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A Brief Introduction to Traffic Modelling with a Closer Look at the Nagel-Schreckenberg Model

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A Brief Introduction to Traffic Modelling
with a Closer Look at the
Nagel-Schreckenberg Model

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Abstract

A literature review is presented that introduces, for educational purposes, the wide scope of research on the modelling of motorway traffic. The fundamentals of traffic modelling are explained, among which the fundamental diagram, different states of traffic, some correlations and standard parameters. Also the notion of bottlenecks is explained. We found four important classes of traffic models, being the microscopic models, following every vehicle separately, the cellular automata models, discretising the problem in such a way that it becomes computationally fast, the macroscopic models, following the dynamics of traffic as a whole, and the class of gas-kinetic models, that add behavioural aspects to the macroscopic approach in terms of acceleration and interactions between different driver-vehicle units. Furthermore, we present the classification of congestion patterns according to Kerner and the newer classification by Schönhof and Helbing, including an explanation of the controversies that have arisen about the former classification. Next to that, we explain the Nagel-Schreckenberg model in more detail and implement it, finding similar results as Nagel and Schreckenberg. This leads to the verification of the cellular automata model, with some clarifications with an educational objective.

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1 Introduction

Motorway traffic has been the topic of research for nearly a century, already starting with a study by Greenshields (1935). Traffic jams occur every day and in many ways, causing the development of a range of different models in combination with the classification of different patterns of congestion. As the scope of traffic modelling has developed into a wide variety of studies and publications, it is a difficult discipline to start with without any guiding principle. Several review articles, such as an extensive article by Helbing (2001), have aimed at making a clear overview, but none of them aims at the completely ignorant reader who understands mathematics, like the author, but does not know anything about traffic modelling. Therefore, the objective of this research is to explain the basics of traffic modelling by presenting a brief introduction to this broad area of research.

Section 2 describes the precise problem statement, after which we present a literature overview in section 3 composed of some fundamental principles in traffic modelling, more information on the notion of bottlenecks, the explanation of various traffic models, an overview of different patterns in congested traffic and the specification of several assumptions, controversies in traffic modelling research and miscellaneous remarks. In section 4, the Nagel-Schreckenberg model is explained in detail, after which section 5 shows its implementation, presenting and explaining the numerically obtained results. Section 6 concludes the study, mentioning a few points of further research that may be used to extend the literature overview.

2 Problem Statement

For educational purposes that aim at introducing motorway traffic modelling to students and ignorant readers, we want to investigate what models already exist. We want to explain the fundamental concepts of traffic modelling, necessary for understanding models and results thereof, introduce the concept of bottlenecks, where traffic is perturbed due to lane closure, merging vehicles or else, and illustrate different kinds of traffic patterns that are found in congestion. We want to give a clear overview of the wide research in traffic modelling, that can be used to get acquainted with the discipline and helps the reader understand the formulas, concepts, model structures and figures used. We aim to combine this in a literature overview that should give the readers a clear view on traffic modelling and gives them the ability to choose an area they want to focus their further research upon.
Apart from the literature overview, we also want to show one of the models in more detail, by implementing it in a numerical simulation. With this, we want to show the reader how a traffic model works, what it looks like and what possible results can be found, with a primarily educational objective. In addition, we want to verify the Nagel-Schreckenberg model (Nagel & Schreckenberg, 1992), to see whether similar results can be produced and to explain this particular model in more detail, aiming at a faster understanding for the readers, such that applying the model raises fewer questions.

Therefore, our general aim is to create a brief overview of the concepts of different models in traffic modelling, with a closer look at the Nagel-Schreckenberg model, in order to introduce traffic modelling to the ignorant reader, including the author, therewith helping the reader to understand the basics of this broad area of expertise.

3 Literature Overview

This literature overview is inspired by and largely, though not completely, based on a review article by Helbing (2001). It introduces the topic of motorway traffic modelling by explaining some fundamentals, such as terminology, measuring techniques, widely-used formulas, an important diagram, different states of traffic, correlations and parameters. Also the threshold concept of a bottleneck is introduced. Moreover, a concise description of four different classes of traffic models follows, each of which is explained using some examples. After that, two categorisations of patterns that occur in congested traffic are explained. Subsequently, some assumptions, remarks and controversies in traffic modelling follow. Apart from the review by Helbing (2001), many other sources have been consulted, also giving rise to other paragraphs, not described in (Helbing, 2001).

3.1 Some Fundamentals of Traffic Modelling

In order to expand on the different classes of models that are used for the modelling of road traffic, there are some fundamental concepts that have to be understood first. This includes some basic jargon on traffic modelling, some widely-used measurement techniques, established correlations between different parameters and the different phases of traffic flow.

First of all, it is important to recognise that any significant traffic situation can be described as a self-driven many-particle system (Helbing, 2001), i.e., the driving force is internal, rather than external, as is the case in for example sailboats, using wind as their external driving force. Instead, the driving force comes from within, which, in case of car traffic, is facilitated by the engines. According to Schweitzer et al. (1998), there has to be an internal source of energy. Naturally, cars run on fuel or charged batteries, which is their energy source. The other aspect to self-driven many-particle systems is the presence of many particles that interact on each other. Generally, traffic situations on motorways contain large numbers of vehicles, except perhaps on weekend days or holidays and during the nights. Therefore, most traffic models focus on regular ‘working’ days. Then, there are indeed many vehicles, each presenting a self-driven particle in the many-particle system.

In many traffic models, different lanes are considered, rather than only modelling a single lane (see, for example, Zhang et al., 2018; Cremer & Ludwig, 1986; Daganzo, 2002a,b). Not only does this contribute to the modelling of the lane changing mechanism, which is especially interesting in the situation where lanes are reduced due to an accident or road works, but it also gives a clear view of the differences between two lanes. For instance, the average velocity on the leftmost lane, when the lane-changing rules that are common in most of Europe are used, is expected to be higher than the velocity on the rightmost lane, due to slower freight traffic being obligated to...
drive on the rightmost lane and overtaking manoeuvres only being allowed on the left side. Such lane changing rules have to be clearly specified to model lane changing.

When lane changes are modelled, the situation of a particle, for trivial reasons called a vehicle in traffic modelling, is more complicated than only time and the one-dimensional position. The time $t$ usually runs from $t = 0$ at the starting point to some time $t = T$ at the end of the model, whereas the one-dimensional position $x$ runs from the start of the modelled road section at $x = 0$ to the end of the road section at some value $x = L$, for $L > 0$. In case of multi-lane models, a vehicle’s position also depends on the current lane $i$ for $i = 1, 2, \ldots, I$, where $I$ is the total number of lanes. This discrete approach to the modelling of lanes makes it possible to consider the different characteristics that arise in the different lanes due to both driving behaviour, velocity restrictions and lane changing rules. Figure 1 shows a sketch of the situation in a model where two lanes are considered. The first lane is indexed $i = 1$ and the second lane has index $i = 2$.

![Figure 1: Example of a situation in a model where multiple lanes are considered, particularly interesting because of the closure of the first lane. In this model, different lane-changing rules are used for the bottleneck region and no lane changing occurs in the third part, where the first lane is reduced. Figure from Zhang et al. (2018).](image)

When we consider all vehicles in the self-driven many-particle system individually, we speak of a microscopic model. Then, every vehicle is indexed. When looking at vehicle $\alpha$, the vehicle in front of $\alpha$ is represented by index $\alpha - 1$, called the leading vehicle. The very first vehicle to leave the road section, i.e., the vehicle that has the highest position $x$, is indexed $\alpha_1$. This situation is sketched in Figure 2.

![Figure 2: Sketch of the indexation of vehicles on a section of one lane of a motorway. Vehicle $\alpha_1$ is the first vehicle to leave the road section. Vehicle $\alpha - 1$ is the leading vehicle of $\alpha$ and vehicle $\alpha$ is the leading vehicle of $\alpha + 1$. Also, vehicle $\alpha_1$ is the leading vehicle of $\alpha_2$. The arrows indicate the upstream and downstream directions.](image)

In traffic literature, there is often reference to upstream and downstream of a position. This is comparable to the stream of a river. Rivers tend to start at high places, for example starting as a melting stream of a glacier, and stream downwards to sea-level to eventually end in a lake or sea. Therefore, when we refer to downstream of a location, it means that we go along with the (traffic) flow. When we refer to upstream of a location, we mean going against the (traffic) flow, up to the origin of the flow. Both are indicated in Figure 2.

Next to position, researchers are also interested in velocity, acceleration and the net or netto distance between two consecutive vehicles, given by the distance from the rear end of a vehicle to the front end of the following vehicle (Gong & Yang, 2009). The net distance is also referred to as the clearance and is sketched in Figure 3.

![Figure 3: Sketch of the netto distance between two consecutive vehicles. The distance from the rear end of a vehicle to the front end of the following vehicle is referred to as the clearance.](image)
There are different measurement techniques used to find the aforementioned information. Roughly, these can be divided into fixed-sensor techniques and the use of one-probe vehicles (Rao & Rao, 2012), obtaining so-called car-following data. Moreover, we have the availability of aerial photography and video recordings, which can arguably be seen as fixed-sensor techniques.

Probe vehicles are vehicles equipped with data-collecting apparatus such as GPS receivers, accelerometers and Distance Measuring Instruments (Rao & Rao, 2012), which are already able to measure quantities as speed, location, acceleration and clearance and lane-changing manoeuvres (Helbing, 2001). Moreover, especially with the data acquisition in cars getting more and more advanced, databases are not only extended by the amount of data, but also by the variety of parameters that can be measured using the extensive equipment installed on cars (Winkler et al., 2020). Governments already collaborate with car companies such as Mercedes Benz to make use of this data (Dutch Ministry of Infrastructure and Water Management, 2022) and other companies use this data for their own purposes, including the development of autonomous cars, of which Tesla is a major example (Dikmen & Burns, 2016). Nowadays, the tracked location of vehicles can be used to show the density on a certain place on the road (Sjöman & Maunu, 2022). Moreover, the ability to connect other devices, such as smartphones, to cars via Bluetooth or otherwise, enhances the collection of data. Other possibilities include the reporting of driving style, intensity of breaking, orientation of the vehicle, alignment of the vehicle (Winkler et al., 2020), camera and radar pictures and videos (Sjöman & Maunu, 2022).

The use of fixed-sensor techniques is already used for a longer time. Examples are techniques based on image sensors or on magnetic sensors, the last of which is designed as a low-cost alternative to inductive loops (Sjöman & Maunu, 2022). Inductive loops can be used at a fixed location $x$. Single induction-loop detectors, for example, measure the number $\Delta N$ of crossing vehicles $\alpha$ during a time interval $\Delta T$. They also measure the times $t_{0, \alpha}$ and $t_{1, \alpha}$ of the vehicle $\alpha$ reaching and leaving the detector (Helbing, 2001), as illustrated in Figure 4. This gives rise to the definition of the time headway $\Delta t_{\alpha} = t_{0, \alpha} - t_{1, \alpha - 1}$, being the time difference between the front of vehicle $\alpha - 1$ and the front of vehicle $\alpha$ reaching the detector (Figure 5). Also the so-called time clearance can be computed, using $s_{\alpha} = d_{\alpha} - l_{\alpha - 1}$, which is the time equivalence of the clearance, also described as the netto time separations. The time clearance can be seen as the time it takes between the moment the rear end of a vehicle leaves the detector and the moment the following vehicle’s front reaches the detector (Figure 6). The vehicle flow, the number of cars $\Delta N$ that pass the detector per time unit $\Delta T$, is given by:

$$Q(x, t) = \frac{\Delta N}{\Delta T}. \quad (1)$$

Additionally, induction loops can measure vehicle velocities $v_{\alpha}$ and vehicle lengths $l_{\alpha}$ (Helbing, 2001). Using this, we can compute the headway, which is the gross distance $d_{\alpha} = v_{\alpha} \Delta t_{\alpha}$. The earlier-mentioned clearance can be computed by $s_{\alpha} = d_{\alpha} - l_{\alpha - 1}$.

The average velocity can be computed by:

$$V(x, t) = \frac{1}{\Delta N} \frac{\sum_{\alpha=a_1}^{a_1+\Delta N-1} v_{\alpha}}{v_{\alpha}}. \quad (2)$$
The vehicle density, given by the number of vehicles per distance, for example in kilometres, can be computed in a two-fold way (Helbing, 2001). One method is by the use of fluid dynamics. Indeed, by defining the flow \( Q(x,t) \) as the average velocity times the density, we find:

\[
Q(x,t) = \rho(x,t)V(x,t),
\]

which (for positive average velocities) can be rewritten in the form:

\[
\rho(x,t) = \frac{Q(x,t)}{V(x,t)}.
\]

Another method was suggested by May (1990), using the time occupancy \( O(x,t) = \frac{1}{L_D} \sum \Delta t_{a} = \frac{1}{L_D} \sum (t_{a}^1 - t_{a}^0) \), the average vehicle length \( L(x,t) \) and the length of the detector \( L_D \). This way, the vehicle density can be computed by:

\[
\rho(x,t) = \frac{O(x,t)}{L(x,t) + L_D}.
\]
### 3.1.1 Fundamental Diagram

Many times in traffic modelling literature reference is made to the *fundamental diagram*. The fundamental diagram is the diagram that results from the relation between the vehicle flow $Q$, the vehicle density $\rho$ and the average velocity $V$, given in Equation 6 and repeated for clarity in Equation 3 which is referred to as the *fundamental equation of traffic flow* (Bramich et al., 2022):

$$ Q(x, t) = \rho(x, t)V(x, t). $$

(6)

However, when the average velocity is assumed to be a function of density only, we can adjust this equation. Fitting the velocity and density to empirical data, we find the following relation (Helbing, 2001):

$$ Q_e(\rho) = \rho V_e(\rho). $$

(7)

Here, $Q_e$ represents the empirical flow and $V_e$ is the fitted empirical velocity-density relation, which is the relation between velocities and corresponding densities found in empirical data. This fitted empirical velocity is found to be monotonically decreasing with the increase of density (Helbing, 2001), as is visible in Figure 7, for which empirical data is used.

![Empirical velocity-density relations](image)

Figure 7: Empirical velocity-density relations for different definitions of the average velocity, where the line uses the definition given in Equation 2. The average velocity monotonically decreases as the density increases. Figure from Helbing (2001).

The fundamental diagram is therefore the relation between the traffic flow $Q$, the density $\rho$ and the average velocity $V$ as a function of the density $\rho$, as can be seen in Figure 8 for which empirical data were used. Helbing (2001) describes the following properties of the fundamental diagram.

a. For low vehicle densities ($\rho = 0$ to $\rho \approx 30$ in Figure 8), the flow-density relation is almost linear and is determined by multiplying the average velocity in free, non-congested traffic $V_0$ by the density $\rho$, i.e., $Q_e(\rho) \approx \rho V_0$. Figure 8 shows this near-linearity approximately until the density $\rho_{cr}$.

b. As the vehicle density $\rho$ grows ($\rho \approx 30$ and higher densities in Figure 8), $V_e(\rho)$ decreases (as seen in Figure 7), until at some point, given by the so-called jam-density $\rho_{jam}$, the traffic flow vanishes, since then $Q_e(\rho) = \rho V_e(\rho) \approx \rho \cdot 0 = 0$. This density is estimated to be between 120 and 200 vehicles per kilometre per lane, but most realistically lies between 140 and 160 vehicles per kilometre per lane. This corresponds to vehicle densities that are so high that traffic is not able to move anymore and thus $Q = 0$. 

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A fundamental diagram typically consists of two different parts, not continuously connected. Helbing (2001) describes the combination of the two branches as a mirror image of the Greek letter lambda. For low densities, we find the almost linear part as described in item (a). Then, at a certain critical density $\rho_{cr}$, follows a breakdown of the vehicle flow. This breakdown announces a slightly decreasing branch for $\rho > \rho_{cr}$, representing congested traffic. This dichotomy in the fundamental diagram allows for a linear flow-density relation $Q_{sep} = \rho V_{sep}$, separating free traffic from congested traffic, as seen in Figure 8. We can also distinguish a density region $\rho_{c1} \leq \rho \leq \rho_{c2} = \rho_{cr}$ in which both free and congested traffic can occur. $\rho_{c2} = \rho_{cr}$ can be seen from the fundamental diagram, as shown in Figure 8, since it is the point where traffic breaks down and traffic flow suddenly drops. For the density $\rho_{c1}$, we can look at Figure 9. Around a flow of 2200 vehicles per hour, we see that the probability of a breakdown in traffic flow increases and around a flow of 2600 vehicles per hour the increase is drastic. Since the probability of a breakdown increases already around $Q_{c1} \approx 2200$ vehicles per hour, the flow $Q_{c1}$ corresponds to $\rho_{c1}$.

Near the critical density we find states of free traffic, where the flow is around its highest. These states are unstable due to the inevitable transition to the second branch of the fundamental diagram, i.e., due to the breakdown of traffic flow that will occur. In fact, $\rho_{c1}$ and $\rho_{c2}$ are such that:

$$
P\{\text{Congestion} \mid \text{Free traffic}\}(Q_e) =
\begin{cases}
0 & \text{for } Q_e \leq Q_{c1} = Q_e(\rho_{c1}); \\
1 & \text{for } Q_e \geq Q_{c2} = Q_e(\rho_{c2}); \\
\in [0,1] & \text{otherwise},
\end{cases}
$$

where $P\{\text{Congestion} \mid \text{Free traffic}\}(Q_e)$ is the probability of a transition from free traffic to congested traffic, also known as the breakdown probability, as a function of the empirical vehicle flow. When $Q_{c1} \leq Q_e \leq Q_{c2}$, this probability increases monotonically (see Figure 9). At the transition from congested traffic to free traffic, the flows usually lie around $Q_e(\rho)$, for $\rho < \rho_{c2}$. In particular, $\rho \approx \rho_{c1}$.

d. The maximum vehicle flow $Q_{max}$ can be found at medium densities, around the critical density $\rho_{cr}$, since the highest possible flow is often reached when the density is at $\rho_{cr}$. After that, the traffic flow breaks down. In Figure 8, the maximum flow is indeed equal to the flow $Q_{cr}$ at the critical density $\rho_{cr}$, i.e., $Q_{max} = Q_{cr} = Q_e(\rho_{cr})$.

e. The flow-density data in the congested part is widely scattered. In Figure 8, these are the empirical flow values corresponding to density values $\rho > \rho_{cr}$. Therefore, no clear relation can be found between flow, density and average velocity as a function of density when traffic is congested.

f. The fundamental diagram should only be fitted in the range of stable traffic flow, since the model results for congested traffic often vary significantly from the fundamental diagram.

Therefore, the fundamental diagram is a useful tool to show how the flow, i.e., the number of vehicles that passes a certain point per timestep, for example in vehicles per hour, develops as the vehicle density, i.e., the number of vehicles per distance, for example in vehicles per kilometer, increases. To summarise, the flow equals $Q = 0$ when $\rho = 0$, since there is no flow of vehicles when there are no vehicles. Then, as the density increases, the flow almost linearly increases according to $Q \approx \rho V_0$, for the average velocity in free traffic $V_0$, as well. At a certain critical density $\rho_{cr}$, that lies between $\rho_{c1}$ and $\rho_{c2}$, the flow breaks down, as the road becomes too full. This leads into a congested situation for higher densities, for which there is no clear relation anymore between the flow and the density. However, sometimes a slightly decreasing flow can be found as the density increases.

This leads to the characteristic form of the fundamental diagram, sometimes also referred to as the flow-density diagram, due to the relation between flow and density. The diagram hence shows
an increasing line on one side and a decreasing line or scattered values on the other side. The two possible lines are disconnected due to the breakdown of flow.

Figure 8: Empirical fundamental diagram. At low densities, between \( \rho = 0 \) and \( \rho = \rho_{cr} \), the flow increases almost linearly, corresponding to free traffic, after which a clear breakdown of traffic flow occurs. For densities \( \rho > \rho_{cr} \), corresponding to congested traffic, the values of the flow are widely scattered. The maximum flow value is \( Q_{max} = Q_{cr} \) at density \( \rho_{cr} \). The dichotomy of the fundamental diagram can be separated into two parts by \( \rho V_{sep} \). Figure from Helbing (2001).

Figure 9: Probability of breakdown of free traffic as a function of the empirical vehicle flow, measured using different time intervals for averaging the probabilities. Figure from Helbing (2001), after Persaud et al. (1998). The original figure from (Persaud et al., 1998) is given in Appendix A.

3.1.2 Different States of Traffic

As is clearly displayed by the fundamental diagram, there are two states that separate congestion from free flow. These are the states of free traffic and congested traffic. Up to the point of density \( \rho_{c1} \), there is free traffic, while after the critical density \( \rho_{cr} \), there is congested traffic. For \( \rho \in (\rho_{c1}, \rho_{cr}) \), the traffic can both be characterised as free and as congested, depending on what happened before. This dependence on the past is called hysteresis.

In case of free traffic, we cannot give an extensive elaboration, since vehicles can all drive their own speed, limited by either the speed limit or, in case of certain regulations, such as German motorways, by the maximum speed of the vehicle or the maximum speed deemed safe by drivers. Therefore, each particle representing a vehicle moves independent of any other particle in the system.
When we turn to congested traffic, it gets more involved. With a higher density, there are trivially more factors to take into account. Velocities of vehicles have to be adjusted to vehicles in front and on other lanes, acceleration of vehicles is limited by vehicles driving in front and lane-changing behaviour is hampered due to the denser traffic driving on the faster lanes. This leads to different and possibly more interesting results. For example, Nagel & Herrmann (1993) and Nagel & Paczuski (1995) showed that congested traffic drives itself towards the critical density $\rho_{cr}$. This is referred to as self-organised criticality. That is, the traffic tries to achieve the highest density possible without leaving the state of free traffic.

As we elaborate on in section 3.4, many different patterns are found in traffic congestion. Roughly, they can be divided into localised congestion and extended congestion. Localised congestion is found in a relatively small section of the road and propagates either downstream (in case of small perturbations of free traffic, where few vehicles are involved and take the congestion along as they move forward) (Hillegas et al., 1974) or upstream (in case of bigger perturbations of free traffic, where many vehicles behind are influenced by the congestion starting in front of them) (Mika et al., 1969). Extended congestion, on the other hand, covers large road segments and is often present for a long period of time.

Localised congestion is also described as wide-moving jams. Wide-moving jams are a type of congestion that expand over a long section of road. Additionally, the location of these jams moves along the road.

Extended congestion occurs mainly near bottlenecks, where it typically starts to form downstream of bottlenecks and subsequently changes direction to move upstream (Helbing, 2001), as is illustrated by the spatio-temporal representation in Figure 10. An important characteristic of extended congestion is the occurrence of synchronisation between different lanes. Due to the extended congestion, the flow in all lanes becomes similar. Therefore, we also speak of synchronised flow (Kerner & Rehborn, 1996) or, due to the resemblance to the flow in a pipe-system, of one-pipe flow (Daganzo, 2002b).

![Figure 10: Spatio-temporal evolution of the vehicle density near a bottleneck. The bottleneck is located at $x = 0$. The density first increases downstream (positive $x$) of the bottleneck, while propagating upstream after a short period of time. Figure from Helbing (2001).](image)

In fact, Kerner & Rehborn (1996) suggested that we consider a threefold categorisation of congestion types rather than the dichotomy of free traffic and congested traffic. These three categories are that of free flow, synchronised flow and wide-moving jams. This categorisation is well-posed, since traffic in free flow can immediately move to either of the other two categories.

Synchronised flow can again be categorised into three different types.
use the following three categories.

1. In *stationary and homogeneous states* the average speed and the flow rate are stationary and homogeneous for a long time interval. This type of traffic is therefore also referred to as *homogeneous congested traffic* \(^{\text{Helbing, 2001}}\).

2. When only the average speed is stationary, we speak of *homogeneous-in-speed states*. This type of traffic is often found immediately downstream of bottlenecks. The traffic shows a great resemblance to free traffic, but the velocity is often lower than desired, as it takes some time to accelerate, partly influenced by vehicles in front that need to accelerate first. \(^{\text{Helbing, 2001}}\), therefore, suggests that this is an indication of a transition from a congested state to a free state of traffic. He introduces the notion of *recovering traffic* for this traffic state. However, \(^{\text{Helbing, 2001}}\) also mentions the disputability of this interpretation, as \(^{\text{Kerner, 1999}}\) showed that homogeneous-in-speed states can be spread over at least 3 km.

3. *Nonstationary and nonhomogeneous states* are referred to by \(^{\text{Helbing, 2001}}\) as *oscillating congested traffic*, i.e., the congestion is repositioned in an oscillatory fashion.

### 3.1.3 Correlations

In free traffic, not many correlations are found. According to \(^{\text{Neubert et al., 1999}}\), there is almost no statistical dependence between individual vehicle velocities in free traffic. However, until the critical breakdown density \(\rho_{\text{cr}}\), we can find a correlation between density and flow. Indeed, as the density gets higher, more traffic flows through each of the lanes of the motorway.

As opposed to free traffic, among congested traffic there can be found numerous correlations between different parameters. First of all, we see that the flow-density relation gets anticorrelated when the density has reached its critical value. As more and more vehicles are present, congestion will occur, strongly counteracting the flow. There is, on the other hand, a correlation between the average velocity in two neighbouring lanes in congested traffic, since this tends to synchronise. This synchronisation is thought to be caused by vehicles changing to the faster lane, therewith causing the faster lane to slow down, until it is at some point synchronised \(^{\text{Lee et al., 1998}}\). Since there are regulations for trucks driving only on the right lane, or having a lower maximum velocity than cars, thus not overtaking on the left lane, the number of trucks is higher on the right lane than on the left lane in many countries. With trucks being long vehicles, compared to cars or motorcycles, this means the density (in terms of the number of vehicles) is larger on the left lane in this synchronised situation.

### 3.1.4 Some Parameters

Oftentimes, traffic congestion is triggered by an either small or bigger perturbation in free traffic. An example of such a small perturbation is a car braking right in front of another car, causing the latter vehicle to slow down as well. Also unexpected lane-changing is such a cause. Bigger perturbations are found, for instance, when a lane is reduced due to road works or when an accident occurs. Yet, many researchers are still intrigued by the notion of so-called *phantom traffic jams*: traffic congestion that seemingly has no cause. Some researchers claim the existence of such jam formation \(^{\text{Treiterer & Taylor, 1966}}\), while others mention that these types of congestion are in fact caused by above-mentioned small perturbations \(^{\text{Daganzo, 2002a}}\). This does not imply an inevitable cause of congestion with every small perturbation. In fact, it highly depends on the situation, even though conditions can be quite comparable \(^{\text{Helbing, 2001}}\). This attests to and substantiates the instability of free traffic.

Many times, the *propagation velocity* of such perturbations is investigated, due to its high importance to the modelling of traffic congestion. It shows at what rate a traffic jam moves from one place to the other, which can help us to prevent further congestion. It is found that small
perturbations often propagate downstream with a certain velocity \( C(\rho) < V \) that is dependent on the density, with \( V \) the average velocity of vehicles. As vehicles move forward with an average velocity \( V \) after the small perturbation has occurred at some position \( x \), they move at such a speed that the vehicles behind it, that are influenced by the deceleration of their predecessors, also already passed position \( x \) before having to slow down, causing the downstream propagation. Large perturbations, on the contrary, propagate upstream. It turns out that the perturbation velocity \( C_0 \) can be looked upon as some sort of natural constant. It typically lies in the interval \( \tilde{C} = [10, 20] \) km/h, independent of which country. The small deviations between countries depend on safe time clearance, which is the desired time clearance to be in a safe situation. The safe time clearance depends not only on the velocity, but also on individual driver behaviour, governmental legislation and general consensus among different nations or regions. The perturbation velocity also depends on the average vehicle length \( \text{[Mika et al., 1969]} \). Think for example of the difference in vehicle lengths in the United States of America compared to those in India or Europe, or of areas where a lot of freight traffic is present, compared to busy city centres with smaller cars. Although this list is not exhaustive, it must be mentioned that weather conditions are also of great impact. Drivers tend to keep more distance and drive in a more calm manner when it is raining and even drive more cautiously when weather conditions are at its worst, in situations of fog, snow or other severe weather.

The propagating, wide-moving jams have several parameters that tend to be universally used. \( \text{[Helbing (2001)]} \) mentions the following:

1. the propagation velocity \( C_0 \);
2. the density \( \rho_{\text{jam}} \) inside jams;
3. the average velocity and flow inside jams, both approaching 0 as \( \rho \) approaches its maximum value, at which traffic jams are fully saturated;
4. the outflow \( Q_{\text{out}} \) from jams, typically around \( \frac{2}{3} \) of the maximum flow in free traffic, and
5. the density \( \rho_{\text{out}} \) of traffic downstream of a congested area. When the congestion propagates through synchronised flow, this density is dictated by the surrounding traffic.

### 3.2 Bottlenecks

An important aspect in traffic modelling is the bottleneck. One could argue that a bottleneck is a changed situation in which an equal number of vehicles to before has to pass a narrower part of the road, as is the case in Figure 1. However, among others, \( \text{[Daganzo (2002a)]} \) argues that bottlenecks can be any kind of place where traffic is congested, thereby holding up the chasing traffic. The chasing traffic has to wait until the vehicles in front of them accelerate before they can accelerate themselves. Using this definition, there are many causes for bottlenecks. These include accidents, lowered speed limits, road construction, steep slopes - where heavy vehicles have a lower velocity than lighter vehicles -, curves, bad road conditions, local bad weather conditions, cars merging, congestion on off-ramps and even moving bottlenecks, when there is an extremely slow vehicle, as found in, for example, exceptional transport. In the situations that are mainly considered in traffic modelling, a bottleneck situation typically means the partial reduction of lanes (Figure 11) or the merging of on-ramps and off-ramps with motorways (Figure 11).

We find that upstream of bottlenecks, congestion is almost always clearly present. \( \text{[Tilch & Helbing (2000)]} \) have also found out that, near bottlenecks, vehicles tend to keep more distance from each other in congested traffic than in free traffic, i.e., the distance \( d_{\text{free}}(v) \) between two consecutive vehicles as a function of the individual velocity \( v \) in free traffic is smaller than the analogous distance \( d_{\text{congested}}(v) \) in congested traffic, though both functions are monotonically increasing with \( v \). Indeed, Figure 12(a) shows that \( d_{\text{free}}(v) < d_{\text{congested}}(v) \), especially at higher velocities.
\( v > 50 \text{ km/h} \). When the distance from the bottleneck is large enough, the functions seem equal, i.e., \( d_{\text{free}}(v) = d_{\text{congested}}(v) \), as shown in Figure 12(b). Right downstream of a bottleneck, however, a small congested area can be found. These are the aforementioned homogeneous-in-speed states, where the average velocity is stationary, but lower than the maximum possible value, since vehicles still have to accelerate. In this type of traffic, there is often a transition from congested to free traffic, which is exactly what can be expected behind a bottleneck.

**Figure 11:** Example of a situation where a consecutive off-ramp and on-ramp form a bottleneck. \( Q_{\text{cong}} \) in the figure corresponds to the bottleneck flow \( Q_{\text{bot}} \). \( Q_{\text{out}} \) corresponds to the discharge flow \( \tilde{Q}_{\text{out}} \). The downstream front of the bottleneck congestion is pinned at the on-ramp. (a) shows that the upstream front is fixed when the bottleneck capacity \( Q_{\text{bot}} \) (here \( Q_{\text{cong}} \)) is not yet reached. (b) shows that the upstream front propagates upstream as \( Q_{\text{bot}} \) (here \( Q_{\text{cong}} \)) is exceeded, i.e., \( Q_{\text{cong}} < Q'_{\text{cong}} \). Figure from Helbing et al. (2009).

**Figure 12:** Average vehicle distances \( d \) as a function of individual vehicle velocity \( v \) for free traffic and congested traffic, from empirical data. (a) shows that vehicles keep increased distances in congested traffic flow (dashed line) in contrast to free flow (regular line) near bottlenecks. (b) shows that no significant difference in distance between vehicles is found between free and congested traffic at a distance from the bottleneck that is large enough. Figure from Helbing (2001), after Tilch & Helbing (2000), with lines fitted for clarity.

An interesting feature of bottlenecks to look at is the **bottleneck flow** \( Q_{\text{bot}} \) (\( Q_{\text{cong}} \) in Figure 11), which is the flow of traffic in a congested part of a bottleneck. These bottleneck flows are not uniquely determined. They might, for example, depend on the flow coming in from on-ramps, i.e., the flow of vehicles that enters a motorway by merging into the right-most lane, or the flow leaving via off-ramps. Directly behind a bottleneck, we find the flow of vehicles leaving the congested flow. This flow is called the **discharge flow**, which we denote by \( \tilde{Q}_{\text{out}} \) (\( Q_{\text{out}} \) in Figure 11). Naturally, since traffic is recovering to a free state, it follows that \( \tilde{Q}_{\text{out}} \geq Q_{\text{bot}} \). Moreover, according to Helbing (2001), the discharge flow of congestion at bottlenecks appears to be higher than the outflow \( Q_{\text{out}} \) of localised congestion, described in section 3.1.2, characterised by wide-moving jams, i.e., \( \tilde{Q}_{\text{out}} \geq Q_{\text{out}} \).
In the extended congestion that is often found at bottlenecks, the downstream end of the congestion tends to be fixed (also referred to as pinned) at the downstream end of the bottleneck region (Figure 11a). In other words, the downstream front of the congestion is fixed at a clear position on the road. The upstream end, instead, extends the congested area by moving upstream when the maximum bottleneck capacity $Q_{\text{bot}}$ is exceeded (Figure 11b). The congestion is narrowed again when the inflow of traffic into the congestion is lower than the capacity $Q_{\text{bot}}$.

### 3.3 Various Traffic Models

In traffic modelling, already by 2001 there were over a hundred different models describing the behaviour of traffic or individual vehicles (Helbing, 2001). These include fairly realistic models, but also purely theoretical models for studying properties and possible phenomena occurring in traffic. These theoretical models are commonly based on a circular ring road, without any on- or off-ramps, where traffic periodically passes a certain spatial point on the ring road. By using the method of explanation by relaxation (Hindriks, 2013), a lot can be learnt about traffic behaviour. The more realistic models can generally be divided into four different classes of models:

1. **Microscopic models**, based on molecular dynamics, also called follow-the-leader models;
2. (lattice gas) cellular automata models, dividing space into several blocks, similar to a grid or lattice;
3. **Macroscopic models**, based on fluid dynamics, and

In the following paragraphs, the nature of each of these four classes of models will be explained and some relevant variations are added.

#### 3.3.1 Microscopic Follow-the-Leader Models

Microscopic models are characterised by two important aspects. On the one hand, there is a microscopic level of modelling, which means that every vehicle $\alpha$ is taken into account. One can compare this to molecular dynamics by considering every vehicle to be a molecule and setting some constraints for the dynamics. On the other hand, this type of models is referred to as follow-the-leader models. This is due to the assumption that the driving behaviour of vehicle $\alpha$ is mainly influenced by the driving behaviour of the vehicle $\alpha-1$ in front of it, called the leading vehicle. Among the factors that contribute to this behaviour, we first and foremost consider the acceleration behaviour of vehicles.

In order to arrive at microscopic models for traffic modelling step-by-step, let us first consider Newton’s second law of motion:

$$m_\alpha \frac{d^2 x_\alpha(t)}{dt^2} = \sum_{\beta \neq \alpha} F_{\alpha\beta}(t),$$  

(9)

where $m_\alpha$ describes the mass of an object $\alpha$ and $x_\alpha(t)$ the location of object $\alpha$ at time $t$, hence $\frac{d^2 x_\alpha(t)}{dt^2}$ the acceleration of object $\alpha$ at time $t$. The summation describes forces that interact between object $\alpha$ and other objects $\beta$ at time $t$.

A **driven** system is a system in which particles are driven by some internal (in case of a self-driven system) or external force, such as fluids influenced by pressure gradients and boundary forces or sand falling through a pipe system (Helbing, 2001). We can apply the relation in Equation 9 to a driven system, but we need to add several factors to fit the system. For example, we should consider driving forces $F_0(x,t)$, such as gravity. Also friction forces $F_{\text{fr}}(t) = -\gamma_\alpha v_\alpha(t)$ should be added, where $\gamma_\alpha$ is a friction coefficient belonging to object $\alpha$ and $v_\alpha$ is the velocity of the same
models, by defining: Helbing (2001) suggests scaling the system, such as vehicles, can be resolved by assuming a distribution of masses. However, Helbing (2001), the masses presence of internal driving forces, opposed to external driving forces, by replacing the energy for the vehicle’s engine, or a battery, with the same purpose. We can provide the internal energy reservoir. In the case of motorway traffic, vehicles contain a fuel tank, providing α driven systems, we must take into account that the driving force is not from outside the system, by Helbing (2001). However, since we are considering self-driven systems, rather than regular This equation already contains many mechanisms that are also present in traffic, as elaborated on 9 form the following equation for a driven, microscopic system: fluctuations that reflect thermal interactions with the environment. These additions to Equation α driven systems, we must take into account that the driving force is not from outside the system, by Helbing (2001). However, since we are considering self-driven systems, rather than regular This equation already contains many mechanisms that are also present in traffic, as elaborated on.

\[
m_a \frac{d^2}{dt^2} x_a(t) = F_0(x_a(t), t) - \gamma_a v_a(t) + \sum_{\beta \neq \alpha} F_{\alpha\beta}(t) + \xi_a(t). \tag{10}
\]

This equation already contains many mechanisms that are also present in traffic, as elaborated on by Helbing (2001). However, since we are considering self-driven systems, rather than regular driven systems, we must take into account that the driving force is not from outside the system, but from within each object \( \alpha \). As Schweitzer et al. (1998) stated, this implies the presence of an internal energy reservoir. In the case of motorway traffic, vehicles contain a fuel tank, providing the energy for the vehicle’s engine, or a battery, with the same purpose. We can provide the presence of internal driving forces, opposed to external driving forces, by replacing \( F_0(x_a(t), t) \) in Equation (10) by an internal driving force \( F_0^\alpha(t) \), depending on the object \( \alpha \). Now, according to Helbing (2001), the masses \( m_\alpha \) are often not well-defined. The heterogeneity of the particles in the system, such as vehicles, can be resolved by assuming a distribution of masses. However, Helbing (2001) suggests scaling \( m_\alpha \) instead, to finally find an equation for microscopic traffic models, by defining:

\[\begin{align*}
\text{a. } & \tau_\alpha := \frac{m_\alpha}{m_\alpha}, \text{ so that } \gamma_\alpha = \frac{m_\alpha}{\tau_\alpha}, \\
\text{b. } & F_0^\alpha(t) := \gamma_\alpha v_\alpha^0(t), \text{ where } v_\alpha^0(t) \text{ is the desired velocity of } \alpha, \text{ i.e., the velocity at which the driver of vehicle } \alpha \text{ wants to drive, taking into account legislative restrictions;}
\text{c. } & F_{\alpha\beta}(t) := m_\alpha f_{\alpha\beta}(t), \text{ where } f_{\alpha\beta}(t) \text{ describes the accelerations of } \alpha \text{ as interactions with another object } \beta, \text{ which can, among other causes, be due to unexpected braking, but also due to driver behaviour of object } \alpha, \text{ and}
\text{d. } & \xi_\alpha(t) := \gamma_\alpha \xi_\alpha(t).
\end{align*}\]

This yields the following equation:

\[
m_\alpha \frac{d^2}{dt^2} x_\alpha(t) = \frac{m_\alpha}{\tau_\alpha} v_\alpha^0(t) - \frac{m_\alpha}{\tau_\alpha} v_\alpha(t) + \sum_{\beta \neq \alpha} m_\alpha f_{\alpha\beta}(t) + \frac{m_\alpha}{\tau_\alpha} \xi_\alpha(t). \tag{11}
\]

Dividing out the mass \( m_\alpha \) and replacing \( \frac{d^2}{dt^2} x_\alpha(t) \) with \( \frac{d}{dt} v_\alpha(t) \), finally gives:

\[
\frac{d}{dt} v_\alpha(t) = \frac{v_\alpha^0(t) - v_\alpha(t) + \xi_\alpha(t)}{\tau_\alpha} + \sum_{\beta \neq \alpha} f_{\alpha\beta}(t). \tag{12}
\]

In this equation, \( \tau_\alpha \) can be considered as a relaxation time. We see that the term \( \frac{v_\alpha^0(t) - v_\alpha(t)}{\tau_\alpha} \) changes as the velocity changes. When the positive velocity \( v_\alpha(t) \) of \( \alpha \) has reached the desired velocity \( v_\alpha^0(t) \), the term vanishes, while it contributes to large changes when the difference between \( v_\alpha(t) \) and \( v_\alpha^0(t) \) is large, i.e., when \( 0 \leq v_\alpha(t) < v_\alpha^0(t) \). However, this change is amended by the fluctuations \( \xi_\alpha(t) \) and the interactions with other vehicles \( f_{\alpha\beta}(t) \).

This forms the basis for microscopic follow-the-leader models. We consider the desired velocity \( v_\alpha^0 \), assumed to be equal at all times - think of the speed limit or another velocity that fits the engine and the size and quality of the vehicle -, the velocity of vehicle \( \alpha \), given by \( v_\alpha(t) \), and the velocity fluctuations \( \xi_\alpha(t) \), that are a representation of changes in velocity by the driver, for example occurring when a driver does not notice their vehicle slowing down due to a different gradient or when a driver suddenly brakes after seeing a bird fly over. The interactions between different particles are given shape by only considering the interaction in terms of acceleration.
between vehicle $\alpha$ and leading vehicle $\alpha - 1$. Hence, microscopic follow-the-leader models are characterised by the following equation:

$$\frac{dv_{\alpha}(t)}{dt} = \frac{v_{\alpha}^0 + \xi_{\alpha}(t) - v_{\alpha}(t)}{\tau_{\alpha}} + f_{\alpha, (\alpha - 1)}(t), \quad (13)$$

where $f_{\alpha, (\alpha - 1)}(t)$ describes the interactions between $\alpha$ and $\alpha - 1$. Usually, these interactions depend on the following quantities:

- a. the relative velocity $\Delta v_{\alpha}(t) := v_{\alpha}(t) - v_{\alpha - 1}(t)$;
- b. the velocity $v_{\alpha}(t)$, since a vehicle adjusts its velocity in order to keep safe distance (depending on velocity) from the leading vehicle, and
- c. either the headway, which is the distance between two vehicles, given by $d_{\alpha}(t) := x_{\alpha - 1}(t) - x_{\alpha}(t)$, or the clearance, which is the distance between the rear end of the leading vehicle and the front of vehicle $\alpha$ (see Figure 3), given by $s_{\alpha}(t) := d_{\alpha}(t) - l_{\alpha - 1}$, with $l_{\alpha - 1}$ being the length of vehicle $\alpha - 1$.

Notice that the desired velocity $v_{\alpha}^0$ does not anymore depend on time, since (in general) this velocity stays the same at all times.

To simplify Equation 13 one could assume identical driving behaviour, so that $v_{\alpha}^0 = v_0$, $\tau_\alpha = \tau$ and $f_{\alpha, (\alpha - 1)}(t) = f(s_{\alpha}(t), v_{\alpha}(t), \Delta v_{\alpha}(t))$. Also, fluctuations could be neglected, to form the equation:

$$\frac{dv_{\alpha}(t)}{dt} = \frac{v_{\alpha} - v_{\alpha}(t) + \tau f(s_{\alpha}(t), v_{\alpha}(t), \Delta v_{\alpha}(t))}{\tau}, \quad (14)$$

This allows us to introduce a velocity $v^e$ that depends on traffic through $s_{\alpha}$, $v_{\alpha}$, $\Delta v_{\alpha}$ and $v_0$ and to which the driver in vehicle $\alpha$ tries to adapt:

$$v^e(s_{\alpha}, v_{\alpha}, \Delta v_{\alpha}) := v_0 + \tau f(s_{\alpha}, v_{\alpha}, \Delta v_{\alpha}). \quad (15)$$

As opposed to Equation 13 this gives a simplified microscopic follow-the-leader model:

$$\frac{dv_{\alpha}(t)}{dt} = \frac{v^e(s_{\alpha}, v_{\alpha}, \Delta v_{\alpha})(t) - v_{\alpha}(t)}{\tau}. \quad (16)$$

Variations on this model include time delays $\Delta t$, where a widely used value is $\Delta t = 1.3$ seconds. According to Dro˙zdziel et al. (2020), the reaction time depends on the type of stimulus, but generally lies between 0.7 and 1.5 seconds. This is the complete process from perceiving a signal and recognising and identifying the object to the time of physically responding by moving the leg (in case of cars) and the time for the vehicle to respond to the leg movement. Dro˙zdziel et al. (2020) found a resulting reaction time of 1.34 seconds. This is not to confuse with only the human part of the reaction time, as investigated by Podoprigora et al. (2020). This value is lower and typically lies between 0.67 and 0.95 seconds for the age groups from 18 to 60 (Podoprigora et al. 2020). This type of traffic models is part of so-called delay differential equations (Helbing 2001).

The described model in Equation 16 does not describe the driving behaviour of a particular vehicle $\alpha$ only, since the behaviour of vehicle $\alpha$ depends on its leading vehicle $\alpha - 1$, which has its own driving behaviour. Therefore, many models base the adaptation of the velocity of vehicle $\alpha$ on the headway $d_{\alpha}$ between two vehicles, rather than on the velocity. This ensures that the driver behaviour of vehicle $\alpha$ now only depends on the location of the leading vehicle, which does not take into account its driving behaviour. Therefore, instead of $v^e(s_{\alpha}, v_{\alpha}, \Delta v_{\alpha})$, they use $v^e_{\prime}(d_{\alpha})$, the so-called distance-dependent velocity. It is sometimes referred to as the optimal velocity, since it is optimal for large headways $d_{\alpha}$.
However, this approach also knows some deficiencies, due to the aforementioned optimal velocity not taking into account the velocity of the leading vehicle. This way, the model causes accidents when fast vehicles approach other vehicles at standstill. Therefore, great care must be taken when using this approach. Solutions taking this issue into account have been proposed by numerous researchers.

### 3.3.2 Cellular Automata Models

The notion of cellular automata models has not been around for a long time, considering that traffic modelling started already by Greenshields’ studies in 1935 (Greenshields, 1935). The first appearance of a cellular automata approach to traffic modelling was in 1986, when Cremer & Ludwig (1986) introduced a model for the dynamic process of traffic flow through urban networks, in order to simulate the movement of cars. They made use of Boolean operators to include movements such as driving at a constant speed, lane changing, overtaking, decelerating and accelerating, queuing and turning at intersections. Already then, they mentioned the most important aspect of cellular automata models: “The computational requirements are rather low with respect to both storage and computation time making it possible to simulate large traffic networks on personal computers.” The importance of efficiency of a model is high, but next to efficiency, cellular automata models prove to exhibit a lot of complex dynamic behaviour. This makes the models highly-functioning in addition to performing efficiently.

Cellular automata models have several important and defining properties. Their main property is the discretisation of space. The road is divided into identical space intervals $j$ of size $\Delta x$, called lattice sites or cells. Some models also make a distinction between different lanes on a motorway (Zhang et al., 2018). Due to their great resemblance to lattices or grids, these models are often referred to as lattice models. Another property is the timestep $\Delta t$, at which the model is updated. At times $t = i\Delta t$, with $i = 0, 1, 2, 3, \ldots$, every lattice site in the discretised space is updated, with the same rules on updating for every site. For each of these cells, the update rules only affect a small number of neighbouring cells. Moreover, each site can only have a finite number of possible states $g(x)$.

Due to these properties, cellular automata models are a perfect fit for parallel computing. The idea of parallel computing is that a problem that takes $t$ time to be solved, can potentially be split up into $p \in \mathbb{N}$ similar smaller problems, that can all be solved in $\frac{t}{p}$ time. If this is indeed possible and all sub-problems are solved in parallel on $p$ different processors, the whole problem is then also solved in $\frac{t}{p}$ rather than $t$ time (Golub & Ortega, 1993). In the case of cellular automata models, the update of each cell is considered a subproblem of the overall problem of updating the traffic flow on a road. This way of computing does require some dependency conditions, since the state of a cell can, for example, not depend on the state of another cell that is currently being computed. Therefore, Bernstein (1966) introduced the following conditions. Let $P_i$ and $P_j$ be problems, $I_i$ and $I_j$ their respective input variables, i.e., to solve the problem, we need the sets $I_i$ and $I_j$, respectively, and $O_i$ and $O_j$ their respective output variables, in our case the updated state $g(x)$ of a lattice site. Then, in order to apply parallel programming, the following conditions should hold:

- a. $I_i \cap O_j = \emptyset$
- b. $I_j \cap O_i = \emptyset$ and
- c. $O_i \cap O_j = \emptyset$.

Indeed, this ensures a very fast simulation when numbers get large. However, the simplifications that are applied to get this model, show less detail than the microscopic follow-the-leader models that were proposed in section 3.3.1. Still, due to complex dynamic behaviour still being possible to be made visible after simulation and due to the fast computation, this model is often favoured...
above microscopic follow-the-leader models.

According to Helbing (2001), the Nagel-Schreckenberg model (Nagel & Schreckenberg, 1992), which can be characterised as a cellular automata model, is an “extremely compact and elegant” model. Nagel and Schreckenberg divide the road into cells \( j \), each having length \( \Delta x \), and discretise time \( t \) using intervals \( i \) that each have a duration of \( \Delta t = 1 \) second. Each cell is assigned a Boolean value describing whether it is occupied or not occupied (empty) by exactly one vehicle. If it is occupied, then the vehicle’s speed at time interval \( i \) is given by \( v_i = \hat{v}_i \frac{\Delta t}{\Delta x} \), with \( \hat{v}_i \in \{0, 1, \ldots, \hat{v}_{\text{max}}\} \) the nondimensionalised integer velocity of the vehicle occupying cell \( i \), for a (scaled) desired velocity \( \hat{v}_{\text{max}} = v_0 \frac{\Delta t}{\Delta x} \), rounded to the nearest integer, and \( v_0 \) the desired velocity corresponding to \( \hat{v}_{\text{max}} \) in Equation (14). Usually, the values \( \Delta x = 7.5 \) metres and \( \hat{v}_{\text{max}} = 5 \) are chosen (Nagel & Schreckenberg, 1992).

Besides the discretisation, the Nagel-Schreckenberg model has four update rules. At every timestep, the following rules have to be obeyed consecutively for every vehicle (Helbing, 2001; Nagel & Schreckenberg, 1992).

1. **(acceleration)** If the maximum velocity \( v_0 \) of a vehicle has not yet been reached, increase the velocity to \( \hat{v}_i' = \hat{v}_i + 1 \). This is a difference in velocity of one cell per timestep, i.e., \( \frac{\Delta x}{\Delta t} \), which therefore gives an acceleration of \( \frac{\Delta x}{(\Delta t)^2} \).

2. **(deceleration)** If the headway, i.e., the number of cells to the next vehicle ahead, is \( \hat{d}_i \leq \hat{v}_i' \), then the velocity is reduced to \( \hat{v}_i'' = \hat{d}_i - 1 \). If \( \hat{d}_i > \hat{v}_i' \), then \( \hat{v}_i'' = \hat{v}_i' \). This rule makes sure that a vehicle does not collide with or overtake the vehicle ahead in the next step, as it cannot bridge the distance between the two vehicles in one timestep.

3. **(randomisation)** The velocity is reduced to \( \hat{v}_i''' = \hat{v}_i'' - 1 \) if \( \hat{v}_i'' - 1 \geq 0 \) or set to \( \hat{v}_i''' = 0 \) otherwise, with probability \( p \). This probability \( p \) is called the slowdown probability, to which Nagel & Schreckenberg (1992) have often given the value \( p = 0.5 \). In city traffic, the value \( p = 0.2 \) is often chosen, in combination with \( \hat{v}_{\text{max}} = 2 \). The slowdown probability reflects individual fluctuations in acceleration, since some drivers are late with accelerating, referred to as imperfect driving by Helbing (2001).

4. **(motion)** Move the vehicle forward by \( \hat{v}_{i+1} = \hat{v}_i''' \) cells.

The order of the consecutive rules is important, as is illustrated in Appendix C. Instead of these four rules, one can merely use the rule of motion and combine the other rules into:

\[
\hat{v}_{i+1} = \max\{0, \min\{\hat{v}_{\text{max}}, \hat{d}_i - 1, \hat{v}_i + 1\} - \xi_i^{(p)}\},
\]

(17)

to find one rule for the update of the velocity \( \hat{v}_{i+1} \) in terms of the previous velocity \( \hat{v}_i \). Here, \( \xi_i^{(p)} \) is a Boolean random variable taking a Bernoulli distribution, i.e.:

\[
\xi_i^{(p)} = \begin{cases} 
1 & \text{w.p. } p; \\
0 & \text{w.p. } 1 - p.
\end{cases}
\]

(18)

As a slight variation to this model, a slow-to-start rule could be added, modelling that vehicles tend to get to speed slower when they are at a standstill. This can be done by using a higher value for \( p \), some value \( p_0 \), when \( \hat{v}_i = 0 \). This could be useful when modelling situations where traffic starts moving again, after having got to a stop in a congestion.

Another variation adds a cruise control rule, which also affects the slowdown probability \( p \), now by setting \( p = 0 \) when \( \hat{v}_i = \hat{v}_{\text{max}} \). Also, rules may be added to model a portion of self-driving autonomous vehicles, which know fewer fluctuations, due to their deterministic driving behaviour. In general, many cellular automata models are based on the Nagel-Schreckenberg model by adding or changing update rules, though they need not be.
3.3.3 Macroscopic Models

Whereas microscopic models focus on individual vehicles and their impact on the system as a whole, macroscopic models simulate the dynamics of a traffic system, looking at the spatial vehicle density $\rho(x,t)$ per lane and the average velocity $V(x,t)$, averaged over a number of vehicles. These values are functions of the location $x$ and time $t$. Since macroscopic models do not have to consider every single vehicle, they are more time-efficient than microscopic traffic models. However, with the rise of cellular automata models, a new scope of efficiency is reached, that also macroscopic traffic models do not attain. Still, they are used for several reasons. First of all, it appears that results of macroscopic models are usually in accordance with empirical data. Also, analytical investigation of these models is fairly easy, on- and off-ramps can be dealt with in a simple manner and there is a possibility to deal with multiple lanes without changing the model too much. Effectively, one can consider a one-lane model and allow for overtaking.

All macroscopic models are based around the idea that, not considering on- and off-ramps and neglecting accidents or vehicle breakdowns, the number of vehicles on the motorway stays constant. Defining the flow $Q$ as in Equation 6, this gives rise to the so-called continuity equation:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = 0. \quad (19)$$

We now introduce the material derivative, also known as the substantial derivative or, most-fittingly, the hydrodynamic derivative, as given in Equation 20.

$$\frac{DV}{Dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x}. \quad (20)$$

Equation 20 describes the changes in time as the spatio-temporal system moves with velocity $V(x,t)$. One can see the temporal changes as though they move along the system with a velocity $V(x,t)$. Using the hydrodynamic derivative and the flow-density relation given in Equation 6, we can rewrite Equation 19 as:

$$\frac{DV}{Dt} \rho(x,t) = -\rho(x,t) \frac{\partial V(x,t)}{\partial x}, \quad (21)$$

where we used that $\frac{\partial \rho(x,t) V(x,t)}{\partial x} = V(x,t) \frac{\partial \rho(x,t)}{\partial x} + \rho(x,t) \frac{\partial V(x,t)}{\partial x}$. From this, we can conclude that density increases in time when velocity decreases along the road. The reverse also holds. In order to use the macroscopic approach, it is of great importance to specify the traffic flow $Q(x,t)$. Several models have been suggested for this specification.

The Lighthill-Whitham model, also known as the Lighthill-Whitham-Richards model or LWR model, is the most popular version, according to Helbing (2001). They proposed the flow to be a function of the density:

$$Q(x,t) = Q_e(\rho(x,t)) = \rho(x,t) V_e(\rho(x,t)). \quad (22)$$

Here, $Q_e$ represents the fundamental diagram and $V_e$ the equilibrium velocity-density relation, i.e., the density-dependent velocity for which $\rho V_e(\rho) = Q_e(\rho)$. They can be fit to empirical data, for which several functions have already been proposed. For example, the measurements by Greenshields (1935) suggested a linear relation given by:

$$V_e(\rho) = V_0(1 - \frac{\rho}{\rho_{jam}}), \quad (23)$$

where $V_0$ and $\rho_{jam}$ are the aforementioned average velocity in free traffic and the density in which the flow vanishes, respectively.
Applying Equation 22 to Equation 19, the continuity equation, and using the fact that \( \frac{\partial \rho V}{\partial x} = V(\rho) \frac{\partial \rho}{\partial x} + \rho \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial x} \), we find the equation:

\[
\frac{\partial \rho}{\partial t} + (V(\rho) + \rho \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial x}) \frac{\partial \rho}{\partial x} = 0,
\]

which can be rewritten in the form:

\[
\frac{\partial \rho}{\partial t} + C(\rho) \frac{\partial \rho}{\partial x} = 0,
\]

describing the propagation of kinematic waves, where the propagation velocity is given by:

\[
C(\rho) = V(\rho) + \rho \frac{\partial V}{\partial \rho}.
\]

This equation gives us insight into the motion of kinematic waves in vehicle traffic. Equation 23 implies that \( \frac{\partial V}{\partial \rho} \leq 0 \), from which it follows that \( C(\rho) \leq V(\rho) \), which means that the kinematic waves propagate slower than the average velocity of vehicles. Due to the non-linearity of the wave equation given in Equation 25 and the kinematic waves propagating slower than the average velocity of vehicles, it gives rise to shock fronts, which are situations where large discontinuous jumps in density can be found (see Figure 13).

Figure 13: Spatio-temporal representation of vehicle density, showing shock waves occurring in the LWR model on a circular road. The initial condition at time \( t = 0 \) minutes is a smooth sinusoidal wave in \( x \), but eventually different shock fronts occur, such as the front starting at approximately \( x = 10 \) km. The shock fronts propagate upstream at a constant velocity, shown by the linearity of this shock front in the spatio-temporal plane. Figure from Helbing (2001).

The development of shock fronts makes it more difficult to numerically solve the Lighthill-Whitham model. Therefore, several variants of this model have been developed that add a small diffusion term, slowing down vehicles when the density gets higher. This causes a bigger spread of vehicles, smoothing the shock fronts. An example of such a diffusion term is the following adjustment for \( Q \):

\[
Q(x, t) = Q_e(\rho(x, t)) - D \frac{\partial \rho}{\partial x},
\]

adding a diffusion term to the flow function defined by Lighthill and Whitham (Equation 22). Applying this to the continuity equation 19 by using Equation 25 yields:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial Q_e(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + C(\rho) \frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial \rho}{\partial x} \right).
\]

Here, \( D \) is a positive diffusion constant. The diffusion term \( \frac{\partial}{\partial x} \left( D \frac{\partial \rho}{\partial x} \right) \) gives a large contribution only if \( \frac{\partial^2 \rho}{\partial x^2} \) is large, i.e., at a location where the curvature of the density as a function of \( x \) is large.
Hence, only in case of shock fronts, which have large local curvature, the diffusion term will contribute to a significant change in density, thus smoothing the shock fronts.

Using the above, the so-called Burgers equation was found [Helbing, 2001], given by:

$$\frac{\partial C(x,t)}{\partial t} + C(x,t) \frac{\partial C(x,t)}{\partial x} = D \frac{\partial^2 C(x,t)}{\partial x^2}, \quad (29)$$

with $C(x,t)$ the propagation velocity given by $C(x,t) = V_0(1 - \frac{2\rho(x,t)}{\rho_{jam}})$. This is considered the most simple adjustment to the Lighthill-Whitham model by adding a diffusion term, hereby avoiding the occurrence of shock waves by taking $D$ large enough. Although the addition of the diffusion term makes it more difficult to solve this equation, it can still be solved exactly, since it turns out to be related to the linear heat equation:

$$\frac{\partial \Psi(x,t)}{\partial t} = D \frac{\partial^2 \Psi(x,t)}{\partial x^2}. \quad (30)$$

According to [Helbing, 2001], Whitham showed that the Cole-Hopf transformation can be used to relate the two equations:

$$C(x,t) = -\frac{2D}{\Psi(x,t)} \frac{\partial \Psi(x,t)}{\partial x}. \quad (31)$$

An important aspect of the Lighthill-Whitham model builds on the assumption that $Q(x,t) = Q_c(\rho(x,t))$, i.e., there are empirical data points that specify the traffic flow in terms of density. However, in the fundamental diagram, we can see that there is no stability in the congested area right after the critical density $\rho_{cr}$ (Figure 8) and thus it makes no sense to define an empirical relation between flow and density. Therefore, many generalisations and alternative approaches have been proposed, to overcome this issue.

There are, however, also macroscopic models that are not based on the Lighthill-Whitham model. An example is the Weidlich-Hilliges model, that makes use of a spatial discretisation of the continuity equation [19] and shows similarities with cellular automata models. This model divides the road into cells $j$ that each have length $\Delta x$ and makes use of the density function $\hat{\rho}(j, t) = \rho(j/\Delta x, t)$. Equation [19] is then discretised in terms of space, such that the following equation arises:

$$\frac{\partial \hat{\rho}(j, t)}{\partial t} + \frac{\hat{Q}(j, t) - \hat{Q}(j - 1, t)}{\Delta x} = 0. \quad (32)$$

The model uses that $\hat{Q}(j, t) = \hat{\rho}(j, t)\hat{V}(j + 1, t)$, which implies that a vehicle in cell $j$ adapts its velocity to the velocity of the vehicle in the next cell, $j + 1$. This assumes that drivers anticipate on what happens in front of them, which is an assumption on driving behaviour. [Helbing, 2001] points out that the Weidlich-Hilliges model is the discrete analogy of Equation 28 with density-dependent diffusion function $D(\rho) = \frac{\Delta x}{\tau(\rho)}(V_c(\rho) - \rho \frac{\partial V_c}{\partial \rho})$.

In general, many macroscopic models are based on the density equation:

$$\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x} = -\rho \frac{\partial V}{\partial x} + D(\rho) \frac{\partial^2 \rho}{\partial x^2} + \xi_1(x,t), \quad (33)$$

and the velocity equation:

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = -\frac{1}{\rho} \frac{\partial P(\rho)}{\partial \rho} \frac{\partial \rho}{\partial x} + v(\rho) \frac{\partial^2 V}{\partial x^2} + \frac{1}{\tau(\rho)}(V_c(\rho) - V) + \xi_2(x,t), \quad (34)$$

where the differences lie in determining the different parameters and functions, with:

a. $D(\rho)$ the density-dependent diffusion coefficient, smoothing the shock fronts;
b. \( \xi_1(x, t), \xi_2(x, t) \) additional source terms, describing the source of individual fluctuations in acceleration due to individual acceleration behaviour;

c. \( P(\rho) =\rho(x, t)\theta(x, t) \) the traffic pressure, where \( \theta(x, t) \) describes the velocity variance, i.e., the variance of the different velocities of vehicles;

d. \( v(\rho) \) a viscosity-like coefficient for the velocity equation, with a similar effect as the diffusion coefficient;

e. \( \tau(\rho) \) the relaxation time, reflecting the time it takes for the model to be significantly perturbed, and

f. \( V_e(\rho) \) the equilibrium velocity or the fitted empirical velocity-density relation.

As can be seen in Equation 21, the Lighthill-Whitham model merely uses the density equation (33) with both the diffusion term \( D(\rho) \) and the additional source term \( \xi_1(x, t) \) set to 0. When a diffusion coefficient is added (Equation 28), still the density equation can be used, now with \( D(\rho) = \frac{2-\rho}{\rho} D \) and \( \xi_1(x, t) = 0. \)

### 3.3.4 Gas-Kinetic Models

Gas-kinetic models show large similarities to the standard macroscopic models. However, their continuity equation differs. The basis for gas-kinetic models is the phase-space density equation, given by:

\[
\tilde{\rho}(x, v, t) = \rho(x, t)\tilde{P}(v, x, t).
\]  
(35)

Here, \( \rho(x, t) \) is the vehicle density at location \( x \) and time \( t \) and \( \tilde{P}(v, x, t) \) is the distribution of velocities \( v \) at location \( x \) and time \( t \), i.e., \( \tilde{P} \) describes how fast different vehicles move at a given location \( x \) and time \( t \). Together they factor to the phase-space density \( \tilde{\rho}(x, v, t) \). Due to Equation 35, gas-kinetic models show a different relation to individual vehicles than standard macroscopic models, since the distribution \( \tilde{P} \) reflects the different velocities that vehicles adopt. As in the macroscopic case, though, we can find a continuity equation, now in terms of the phase-space density:

\[
\frac{D_v\tilde{\rho}}{Dt} = \frac{\partial \tilde{\rho}}{\partial t} + v \frac{\partial \tilde{\rho}}{\partial x} = \left( \frac{d\tilde{\rho}}{dt} \right)_{\text{acc}} + \left( \frac{d\tilde{\rho}}{dt} \right)_{\text{int}},
\]  
(36)

where the most significant point of attention is the presence of the acceleration and interaction terms. The changes of the phase-space density over time in the acceleration behaviour of drivers and the interaction between different drivers together form the phase-space density system that moves with velocity \( v \). The difficulty of gas-kinetic models is in describing the two terms.

Prigogine, who first came up with the above model [Helbing, 2001] proposed to describe the acceleration behaviour by relaxing the velocity distribution \( P(v, x, t) \) to a distribution \( \tilde{P}_0(v) \). This is the desired distribution, which shows the distribution of desired velocities among drivers, since not every driver wants to drive at the same velocity. According to Tilch [Helbing, 2001], one can measure the distribution \( \tilde{P}_0 \) by using the velocity distribution of vehicles with large clearances. The relaxation gives rise to the following acceleration behaviour term:

\[
\left( \frac{d\tilde{\rho}}{dt} \right)_{\text{acc}} = \frac{\rho(x, t)}{\tau(\rho(x, t))} \left( \tilde{P}_0(v) - \tilde{P}(v, x, t) \right).
\]  
(37)

Here, \( \tau(\rho) \) is a relaxation time that depends on the density. This is the time it approximately takes for a perturbation to come through in the system.
To describe the interactions between different vehicles, one should consider the following difference of integrals:

\[
\left( \frac{d\tilde{p}}{dt} \right)_{\text{int}} = \int_{w>v} (1 - \tilde{p}(\rho)) |w - v| \tilde{p}(x, w, t) \tilde{p}(x, v, t) \, dw \\
- \int_{w<v} (1 - \tilde{p}(\rho)) |w - v| \tilde{p}(x, w, t) \tilde{p}(x, v, t) \, dw.
\]  

Here, \( \tilde{p}(\rho) \) represents the probability that a vehicle with velocity \( w \) overtakes a vehicle with velocity \( v \) in case that \( w > v \) and vice versa if \( v > w \). This probability is density-dependent, since it is less apparent to overtake when the road is full of vehicles than when the only vehicle around is a truck one wants to overtake. Hence, \( 1 - \tilde{p}(\rho) \) is the probability that a vehicle with velocity \( w \) stays behind a vehicle with velocity \( v \). In case the vehicle is faster than the velocity \( v \) at which the system moves, i.e., \( w > v \), this means the vehicle has to slow down. Therefore, it gives a change in phase-space density, since there is an interaction between the vehicles with velocities \( w \) and \( v \). On the contrary, when \( w < v \), i.e., the vehicle with velocity \( w \) drives slower than the velocity \( v \) at which the system moves, the vehicle slows down other vehicles, causing the minus sign in the above equation. The integrals also depend on the relative velocity of the vehicles, which is the absolute difference between the velocities \( v \) and \( w \), and on \( \tilde{p}(x, w, t) \) and \( \tilde{p}(x, v, t) \), which represent how often vehicles with velocities \( w \) and \( v \) respectively meet at point \( x \).

It is necessary to notice that \( \tilde{P}(v, x, t) \) is a \textit{probability density function} (pdf). A main characteristic of pdf’s \( f(x) \) is that:

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1.
\]  

The \textit{expected value} is given by:

\[
E = \int_{-\infty}^{\infty} x f(x) \, dx.
\]  

If we now look at probability density function \( \tilde{P}(v, x, t) \), we can find the following expected value:

\[
V = \int_{-\infty}^{\infty} w \tilde{P}(w, x, t) \, dw,
\]  

which is the expected value of the velocities. We can therefore use this as the average velocity and can rewrite it as follows, using Equation 35:

\[
V(x, t) = \int_{-\infty}^{\infty} w \tilde{P}(w, x, t) \, dw = \int_{-\infty}^{\infty} w \frac{\tilde{p}(x, w, t)}{\rho(x, t)} \, dw.
\]  

By using Equation 42, Equation 35 and the main characteristic of pdf’s given in Equation 39, we can rewrite the interaction term in Equation 38

\[
\left( \frac{d\tilde{p}}{dt} \right)_{\text{int}} = \int_{w>v} (1 - \tilde{p}(\rho)) |w - v| \tilde{p}(x, w, t) \tilde{p}(x, v, t) \, dw \\
- \int_{w<v} (1 - \tilde{p}(\rho)) |w - v| \tilde{p}(x, w, t) \tilde{p}(x, v, t) \, dw \\
= \int_{-\infty}^{\infty} (1 - \tilde{p}(\rho)) (w - v) \tilde{p}(x, w, t) \tilde{p}(x, v, t) \, dw \\
= (1 - \tilde{p}(\rho)) \tilde{p}(x, v, t) \rho(x, t) \int_{-\infty}^{\infty} (w - v) \frac{\tilde{p}(x, w, t)}{\rho(x, t)} \, dw \\
= (1 - \tilde{p}(\rho)) \tilde{p}(x, v, t) \rho(x, t) \left( \int_{-\infty}^{\infty} w \frac{\tilde{p}(x, w, t)}{\rho(x, t)} \, dw - v \int_{-\infty}^{\infty} \frac{\tilde{p}(x, w, t)}{\rho(x, t)} \, dw \right) \\
= (1 - \tilde{p}(\rho)) \tilde{p}(x, v, t) \rho(x, t) (V(x, t) - v) \int_{-\infty}^{\infty} \tilde{P}(w, x, t) \, dw \\
= (1 - \tilde{p}(\rho)) \tilde{p}(x, v, t) \rho(x, t) (V(x, t) - v),
\]
to find that:
\[
\left( \frac{d\hat{\rho}}{dt} \right)_{\text{int}} = (1 - \hat{p}(\rho)) \hat{\rho}(x, v, t) \rho(x, t) (V(x, t) - v).
\] (43)

As is the case with many gas-kinetic models, the above model proposed by Prigogine can be described as a macroscopic model by specifying the parameters and functions in the density equation \(33\) and velocity equation \(34\). According to Helbing (2001), by noticing that:
\[
\int_{-\infty}^{\infty} \hat{\rho}(x, v, t)dv = \int_{-\infty}^{\infty} \hat{\rho}(x, t) \hat{P}(v, x, t)dv = \rho(x, t) \int_{-\infty}^{\infty} \hat{P}(v, x, t)dv = \rho(x, t),
\] (44)
integrating over the phase-space continuity equation using the acceleration and interaction terms in Equation \(37\) and Equation \(43\) respectively, one can find the density and velocity equations for macroscopic models with the following parameter values:

a. \(D(\rho) = 0;\)
b. \(\xi_1(t) = \xi_2(t) = 0;\)
c. \(P(\rho, \theta) = \rho \theta,\) with \(\theta(x, t) = \int_{-\infty}^{\infty} (v - V(x, t))^2 \hat{P}(v, x, t)dv\) the velocity variance;
d. \(v(\rho) = 0;\)
e. \(\tau(\rho)\) as given in the acceleration term in Equation \(37\) and
   f. \(V_e(\rho, \theta) = V_0 - \tau(\rho)(1 - \hat{p}(\rho))\rho \theta,\) where \(V_0 = \int_{-\infty}^{\infty} \hat{P}_0(v)dv\) is the average desired velocity.

In fact, Prigogine was the first one to relate the functions \(P\) and \(V_e\) mathematically to the density and velocity variance, rather than assuming the functions based on observations. However, the relations only hold for small densities (Helbing 2001). Apart from that, Prigogine found that from a certain critical density \(\rho_c\), on, a transition from a free state of traffic to a congested state takes place, which can be compared to a phase transition from a gaseous phase to a liquid phase, whence the name of gas-kinetic models. This transition can also be seen in the fundamental diagram (Figure 8).

Many have made adaptations to Prigogine’s model by changing the acceleration term, adding correlations between the velocities of successive vehicles, introducing interactions between vehicles in neighbouring lanes and more. Among these modifications is the contribution of Paveri-Fontana, who argued that the desired velocities should depend on the driver personalities (Helbing 2001). In general, two cases are distinguished: (i) aggressive drivers, that drive fast, and (ii) timid drivers, that drive slowly, also referred to as rabbits and slugs, respectively (Daganzo 1995).

The different driver personalities were represented by Paveri-Fontana by individual desired velocities \(v_0\). In order to process this change, Paveri-Fontana introduced an extended phase-space density \(\rho_e(x, v, v_0, t)\) with its corresponding gas-kinetic continuity equation, analogous to Equation \(36\). After integrating over \(v_0\), the phase-space density equation is found again. Next to this, the acceleration term was changed, since it showed discontinuous velocity jumps. For this, Paveri-Fontana used an approach showing similarities to the microscopic equation for the change in velocity, given in Equation \(16\), resulting in the following term for the acceleration behaviour:
\[
\left( \frac{d\hat{\rho}}{dt} \right)_{\text{acc}} = -\frac{\partial}{\partial v} \left( \hat{\rho}(x, v, t) \frac{\tilde{V}_0(v, x, t) - v}{\tau(\rho(x, t))} \right),
\] (45)
where \(\tilde{V}_0(v, x, t) = \int_{-\infty}^{\infty} v_0 \rho_{e}(x, v, v_0, t)dv_0\) is the average desired velocity at location \(x\) and time \(t\).

In contrast to \(V_0\), \(\tilde{V}_0\) depends on velocity as well, taking into account that fast vehicles have a different desired velocity than slower vehicles.
3.4 Patterns in Congested Traffic

Using traffic models as described above, several patterns can be recognised in congestion. These patterns clarify in which sense vehicles are moving during a particular type of congestion. Traffic can only move from certain patterns to other patterns and by investigating these patterns, it is easier to predict the occurrence of traffic jams on certain road segments, enabling regulatory bodies to make changes in infrastructure for the prevention of congestion. In traffic modelling, there are two categorisations that are widely used.

3.4.1 Three-Phase Traffic Theory

The first categorisation is based on the three different states of traffic, that were elaborated on in section 3.1.2. Based on these states of free traffic, localised congestion (also known as wide-moving jams) and extended congestion (also known as synchronised flow) Kerner introduced the three-phase traffic theory [Kerner & Klenov, 2003], which describes transitions between the three different states. It provides room for introducing more specific congestion patterns and does not make use of the fundamental diagram, that was used before introducing this categorisation. Kerner & Klenov (2003) found that the fundamental diagram does not entirely fit to the empirical data, mostly due to the instability of traffic flow in congested states (Helbing, 2001). To overcome this problem with the, more theoretical, fundamental diagram, referred to as the fundamental diagram approach, Kerner & Klenov (2003) described the different possibilities of transitions. In order to do so, the following notation was introduced:

a. \(F\) denotes the state of free traffic;

b. \(S\) denotes the state of synchronised traffic;

c. \(J\) denotes the state of wide-moving jams and

d. \(A \rightarrow B\) denotes the transition from state \(A\) to state \(B\), for \(A, B \in \{F, S, J\}\).

Also, two main patterns specifying the traffic states were introduced [Kerner, 2002]:

a. the general pattern (GP) (Figure 14) is the pattern of congestion that is caused by a bottleneck in isolation, i.e., far enough from other bottlenecks, characterised by synchronised flow occurring upstream of the bottleneck and wide-moving jams occurring spontaneously inside the synchronised flow, and

b. the synchronised flow pattern (SP) (Figure 15) is the pattern of congestion caused by a bottleneck in isolation that exclusively shows synchronised flow, i.e., no wide-moving jams occur.

Near isolated bottlenecks, the general pattern is the type of congestion that occurs most frequently, while the synchronised flow pattern shows more variations. Indeed, these two patterns contain several special cases, depending on all kinds of parameters, such as vehicle density, bottleneck specifications and road capacity. A first specification is the alternating general pattern (AGP), in which traffic is not only found in a congested state, but it alternates between free traffic and synchronised traffic, explaining the nomenclature [Kerner, 2002]. The general pattern can also show a variation that starts off as GP, but eventually reaches a state of synchronised flow that it cannot leave anymore. This variation is called dissolving general pattern (DGP). Figure 14 shows a spatio-temporal representation of GP and DGP.

The synchronised flow pattern also shows several special cases, starting with the moving synchronised flow pattern (MSP) (Figure 15(e)). This pattern occurs when both the upstream and downstream front of the synchronised flow pattern are moving upstream. There is an important difference between the moving synchronised flow pattern and wide-moving jams, which is the velocity. Indeed, MSP shows velocities around 40 to 70 km/h, which is higher than the average
velocity in a wide-moving jam (Kerner 2002). Then, at some point, if and when the moving synchronised flow pattern reaches a bottleneck, it can form a localised synchronised flow pattern (LSP) (Figure 15(d)). The most important feature of the localised synchronised flow pattern is that the downstream front of the SP is localised to the position of the bottleneck (Kerner 2002). The upstream front is located at some time-dependent distance $L_{\text{LSP}}$ upstream of the downstream front, often varying from 0.5 km to 10 km (Kerner & Klenov 2003). Next to LSP and MSP, there is also the widening synchronised flow pattern (WSP) (Figure 15(c)), which is characterised by the upstream front continuously moving upstream, with the downstream front (as in LSP) fixed at the bottleneck. The movement of the upstream front, hence, results in the widening of the area of synchronised flow. As is the case with GP, also SP knows a pattern in which free flow is alternated with congestion. In this case, this congestion only contains synchronised flow, alternated with free flow, whence the name alternating synchronised flow pattern (ASP) (Kerner & Klenov 2003).

**Figure 14:** Spatio-temporal velocity representations of the general pattern (GP). (g) shows synchronised flow occurring upstream of the on-ramp bottleneck and wide jams spontaneously occurring inside the synchronised flow, with a constant upstream propagation velocity. (h) shows a dissolving general pattern (DGP), where the wide-moving jams dissolve after some time. Figure from Kerner & Klenov (2003).

It is important to notice that a real motorway does not just have a single bottleneck, but contains a vast number of them, either fixed (in case of on- and off-ramps) or incidental (in case of accidents, road works etc.). Hence, not only do the different patterns constantly merge into other patterns (Figure 16), but the patterns also live next to each other (Figure 27), i.e., they can appear simultaneously (Kerner 2002). The presence of several bottlenecks close to each other gives rise to a different type of pattern that extends over a larger segment of the road, covering several bottlenecks. This traffic pattern is called the expanded congested pattern (EP) (Figure 17). Either the expanded congested pattern is similar to a general pattern (GP), or a synchronised flow pattern (SP) occurs over a large area, covering several bottlenecks. It is important to consider this type of pattern, since these bottlenecks are not isolated and therefore together contribute to a type of congestion that can differ from the pattern that both bottlenecks would individually produce. EP can consist of any combination of aforementioned congestion patterns (Kerner & Klenov 2003). The different patterns can then non-linearly interact with each other, resulting in a possibly unexpected expanded congested pattern (Kerner & Klenov 2003).
Figure 15: Spatio-temporal velocity representations of the synchronised flow pattern (SP). (c) shows the widening synchronised flow pattern (WSP), where the downstream front is located at the on-ramp bottleneck, while the upstream front keeps propagating upstream, hereby widening the synchronised flow pattern. (d) shows the localised synchronised flow pattern (LSP), where the downstream front of congestion is localised at the position of the on-ramp bottleneck. The upstream front is located at $L_{\text{LSP}}(t)$, with $L_{\text{LSP}}(t) < 10$ km for all $t > t_0$. (e) shows the moving synchronised flow pattern (MSP), where both the upstream and the downstream front of the congestion propagate upstream over time. The velocity is still significantly higher than the wide-moving jam shown in Figure 14(h). Figure from Kerner & Klenov (2003).

Figure 16: Spatio-temporal velocity representation of different congestion patterns merging into each other. From approximately time 7:00 to 9:00 the general pattern (GP) occurs, after which the congestion merges into a localised synchronised flow pattern (LSP) till approximately 9:30, where it merges into a dissolving general pattern (DGP). Around 10:30 traffic merges to a localised synchronised flow pattern (LSP) again. Figure from Kerner & Klenov (2003).
3.4.2 Empirically-Substantiated Traffic States

Although widely used, there has also been a lot of criticism on the congested traffic states as described by Kerner (2002) and Kerner & Klenov (2003). The controversy is of such size that not only the classification of congested traffic states, but also the three-phase traffic theory as a whole is criticised. The main objection raised by Schönhof & Helbing (2007) is that the congestion state of the moving synchronised flow pattern (MSP) is not easily distinguishable from wide-moving jams, which do not belong to synchronised traffic. According to Schönhof & Helbing (2007) these patterns are too much alike to be classified as two different traffic phases, therewith refuting the three-phase traffic theory.

In order to make an alternative and, according to Helbing et al. (2009), simpler distinction between congested traffic states and in order to cope with the unexpected complexity of traffic states, Helbing et al. (2009) and Schönhof & Helbing (2007) came up with an empirical classification. Studying data from the German autobahn A5 near Frankfurt am Main, a road segment that was also extensively used by Kerner for empirical observations, five types of congested traffic have been found (Schönhof & Helbing, 2007). However, the state of free traffic (previously denoted \( F \)) is subsumed from Kerner (2002). Depending on the average velocity \( V(x,t) \), traffic belongs to the state of free traffic (FT) (Figure 18) when \( V(x,t) > V_{\text{crit}} \) for some critical velocity \( V_{\text{crit}} \). In any other situation, we speak of congested traffic. For German motorways, a value \( V_{\text{crit}} \approx 80 \text{ km/h} \) is often used.

![Figure 17](image1.png)

**Figure 17:** Spatio-temporal velocity representations of the expanded congested pattern (EP). (a) shows an expanded congested pattern (EP) consisting of two general patterns (GPs), starting at on-ramp \( U \) and \( D \), respectively. (b) shows an expanded congested pattern (EP), consisting of a general pattern (GP) at on-ramp \( U \) and a widening synchronised flow pattern (WSP) at on-ramp \( D \). Figure from Kerner & Klenov (2003).

![Figure 18](image2.png)

**Figure 18:** Spatio-temporal velocity representation of free traffic (FT). At the on-ramp, velocity only falls slightly below the maximum velocity in this spatio-temporal segment, not exceeding the critical velocity \( V_{\text{crit}} = 80 \text{ km/h} \). Figure from Kerner & Klenov (2003).
The congested traffic is divided into *localised clusters* (LC), when only for a short section of the motorway it holds that $V(x,t) < V_{\text{crit}}$ and the length of this section is approximately stable or stabilises over time, and *spatially extended congestion states* otherwise. Localised clusters are subdivided into *pinned localised clusters* (PLC), staying at a fixed location, and *moving localised clusters* (MLC), propagating upstream, with a characteristic speed $C_0$. In the extended congestion, we distinguish *stop-and-go waves* (SGW), *oscillating congested traffic* (OCT) and *homogeneous congested traffic* (HCT). What characterises the five types of congestion, PLC, MLC, SGW, OCT and HCT, is that they are spatio-temporal patterns, i.e., the patterns depend both on space and on time (Helbing et al., 2009). Moreover, all the patterns that were found in empirical data gathered from the German A5 (Figure 19) can be reproduced by simulation (Helbing et al., 2009), as seen in Figure 20. The results that were simulated by Helbing et al. (2009) are shown in Appendix B.

**Figure 19:** Spatio-temporal velocity representations of the five types of congestion patterns and an additional pattern found by Helbing et al. (2009) in (d), based on empirical data of the A5 motorway near Frankfurt am Main. Velocities are displayed upside down, for a better illustration of the traffic patterns. (a) shows two moving localised clusters (MLCs). (b) shows stop-and-go waves (SGW). (c) shows oscillating congested traffic (OCT). (d) shows a pattern that is not part of the characterisation of five types of traffic patterns and looks similar to the widening synchronised flow pattern (WSP), as classified by Kerner (2002). (e) shows a pinned localised cluster (PLC). (f) shows homogeneous congested traffic (HCT). Figure from Helbing et al. (2009).

**Figure 21** shows a pinned localised cluster (PLC). A PLC most often occurs when the vehicle density gets higher (for example during rush hours) and there is a bottleneck (such as an on-ramp). An important aspect of pinned localised clusters is the fixed location to the bottleneck and the limitation of the spatial extension (Schönhof & Helbing, 2007). Both upstream and downstream of the PLC, one can find free traffic. When the vehicle density gets past a specific point, the pinned localised cluster starts extending, which gives rise to another, congested traffic pattern.

In **Figure 22** we see a moving localised cluster (MLC). The width of an MLC is limited, as we also see in pinned localised clusters. However, moving localised clusters propagate upstream, rather than being fixed at a certain location. They are often found at a bottleneck, when a large perturbation occurs. The upstream propagation velocity typically lies around $C_0 \approx 16$ km/h (Schönhof & Helbing, 2007).
Figure 20: Spatio-temporal velocity representations of the five types of congestion patterns, simulated with an on-ramp at $x = 0$. The simulated patterns match the empirically found patterns in Figure 19. Velocities are displayed upside down, for a better illustration of the traffic patterns. In the following, the letter behind each pattern refers to the empirically found pattern from Figure 19 it can be compared to. From left to right, the top row shows a moving localised cluster (MLC) (a), a pinned localised cluster (PLC) (e) and stop-and-go waves (SGW) (b). The bottom row shows oscillating congested traffic (OCT) (c) and homogeneous congested traffic (HCT) (f). Figure from Schönhof & Helbing (2007).

Figure 21: Spatio-temporal velocity representation of a pinned localised cluster (PLC), with an on-ramp at $x = 481.3$ km. Velocities are displayed upside down, for a better illustration of the traffic pattern. The PLC is fixed to the on-ramp bottleneck and only slightly expands spatially. Both upstream and downstream of the pinned localised cluster free traffic is found. Figure from Schönhof & Helbing (2007).

Figure 23 displays homogeneous congested traffic (HCT). This type of congestion does not occur frequently, compared to the other types of congestion, that constantly emerge in traffic. It was especially seen when serious accidents had taken place and occurs specifically at the closing of one or more lanes or when closing even all lanes (Schönhof & Helbing, 2007). Also in heavy holiday traffic homogeneous congested traffic can emerge. Homogeneous congested traffic is mostly related to low and approximately constant - hence the homogeneity - vehicle velocities. It extends over a long section of a motorway. In most cases, the downstream end of the HCT is located just downstream of the upstream end of a bottleneck, as happens right behind a closed lane, where more lanes open and traffic typically accelerates. Hence, downstream of the bottleneck, we typically find free traffic. The upstream end of the congestion moves upstream with a velocity $C_0 \approx 16$ km/h, thus extending the homogeneous congested traffic.
Figure 22: Spatio-temporal velocity representation of a moving localised cluster (MLC), with an on-ramp at $x = 481$ km. Velocities are displayed upside down, for a better illustration of the traffic pattern. The MLC propagates upstream and shows almost no spatial expansion. At the on-ramp bottleneck, a pinned localised cluster (PLC) emerges from the moving localised cluster. The MLC is probably caused by overtaking trucks, keeping up the traffic from behind (Schönhof & Helbing, 2007). Figure from Schönhof & Helbing (2007).

Figure 23: Spatio-temporal velocity representation of homogeneous congested traffic (HCT), with an accident occurring at $x = 478.736$ km around time 19:15. Velocities are displayed upside down, for a better illustration of the traffic pattern. The HCT starts at the location of the accident, where it is present for a long period of time and propagates upstream. Downstream of the congestion front, i.e., behind the location of the accident, free traffic is found. Also, a pinned localised cluster (PLC) is found at the location of the accident for a short time, as vehicles tend to drive more slowly for a short period of time when the emergency lane is still occupied. Figure from Schönhof & Helbing (2007).

In Figure 24, the pattern of oscillating congested traffic (OCT) is depicted. The development and growth of OCT is similar to that of homogeneous congested traffic (HCT), but contrarily, the vehicle velocity is not constant. It oscillates with approximately constant frequency and amplitude. As is the case with a moving localised cluster (MLC) and with homogeneous congested traffic, the oscillations in oscillating congested traffic propagate upstream with a velocity around 16 km/h. OCT often emerges from a perturbation, but it can also be triggered by the traffic density exceeding a certain value. Around OCT one can often find free traffic (FT).

Figure 25 shows the pattern of stop-and-go waves (SGW). Stop-and-go waves, also sporadically referred to as start-stop waves (Nagel & Schreckenberg, 1992), are characterised by segments of congestion, with free traffic in between. It shows resemblances with both extended traffic, in the
form of oscillating congested traffic, and localised traffic. Each part of congestion is localised and propagates upstream, similar to a moving localised cluster (MLC). Stop-and-go waves can therefore be considered as a sequence of MLCs (Schönhof & Helbing, 2007). The distance between consecutive jams differs significantly. On the other hand, one can consider stop-and-go waves as a special case of oscillating congested traffic, with a large amplitude and no typical wave-length. Since in some cases stop-and-go waves are triggered by small perturbations in traffic flow (Schönhof & Helbing, 2007), one can also distinguish a sixth congestion pattern, named triggered stop-and-go traffic (TSG) (Zhang et al., 2018; Kerner, 2002; Helbing, 2001). Furthermore, stop-and-go waves have been compared to several phenomena, such as the forming of waves in shallow water, the clogging of sand that falls through a vertical pipe and the falling of lead spherical objects through a fluid column (Helbing, 2001).

Figure 24: Spatio-temporal velocity representation of oscillating congested traffic (OCT). Velocities are displayed upside down, for a better illustration of the traffic pattern. The OCT starts due to an obstruction on the fast lane between $x = 486.0$ km and $x = 486.9$ km. The waves propagate upstream, while the area around the oscillating congested traffic shows free traffic (FT). Figure from Schönhof & Helbing (2007).

Figure 25: Spatio-temporal velocity representation of stop-and-go waves (SGW). Velocities are displayed upside down, for a better illustration of the traffic pattern. The SGW start due to a small perturbation that is due to an uphill gradient between $x = 474$ km and $x = 472$ km around time 14:00 and stop forming around 15:10. The MLC-like waves propagate upstream. The distance between the waves is different between all waves and in between the waves free traffic (FT) is found. Figure from Schönhof & Helbing (2007).
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Figure 26: Spatio-temporal velocity representation of triggered stop-and-go waves (TSG) in two lanes, from simulations using a lattice hydrodynamic model. Around lattice $x = 300$ a bottleneck is located, where traffic from the first lane (a) has to merge with traffic from the second lane (b). This trigger causes stop-and-go waves to perturbate upstream. Figure from Zhang et al. (2018).

The majority of all congested traffic patterns that were empirically found by Schönhof & Helbing (2007) can be considered as a combination of the six aforementioned traffic states: free traffic (FT), pinned localised clusters (PLC), moving localised clusters (MLC), stop-and-go waves (SGW), oscillating congested traffic (OCT) and homogeneous congested traffic (HCT). It is, moreover, important to notice that these traffic states can coexist at different locations (see Figure 27) and, by metastability, similar conditions can give rise to different traffic states (Schönhof & Helbing, 2007).

Comparing the classifications from Schönhof & Helbing (2007) and Helbing et al. (2009) to those of Kerner (2002) and Kerner & Klenov (2003), we find some correspondences. For example, homogeneous congested traffic (HCT) (Figure 23) may correspond to synchronised traffic flow with stationary and homogeneous states, as described in section 3.1.2. Oscillating congested traffic (OCT) (Figure 24), on the other hand, is comparable to the nonstationary and nonhomogeneous states, as described in the same paragraph. These traffic states correspond to the general pattern (GP) (Figure 14) (Helbing et al., 2009). The homogeneous-in-speed states contain both characteristics of free traffic and of congested traffic. Also, wide-moving jams or moving synchronised patterns (MSP) (Figure 15(e)) correspond largely with moving localised clusters (MLC) (Figure 22) and hence stop-and-go waves (SGW) (Figure 25) correspond to a sequence of moving synchronised flow patterns (MSPs). Furthermore, we see that the pinned localised cluster (PLC) (Figure 21) is related to the localised synchronised pattern (LSP) (Figure 15(d)). The widening synchronised pattern (WSP) (Figure 15(c)) and dissolving general pattern (DGP) (Figure 14(h)) do not seem to occur in the data from the German A5 (Helbing et al., 2009). Still WSP is simulated by Helbing et al. (2009), as shown in Appendix B.
Figure 27: Spatio-temporal velocity representation of different traffic states occurring simultaneously, using real data. (a) shows homogeneous congested traffic (HCT) occurring after an accident at $x = 487.5$ km around time 17:13. The HCT merges into oscillating congested traffic (OCT). The stop-and-go waves (SGW), that were present before the accident occurred, dissolve due to the lower density (as a consequence of the HCT) at locations after the accident. (b) shows oscillating congested traffic (OCT) starting at a bottleneck around $x = 480$ km, quickly merging into (triggered) stop-and-go waves. The stop-and-go waves dissolve the OCT pattern that was present between times 7:00 and 9:00 between locations $x = 465$ km and $x = 470$ km. The SGW also cause an accident at $x = 478.33$ km, triggering a new OCT pattern. Due to this accident, the density falls, dissolving the OCT that was initially started at the bottleneck around $x = 480$ km. Figure from Helbing et al. (2009).

3.5 Assumptions, Remarks and Controversies

As in any form of mathematical modelling, traffic modelling cannot exist without assumptions. First of all, in many theoretical studies of traffic, a circular ring road is used, without any on- or off-ramps. This avoids the use of boundary conditions and ensures the total flow to stay equal at all times, i.e., there is no inflow nor outflow. Naturally, no such situations emerge in real traffic. Still, it is useful to study these systems, because of the absence of impurities. Also, the above models are all based on motorways, i.e., one-way traffic without intersections and merely dealing with on- and off-ramps. When focussing on city traffic, other traffic dynamics occur, beyond the scope of this report.

Apart from this, many models are based on vehicles of similar length. Indeed, the length difference among different cars is often negligible, but in reality, traffic can roughly be divided in long and short vehicles. An important consideration is that among the long vehicles one can find trucks, busses and cars towing a caravan or other trailer, often having a lower desired and permitted velocity than the small vehicles, among which we find cars, motorcycles and more. According to Helbing (2001) this point “may be quite relevant for the explanation of some observed phenomena”. This is partly due to the alternating fraction of trucks on motorways throughout the day, as seen in Figure 28 making a different contribution to traffic flow throughout the day. However, also considering the differences between cars and trucks in (desired) velocities (Figure 29) and in time headways (Figure 30), i.e., the difference in time between two vehicles, raises the aforementioned statement.

Not only the length of vehicles is important, but also the behaviour of their drivers affects the traffic flow, which can already be seen from the differences between truck drivers, that have to
manage a large, heavy, slow-to-respond vehicle, and regular car drivers, that have a more agile vehicle, not to speak of motorcycles. Although many models consider driver behaviour to be similar for every so-called driver-vehicle unit, doing so is, therefore, a very simplified version of reality. A way to address this issue is to consider two classes of drivers. Examples are trucks (long vehicles) and cars (short vehicles) or rabbits (aggressive drivers) and slugs (timid drivers) (Helbing, 2001).

![Figure 28](image1.png)

**Figure 28:** The fraction of trucks on a motorway with time. Figure from Helbing (2001).

![Figure 29](image2.png)

**Figure 29:** Velocity differences between cars and trucks, as a function of vehicle density. Figure from Helbing (2001).

![Figure 30](image3.png)

**Figure 30:** The distribution of time-headways $\Delta t$ for trucks, cars and both long and short vehicles combined. On the left, the distribution is shown for a (low) density $\rho = 10$ to $\rho = 20$ vehicles/kilometre. On the right, the density is $\rho > 30$ vehicles/kilometre, corresponding to congested traffic. Figure from Helbing (2001).
Another point of attention is that many models are based on or modelled as one-lane roads, while most motorways have at least two and possibly more lanes. Macroscopic and gas-kinetic models can integrate lane-changing by adding a lane-changing term (Helbing, 2001). However, empirical data is often gathered from the complete cross-section of roads rather than the specific lanes and it is hard to empirically find out the number of lane changes. This is partly due to asymmetric rules on lane changing, as many countries only allow for overtaking on one side. Moreover, when traffic needs to weave, i.e., two lanes have to merge, drivers tend to behave differently, since smaller gaps are acceptable. The gaps also depend on the speed of vehicles, speed differences between different vehicles and, again, driver behaviour.

Next to the aforementioned assumptions, the models also contain some shortcomings. According to Helbing (2001), there are certain effects that occur in reality, but are not or barely visible in the aforementioned models. This includes the anticipation behaviour of drivers, the response to blinkers, avoidance of driving side-by-side, the adaptation of velocity to surrounding traffic, tolerance of small clearances for a short period of time, the avoidance of the slowest lane, the attention or concentration of drivers, changes in driver strategies or behaviour and the effect of road conditions, weather conditions, cruise control, autonomous cars etc. As is the case with lane changing, these effects are relatively hard to measure empirically or to prove or disprove.

Besides the commentary about models, Helbing (2001) also stresses the statement that the fundamental diagram should only be fitted in the range of stable traffic flow. However, in most situations traffic cannot be considered stable. The fundamental diagram should thus be considered as a purely theoretical, yet very useful tool. Schönhof & Helbing (2007) stress that different traffic patterns are dependant on the day of the week. As is substantiated by Figure 28, they even depend on the time of the day.

In traffic modelling, also some controversies have arisen. When Kerner & Klenov (2003) wanted to overcome the problems that made the fundamental diagram not applicable to situations where traffic is not stable, the three-phase traffic theory was introduced. However, this theory is questioned by Schönhof & Helbing (2007). In their introduction, they list some standpoints, substantiated by different traffic research, which they argue to be “mutually inconsistent”. The first of these points is that the Lighthill-Whitham model, as described in section 3.3.3, is sufficient for understanding observed traffic phenomena. On the other hand, Kerner argues that any model that makes use of the fundamental diagram, including the Lighthill-Whitham model, is wrong. Several other standpoints are taken as well, leading Schönhof & Helbing (2007) to state that “this controversy may be considered so fundamental that it questions traffic flow modeling as a scientific discipline”. Moreover, they mention the confusion raised by the many different terms for the same observations or traffic states and the questioning of theories for the wrong reasons. To make a distinction between better and lesser models, Helbing (2001) introduced some criteria a model should satisfy, stating that a good traffic model should:

a. have only a few variables and parameters, all having an intuitive meaning, easy to measure and with realistic values;

b. reproduce all known features of traffic flows in a qualitative manner, including the aforementioned traffic patterns from section 3.4.2;

c. be theoretically consistent and make new predictions that can be verified or disproved;

d. not lead to vehicle collisions nor should it exceed the maximum vehicle density, and

e. make a fast numerical simulation possible.
4 Model

In order to show the results of a traffic model, we will implement the Nagel-Schreckenberg model (Nagel & Schreckenberg, 1992), as described in section 3.3.2. We divide a road into \( L \) sites of equal length. Each site is either occupied by a single vehicle or not occupied (empty). Each vehicle is assigned a nondimensional integer velocity \( \hat{v}_i \in \{0, 1, \ldots, \hat{v}_{\text{max}}\} \) for some maximum nondimensional velocity \( \hat{v}_{\text{max}} \). As described in section 3.3.2, the nondimensionalised integer velocity \( \hat{v}_i \) can be scaled to the real velocity \( v_i \) through:

\[
v_i = \hat{v}_i \frac{\Delta x}{\Delta t}, \tag{46}
\]

where \( \Delta x \) is the difference in the number of sites and \( \Delta t \) is the difference in the number of timesteps. In general \( \hat{v}_{\text{max}} = 5 \) is chosen, corresponding to approximately 135 km/h, as elaborated on below. The total number of vehicles on the road is \( N \). We start with an initial configuration, i.e., \( N \) vehicles are divided over the \( L \) sites, each having an initial velocity \( \hat{v}_0^i \). We consider two different initial configurations:

1. a uniform initial configuration and
2. a random initial configuration.

The uniform initial configuration assumes that vehicles are moving when the simulation starts. Vehicles are all assigned the velocity \( \hat{v}_i = \hat{v}_{\text{max}} - 1 \) and are approximately uniformly distributed over the \( L \) sites. More specifically, with the density given by \( \rho = \frac{N}{L} \), vehicle \( i \) is placed at site:

\[
\left( \frac{1}{\rho} \right) (i - 1) + 1, \tag{47}
\]

where \( \left( \frac{1}{\rho} \right) \) equals \( \frac{1}{\rho} \) rounded off to the nearest integer. Note that at high densities, the distance between consecutive vehicles gets lower. Hence, at some point, the initial velocity \( \hat{v}_0^i = \hat{v}_{\text{max}} - 1 \) would facilitate collisions, as the distance between consecutive vehicles becomes lower than the distance driven by the vehicles in one step. Therefore, this initial configuration should only be used for densities that are low enough.

The random initial configuration assumes that vehicles are at a standstill, i.e., \( \hat{v}_0^i = 0 \) for all vehicles \( i \). They are randomly distributed over the road. Therefore, it can happen that several vehicles are located directly behind each other and, thus, it is important that \( \hat{v}_0^i = 0 \) for all vehicles \( i \). Then, vehicles will accelerate only if there is no vehicle immediately ahead of them. This initial configuration is used by Nagel & Schreckenberg (1992) in their model, where results are only collected after a number of timesteps, so that the system represents a more realistic initial situation. After all, it is not realistic to consider vehicles with no velocity at random places on a motorway.

With every timestep, the configuration is updated, using four rules. These rules can be executed in parallel for all vehicles, which, as described in section 3.3.2, enables fast computation. For every vehicle, first the velocity is adjusted according to the three velocity rules. Then, the vehicle is moved forward exactly the amount of steps given by the newly-computed velocity, in accordance with the rule of motion. The four consecutive rules, as described in section 3.3.2, have to be executed for every vehicle in the given order. Helbing (2001), who calls the Nagel-Schreckenberg model “extremely compact and elegant”, fails to recognise the importance of the order of execution of these rules. In Appendix C, we give an example of a situation in which the interpretation of the Nagel-Schreckenberg model by Helbing (2001) results in a collision of vehicles, thereby violating the definition of the model, which states that a site can only be occupied by one vehicle at a time. Following the order of execution as given by Nagel & Schreckenberg...
Algorithm 1 Vehicle Update

if $\hat{v}_i < \hat{v}_{\text{max}}$ then
 ▷ Acceleration
  check all $\hat{v}_i + 1$ sites ahead
  if all sites are empty then
    $\hat{v}_i \leftarrow \hat{v}_i + 1$
  end if
endif

if a site ahead is occupied at distance $d_i \leq \hat{v}_i$ then
  ▷ Deceleration
  $\hat{v}_i \leftarrow d_i - 1$
end if

if $\hat{v}_i > 0$ then
  ▷ Randomisation
  with probability $p$:
  $\hat{v}_i \leftarrow \hat{v}_i - 1$
end if

move vehicle forward $\hat{v}_i$ sites
  ▷ Motion

(1992), this yields Algorithm 1 that should be applied to every vehicle in every timestep.

An important aspect of the model is the set of boundary conditions that is used. We consider two cases:

1. a closed system and
2. an open system.

The closed system can be viewed as a circular road, much like a racing circuit or a ring road without any incoming or outgoing traffic. Once a vehicle is at the end of the road, i.e., site $L$, it continues at the start, i.e., site 1. For example, when a vehicle is at site $L - 1$ and has to move 5 cells, its next position is site 4. As mentioned in section 3.3, the use of such circular roads in traffic models is fairly common, as it already shows realistic behaviour, yet, no boundary conditions have to be used. A useful feature of such a closed system is that the number of vehicles stays equal in every timestep, whence maintaining a fixed global vehicle density, given by:

$$\rho = \frac{N}{L}. \quad (48)$$

Therefore, the density can be adjusted by either changing the number of vehicles or the length of the road section. When considering a real road segment, the length $L$ is often fixed, so that the density can only be adjusted by changing $N$.

The open system, on the other hand, does need boundary conditions. There are both conditions for vehicles entering the system and for vehicles leaving the system. [Nagel & Schreckenberg (1992)] proposed to occupy the leftmost site with a vehicle of velocity 0 whenever the site is empty. It is suggested that this condition represents the start of a bottleneck situation, where vehicles have to merge from several fully-saturated lanes to only one lane, hence starting with a velocity of 0. Then, the last six sites are removed from any configuration, representing the end of a bottleneck, where vehicles can fully accelerate to multiple lanes. Since the vehicles are removed, this is an open boundary condition. However, since we use $\hat{v}_{\text{max}} = 5$, vehicles can move at most 5 sites every timestep. Therefore, we choose to change this boundary condition such that the last 5, rather than 6, vehicles are removed.

Seeing that [Nagel & Schreckenberg (1992)] do not mention which initial configuration is used for the open system, we consider both the case where there is a random initial configuration, equal to the initial configuration of the closed system, and the case where there is a zero initial
configuration, i.e., there are initially no vehicles in the system. The boundary condition that constantly supplies site 1 then provides an uninterrupted inflow of vehicles.

As described above, the road is divided into $L$ separate sites. According to Nagel & Schreckenberg (1992), in a complete jam, i.e., an area of congestion that is fully saturated, each car occupies approximately 7.5 m of the road surface. Therefore, each site represents a road segment of 7.5 m. Indeed, when the vehicle density $\rho$ is equal to 1 vehicle per site, this agrees with a fully saturated jam. The average velocity in free traffic should, according to Nagel & Schreckenberg (1992), correspond to the velocity that belongs to 4.5 sitelengths per timestep. They propose 120 km/h as the average velocity in free traffic. With this, we can find the following computation for the actual timestep:

\[
\frac{7.5 \text{ m/site} \cdot 4.5 \text{ sites/timestep}}{\frac{120}{3600} \text{ s/m}} = \frac{337.5 \text{ s}}{1 \text{ timestep}} \approx 1 \text{ second/timestep. (49)}
\]

Therefore, a timestep in the Nagel-Schreckenberg model represents approximately 1 second in reality. Using Equation 46 and filling in the values $\Delta x = 7.5 \text{ m}$ and $\Delta t = 1 \text{ s}$, we find that the real velocity $v_i$ can be computed by:

\[
v_i = \hat{v}_i \frac{7.5 \text{ m}}{1 \text{ s}} = 7.5 \cdot \hat{v}_i \text{ m/s} = 7.5 \cdot 3.6 \cdot \hat{v}_i \text{ km/h} = 27 \cdot \hat{v}_i \text{ km/h. (50)}
\]

This shows that $v_{\text{max}} = 27 \cdot \hat{v}_{\text{max}} = 27 \cdot 5 = 135 \text{ km/h}$ and, indeed, for $\hat{v}_i = 4.5$, we find a corresponding real velocity of $v_i = 27 \cdot \hat{v}_i = 27 \cdot 4.5 = 121.5 \approx 120 \text{ km/h}$. However, whereas Nagel & Schreckenberg (1992) use a maximum velocity that corresponds to $\hat{v}_i = 4.5$ of 120 km/h, as is a reachable velocity on German motorways, this velocity cannot be reached everywhere. In the Netherlands, for example, the maximum velocity during the day is 100 km/h. Hence, the same computation as Equation 49 gives a value of 1.215 seconds per timestep.

Apart from the values for $\hat{v}_{\text{max}}$ and the length of each site, there are more parameters in the model. Most importantly, the slowdown probability $p$ is often set to $p = 0.5$. This probability is of great importance, taking into account the individual fluctuations in acceleration, especially the deceleration or braking without a major cause. Without $p$, the model would be completely deterministic, causing a stationary traffic pattern shifting upstream along the road.

5 Numerical Simulation

For our simulation, we implemented the model described in section 4, using MATLAB version R2021a and aimed (for verification) at producing figures with similar patterns as the figures provided by Nagel & Schreckenberg (1992). All code can be found in Appendix E.

Unless otherwise specified, we use the following parameter values:

- a. the slowdown probability $p = 0.5$;
- b. the nondimensional integer maximum velocity $\hat{v}_{\text{max}} = 5$ and
- c. the vehicle density $\rho = \frac{N}{L} = 0.1$ vehicles per site.

5.1 Uniform Initial Configuration

We portray the dynamics of the model in a spatial-temporal representation, in the form of a heatmap. The velocity of each vehicle determines the colour and the configuration in terms of vehicle position is given for every timestep. When using the uniform initial configuration, as described in section 4, using the aforementioned parameter values, two results are found, shown
in Figure 31 and Figure 32 that show different traffic patterns, as the model is nondeterministic and hence the same input can give different results. Indeed, as explained below, Figure 31 shows free traffic, while Figure 32 shows some congestion. Here, we used a number of sites \( L = 10^2 \) and \( T = 10^2 \) timesteps, in order to display a clear view of the traffic dynamics that the Nagel-Schreckenberg model provides. The MATLAB code used for this representation can be found in Appendix E.1.

For every time step, all \( L = 10^2 \) sites are displayed. Either a site is empty, in which case it is white and is assigned the value \(-1\), or it is occupied, in which case it is coloured and assigned the value \( \hat{v}_i \in \{0, 1, \ldots, \hat{v}_{\text{max}} = 5\} \), i.e., the velocity of the vehicle currently occupying site \( i \). Note, particularly, that a value \( \hat{v}_i = 0 \) has been assigned the colour black. For higher velocities, it changes from blue to green, so that one can immediately see areas of congestion, since they contain mostly sites coloured black or blue. When starting at timestep \( t = 0 \), one can choose an occupied site and follow the trail of the vehicle that occupies it by moving the number of sites given by the value of \( \hat{v}_i \) at the considered timestep. Note, however, that this representation shows the updated velocity of the given timestep \( t \) in the configuration belonging to that timestep, while the motion that is computed in timestep \( t \) is only visible in the configuration of the next timestep, \( t + 1 \). For example, if the velocity in timestep \( t = 1 \) s is updated from \( \hat{v}_i = 4 \) to \( \hat{v}_i = 5 \), site \( i \) is given the value \( \hat{v}_i = 5 \) and the configuration at timestep \( t = 2 \) s has site \( i + 5 \) occupied, with the value that is computed in timestep \( t = 2 \) s. This is in agreement with the representation used by \cite{Nagel1992}. The drawback of this representation is that at timestep \( t = 0 \) s, the velocity is not yet updated, which does not influence the representation used by \cite{Nagel1992}, since they do not show the initial configuration. Since we do show the initial configuration, the vehicle does not seem to move according to the rules at the first time step. Therefore, in order to follow the trail of a vehicle at site \( i \), starting at timestep \( t = 0 \) s, one should follow the trail of the vehicle that occupies site \( i \) again in timestep \( t = 1 \) s. After that, the trail can be followed in the regular manner. Also note that the simulation shows a circular road, which implies that a vehicle that reaches the end of the road returns around the beginning of the road in the next timestep.

**Figure 31**: Heatmap for closed system with uniform initial configuration, with \( p = 0.5, \rho = 0.1, L = 10^2 \) and \( T = 10^2 \), where, despite using the same values as Figure 32, free traffic is found, due to the model being nondeterministic. The initial uniform situation is at \( t = 0 \) at the top of the figure.

In Figure 31, we find free traffic. All values \( \hat{v}_i \) of vehicles occupying a site are at least 3 and vehicles continually accelerate to the maximum velocity of \( \hat{v}_{\text{max}} = 5 \), corresponding to approx-
imately 135 km/h, computed using Equation 50. The figure also clearly shows how the model is nondeterministic, rather than deterministic. Namely, when following the trail of a vehicle, the velocity of this vehicle decreases at random, without any given cause of vehicles ahead. This is particularly visible by the neon green sites that change into darker green sites in the next step and possibly re-change to neon green in the subsequent timestep.

Figure 32 shows a clear area of congestion, characterised by the black and dark blue cells of value 0 and 1, respectively. Here, following trails of vehicles, one sees that vehicles have to reduce velocity to 1 or 0 and need one or more timesteps to be able to accelerate again. This acceleration is clearly visible by the change of colours. In the meantime, other vehicles approach the already slowly moving vehicles, which implies that they also need to slow down. This results in the backward, i.e., upstream motion of the traffic jam, which is exactly what we expect to happen in real traffic (Nagel & Schreckenberg, 1992).

Figure 32: Heatmap for closed system with uniform initial configuration, with \( p = 0.5, \rho = 0.1, L = 10^2 \) and \( T = 10^2 \), where, despite using the same values as Figure 31, an area of congestion is found, due to the model being nondeterministic. The initial uniform situation is at \( t = 0 \) at the top of the figure.

5.2 Random Initial Configuration

When using the random initial configuration, as described in section 4, for which the code used can be found in Appendix E.2, the results are very similar to the results by Nagel & Schreckenberg (1992), as found in Figure 34 and Figure 36. Setting the density to \( \rho = 0.03 \) vehicles per site, we see free traffic once the vehicles have come to their maximum velocity (Figure 33). Their distance to each other is so large that the vehicles do not influence the velocity of the other vehicles. Note that especially the values of the velocities show a similar pattern in both Figure 33 and Figure 34, where vehicles alternate between values \( \hat{v}_i = 4 \) and \( \hat{v}_i = 5 \) and only sporadically have the same value for more than 5 timesteps. However, since the vehicles in Figure 34 are located closer to each other, there occasionally is a vehicle with velocity \( \hat{v}_i = 3 \) present. Still, no congestion is formed. One should note the difference between the two representations in Figure 33 and Figure 34 as the representation of Nagel & Schreckenberg (1992) is an excerpt from a modelled circular road, while the heatmaps are a representation of the complete circular road. Hence, after leaving one of the last positions with a velocity \( \hat{v}_i \), vehicles enter the diagram in the next timestep at one of the first positions, corresponding to \( i + \hat{v}_i - L \). For example, a vehicle that is at position 99 at timestep \( t \) with velocity \( \hat{v}_i = 3 \), is at position 99 + 3 - 100 = 2 at timestep \( t + 1 \).
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Figure 33: Heatmap for closed system with random initial configuration, with $p = 0.5$, $\rho = 0.03$, $L = 10^2$ and $T = 10^2$. Once the vehicles have come to their maximum velocity, we see free traffic. The initial random configuration is at $t = 0$ at the top of the figure.

Figure 34: Simulated traffic with density $\rho = 0.03$. Empty sites are represented by a dot and sites which are occupied by a car are represented by the integer number of its velocity. Each line shows the configuration of one timestep, using the same rules as used for the heatmaps in this section. Figure from Nagel & Schreckenberg (1992).

Changing the density to $\rho = 0.1$ vehicle per site immediately results in a clear congestion pattern. In Figure 35 such a pattern occurs at the beginning of the road, where vehicles of maximum velocity $\hat{v}_i = \hat{v}_{\text{max}} = 5$ and of velocity $\hat{v}_i = 4$ have to slow down to $\hat{v}_i = 0$, causing a standstill. Vehicles approaching from behind awaits the same deceleration, after which they can only start accelerating again after several timesteps (visible by the vertical length of the sites coloured black). Especially the upstream motion of the traffic jam is again an important and realistic re-
sult. Comparing the heatmap to Figure 36, a similar congestion pattern is found, also showing
the upstream motion and approaching vehicles that have to slow down from higher velocities to
\( \dot{v}_i = 0 \). Similar to Figure 35, vehicles can clearly be seen accelerating from the traffic jam to reach
free traffic again.

Figure 35: Heatmap for closed system with random initial configuration, with \( p = 0.5, \rho = 0.1, L = 10^2 \) and
\( T = 10^2 \). The initial random configuration is at \( t = 0 \) at the top of the figure.

Figure 36: Simulated traffic with density \( \rho = 0.1 \). Empty sites are represented by a dot and sites which are
occupied by a car are represented by the integer number of its velocity. Each line shows the configuration of
one timestep, using the same rules as used for the heatmaps in this section. Figure from Nagel & Schreckenber[1992].
When we choose an even higher vehicle density $\rho = 0.35$ vehicles per site, we see that traffic jams occur more frequently (Figure 37), which can be expected. Many traffic jams follow each other shortly, resulting in so-called start-stop traffic. The intensity of the traffic jams is also higher than before, as the vertical length of the consecutive sites coloured black is higher than in Figure 35.

Figure 37: Heatmap for closed system with random initial configuration, with $p = 0.5, \rho = 0.35, L = 10^2$ and $T = 10^2$. The initial random configuration is at $t = 0$ at the top of the figure.

Figure 38: Heatmap for closed system with random initial configuration, with $p = 0.5, \rho = 0.9, L = 10^2$ and $T = 10^2$. The initial random configuration is at $t = 0$ at the top of the figure.

A density that was not explored by Nagel & Schreckenberg (1992) using a spatio-temporal representation is $\rho = 0.9$ vehicles per site, which is particularly interesting. At such a high density, traffic is already completely congested and almost reaches a fully saturated jam, i.e., almost all sites are filled, so that vehicles all use up approximately 7.5 metres of the road. As Figure 38 shows, traffic is almost completely at a standstill, apart from a small number of vehicles that accelerate to the lowest possible still-moving velocity $\hat{v}_i = 1$ and then immediately braking to
\( \dot{v}_i = 0 \) again. This is comparable to a situation of a congested area in which vehicles are almost all at a standstill, while sometimes a possibility occurs in which a vehicle can move forward for only a short distance. The time that vehicles are at a standstill is, therefore, rather long. In Figure 38 for which the code can be found in Appendix E.2.1 there is, for instance, a vehicle that stays at position 49 for 59 timesteps, which corresponds to almost a minute of not moving at all. This is possible for very heavy congestion, such as HCT, described in section 3.4.2 but in the most common congestion patterns, such times do not occur, as traffic still tends to move slightly. Since \( L = 10^2 \), the part of the road showing this congestion is approximately \( 7.5 \cdot 10^2 = 750 \) metres long, which is a realistic length to show such congestion. However, this kind of congestion is now largely influenced by the road being circular, as vehicles have no opportunity to leave the congestion. Still, it shows a pattern that can be compared to realistic situations of traffic congestion, as explained above.

5.3 Slowdown Probability

As has been mentioned in section 4, an important aspect of the Nagel-Schreckenberg model is that it is nondeterministic. Indeed, it depends on a slowdown probability \( p \), that takes into account the individual fluctuations in acceleration that do not depend on the vehicle in front, but have other causes. Examples are seeing braking lights when the vehicle in front does not brake, but merely touches the pedal, being distracted by something at the side of the road, such as a billboard or a bird of prey, or having a shock reaction due to a vehicle unexpectedly overtaking or driving very fast. Many more examples can be the cause of such fluctuations, but their most important characteristic is that only individual vehicles are influenced.

![Figure 39: Heatmap for closed system with random initial configuration, with \( p = 0, \rho = 0.1, L = 10^2 \) and \( T = 10^2 \). The initial random configuration is at \( t = 0 \) at the top of the figure.](image)

If we would not take into account such fluctuations, i.e., \( p = 0 \), the model would become deterministic, so that it is always clear what pattern will arise. This is illustrated by Figure 39 and Figure 40 where \( p = 0 \). The code used for these figures can be found in Appendix E.2. When the density is \( \rho = 0.1 \) vehicles per site (Figure 39), almost immediately a pattern is started where every vehicle moves at the maximum speed \( \dot{v}_{\text{max}} \) and keeps having the same velocity. This raises the hypothesis that at densities that are low enough, no traffic congestion would occur when large perturbations are not taken into account. In other words, there are no individual fluctuations, so that there is no reason to slow down and hence no traffic congestion. However, this is
not at all realistic in traffic where vehicles are being driven by humans, since these fluctuations will always be part of human nature. It could be an interesting case for autonomous vehicles. On the other hand, when the density is higher, for example $\rho = 0.35$ vehicles per site, as in Figure 40 we find other patterns. We see that these patterns are shifted backward with every timestep, thus propagating upstream, since the vehicle that is behind has to adjust to the leading vehicle. This linearity is not a realistic representation of real traffic.

Figure 40: Heatmap for closed system with random initial configuration, with $p = 0, \rho = 0.35, L = 10^2$ and $T = 10^2$. The initial random configuration is at $t = 0$ at the top of the figure.

In order for the slowdown probability to have a realistic value, it cannot be too high. Indeed, when we choose $p = 0.9$, we see a pattern occurring where vehicles take many timesteps to accelerate from $\hat{v}_i = 0$ to $\hat{v}_i = 2$ and are at a standstill for a long time, with no clear reason, as illustrated in Figure 41 for which the code can be found in Appendix E.2. At a higher density
of \( \rho = 0.35 \) vehicles per site, as used in Figure 42, for which the code can be found in E.2.2, the traffic is largely jammed, where vehicles take more time to start driving after being at a standstill and almost no high velocities are reached, in contrast to the more diverse traffic in Figure 37. Therefore, the widely-used value \( p = 0.5 \) indeed seems reasonable and realistic.

**Figure 42:** Heatmap for closed system with random initial configuration, with \( p = 0.9, \rho = 0.35, L = 10^2 \) and \( T = 10^2 \). The initial random configuration is at \( t = 0 \) at the top of the figure.

**Figure 43:** Heatmap for open system with random initial configuration, with \( p = 0.5, \rho = 0.03, L = 10^2 \) and \( T = 10^2 \). The initial random configuration is at \( t = 0 \) at the top of the figure.

### 5.4 Bottleneck Situation

In the real-life traffic situation where a bottleneck occurs in the sense that two lanes merge into one lane and after a certain distance they form multiple lanes again, an open system of the Nagel-Schreckenberg model can be used, as described in section 4. Nagel & Schreckenberg (1992) use the assumption that four lanes are formed after the bottleneck, but already similar patterns will
be seen when the motorway rather resizes to two lanes. In the representation of this system, the
diagram represents the one-lane bottleneck situation, whereas the boundary conditions mimick the
behaviour caused by the multiple lanes outside the bottleneck situation. When we use the
same random initial configuration as before, we then get results that are given in Figure 43, Figure 44 and Figure 45 for starting densities $\rho = 0.03$ vehicles per site, $\rho = 0.1$ vehicles per site and $\rho = 0.35$ vehicles per site, respectively. The code for these figures is given in Appendix E.2.3. At low density, already some small perturbations occur, in contrast to the closed system in Figure 33. At density $\rho = 0.1$ vehicles per site, we can clearly see the slow and, at some places, slightly congested traffic near the start of the bottleneck, where vehicles have had to merge and should start driving. At higher density, one can determine an area forming free traffic near the end of the bottleneck region, where vehicles can accelerate into a wider part of the motorway. This is particularly different from Figure 37, where vehicles that reach the end of the road segment are still influenced by vehicles at the start of the road segment, due to the periodic nature of the closed system.

When we use the zero initial configuration, i.e., no vehicles in the initial configuration, it is important to notice that there is no starting density, as there is no vehicle at timestep 0, i.e., $\rho = 0$. In Figure 46, for which the code can be found in Appendix E.3, we see that near the start of the bottleneck, there are slightly more areas of small congestion. However, it does not give rise to a significant difference with the representations where we used a random initial configuration. Still, there clearly is no congestion after site 10, i.e., only free traffic is found, largely corresponding to Figure 43, where there is a very low density. Therefore, using the zero initial configuration does not seem a realistic way to model a bottleneck. After all, traffic realistically does not drive at maximum velocity through bottlenecks, unless the density is low and it is permitted. In Appendix D a different heatmap is presented, where the slowdown probability is set to $p = 0.9$.

![Figure 44: Heatmap for open system with random initial configuration, with $p = 0.5$, $\rho = 0.1$, $L = 10^2$ and $T = 10^5$. The initial random configuration is at $t = 0$ at the top of the figure.](image-url)
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Figure 45: Heatmap for open system with random initial configuration, with $p = 0.5$, $\rho = 0.35$, $L = 10^2$ and $T = 10^2$. The initial random configuration is at $t = 0$ at the top of the figure.

Figure 46: Heatmap for open system with zero initial configuration, with $p = 0.5$, $L = 10^2$ and $T = 10^2$. The zero initial configuration is at $t = 0$ at the top of the figure.

5.5 Fundamental Diagram

Using the Nagel-Schreckenberg model, a fundamental diagram can be created for the closed system, where vehicle flow is depicted in terms of vehicle density. However, since a closed system does not pose a realistic scenario, Nagel & Schreckenberg (1992) chose to mimic real conditions by measuring the density and flow at a certain point. Therefore, a random position $i^*$ is chosen as a measuring point. Then, the simulation is run for several timesteps, where for each timestep $t$ a variable $n_f(t)$ denotes whether site $i^*$ is occupied or not, i.e.,

$$n_f(t) = \begin{cases} 1 & \text{if site } i^* \text{ is occupied at timestep } t; \\ 0 & \text{if site } i^* \text{ is empty at timestep } t. \end{cases}$$

(51)
Next, the values are averaged over a time period $T$. However, since the first timesteps are largely influenced by the initial configuration, data is only collected after $t_0$ timesteps, where we choose $t_0 = L \cdot 10$ seconds, after Nagel & Schreckenberg (1992). This yields the following expression for the density $\rho_T$, averaged over $T$ timesteps after $t_0$ initial timesteps:

$$
\rho_T = \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} n_t(t).
$$

(52)

To measure the vehicle flow, we use a similar approach. Using the same simulation, we start measuring for $T$ timesteps, after the initial $t_0$ timesteps. In every timestep, the variable $n_{i^*,i^*+1}(t)$ is updated, where $n_{i^*,i^*+1}(t)$ denotes the detection of a vehicle between sites $i^*$ and $i^* + 1$ at timestep $t$, i.e.,

$$
n_{i^*,i^*+1}(t) = \begin{cases} 
1 & \text{if a motion is detected between site } i^* \text{ and } i^* + 1 \text{ at timestep } t; \\
0 & \text{otherwise.}
\end{cases}
$$

(53)

We can concretise this even further, as follows, where $i$ denotes the site currently being evaluated, $i'$ denotes the position of the same vehicle at the next timestep and $i^*$ still marks the measuring point:

$$
n^i_{i^*,i^*+1}(t) = \begin{cases} 
1 & \text{if } i \leq i^* < i' \text{ at timestep } t; \\
0 & \text{otherwise.}
\end{cases}
$$

(54)

Hence, the value of $n^i_{i^*,i^*+1}(t)$ is set to 1 if and only if both site $i$ occurs before or exactly at the measuring point at position $i^*$ and the vehicle’s position $i'$ in the next timestep occurs strictly after the measuring point at $i^*$. Therefore, it always detects motion between site $i^*$ and $i^* + 1$. This, then, yields the following expression for the flow $q_T$, averaged over $T$ timesteps after $t_0$ initial timesteps:

$$
q_T = \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} n^i_{i^*,i^*+1}(t).
$$

(55)

Being able to measure densities and corresponding flows in the model, we can represent this information in a fundamental diagram. In order to do so, we need different flow-density value pairs. Therefore, we compute different pairs $\rho_T$ and $q_T$ by changing the number of vehicles in the system. We increment the number of vehicles $N_t$ starting with $N = 0$ (naturally corresponding to a vehicle density $\rho_T = 0$ vehicles per site), and ending with $N = L$ (corresponding to a vehicle density $\rho_T = 1$ vehicle per site). When choosing $L = 10^4$ and $T = 10^2$, incrementing $N$ by 100 vehicles yields $\frac{L}{T} = \frac{10^4}{10^2} = 100$ flow-density value pairs. Each such value pair is plotted in a flow-density diagram, as seen in Figure 47. The MATLAB code used for all figures produced can be found in Appendix E.4.

Choosing a shorter road section $L = 10^3$ improves the computation time, allowing to use a smaller incrementation size of $N_t$ by 1 vehicle every step. Therefore, for all possible numbers of vehicles present in the system, the density and flow are averaged over $T = 10^2$ timesteps, resulting in $\frac{L}{T} = \frac{10^3}{10^2} = 1000$ points in the diagram, depicted in Figure 48. The fundamental diagram provided by Nagel & Schreckenberg (1992), as depicted in Figure 49, shows a very similar scatter plot. The dots display the flow-density values for a road of length $L = 10^3$, also averaged over $T = 10^2$ timesteps. However, there are fewer than 1000 points. Comparing the two plots to each other, a similar pattern occurs. As we can expect from a fundamental diagram, the flow quickly increases as the density increases. Then, at some point, the flow stops increasing and more slowly decreases toward a flow of $q_T = 0$ vehicles per site per timestep at density $\rho_T = 1$ vehicle per site, which is a realistic value, since traffic cannot move when fully saturated. Theoretically, traffic could move when fully saturated, but due to the behaviour of human beings, small perturbations are inevitable, while there is no room for perturbations in fully saturated
traffic. The diagram shows values that are in line of what we expect from fundamental diagrams, as elaborated on in section 3.1.1. However, not only does the plot in Figure 48 show similarities with the simulated data from Nagel & Schreckenberg (1992), but it also fits empirical data, such as the empirical fundamental diagram (EFD) in Figure 50, retrieved from Nagel & Schreckenberg (1992), and the EFD in Figure 51, retrieved from Bramich et al. (2022).

Figure 47: Fundamental diagram, with $p = 0.5$, $L = 10^4$ and $T = 10^2$, displaying 100 points.

Since $\rho_T$ is an average over $T$ timesteps, we can take the limit of $T$ to infinity, to find the real value of $\rho$, i.e., the number of vehicles per site $\rho = \frac{N}{L}$, as initiated in the random initial configuration. We find that:

$$\lim_{T \to \infty} \rho_T = \rho.$$  \hspace{1cm} (56)

This is illustrated by Figure 52, where the same number of sites $L = 10^3$ and the same incrementation size of 1 vehicle every step are chosen, but the values are averaged over a factor 1000 more timesteps, i.e., $T = 10^3$. Now, the averages lay closer to each other and almost form a line, which is very similar to the line in Figure 49, where $L = 10^4$ and $T = 10^6$. We see the mirrored lambda-like, nearly bilinear shape of the fundamental diagram, as described in section 3.1.1. A clear transition from free traffic, visible by the linear growth from density $\rho_T = 0$ vehicles per site to approximately $\rho_T = 0.07$ vehicles per site, to congested traffic can be seen in the diagram. At the critical density $\rho_T^{cr} \approx 0.07$ vehicles per site, the traffic breaks down into congestion, as is to be expected (see section 3.1.1). The maximum value of the flow lies around $q_T^{max} \approx 0.33$ vehicles per site per timestep. The value of $\rho_T^{cr}$ and $q_T^{max}$ are difficult to predict beforehand. According to Nagel & Schreckenberg (1992), it depends on the size of the system, which we interpret as the size of $L$. Indeed, in Figure 47, where $L = 10^4$ rather than $L = 10^3$, the critical density seems to be higher than 0.1 vehicles per site and the maximum flow value is higher than 0.35 vehicles per site per timestep. For the deterministic variant of the model, where $p = 0$, the size of the system should not matter for the value of the critical density, according to Nagel & Schreckenberg (1992). One should, however, note that the congested area displayed in the fundamental diagram is not reliable, since congested traffic is not stable, as explained in section 3.1.1. However, in spite of congested traffic not necessarily relating to the fundamental diagram, the Nagel-Schreckenberg model does display the fundamental diagram in the expected manner.
Figure 48: Fundamental diagram, with $p = 0.5$, $L = 10^3$ and $T = 10^2$, displaying 1000 points.

Figure 49: Traffic flow in terms of density, with $L = 10^4$, $T = 10^2$ (dots) and $T = 10^6$ (line), from simulations. Figure from Nagel & Schreckenberg (1992).
Figure 50: Empirical fundamental diagram (EFD), displaying traffic flow in terms of occupancy - the percentage of the road that is covered by vehicles - from measurements in reality. Figure from Nagel & Schreckenberg (1992).

Figure 51: Empirical fundamental diagram (EFD), displaying traffic flow in terms of occupancy - the percentage of the road that is covered by vehicles - from measurements in reality. Figure from Bramich et al. (2022).
6 Conclusions and Recommendations

We have given a broad overview of motorway traffic modelling, including fundamental concepts, four widely-used classes of models and two classifications of congestion patterns. We found that the concept of a fundamental diagram is indeed a fundamental part of traffic modelling, we came across numerous mentions of bottleneck situations and we have concluded that it is important to know different states of traffic. Moreover, we found that several classifications of congestion patterns are used, causing controversies among researchers. The two most important classifications are that by Kerner (1999), refuted through several studies, and that by Schönhof & Helbing (2007), overcoming the problems raised by Kerner’s classification. Next to that, four main classes of traffic models were found, each containing a wide collection of different models, whether or not based on the same initial model. One of those classes is the class of cellular automata models, among which the Nagel-Schreckenberg model (Nagel & Schreckenberg, 1992) is the most well-known. After implementation of this model, we found that similar results can be produced, using more advanced, coloured representations. Therefore, we conclude that our representation of the Nagel-Schreckenberg model is verified, in the sense that all figures can be reproduced. Further research can be done on reproducing the different congestion patterns explained in section 3.4. Also, the literature overview could be extended by looking at the implementation of different types of driver behaviour, applying the distinction between long and short vehicles, adding lane-changing terms and dealing with new behaviour that arises due to autonomous driving.

Altogether, we have given a concise overview of the fundamental concepts in traffic modelling and introduced several different models, with an educational approach to explaining, and we examined the Nagel-Schreckenberg model in more detail, so that implementing it raises fewer questions. We are confident that reading this report gives a better view of traffic modelling, enabling the reader to judiciously choose a subject or class of models to move their focus to.

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### A Breakdown Probability

The original figure that Figure 9 was based on, is given in Figure 53. Helbing (2001) changed some values, changed the layout for clarity and added the dotted line with “$Q_c^2$” to better accompany the text of the review article. Still, the same conclusions can be drawn from both figures.

![Figure 53: Probability of breakdown of free traffic as a function of the empirical vehicle flow, measured using different time intervals for averaging the probabilities. Figure from Persaud et al. (1998).](image)

### B Simulated Traffic Patterns

The simulations that were made to fit the patterns arising from the empirically found traffic patterns in Figure 19 are shown in Figure 54.
Figure 54: Spatio-temporal velocity representations of the five types of congestion patterns and an additional pattern found by Helbing et al. (2009) in (d), simulated in accordance with the empirical results from Figure 19. An on-ramp is located at $x = 0$. Velocities are displayed upside down, for a better illustration of the traffic patterns. (a) shows a moving localised cluster (MLC). (b) shows stop-and-go waves (SGW). (c) shows oscillating congested traffic (OCT). (d) shows a widening synchronised flow pattern (WSP), as classified by Kerner (2002). (e) shows a pinned localised cluster (PLC). (f) shows homogeneous congested traffic (HCT). Figure from Helbing et al. (2009).

C Critique

Although Helbing (2001) has called the Nagel-Schreckenberg model “extremely compact and elegant”, the article does not fully correctly describe the model. It does not mention the importance of the order of execution of the four rules. In fact, the rules are presented in a different order than required. Whereas Helbing (2001) first mentions the rule of motion, followed by the three velocity-adaptation rules, the Nagel-Schreckenberg model should execute the rule of motion after having adjusted the velocity (Nagel & Schreckenberg, 1992). In particular, when first executing the rule of motion, collisions can occur. In other words, two vehicles can move to the same site in the same timestep, violating the definition of the model, which states that a site can only be occupied by one vehicle. An example is illustrated in Figure 55 for which the MATLAB code can be found in Appendix C.1. Here, the position is the considered site, with $L = 25$, and for each timestep the current configuration is depicted. A site with $-1$ reflects that the site is empty. The other depicted values represent the nondimensional velocity of the vehicle that occupies the cell, with $\hat{v}_{\text{max}} = 5$. To produce Figure 55 a specifically chosen random initial configuration with 4 vehicles is used. Then, for every timestep, the rule of motion is applied first, after which the velocity is updated. Note that at timestep 4, the vehicle at position 8 has velocity 4, hence has to move 4 sites in the next timestep, ending at position 12. However, the vehicle that is at position 12 at timestep 4 has velocity 0, meaning that it is still fixed to position 12 in the next timestep. Thus, the two vehicles collide in timestep 5 at position 12. One could quickly see this by noting that the initial configuration at timestep 0 contains 4 vehicles, while only 3 vehicles are left at timestep 5. Therefore, the Nagel-Schreckenberg model as presented by Helbing (2001) suggests an order of execution that is not allowed and hence incorrect.
C.1 MATLAB Code for Example

The code below is used to create Figure 55.

```matlab
1 clear all
2 clc
3
4 % HEATMAP EXAMPLE CRITIQUE HELBING
5 % Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
6 % Closer Look at the Nagel-Schreckenberg Model
7 % Mathieu de Ridder, July 28, 2023
8 % Eindhoven University of Technology (TU/e)
9
10 speeds = [-1, 0, -1, -1, -1, -1, -1, -1, -1, -1, -1, 0, 0, -1, -1, ... -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]; % initial configuration
11 speeds = [speeds; -1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 0, 0, 1, ... -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]; % first timestep
12 speeds = [speeds; -1, -1, 2, -1, -1, -1, -1, -1, -1, -1, -1, 0, 0, -1, ... -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]; % second timestep
13 speeds = [speeds; -1, -1, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]; % third timestep
14 speeds = [speeds; -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]; % fourth timestep
15 speeds = [speeds; -1, -1, 3, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1]; % fifth timestep
```

Figure 55: Incorrectly applied update rules, causing a collision of vehicles.
D Open System with High Slowdown Probability

When the slowdown probability is set to $p = 0.9$, we immediately see in Figure 56 a different pattern from Figure 46. Due to this high probability, vehicles take a long time to leave the start of the bottleneck and then, again, take a long time to accelerate to higher velocities than $\hat{v}_i = 1$. This quickly gives rise to a larger jam, where vehicles have to wait for a long time before they can start driving again. This leads to a system where, in the case of Figure 56, for which the code can be found in Appendix D.1, only three vehicles leave the system in $T = 10^2$ timesteps.

Figure 56: Heatmap for open system with zero initial configuration, with $p = 0.9$, $L = 10^2$ and $T = 10^2$. The zero initial configuration is at $t = 0$ at the top of the figure.

D.1 MATLAB Code for Heatmap

The code below is used to create Figure 56:

```matlab
1 clear all
2 clc
3
4 % HEATMAP USING ZERO INITIAL CONFIGURATION IN OPEN SYSTEM
5 % Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
6 % Closer Look at the Nagel-Schreckenberg Model
```
\[
L = 10^2; \quad \% \text{number of sites, approximately } 7.5 \text{ m each}
\]
\[
N = 10; \quad \% \text{total number of cars}
\]
\[
\rho = N/L; \quad \% \text{density of vehicles per site}
\]
\[
v_{\text{max}} = 5; \quad \% \text{maximum integer velocity in sites per timestep}
\]
\[
p = 0.9; \quad \% \text{slowdown probability}
\]

\[
\text{positions} = \text{zeros}(1, L); \quad \% \text{positions}(i) = 1 \text{ if occupied, 0 otherwise}
\]
\[
\text{velocities} = -1*\text{ones}(1, L); \quad \% \text{values between } -1, 0, \ldots, v_{\text{max}}; -1 \text{ empty}
\]
\[
\text{speeds} = \text{velocities}; \quad \% \text{data for heatmap}
\]

\[
\% \text{SIMULATION (T time-steps)}
\]
\[
T = 100; \quad \% \text{number of timesteps}
\]
\[
\text{for } t = 1:T \quad \% \text{update using 4 rules}
\]
\[
\text{positions}\_\text{updated} = \text{zeros}(1, L); \quad \% \text{array for new positions}
\]
\[
\text{velocities}\_\text{updated} = -1*\text{ones}(1, L); \quad \% \text{array for new velocities}
\]
\[
\text{for } i = 1:L-5 \quad \% \text{traverse over first } L-5 \text{ sites only}
\]
\[
\text{if } (i == 1) \&\& (\text{positions}(i) == 0) \quad \% \text{site 1 empty}
\]
\[
\text{positions}(i) = 1; \quad \% \text{place new vehicle}
\]
\[
\text{velocities}(i) = 0; \quad \% \text{assign velocity 0}
\]
\[
\end
\]
\[
\% \text{only update when site is occupied}
\]
\[
\text{v}_{\text{i}} = \text{velocities}(i); \quad \% \text{velocity of vehicle at site } i
\]
\[
\text{if } v_{\text{i}} < v_{\text{max}} \quad \% \text{first condition for acceleration}
\]
\[
\text{count} = 0; \quad \% \text{counter for free sites in front}
\]
\[
\text{for } j = 1:v_{\text{i}} + 1 \quad \% \text{check whether distance is larger than } v_{\text{i}} + 1
\]
\[
\text{site}_{\text{cand}} = i + j; \quad \% \text{candidate for new site}
\]
\[
\text{if } \text{positions(site}_{\text{cand}}) == 0 \quad \% \text{candidate is empty}
\]
\[
\text{count} = \text{count} + 1; \quad \% \text{count free site}
\]
\[
\end
\]
\[
\% \text{Deceleration}
\]
\[
\text{v}_{\text{i2}} = \text{velocities}(i); \quad \% \text{updated velocity}
\]
\[
\text{for } k = 1:v_{\text{i2}} \quad \% \text{check all } v_{\text{i}} \text{ sites in front}
\]
\[
\text{site}_{\text{cand2}} = i + k; \quad \% \text{candidate for new site}
\]
\[
\text{if } \text{positions(site}_{\text{cand2}}) == 1 \quad \% \text{site occupied}
\]
\[
\text{velocities}(i) = k - 1; \quad \% \text{vehicle decelerates}
\]
\[
\text{break} \quad \% \text{first vehicle ahead determines new velocity}
\]
\[
\end
\]
\[
\% \text{Randomisation}
\]
\[
\text{if } \text{rand} <= p \quad \% \text{velocity is reduced with probability } p
\]
\[
\text{v}_{\text{i3}} = \text{velocities}(i); \quad \% \text{updated velocity}
\]
A Brief Introduction to Traffic Modelling

if v_i3 > 0 % velocity cannot be reduced if <= 0
    velocities(i) = v_i3 - 1; % vehicle slows down
end

% Motion
v_new = velocities(i); % updated velocity
new_position = i + v_new; % new position
positions_updated(new_position) = 1; % update position
velocities_updated(new_position) = v_new; % update velocity
end

speeds = [speeds; velocities]; % add velocity data for heatmap

for i = L-4:L % delete vehicles in last 5 sites
    positions_updated(i) = 0; % remove possible vehicle
    velocities_updated(i) = -1; % set velocity to -1
end

positions = positions_updated; % updated positions for all vehicles
velocities = velocities_updated; % updated velocities for all vehicles
end

% HEATMAP
times = 0:T; % array of all timesteps
sites = 1:L; % array of all sites
custom_color = [1 1 1; 0 0 0; 0 0 1; 0 0.4 0.8; 0 0.6 0.7; 0 0.8 0.6;... 0 1 0.5]; % colour scheme
h = heatmap(sites, times, speeds); % heatmap
h.GridVisible = 'off'; % no grid
h.Title = {'Spatio-temporal representation','p=0.9'}; % title
h.XLabel = 'position'; % label for horizontal axis
h.YLabel = 'timestep'; % label for vertical axis
h.Colormap = custom_color; % use given colour scheme

E MATLAB Code

All MATLAB code used to do the simulation and create figures can be found below.

E.1 Heatmap with Uniform Initial Configuration

The code below is used to create Figure 31 and Figure 32.
clear all
clc

% HEATMAP USING UNIFORM INITIAL CONFIGURATION
% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)
L = 10^2; % number of sites, approximately 7.5 m each
N = 10; % total number of cars
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.5; % slowdown probability

positions = zeros(1, L); % positions(i) == 1 if occupied, 0 otherwise
velocities = -1*ones(1, L); % values between -1, 0, ..., v_max; -1 empty

% INITIAL CONFIGURATION: t = 0
for k = 1:N % add all vehicles to configuration
    vehicle = round(1/rho) * (k - 1) + 1; % position of vehicle
    positions(vehicle) = 1; % approximately uniform
    velocities(vehicle) = v_max - 1; % sites/timestep
end

% SIMULATION (T time-steps)
speeds = velocities; % data for heatmap
T = 100; % number of timesteps
for t = 1:T % update using 4 rules
    positions_updated = zeros(1, L); % array for new positions
    velocities_updated = -1*ones(1, L); % array for new velocities
    for i = 1:L % traverse over all sites
        if positions(i) == 1 % only update when site is occupied
            % Acceleration
            v_i = velocities(i); % velocity of vehicle at site i
            if v_i < v_max % first condition for acceleration
                count = 0; % counter for free sites in front
                for j = 1:v_i + 1 % check wether distance is larger than % v_i + 1
                    site_cand = i + j; % candidate for new site
                    if site_cand > L % exceeds array
                        site_cand = site_cand - L; % circular road
                    end
                    if positions(site_cand) == 0 % candidate is empty
                        count = count + 1; % count free site
                    end
                end
                if count == v_i + 1 % all v_i + 1 sites empty
                    velocities(i) = v_i + 1; % vehicle accelerates
                end
            end
            % Deceleration
            v_i2 = velocities(i); % updated velocity
            for k = 1:v_i2 % check all v_i sites in front
                site_cand2 = i + k; % candidate for new site
                if site_cand2 > L % exceeds array
                    site_cand2 = site_cand2 - L; % circular road
                end
            end
        end
    end
end
if positions(site_cand2) == 1  % site occupied
velocities(i) = k - 1;  % vehicle decelerates
    break  % first vehicle ahead determines new velocity
end

% Randomisation
if rand <= p  % velocity is reduced with probability p
    v_i3 = velocities(i);  % updated velocity
    if v_i3 > 0  % velocity cannot be reduced if <= 0
        velocities(i) = v_i3 - 1;  % vehicle slows down
    end
end

% Motion
v_new = velocities(i);  % updated velocity
new_position = i + v_new;  % new position
if new_position > L  % exceeds array
    new_position = new_position - L;  % circular road
end
positions_updated(new_position) = 1;  % update position
velocities_updated(new_position) = v_new;  % update velocity
end

speeds = [speeds; velocities];  % add velocity data for heatmap
positions = positions_updated;  % updated positions for all vehicles
velocities = velocities_updated;  % updated velocities for all
% vehicles
end

% HEATMAP
times = 0:T;  % array of all timesteps
sites = 1:L;  % array of all sites
custom_color = [1 1 1; 0 0 0; 0 0 1; 0 0.4 0.8; 0 0.6 0.7; 0 0.8 0.6;...
    0 1 0.5];  % colour scheme
h = heatmap(sites, times, speeds);  % heatmap
h.GridVisible = off;  % no grid
h.Title = {'Spatio-temporal representation', 'p=0.5, \rho=0.1'};  % title
h.XLabel = 'position';  % label for horizontal axis
h.YLabel = 'timestep';  % label for vertical axis
h.Colormap = custom_color;  % use given colour scheme

E.2 Heatmap with Random Initial Configuration

The code below is used to create Figure 33, Figure 35 and Figure 37, by changing the value of N to 3, 10 and 35, respectively, and for Figure 39, Figure 40 and Figure 41, by changing the value of p to 0, with the value of N changed to 10 and 35, respectively, and changing the value of p to 0.9, with the value of N changed to 10.
% HEATMAP USING RANDOM INITIAL CONFIGURATION
% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)

L = 10^2; % number of sites, approximately 7.5 m each
N = 10; % total number of cars
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.5; % slowdown probability

positions = zeros(1, L); % positions(i) == 1 if occupied, 0 otherwise
velocities = -1*ones(1, L); % values between -1, 0, ..., v_max; -1 empty

% ALTERNATIVE INITIAL CONFIGURATION (RANDOM): t = 0
k = N; % place all vehicles
while k > 0 % continue until all vehicles are placed
    vehicle = randi([1, L]); % random position
    if positions(vehicle) == 0 % only place vehicle if site not occupied
        positions(vehicle) = 1; % place vehicle
        velocities(vehicle) = 0; % set velocity to 0
        k = k - 1; % go to next vehicle
    end
end

speeds = velocities; % data for heatmap

% SIMULATION (T time-steps)
T = 100; % number of timesteps
for t = 1:T % update using 4 rules
    positions_updated = zeros(1, L); % array for new positions
    velocities_updated = -1*ones(1, L); % array for new velocities
    for i = 1:L % traverse over all sites
        if positions(i) == 1 % only update when site is occupied
            % Acceleration
            v_i = velocities(i); % velocity of vehicle at site i
            if v_i < v_max % first condition for acceleration
                count = 0; % counter for free sites in front
                for j = 1:v_i + 1 % check weather distance is larger than
                    % v_i + 1
                    site_cand = i + j; % candidate for new site
                    if site_cand > L % exceeds array
                        site_cand = site_cand - L; % circular road
                    end
                    if positions(site_cand) == 0 % candidate is empty
                        count = count + 1; % count free site
                    end
                end
                if count == v_i + 1 % all v_i + 1 sites empty
                    velocities(i) = v_i + 1; % vehicle accelerates
                end
            end
        end
    end
% Deceleration
v_i2 = velocities(i); % updated velocity
for k = 1:v_i2 % check all v_i sites in front
    site_cand2 = i + k; % candidate for new site
    if site_cand2 > L % exceeds array
        site_cand2 = site_cand2 - L; % circular road
    end
    if positions(site_cand2) == 1 % site occupied
        velocities(i) = k - 1; % vehicle decelerates
        break % first vehicle ahead determines new velocity
    end
end

% Randomisation
if rand <= p % velocity is reduced with probability p
    v_i3 = velocities(i); % updated velocity
    if v_i3 > 0 % velocity cannot be reduced if <= 0
        velocities(i) = v_i3 - 1; % vehicle slows down
    end
end

% Motion
v_new = velocities(i); % updated velocity
new_position = i + v_new; % new position
if new_position > L % exceeds array
    new_position = new_position - L; % circular road
end
positions_updated(new_position) = 1; % update position
velocities_updated(new_position) = v_new; % update velocity
end

speeds = [speeds; velocities]; % add velocity data for heatmap
positions = positions_updated; % updated positions for all vehicles
velocities = velocities_updated; % updated velocities for all vehicles
end

% HEATMAP
times = 0:T; % array of all timesteps
sites = 1:L; % array of all sites
custom_color = [1 1 1; 0 0 0; 0 0 1; 0 0.4 0.8; 0 0.6 0.7; 0 0.8 0.6;... 0 1 0.5]; % colour scheme
h = heatmap(sites, times, speeds); % heatmap
h.GridVisible = 'off'; % no grid
h.Title = {'Spatio-temporal representation','p=0.5, \rho=0.1'}; % title
h.XLabel = 'position'; % label for horizontal axis
h.YLabel = 'timestep'; % label for vertical axis
h.Colormap = custom_color; % use given colour scheme
E.2.1 Heatmap with Random Initial Configuration at High Density

The code below is used to create Figure 38.

```matlab
clear all
clc

% HEATMAP USING RANDOM INITIAL CONFIGURATION AT HIGH DENSITY
% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)

L = 10^2; % number of sites, approximately 7.5 m each
N = 90; % total number of cars
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.5; % slowdown probability

positions = zeros(1, L); % positions(i) == 1 if occupied, 0 otherwise
velocities = -1*ones(1, L); % values between -1, 0, ..., v_max; -1 empty

% ALTERNATIVE INITIAL CONFIGURATION (RANDOM): t = 0
k = N; % place all vehicles
while k > 0 % continue until all vehicles are placed
    vehicle = randi([1, L]); % random position
    if positions(vehicle) == 0 % only place vehicle if site not occupied
        positions(vehicle) = 1; % place vehicle
        velocities(vehicle) = 0; % set velocity to 0
        k = k - 1; % go to next vehicle
    end
end

speeds = velocities; % data for heatmap

% SIMULATION (T time-steps)
T = 100; % number of timesteps
for t = 1:T % update using 4 rules
    positions_updated = zeros(1, L); % array for new positions
    velocities_updated = -1*ones(1, L); % array for new velocities
    for i = 1:L % traverse over all sites
        if positions(i) == 1 % only update when site is occupied
            % Acceleration
            v_i = velocities(i); % velocity of vehicle at site i
            if v_i < v_max % first condition for acceleration
                count = 0; % counter for free sites in front
                for j = 1:v_i + 1 % check whether distance is larger than v_i + 1
                    site_cand = i + j; % candidate for new site
                    if site_cand > L % exceeds array
                        site_cand = site_cand - L; % circular road
                    end
                    if positions(site_cand) == 0 % candidate is empty
                        count = count + 1; % count free site
                    end
                end
                if count == v_i + 1 % all sites are free
                    velocities_updated(i) = v_i + 1; % increase velocity by 1
                else
                    velocities_updated(i) = v_i; % no change in velocity
                end
            end
        end
    end

end
```

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end end
if count == v_i + 1 % all v_i + 1 sites empty
   velocities(i) = v_i + 1; % vehicle accelerates
end end

% Deceleration
v_i2 = velocities(i); % updated velocity
for k = 1:v_i2 % check all v_i sites in front
   site_cand2 = i + k; % candidate for new site
   if site_cand2 > L % exceeds array
      site_cand2 = site_cand2 - L; % circular road
   end
   if positions(site_cand2) == 1 % site occupied
      velocities(i) = k - 1; % vehicle decelerates
      break % first vehicle ahead determines new velocity
   end
end

% Randomisation
if rand <= p % velocity is reduced with probability p
   v_i3 = velocities(i); % updated velocity
   if v_i3 > 0 % velocity cannot be reduced if <= 0
      velocities(i) = v_i3 - 1; % vehicle slows down
   end
end

% Motion
v_new = velocities(i); % updated velocity
new_position = i + v_new; % new position
if new_position > L % exceeds array
   new_position = new_position - L; % circular road
end
positions_updated(new_position) = 1; % update position
velocities_updated(new_position) = v_new; % update velocity
end
end

speeds = [speeds; velocities]; % add velocity data for heatmap
positions = positions_updated; % updated positions for all vehicles
velocities = velocities_updated; % updated velocities for all % vehicles

% HEATMAP
times = 0:T; % array of all timesteps
sites = 1:L; % array of all sites
custom_color = [1 1 1; 0 0 0; 0 0 1]; % colour scheme
h = heatmap(sites, times, speeds); % heatmap
h.GridVisible = 'off'; % no grid
h.Title = {'Spatio-temporal representation','p=0.5, \rho=0.9'}; % title
h.XLabel = 'position'; % label for horizontal axis
h.YLabel = 'timestep'; % label for vertical axis
E.2.2 Heatmap with Random Initial Configuration at High Slowdown Probability

The code below is used to create Figure 42.

```matlab
% HEATMAP USING RANDOM INITIAL CONFIGURATION AT HIGH SLOWDOWN PROBABILITY
% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)

L = 10^2; % number of sites, approximately 7.5 m each
N = 35; % total number of cars
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.9; % slowdown probability

positions = zeros(1 , L); % positions(i) == 1 if occupied, 0 otherwise
velocities = -1*ones(1 , L); % values between -1, 0, ..., v_max; -1 empty

% ALTERNATIVE INITIAL CONFIGURATION (RANDOM): t = 0
k = N; % place all vehicles
while k > 0 % continue until all vehicles are placed
    vehicle = randi([1 ,L]); % random position
    if positions(vehicle) == 0 % only place vehicle if site not occupied
        positions(vehicle) = 1; % place vehicle
        velocities(vehicle) = 0; % set velocity to 0
        k = k -1; % go to next vehicle
    end
end

speeds = velocities; % data for heatmap

% SIMULATION (T time-steps)
T = 100; % number of timesteps
for t = 1:T % update using 4 rules
    positions_updated = zeros(1,L); % array for new positions
    velocities_updated = -1*ones(1,L); % array for new velocities
    for i = 1:L % traverse over all sites
        if positions(i) == 1 % only update when site is occupied
            % Acceleration
            v_i = velocities(i); % velocity of vehicle at site i
            if v_i < v_max % first condition for acceleration
                count = 0; % counter for free sites in front
                for j = 1:v_i + 1 % check whether distance is larger than
                    site_cand = i + j; % candidate for new site
                    if site_cand > L % exceeds array
                        site_cand = site_cand - L; % circular road
```
if positions(site_cand) == 0  % candidate is empty
    count = count + 1;  % count free site
end

if count == v_i + 1  % all v_i + 1 sites empty
    velocities(i) = v_i + 1;  % vehicle accelerates
end

% Deceleration
v_i2 = velocities(i);  % updated velocity
for k = 1:v_i2  % check all v_i sites in front
    site_cand2 = i + k;  % candidate for new site
    if site_cand2 > L  % exceeds array
        site_cand2 = site_cand2 - L;  % circular road
    end
    if positions(site_cand2) == 1  % site occupied
        velocities(i) = k - 1;  % vehicle decelerates
        break  % first vehicle ahead determines new velocity
    end
end

% Randomisation
if rand <= p  % velocity is reduced with probability p
    v_i3 = velocities(i);  % updated velocity
    if v_i3 > 0  % velocity cannot be reduced if <= 0
        velocities(i) = v_i3 - 1;  % vehicle slows down
    end
end

% Motion
v_new = velocities(i);  % updated velocity
new_position = i + v_new;  % new position
if new_position > L  % exceeds array
    new_position = new_position - L;  % circular road
end
positions_updated(new_position) = 1;  % update position
velocities_updated(new_position) = v_new;  % update velocity

speeds = [speeds; velocities];  % add velocity data for heatmap
positions = positions_updated;  % updated positions for all vehicles
velocities = velocities_updated;  % updated velocities for all vehicles
end

% HEATMAP
times = 0:T;  % array of all timesteps
sites = 1:L;  % array of all sites
custom_color = [1 1 1; 0 0 0; 0 0 1; 0 0.4 0.8; 0 0.6 0.7];  % colour scheme
h = heatmap(sites, times, speeds);  % heatmap
E.2.3 Heatmap for Open System with Random Initial Configuration

The code below is used to create Figure 43, Figure 44 and Figure 45 by changing the value of $N$ to 3, 10 and 35, respectively.

```matlab
clear all
cclc

% HEATMAP USING RANDOM INITIAL CONFIGURATION IN OPEN SYSTEM
% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)

L = 10^2; % number of sites, approximately 7.5 m each
N = 10; % total number of cars
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.5; % slowdown probability

positions = zeros(1, L); % positions(i) == 1 if occupied, 0 otherwise
velocities = -1*ones(1, L); % values between -1, 0, ..., v_max; -1 empty

% ALTERNATIVE INITIAL CONFIGURATION (RANDOM): t = 0
k = N; % place all vehicles
while k > 0 % continue until all vehicles are placed
    vehicle = randi([1,L]); % random position
    if positions(vehicle) == 0 % only place vehicle if site not occupied
        positions(vehicle) = 1; % place vehicle
        velocities(vehicle) = 0; % set velocity to 0
    end
    k = k -1; % go to next vehicle
end

speeds = velocities; % data for heatmap

% SIMULATION (T time-steps)
T = 100; % number of timesteps
for t = 1:T % update using 4 rules
    positions_updated = zeros(1, L); % array for new positions
    velocities_updated = -1*ones(1, L); % array for new velocities
    for i = 1:L-5 % traverse over first L-5 sites only
        if (i == 1) && (positions(i) == 0) % site 1 empty
            positions(i) = 1; % place new vehicle
            velocities(i) = 0; % assign velocity 0
        end
        if positions(i) == 1 % only update when site is occupied
            positions_updated(i) = 1; % place new vehicle
            velocities_updated(i) = 0; % assign velocity 0
        end
    end
end
```
% Acceleration
v_i = velocities(i); % velocity of vehicle at site i
if v_i < v_max % first condition for acceleration
count = 0; % counter for free sites in front
for j = 1:v_i + 1 % check whether distance is larger than
    site_cand = i + j; % candidate for new site
    if positions(site_cand) == 0 % candidate is empty
        count = count + 1; % count free site
    end
end
if count == v_i + 1 % all v_i + 1 sites empty
    velocities(i) = v_i + 1; % vehicle accelerates
end
end

% Deceleration
v_i2 = velocities(i); % updated velocity
for k = 1:v_i2 % check all v_i sites in front
    site_cand2 = i + k; % candidate for new site
    if positions(site_cand2) == 1 % site occupied
        velocities(i) = k - 1; % vehicle decelerates
        break % first vehicle ahead determines new velocity
    end
end

% Randomisation
if rand <= p % velocity is reduced with probability p
    v_i3 = velocities(i); % updated velocity
    if v_i3 > 0 % velocity cannot be reduced if <= 0
        velocities(i) = v_i3 - 1; % vehicle slows down
    end
end

% Motion
v_new = velocities(i); % updated velocity
new_position = i + v_new; % new position
positions_updated(new_position) = 1; % update position
velocities_updated(new_position) = v_new; % update velocity
end
end

speeds = [speeds; velocities]; % add velocity data for heatmap
for i = L-4:L % delete vehicles in last 5 sites
    positions_updated(i) = 0; % remove possible vehicle
    velocities_updated(i) = -1; % set velocity to -1
end
positions = positions_updated; % updated positions for all vehicles
velocities = velocities_updated; % updated velocities for all vehicles
% HEATMAP

% array of all timesteps

% array of all sites

custom_color = [1 1 1; 0 0 0; 0 0 1; 0 0.4 0.8; 0 0.6 0.7; 0 0.8 0.6;...
0 1 0.5];

h = heatmap(sites, times, speeds);

h.GridVisible = 'off';

h.Title = {'Spatio-temporal representation', 'p=0.5, \rho=0.1'};

h.XLabel = 'position';

h.YLabel = 'timestep';

h.Colormap = custom_color;

E.3 Heatmap for Open System with Zero Initial Configuration

The code below is used to create Figure 46.

clear all
clc

% HEATMAP USING ZERO INITIAL CONFIGURATION IN OPEN SYSTEM

% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)

L = 10^-2; % number of sites, approximately 7.5 m each
N = 10; % total number of cars
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.5; % slowdown probability

positions = zeros(1, L); % positions(i) == 1 if occupied, 0 otherwise
velocities = -1*ones(1, L); % values between -1, 0, ..., v_max; -1 empty
speeds = velocities; % data for heatmap

% SIMULATION (T time-steps)

T = 100; % number of timesteps
for t = 1:T % update using 4 rules
    positions_updated = zeros(1, L); % array for new positions
    velocities_updated = -1*ones(1, L); % array for new velocities
    for i = 1:L-5 % traverse over first L-5 sites only
        if (i == 1) && (positions(i) == 0) % site 1 empty
            positions(i) = 1; % place new vehicle
            velocities(i) = 0; % assign velocity 0
        end
        if positions(i) == 1 % only update when site is occupied
            % Acceleration
            v_i = velocities(i); % velocity of vehicle at site i
            if v_i < v_max % first condition for acceleration
                count = 0; % counter for free sites in front
                for j = 1:v_i + 1 % check whether distance is larger than v_i + 1
...
site_cand = i + j; % candidate for new site
if positions(site_cand) == 0 % candidate is empty
    count = count + 1; % count free site
end
if count == v_i + 1 % all v_i + 1 sites empty
    velocities(i) = v_i + 1; % vehicle accelerates
end

% Deceleration
v_i2 = velocities(i); % updated velocity
for k = 1:v_i2 % check all v_i sites in front
    site_cand2 = i + k; % candidate for new site
    if positions(site_cand2) == 1 % site occupied
        velocities(i) = k - 1; % vehicle decelerates
        break % first vehicle ahead determines new velocity
    end
end

% Randomisation
if rand <= p % velocity is reduced with probability p
    v_i3 = velocities(i); % updated velocity
    if v_i3 > 0 % velocity cannot be reduced if <= 0
        velocities(i) = v_i3 - 1; % vehicle slows down
    end
end

% Motion
v_new = velocities(i); % updated velocity
new_position = i + v_new; % new position
positions_updated(new_position) = 1; % update position
velocities_updated(new_position) = v_new; % update velocity
end

speeds = [speeds; velocities]; % add velocity data for heatmap
for i = L-4:L % delete vehicles in last 5 sites
    positions_updated(i) = 0; % remove possible vehicle
    velocities_updated(i) = -1; % set velocity to -1
end

positions = positions_updated; % updated positions for all vehicles
velocities = velocities_updated; % updated velocities for all vehicles
end

% HEATMAP
times = 0:T; % array of all timesteps
sites = 1:L; % array of all sites
custom_color = [1 1 1; 0 0 0; 0 0 1; 0 0.4 0.8; 0 0.6 0.7; 0 0.8 0.6;...
               0 1 0.5]; % colour scheme
h = heatmap(sites, times, speeds); % heatmap
E.4 Fundamental Diagrams

The code below is used to create Figure 47, Figure 48 and Figure 52, by changing the value of $L$ to $10^4$, the value of $T$ to $10^2$ and the value of incr to 100 for Figure 47, $L = 10^3$, $T = 10^2$ and incr = 1 for Figure 48 and $L = 10^3$, $T = 10^5$ and incr = 1 for Figure 52.

```matlab
clear all
clc

% FLOW-DENSITY DIAGRAM
% Bachelor Final Project: A Brief Introduction to Traffic Modelling with a
% Closer Look at the Nagel-Schreckenberg Model
% Mathieu de Ridder, July 28, 2023
% Eindhoven University of Technology (TU/e)
L = 10^3; % number of sites, approximately 7.5 m each
rho = N/L; % density of vehicles per site
v_max = 5; % maximum integer velocity in sites per timestep
p = 0.5; % slowdown probability
measuring_point = randi([1,L]); % site where density and flow are
% measured
incr = 1; % value with which number of vehicles is increased
rho_T = zeros([1,L/incr+1]); % array for density data
q_T = zeros([1,L/incr+1]); % array for flow data

% SIMULATION (from 0 to L vehicles)
for N = 0:incr:L % increment by incr from 0 to L vehicles
    positions = zeros(1,L); % positions(i) == 1 if occupied, 0 otherwise
    velocities = -1*ones(1,L); % values between -1, 0, ..., v_max; -1 % empty

    % ALTERNATIVE INITIAL CONFIGURATION (RANDOM): t = 0
    k = N; % place all vehicles
    while k > 0 % continue until all vehicles are placed
        vehicle = randi([1,L]); % random position
        if positions(vehicle) == 0 % only place vehicle if site not % occupied
            positions(vehicle) = 1; % place vehicle
            velocities(vehicle) = 0; % set velocity to 0
            k = k-1; % go to next vehicle
        end
    end

    % SIMULATION (t_0 + T time-steps)
    t_0 = 10*L; % start collecting data after timestep t_0
```

The code is set to create fundamental diagrams as requested, with adjustments to parameters for each figure as specified. The fundamental diagram represents the relationship between flow and density, which is crucial for understanding traffic dynamics.
\[ T = 10^{-2}; \quad \% \ \text{number of timesteps for data} \]

occupation = zeros(1, t_0+T); \quad \% \ \text{array for data to check whether site is occupied}

motion = zeros(1, t_0+T); \quad \% \ \text{array for data to detect motion through site}

% Timesteps without collecting data
for t = 1: t_0 \quad \% update using 4 rules
    positions_updated = zeros(1, L); \quad \% array for new positions
    velocities_updated = -1*ones(1, L); \quad \% array for new velocities
    for i = 1:L \quad \% traverse over all sites
        if positions(i) == 1 \quad \% only update when site is occupied
            \% Acceleration
            v_i = velocities(i); \quad \% velocity of vehicle at site i
            if v_i < v_max \quad \% first condition for acceleration
                count = 0; \quad \% counter for free sites in front
                for j = 1: v_i + 1 \quad \% check whether distance is larger than v_i + 1
                    site_cand = i + j; \quad \% candidate for new site
                    if site_cand > L \quad \% exceeds array
                        site_cand = site_cand - L; \quad \% circular road
                    end
                    if positions(site_cand) == 0 \quad \% candidate is empty
                        count = count + 1; \quad \% count free site
                    end
                end
                if count == v_i + 1 \quad \% all v_i + 1 sites empty
                    velocities(i) = v_i + 1; \quad \% vehicle accelerates
                end
            end
            \% Deceleration
            v_i2 = velocities(i); \quad \% updated velocity
            for k = 1: v_i2 \quad \% check all v_i sites in front
                site_cand2 = i + k; \quad \% candidate for new site
                if site_cand2 > L \quad \% exceeds array
                    site_cand2 = site_cand2 - L; \quad \% circular road
                end
                if positions(site_cand2) == 1 \quad \% site occupied
                    velocities(i) = k - 1; \quad \% vehicle decelerates
                    break \quad \% first vehicle ahead determines new velocity
                end
            end
            \% Randomisation
            if rand <= p \quad \% velocity is reduced with probability p
                v_i3 = velocities(i); \quad \% updated velocity
                if v_i3 > 0 \quad \% velocity cannot be reduced if <= 0
                    velocities(i) = v_i3 - 1; \quad \% vehicle slows down
                end
            end
        end
    end
end
% Motion
v_new = velocities(i);  % updated velocity
new_position = i + v_new;  % new position
if new_position > L  % exceeds array
    new_position = new_position - L;  % circular road
end
positions_updated(new_position) = 1;  % update position
velocities_updated(new_position) = v_new;  % update velocity
end

t = t_0 +1: t_0 +T  % update using 4 rules
for i = 1:L  % traverse over all sites
    if positions(i) == 1  % only update when site is occupied
        % Acceleration
        v_i = velocities(i);  % velocity of vehicle at site i
        if v_i < v_max  % first condition for acceleration
            count = 0;  % counter for free sites in front
            for j = 1: v_i + 1  % check wether distance is larger than v_i + 1
                site_cand = i + j;  % candidate for new site
                if site_cand > L  % exceeds array
                    site_cand = site_cand - L;  % circular road
                end
                if positions(site_cand) == 0  % candidate is empty
                    count = count + 1;  % count free site
                end
            end
            if count == v_i + 1  % all v_i + 1 sites empty
                velocities(i) = v_i + 1;  % vehicle accelerates
            end
        end
        % Deceleration
        v_i2 = velocities(i);  % updated velocity
        for k = 1: v_i2  % check all v_i sites in front
            site_cand2 = i + k;  % candidate for new site
            if site_cand2 > L  % exceeds array
                site_cand2 = site_cand2 - L;  % circular road
            end
            if positions(site_cand2) == 1  % site occupied
                end
            end
        end
    end
end
velocities(i) = k - 1;  % vehicle decelerates
break  % first vehicle ahead determines new % velocity
end
end

% Randomisation
if rand <= p  % velocity is reduced with probability p
v_i3 = velocities(i);  % updated velocity
if v_i3 > 0  % velocity cannot be reduced if <= 0
velocities(i) = v_i3 - 1;  % vehicle slows down
end
end

% Motion
v_new = velocities(i);  % updated velocity
new_position = i + v_new;  % new position
if new_position > L  % exceeds array
new_position = new_position - L;  % circular road
end
if (new_position >= measuring_point + 1) && ...
    (i <= measuring_point)  % vehicle moves past
    % measuring point
    motion(t) = 1;  % motion detected
end
positions_updated(new_position) = 1;  % update position
velocities_updated(new_position) = v_new;  % update % velocity
end
if i == measuring_point  % check whether measuring point is
    % occupied
    occupation(t) = positions_updated(i);  % 1 if occupied, 0
    % otherwise
end
positions = positions_updated;  % updated positions for all % vehicles
velocities = velocities_updated;  % updated velocities for all % vehicles
end
rho_T(N+1) = 1/T * sum(occupation(t_0+1:end));  % density averaged % over T timesteps
q_T(N+1) = 1/T * sum(motion(t_0+1:end));  % flow averaged over T % timesteps
end

% SCATTERPLOT
scatter(rho_T, q_T,'.')  % scatterplot
xlabel('vehicle density')  % label for horizontal axis
ylabel('vehicle flow')  % label for vertical axis
title('Flow-density diagram')  % title