Solution to Problem 64-6*: Gravitational attraction

elementary methods [1] and the left-hand side is the result given by a general formula which was developed for the nonlinear stockout cost case.

Verify (a). (The proof of (b) is not asked for as apparently it requires too many detailed calculations.)

REFERENCE

[1] G. HADLEY AND T. M. WHITIN, *Analysis of Inventory Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1963.

Problem 65–11*, *Limiting Value of a Solution of an Integral Equation*, by W. T. MOODY (Denver, Colorado).

The function Φ is defined by the integral equation

$$\Phi(y) = \frac{4P}{\pi} \frac{ya^2}{(y^2 + a^2)^2} + \frac{16}{\pi^2} \int_0^\infty \Phi(v) \frac{yv^2}{(y^2 - v^2)^3} \left[(y^2 + v^2) \log\left(\frac{y}{v}\right) - y^2 + v^2 \right] dv.$$

Numerical computations indicate that

$$\lim_{y \to 0} \Phi(y) = \tfrac{1}{2}.$$

Show analytically whether this is true or not. If not true, find what limit the function does approach.

The equation arises in connection with determination of stresses in 90° sector of an elastic plate subject to a concentrated load P acting normal to one of its boundaries at a distance a from the vertex.

SOLUTIONS

Problem 63–9. See the research paper, *An optimal search problem*, by WALLACE FRANCK, this issue, pp. 503–512.

Late solutions:

Problem 64–1 was also solved by DAVID PHILLIPS (Argonne National Laboratory).

Problem 64–6*, *Gravitational Attraction*, by MORTON L. SLATER (Sandia Corporation).

Determine the gravitational attraction between a uniform solid torus and a unit mass particle located on its axis.

Solution by C. J. BOUWKAMP (Technological University, Eindhoven, Netherlands).

Let a be the radius of the cross-section, b the radius of the central line, and z the distance from the unit mass particle to the central plane of the torus. Assume the torus to be of unit mass density. Then the total mass of the torus is

(1) $$M = 2\pi^2 a^2 b.$$

Further, let ρ and ϕ denote polar coordinates in the plane of the cross-section with center located in the central line. Then the gravitational potential of the torus at the location of the unit mass particle is

$$V = 2\pi \int_0^a \rho\, d\rho \int_{-\pi}^{\pi} d\phi \frac{b + \rho \cos \mu}{\sqrt{(z - \rho \sin \phi)^2 + (b + \rho \cos \phi)^2}}.$$

We have

(2) $$V = \frac{\partial}{\partial b} W,$$

where

(3) $$W = 2\pi \int_0^a \rho\, d\rho \int_{-\pi}^{\pi} d\phi [(z - \rho \sin \phi)^2 + (b + \rho \cos \phi)^2]^{1/2}.$$

The expression between square brackets can be transformed into

$$z^2 + b^2 + \rho^2 - 2\rho\sqrt{z^2 + b^2} \sin \left(\phi - \tan^{-1} \frac{b}{z}\right).$$

Since in (3) the integration with respect to ϕ is over a full period of the sine function, we can drop the phase factor $\tan^{-1}(b/z)$ and we can also replace the sine by the cosine. Hence,

(4) $$W = 2\pi \int_0^a \rho\, d\rho \int_{-\pi}^{\pi} d\phi [z^2 + b^2 + \rho^2 - 2\rho\sqrt{z^2 + b^2} \cos \phi]^{1/2}.$$

If we now set

$$u = \sqrt{z^2 + b^2}, \qquad v = \frac{\rho}{u} = \frac{\rho}{\sqrt{z^2 + b^2}},$$

(4) is transformed into

(5) $$W = 2\pi u \int_0^a \rho\, d\rho \int_{-\pi}^{\pi} d\phi [1 - 2v \cos \phi + v^2]^{1/2}.$$

The second integral in (5) can be expressed in terms of complete elliptic integrals, or better still, in terms of a hypergeometric function, viz.,

$$\int_{-\pi}^{\pi} d\phi\, [1 - 2v \cos \phi + v^2]^{1/2} = 4(1 + v) E\left(\frac{2\sqrt{v}}{1 + v}\right)$$
$$= 4[2E(v) - (1 - v^2)K(v)] = 2\pi F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; v^2).$$

The remaining integration over ρ can now easily be performed:

$$W = 2\pi^2 a^2 u F(-\tfrac{1}{2}, -\tfrac{1}{2}; 2; a^2/u^2).$$

In view of (1) and (2), we then have

$$\frac{V}{M} = \frac{1}{u} F\left(-\frac{1}{2}, \frac{1}{2}; 2; \frac{a^2}{u^2}\right).$$

Since the attractive force F equals $-\partial V/\partial z$, the required force is then found to be given by

$$\frac{F}{M} \frac{z}{(z^2 + b^2)^{3/2}} F\left(-\frac{1}{2}, \frac{3}{2}; 2; \frac{a^2}{z^2 + b^2}\right).$$

Also solved by T. C. ANDERSON (Lockheed Missiles and Space Co.) in terms of elliptic functions directly from the force integral.

Problem 64–7, *An Asymptotic Series*, by N. G. DE BRUIJN (Technological University, Eindhoven, Netherlands).

Let $\phi(x)$ be infinitely often differentiable for $x \geqq 0$, and let

$$\int_0^\infty | \phi^{(k)}(x) | \, dx$$

be convergent for each $k = 0, 1, 2, \cdots$. Define

$$F(t) = \sum_{n=1}^{\infty} n^{-1}\phi(nt), \qquad t > 0.$$

Show that $F(t) + \phi(0) \log t$ has an asymptotic development in the form of an asymptotic series $\sum_{n=0}^{\infty} c_n t^n$ if $t > 0$, $t \to 0$.

Solution by the proposer.

Introducing a positive constant λ, we put $\phi_1(x) = \phi(x) - \phi(0)e^{-\lambda x}$. Then ϕ_1 still has the properties attributed to ϕ, and moreover $\phi_1(0) = 0$. We put $x^{-1}\phi_1(x) = \eta(x)$, and we apply the Euler-Maclaurin sum formula to $\sum_0^\infty \eta(nt)$ (we can apply it to the infinite series since $\eta^{(k)}(x) \to 0 (x \to \infty)$ for each k, and $\int_0^\infty |\eta^{(k)}(x)| \cdot dx \to \infty$):

$$\sum_1^\infty \eta(nt) = t^{-1} \int_0^\infty \eta(x) - \frac{1}{2}\eta(0) - \sum_{k=1}^{m} \frac{B_{2k}\, t^{2k-1} \eta^{(2k-1)}(0)}{(2k)!}$$
$$- t^{2m} \int_0^\infty \frac{\eta^{(2m)}(x) B_{2m}(tx - [tx])}{(2m)!} \, dx.$$

Thus we obtain the following asymptotic series for $F(t) + \phi(0) \log t$:

$$\int_0^\infty x^{-1}(\phi(x) - \phi(0)e^{-\lambda x}) \, dx - \phi(0) \log \frac{1 - e^{-\lambda t}}{\lambda t}$$
$$- \frac{1}{2} t\eta(0) - \sum_{k=1}^{\infty} \frac{B_{2k}\, t^{2k} \eta^{2k-1}(0)}{(2k)!}.$$

We finally want to get rid of λ. We note that $\int_0^\infty x^{-1}(e^{-x} - e^{-\lambda x}) \, dx = \log \lambda$ and