

Solution to Problem 64-6*: Gravitational attraction

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elementary methods [1] and the left-hand side is the result given by a general formula which was developed for the nonlinear stockout cost case.

Verify (a). (The proof of (b) is not asked for as apparently it requires too many detailed calculations.)

REFERENCE

- [1] G. HADLEY AND T. M. WHITIN, *Analysis of Inventory Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1963.

*Problem 65-11**, *Limiting Value of a Solution of an Integral Equation*, by W. T. MOODY (Denver, Colorado).

The function Φ is defined by the integral equation

$$\Phi(y) = \frac{4P}{\pi} \frac{ya^2}{(y^2 + a^2)^2} + \frac{16}{\pi^2} \int_0^\infty \Phi(v) \frac{yv^2}{(y^2 - v^2)^3} \left[(y^2 + v^2) \log \left(\frac{y}{v} \right) - y^2 + v^2 \right] dv.$$

Numerical computations indicate that

$$\lim_{y \rightarrow 0} \Phi(y) = \frac{1}{2}.$$

Show analytically whether this is true or not. If not true, find what limit the function does approach.

The equation arises in connection with determination of stresses in 90° sector of an elastic plate subject to a concentrated load P acting normal to one of its boundaries at a distance a from the vertex.

SOLUTIONS

Problem 63-9. See the research paper, *An optimal search problem*, by WALLACE FRANCK, this issue, pp. 503-512.

Late solutions:

Problem 64-1 was also solved by DAVID PHILLIPS (Argonne National Laboratory).

*Problem 64-6**, *Gravitational Attraction*, by MORTON L. SLATER (Sandia Corporation).

Determine the gravitational attraction between a uniform solid torus and a unit mass particle located on its axis.

Solution by C. J. BOUWKAMP (Technological University, Eindhoven, Netherlands).

Let a be the radius of the cross-section, b the radius of the central line, and z the distance from the unit mass particle to the central plane of the torus. As-

sume the torus to be of unit mass density. Then the total mass of the torus is

$$(1) \quad M = 2\pi^2 a^2 b.$$

Further, let ρ and ϕ denote polar coordinates in the plane of the cross-section with center located in the central line. Then the gravitational potential of the torus at the location of the unit mass particle is

$$V = 2\pi \int_0^a \rho \, d\rho \int_{-\pi}^{\pi} d\phi \frac{b + \rho \cos \mu}{\sqrt{(z - \rho \sin \phi)^2 + (b + \rho \cos \phi)^2}}.$$

We have

$$(2) \quad V = \frac{\partial}{\partial b} W,$$

where

$$(3) \quad W = 2\pi \int_0^a \rho \, d\rho \int_{-\pi}^{\pi} d\phi [(z - \rho \sin \phi)^2 + (b + \rho \cos \phi)^2]^{1/2}.$$

The expression between square brackets can be transformed into

$$z^2 + b^2 + \rho^2 - 2\rho\sqrt{z^2 + b^2} \sin\left(\phi - \tan^{-1}\frac{b}{z}\right).$$

Since in (3) the integration with respect to ϕ is over a full period of the sine function, we can drop the phase factor $\tan^{-1}(b/z)$ and we can also replace the sine by the cosine. Hence,

$$(4) \quad W = 2\pi \int_0^a \rho \, d\rho \int_{-\pi}^{\pi} d\phi [z^2 + b^2 + \rho^2 - 2\rho\sqrt{z^2 + b^2} \cos \phi]^{1/2}.$$

If we now set

$$u = \sqrt{z^2 + b^2}, \quad v = \frac{\rho}{u} = \frac{\rho}{\sqrt{z^2 + b^2}},$$

(4) is transformed into

$$(5) \quad W = 2\pi u \int_0^a \rho \, d\rho \int_{-\pi}^{\pi} d\phi [1 - 2v \cos \phi + v^2]^{1/2}.$$

The second integral in (5) can be expressed in terms of complete elliptic integrals, or better still, in terms of a hypergeometric function, viz.,

$$\begin{aligned} \int_{-\pi}^{\pi} d\phi [1 - 2v \cos \phi + v^2]^{1/2} &= 4(1 + v)E\left(\frac{2\sqrt{v}}{1 + v}\right) \\ &= 4[2E(v) - (1 - v^2)K(v)] = 2\pi F\left(-\frac{1}{2}, -\frac{1}{2}; 1; v^2\right). \end{aligned}$$

The remaining integration over ρ can now easily be performed:

$$W = 2\pi^2 a^2 u F\left(-\frac{1}{2}, -\frac{1}{2}; 2; a^2/u^2\right).$$

In view of (1) and (2), we then have

$$\frac{V}{M} = \frac{1}{u} F\left(-\frac{1}{2}, \frac{1}{2}; 2; \frac{a^2}{u^2}\right).$$

Since the attractive force F equals $-\partial V/\partial z$, the required force is then found to be given by

$$\frac{F}{M} \frac{z}{(z^2 + b^2)^{3/2}} F\left(-\frac{1}{2}, \frac{3}{2}; 2; \frac{a^2}{z^2 + b^2}\right).$$

Also solved by T. C. ANDERSON (Lockheed Missiles and Space Co.) in terms of elliptic functions directly from the force integral.

Problem 64-7, An Asymptotic Series, by N. G. DE BRUIJN (Technological University, Eindhoven, Netherlands).

Let $\phi(x)$ be infinitely often differentiable for $x \geq 0$, and let

$$\int_0^\infty |\phi^{(k)}(x)| dx$$

be convergent for each $k = 0, 1, 2, \dots$. Define

$$F(t) = \sum_{n=1}^{\infty} n^{-1} \phi(nt), \quad t > 0.$$

Show that $F(t) + \phi(0) \log t$ has an asymptotic development in the form of an asymptotic series $\sum_{n=0}^{\infty} c_n t^n$ if $t > 0$, $t \rightarrow 0$.

Solution by the proposer.

Introducing a positive constant λ , we put $\phi_1(x) = \phi(x) - \phi(0)e^{-\lambda x}$. Then ϕ_1 still has the properties attributed to ϕ , and moreover $\phi_1(0) = 0$. We put $x^{-1}\phi_1(x) = \eta(x)$, and we apply the Euler-Maclaurin sum formula to $\sum_0^\infty \eta(nt)$ (we can apply it to the infinite series since $\eta^{(k)}(x) \rightarrow 0(x \rightarrow \infty)$ for each k , and $\int_0^\infty |\eta^{(k)}(x)| \cdot dx \rightarrow \infty$):

$$\begin{aligned} \sum_1^\infty \eta(nt) &= t^{-1} \int_0^\infty \eta(x) dx - \frac{1}{2} \eta(0) - \sum_{k=1}^m \frac{B_{2k} t^{2k-1} \eta^{(2k-1)}(0)}{(2k)!} \\ &\quad - t^{2m} \int_0^\infty \frac{\eta^{(2m)}(x) B_{2m}(tx - [tx])}{(2m)!} dx. \end{aligned}$$

Thus we obtain the following asymptotic series for $F(t) + \phi(0) \log t$:

$$\begin{aligned} \int_0^\infty x^{-1}(\phi(x) - \phi(0)e^{-\lambda x}) dx - \phi(0) \log \frac{1 - e^{-\lambda t}}{\lambda t} \\ - \frac{1}{2} t \eta(0) - \sum_{k=1}^{\infty} \frac{B_{2k} t^{2k} \eta^{(2k-1)}(0)}{(2k)!}. \end{aligned}$$

We finally want to get rid of λ . We note that $\int_0^\infty x^{-1}(e^{-x} - e^{-\lambda x}) dx = \log \lambda$ and