
<td>479.50</td>
<td>2.04</td>
<td>103.57</td>
<td>1.083 / 0.908</td>
<td>527.64</td>
</tr>
<tr>
<td>( g(\mathcal{F}) )</td>
<td>97.82 / 83.09</td>
<td>461.69</td>
<td>1.97</td>
<td>103.42</td>
<td>1.085 / 0.909</td>
<td>224.87</td>
</tr>
</tbody>
</table>

TABLE VII: Testing results under various involved POCs

<table>
<thead>
<tr>
<th>Measure</th>
<th>( P_{RD} / P_R ) (%)</th>
<th>( V(\mathcal{R}) ) (MW)</th>
<th>( \alpha )</th>
<th>max{( \hat{P}_{ij} )} (% )</th>
<th>max/min{( v_j )} (p.u.)</th>
<th>Time expense (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta V(\mathcal{R}) )</td>
<td>98.89 / 96.85</td>
<td>1105.20</td>
<td>20.77</td>
<td>101.80</td>
<td>1.017 / 0.958</td>
<td>1045.50</td>
</tr>
<tr>
<td>( V(\mathcal{F}) )</td>
<td>99.43 / 88.21</td>
<td>994.28</td>
<td>18.68</td>
<td>101.38</td>
<td>1.008 / 0.962</td>
<td>276.97</td>
</tr>
<tr>
<td>( V(C(\mathcal{F}, \mathcal{O})) )</td>
<td>99.15 / 94.94</td>
<td>1077.31</td>
<td>20.24</td>
<td>101.50</td>
<td>1.010 / 0.957</td>
<td>523.70</td>
</tr>
<tr>
<td>( g(\mathcal{F}) )</td>
<td>98.88 / 93.09</td>
<td>1052.84</td>
<td>19.78</td>
<td>101.59</td>
<td>1.010 / 0.957</td>
<td>225.90</td>
</tr>
</tbody>
</table>

Fig. 11: Congestion management through power smoothing

B. Concept application in grid congestion management

As discussed in Section I, the multidimensional hosting capacity region is not limited to grid power delivery potential exploitation, but also applicable to help solve grid congestion management problems. As shown in Fig. 11, inheriting grid parameters from the case in Fig. 7, power smoothing for grid congestion management has been widely investigated, where energy storage equipment inject regulating power to compensate for excessive or insufficient power from existing installations in the same POC [40], [41]. Accordingly, the lowest absolute sum of regulating power should be answered for energy storage operation cost optimization.

\[
\begin{align*}
\min & \sum_{i \in \{2,4,7\}} |P_i - \hat{P}_i| \\
\text{s.t.} & \quad (1b) - (1h) \\
\forall i \notin \{2, 4, 7\}, P_{ij} &= \sum_{k:(j, k) \in E} P_{jk} + r_{ij}l_{ij} - P_j \\
\forall i \in \{2, 4, 7\}, P_{ij} &= \sum_{k:(j, k) \in E} P_{jk} + r_{ij}l_{ij} - (\hat{P}_j - P_i)
\end{align*}
\]

In order to verify the evolved concept benefit on solving this problem, we collect randomly selected infeasible points generated at Section IV-A as testing scenarios. Under each of them, the lowest absolute sum of regulating power will be computed. Intuitively, with feasible region unknown, the monolithic optimization model can be written as (7) based on DistFlow model, where \( P_i \) for \( i \notin \{2, 4, 7\} \) are constant parameters as Presumption 3. \( \hat{P}_i \) match 3D testing point coordinate values. \( P_i \) for \( i \in \{2, 4, 7\} \) are decision variables about regulating power.

\[
\begin{align*}
\min & \sum_{i \in \{2,4,7\}} |\hat{P}_i - P_i| \\
\text{s.t.} & \quad (P_2, P_4, P_7) \in \mathcal{R}
\end{align*}
\]

For remaining 721 points outside the region, we compare corresponding optimal storage power separately from (7) and (8), whose distributions are close as shown in Fig. 12. Respective kernel distribution estimation plots are also provided for smoothly visualizing such distribution [42]. Due to region derivation loss and numerical computation error, the optimization result deviation is inevitable. Through calculating absolute values of those deviations, respective deviation distribution is derived and plotted as well in Fig. 12, where most deviation are observed between the range of 0-300kW.

Quantitatively, the average deviation is up to 105.05kW, which accounts for 5.63% of the average of optimal total storage power through monolithic optimization. Such optimization gap between these two models is acceptable in industrial practices, thus proving the validity of "hosting capacity region"
concept application in grid congestion management. Besides, the average solving time for (7) is 289.30ms and that for (8) is 121.62ms. Such concept can save over 58.0% computation time during the congestion management decision, thereby validating its computation acceleration property.

V. CONCLUSION

Aiming for MV grid congestion management, this paper has presented a novel scheme for multidimensional hosting capacity region derivation. As long as communication between POCs is available, this region can help further exploit grid power delivery potential. Through the creation of a heuristic convex region and repeated region intersections, the derived region can be theoretically guaranteed conservative, indicating meaning that all points within it are viable for the precise DistFlow model. Regarding the verification part, such guarantee has been numerically verified, and low region loss can be observed simultaneously as $P_{ji}$ remains greater than 92.50%. Among four proposed measures for facet selection, $\Delta V(R)$ contributes to the most accurate region, and $g(SF)$ owns the highest computation speed. Through applying the derived region in grid congestion management, we further validate the concept effectiveness and computational acceleration benefits when making optimal storage capacity decisions.

The dimensionality of the proposed hosting capacity region derivation scheme is currently limited by available computational resources. In the future, a hosting capacity region with higher dimensionality on a more extensive testing case would be exploited, where computational efficiency will be enhanced by reducing the number of vertices. Furthermore, numerical error correction and a distributed computing framework will be incorporated to increase computational accuracy and speed.

APPENDIX A

LEMMMA, PROPOSITION AND REMARK PROOF

1) Proof of Lemma 1:

Proof. Let $P_j$ and $P_{ij}$ be defined to represent all $P_j$ and $P_{ij}$ in a radial grid with one slack bus and $j$ POCs.

$$P_j = [P_1, P_2, ..., P_j]^T, \ P_{ij} = [P_{01}, P_{12}, ..., P_{ij}]^T \ (9)$$

Based on (2a)-(2b), we can write $P_j = MP_{ij}$, where $M$ is a $j \times j$ matrix. Now first we prove that $M$ is invertible.

Base case: If $j = 1$, $M = [-1]$, which is obviously invertible.

Induction step: Suppose $j \geq 1$, $M$ is invertible. Let $k = j + 1$. Given a $k$-POC case, we write (10), where $0 \leq m \leq j$.

$$P_k = [P_1, P_2, ..., P_j, P_k]^T = [P_j^T \ P_{j,k}]^T \ (10)$$

$$P_{mk} = [P_{01}, P_{12}, ..., P_{ij}, P_{mk}]^T = [P_{ij}^T \ P_{mk}]^T$$

Let $P_k = MP_{mk}$, it can be derived that

$$M = \begin{bmatrix} M & e \\ 0 & -1 \end{bmatrix} \ (11)$$

where $M$ is supposed invertible in a $j$-POC case, which is generated by removing the terminal POC $k$ in original case. $e$ is a zero $j \times 1$ vector except $m$th element is set to 1. $0$ is a zero $1 \times j$ vector.

Let $Det(M)$ denote the matrix determinant. As $Det(M) \neq 0$, $|Det(M)| = |Det(M)| \neq 0$, thus proving $M$ is invertible.

Conclusion: Given $j \geq 1$, $M$ is invertible.

We define $Q_j$ and $Q_{ij}$ likewise. It can be found that $Q_j = MQ_{ij}$. Therefore, based on invertible $M$, $f(E_{ij}) \rightarrow C_j$ is a linear bijective mapping.

2) Proof of Proposition 1:

Proof. First we prove $C_{ij} \subset C_{ij}^{IV}$, where $C_{ij} = C_{ij} \cap C_{ij}^{IV}$. For a point in $C_{ij}$, based on (2c) or (3), we can calculate $v_j^{IV}$ and $v_j^{II}$ in chain. We can calculate $v_j^{II}$ likewise by (1g) in Model-III. In a $j$-POC grid, facing any point in $C_{ij}$, based on Model-III, we claim that all calculated $v_j^{IV}$ still meet (12).

$$v_j^{IV} \leq v_j^{II} \leq v_j^{I} \ (12)$$

Base case: If $j = 1$, it naturally meet (12).

Induction step: Suppose $j \geq 1$, all $v_j^{IV}$ meets (12). Let $k = j + 1$. Given a $k$-POC case, we select a terminal POC numbered as $k$, whose parent POC is $m$. Removing POC $k$, we have all remaining $v_j^{IV}$ meet (12). Moreover, we have

$$v_j^{III} = v_j^{IV} - 2(mkP_{mk} + x_mkQ_{mk}) + z_{mk}^2l_{mk} \leq v_j^{III} - 2(mkP_{mk} + x_mkQ_{mk}) + z_{mk}^2l_{mk} \leq v_j^{III} - 2(mkP_{mk} + x_mkQ_{mk}) + z_{mk}^2l_{mk} = v_j^{III} - z_{mk}^2 = v_j^{IV} - z_{mk}^2 \ (13)$$

Likewise, we also have $v_j^{IV} \geq v_j^{I}$.

Conclusion: Given $j \geq 1$, for any point in $C_{ij}$, respective $v_j^{IV}$ and $v_j^{II}$ based on Model-III still meet (12).

Combining the above conclusion with (2e) in Model-I and Model-II, we can state that all $v_j^{IV}$ keep meeting (1g).

Furthermore, based on (1d) in Model-III and (2e) in Model-I, we have

$$l_{ij}^{IV} = \frac{P_{ij}^2 + Q_{ij}^2}{v_j^{IV}} \leq \frac{P_{ij}^2 + Q_{ij}^2}{v_j^{I}} \leq l_{ij}^{max} \ (14)$$

In summary, for any point in $C_{ij}$, respective $v_j$ and $l_{ij}$ calculated from (1c)-(1d) in Model-III are guaranteed to meet constraints (1g)-(1h) in Model-III. Therefore, $C_{ij} \subset C_{ij}^{IV}$.

Combined with Lemma 1 and Corollary 1-2, we can bijectively mapping such relationship in $E_{ij}$ domain, thus $C_{ij} \subset C_{ij}^{IV}$.

3) Proof of Proposition 2:

Proof. First we focus on $P_j$ part. In a $j$-POC grid, we set $x_{ij}$ to zero, so that (1b) in Model-IV is equivalent to (2b) in Model-III. We claim that $\bigcap_{(i,j) \in E_{ij}} \{R_j\} \subset C_{ij}^{IV}$, where $R_j = C_{ij}^{IV} \cap \{C_{ij}^{II} \oplus r_{ij}^{max}\}$. The subscript $o$ and $s$ distinguish the bijective feasible point in $C_{ij}^{IV}$ and $C_{ij}^{II} \oplus r_{ij}^{max}\) and $s$.

Base case: If $j = 1$, $P_j^{IV} = -P_{01} + r_{01}l_{01}$. In original and shifted model, $P_j^o = -P_{01} = P_{01}^{IV}$, $P_j^s = -P_{01} + r_{01}l_{01}^{max}$. Therefore, $P_j^o \leq P_j^{IV} \leq P_j^s$. In a convex set, $P_j^o$ and $P_j^s$ are feasible, $P_j^{IV}$ must be feasible. The claim holds when $j = 1$.

Induction step: Suppose $j \geq 1$, the claim still holds. Let $k = j + 1$. Given a $k$-POC case, we select a terminal POC numbered as $k$, whose parent POC is $m$. Removing POC $k$, for any point in $\bigcap_{(i,j) \in E_{ij}} \{R_j\} \subset C_{ij}^{IV}$, we can guarantee...
the existence of $[P_j^T, P_{kj}^T]_T$ and $[P_j^T, P_{kj}^T]_T$ in original and shifted model. As $P_{j\alpha} \leq P_{j\beta} \leq P_{j\gamma}$, using the set convexity, $[P_j^T, P_{k}^TV]_T$ must meet (1a) in Model-IV. As this point meets naturally all constraints in Model-III, it is proved to be feasible for Model-IV. Therefore, $\bigcup_{(i,j) \in E} \{\mathcal{R}_j\} \subset C_j^{IV}$.

Conclusion: Given $j \geq 1$, if $x_{ij} = 0$, $\bigcup_{(i,j) \in E} \{\mathcal{R}_j\} \subset C_j^{IV}$.

Following the same philosophy above, when we can reconsider $x_{ij} \in \mathcal{R}_j$ item in (1b), we have $\bigcup_{(i,j) \in E} \{\mathcal{R}_j\} \subset C_j^{IV}$.

4) Proof of Remark 1:

Proof. Referring to Proposition 1, $\mathcal{C}_j \subset C_j^{IV}$, thus

$$
(\mathcal{C}_j + r_{ij}^{max}) \subset (C_j^{IV} + r_{ij}^{max})
$$

Therefore, we can derive that $\mathcal{R}_j \subset \mathcal{R}_{ij}$. Combining Proposition 2, we write $\bigcup_{(i,j) \in E} \{\mathcal{R}_j\} \subset C_j^{IV}$.

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