Fast computation of the dual Gabor window in the case of the quincunx lattice

Fast Computation of the Dual Gabor Window in the Case of the Quincunx Lattice

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Abstract—**In this paper we present an attractive computation technique for approximating the dual window for the Gabor transform in the case of a quincunx lattice. This computation technique is based on a Wexler-Raz identity for the quincunx lattice and the use of biorthogonal-like functions. The method avoids the estimation of the frame bounds.**

Furthermore, in an example we compare the dual windows for the rectangular lattice with the dual windows for the quincunx lattice in the case of a Gaussian window for several choices of oversampling.

I. Introduction

Some methods are known to compute the dual window for the Gabor transform in the case of a rectangular lattice. There are two main methods to compute the dual window: approximation via the inverse frame operator [1] and the Zak transform method [2].

In the Zak transform method, one forms a set of linear equations of finite dimension, and then obtains the dual window by solving that set of linear equations. Recently, the connection between the Zak transform and the Gabor transform has been shown in the case of the quincunx lattice [3]. This quincunx lattice is depicted in Fig. 1.

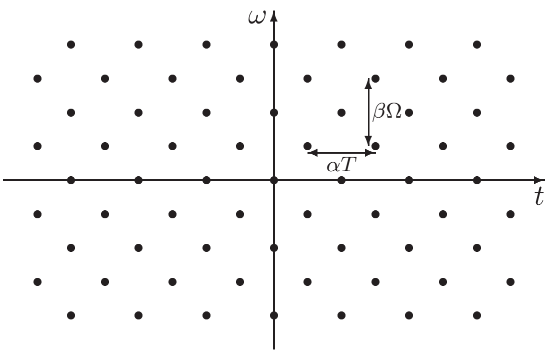

Fig. 1. The quincunx lattice.

The main disadvantage in using frames is that in most cases the frame bounds have to be estimated, which is in general very difficult.

A very attractive computation technique for approximating the dual window in the case of a rectangular lattice is presented in [4]. It is shown that the dual window can be approximated by a finite linear combination of functions that depend on the window. In this paper we extend this idea to the case of a quincunx lattice. Furthermore, we compare the dual windows for the rectangular lattice with the dual windows for the quincunx lattice. In particular, we compare the L_2 norms of the difference between the dual window and the optimum dual window for different choices of oversampling for the rectangular and the quincunx case.

II. Gabor's signal expansion

The Gabor expansion on a quincunx lattice of $L_2(\mathbb{R})$-signals $\varphi(t)$ into a discrete set of properly shifted and modulated elementary signals is defined as

$$\varphi(t) = \sum_{mk} a_{mk} g_{mk}(t) \tag{1}$$

with the time and frequency shifted window (see Fig. 1)

$$g_{mk}(t) = \tfrac{1}{2}(1 + (-1)^{m+k}) g(t - \tfrac{1}{2} m\alpha T) e^{jk\frac{1}{2}\beta\Omega t} \tag{2}$$

where the time shift $\frac{1}{2}\alpha T$ and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\frac{1}{2}\alpha\beta = 1/o \leq 1$, with o the oversampling rate, and where m and k may take all integer values. The expansion coefficients a_{mk} follow from sampling the windowed Fourier transform on the quincunx lattice with the dual window $\gamma(t)$

$$a_{mk} = <\varphi, \gamma_{mk}> = \int_{-\infty}^{\infty} \varphi(t) \gamma_{mk}^*(t)\, dt \tag{3}$$

with $\gamma_{mk}(t)$ the shifted and modulated versions of the dual window $\gamma(t)$ [cf. Eq. (2)].

The discrete set of shifted and modulated versions $g_{mk}(t)$ is said to constitute a frame if there exist $A > 0$

ISBN: 90-73461-15-4

and $B < \infty$ such that, for all $\varphi \in L_2(\mathbb{R})$ [1]

$$A\,\|\varphi\|^2 \leq \sum_{mk} |<\varphi, g_{mk}>|^2 \leq B\,\|\varphi\|^2 . \tag{4}$$

A frame is called snug if the ratio B/A is close to 1, and called tight if $B/A = 1$.

The frame operator S is defined as

$$S\varphi = \sum_{mk} <\varphi, g_{mk}>\, g_{mk}.$$

If the set g_{mk} satisfies relation (4), then it can be shown that the coefficients a_{mk} are the least-squares choice with $\gamma_{mk} = S^{-1} g_{mk}$. The dual window $\gamma(t)$ can be approximated with a geometrically convergent series for S^{-1}

$$\gamma(t) = \frac{2}{A+B} \sum_{k=0}^{\infty} \left(I - \frac{2}{A+B} S \right)^k g(t). \tag{5}$$

However, the frame bounds A and B are in general difficult to determine. As a consequence, the approximation (5) is not a very suitable method in practice.

A very attractive computation technique for approximating the dual window in the case of a rectangular lattice is presented in [4]. We extend this idea for the case of sampling on a quincunx lattice.

III. Wexler-Raz identity in the case of a rectangular lattice

The shifted and modulated versions of the window $g(t)$ in the case of a rectangular lattice look like

$$g'_{mk}(t) = g(t - m\alpha T) e^{jk\beta\Omega t}$$

where the time shift αT and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\alpha\beta = 1/o \leq 1$, with o the oversampling rate, and where m and k may take all integer values.

The technique in [4] is based on the Wexler-Raz identity [5] which is extended to $L_2(\mathbb{R})$-functions in [6] for the case of a rectangular lattice. The author shows that if $g, \gamma \in L_2(\mathbb{R})$ both have a finite upper frame bound then g and γ are dual if and only if

$$<\gamma,\, g'^{(mk)}> = \alpha\beta\, \delta[m]\delta[k] \tag{6}$$

with

$$g'^{(mk)}(t) = g(t - k\frac{T}{\beta}) e^{jm\Omega t/\alpha} \tag{7}$$

and where δ denotes the Kronecker delta. The windows $g'^{(mk)}$ are shifted and modulated versions of the window $g(t)$ on the dual lattice $(2\pi k/\beta\Omega, 2\pi m/\alpha T) = (kT/\beta, m\Omega/\alpha)$. In [7] it is shown that this γ in Eq. (6) is the dual window of g with the minimal L_2-norm.

IV. Wexler-Raz identity in the case of a quincunx lattice

The following Wexler-Raz identity in the case of a quincunx lattice can be derived by combining Eqs. (1) and (3)

$$<\gamma, g^{(mk)}> = \tfrac{1}{2}\alpha\beta\, \delta[m]\delta[k] \tag{8}$$

with [cf. Eq. (7)]

$$g^{(mk)}(t) = \tfrac{1}{2}(1 + (-1)^{m+k}) g(t - k\frac{T}{\beta}) e^{jm\Omega t/\alpha}.$$

It can be shown that this identity is equivalent to

$$<g^{(m''k'')}, \gamma^{(m'k')}> = \tfrac{1}{2}\alpha\beta\, \delta[m' - m'']\delta[k' - k'']. \tag{9}$$

The sequences $\{g^{(m''k'')}\}$ and $\{\gamma^{(m'k')}\}$ are said to be biorthogonal [8] if this identity is fulfilled for every m', m'', k', and k''. Very similar to the proof in [4] it can be shown that the given function $g(t)$ and a biorthogonal function $\gamma(t)$ satisfying Eq. (9) with $\frac{1}{2}\alpha\beta \leq 1$, form a pair of synthesis and analysis functions (window and dual window). The biorthogonal function $\gamma(t)$ belongs to the subspace of $L_2(\mathbb{R})$ spanned by $\{g^{(mk)}\}$ and is the dual window of $g(t)$. Thus the dual window has the form

$$\gamma(t) = \sum_{mk} c_{mk} g^{(mk)}(t).$$

It is possible to approximate $\gamma(t)$ by finite linear combinations of elements in $\{g^{(mk)}\}$

$$\gamma(t) = \sum_{m=-N}^{N} \sum_{k=-N}^{N} c_{mk} g^{(mk)}(t) \tag{10}$$

which is possible in the case of oversampling and when the function $g(t)$ has a finite support and a fast decay. For convenience, let $g^{(mk)} = g^{(i)}$ and $c_{mk} = c_i$ for $i = 0 \ldots (2N+1)^2 - 1$. Then the jth element in the vector $\boldsymbol{a}$ equals

$$a_j = <\gamma, g^{(j)}> = \sum_{i=0}^{(2N+1)^2-1} c_i <g^{(i)}, g^{(j)}> .$$

Thus the vector $\boldsymbol{c}$ with the coefficients c_i is equal to $\boldsymbol{c} = (\boldsymbol{G}^T)^{-1}\boldsymbol{a}$ with $\boldsymbol{G}$ the Gram-matrix. The vector $\boldsymbol{a}$ contains $(2N+1)^2 - 2$ zeros and the constant $\frac{1}{2}\alpha\beta$, which follow directly from the Wexler-Raz identity (8).

V. Example

In this section we determine some dual windows $\gamma(t)$ for the given Gaussian window $g(t) = 2^{1/4} T^{-1/2} e^{-\pi(t/T)^2}$ with $||g|| = 1$, and compare the dual

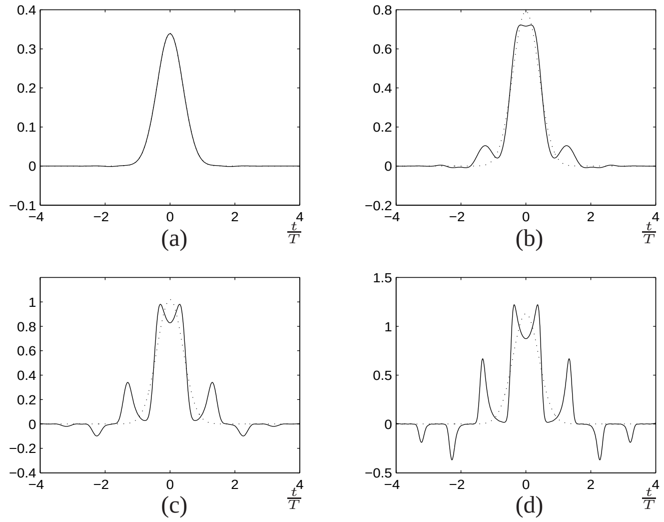

Figure 2: The dual windows (solid line) of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} T^{-\frac{1}{2}} \exp(-\pi(t/T)^2)$ and the optimum windows $\gamma_{opt}(t)$ (dotted line) for different values of oversampling, and the difference of the dual window and the optimum dual window in the L_2 norm sense in the case of the quincunx lattice:

(a) $\alpha = \sqrt{\frac{4\sqrt{3}}{7}}$, $\beta = !\frac{1}{3}\sqrt{3}\alpha$, $p/q = 7/2$, $||\gamma - \gamma_{opt}|| = 0.0012$, $N = 1$

(b) $\alpha = \sqrt{\frac{4\sqrt{3}}{5}}$, $\beta = \frac{1}{3}\sqrt{3}\alpha$, $p/q = 5/2$, $||\gamma - \gamma_{opt}|| = 0.0105$, $N = 2$

(c) $\alpha = \sqrt{\frac{4\sqrt{3}}{3}}$, $\beta = \frac{1}{3}\sqrt{3}\alpha$, $p/q = 3/2$, $||\gamma - \gamma_{opt}|| = 0.1092$, $N = 3$

(d) $\alpha = \sqrt{\frac{12\sqrt{3}}{7}} \beta = \frac{1}{3}\sqrt{3}\alpha$, $p/q = 7/6$, $||\gamma - \gamma_{opt}|| = 0.3191$, $N = 6$

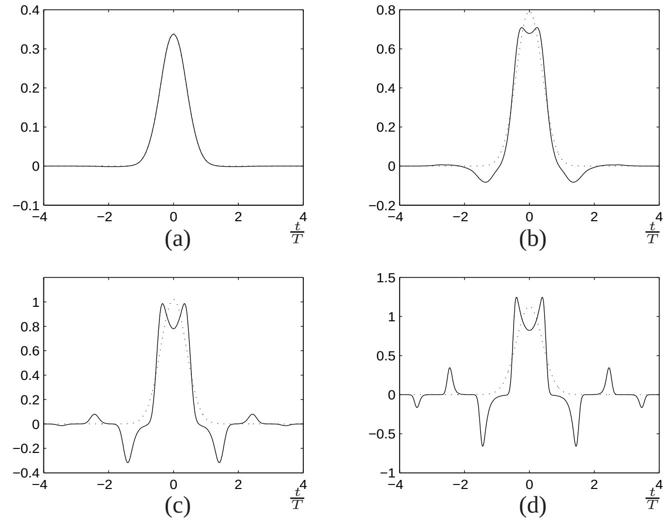

Figure 3: The dual windows (solid line) of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} T^{-\frac{1}{2}} \exp(-\pi(t/T)^2)$ and the optimum windows $\gamma_{opt}(t)$ (dotted line) for different values of oversampling, and the difference of the dual window and the optimum dual window in the L_2 norm sense in the case of the rectangular lattice:

(a) $\alpha = \beta = !\sqrt{2/7}$, $p/q = 7/2$, $||\gamma - \gamma_{opt}|| = 0.0023$, N=1

(b) $\alpha = \beta = \sqrt{2/5}$, $p/q = 5/2$, $||\gamma - \gamma_{opt}|| = 0.0158$, N=2

(c) $\alpha = \beta = \sqrt{2/3}$, $p/q = 3/2$, $||\gamma - \gamma_{opt}|| = 0.1299$, N=3

(d) $\alpha = \beta = \sqrt{6/7}$, $p/q = 7/6$, $||\gamma - \gamma_{opt}|| = 0.3415$, N=6

windows for the quincunx case with the dual windows for the rectangular case. As a measure we take the L_2 norm of the difference of the dual window $\gamma(t)$ and the optimum dual window $\gamma_{opt}(t)$, which is proportional to the window $g(t)$, thus we determine $||\gamma - cg||$. One can show that this norm has a minimum if $c = \frac{1}{2}\alpha\beta/||g||^2$ in the case of the quincunx lattice and if $c = \alpha\beta/||g||^2$ in the case of the rectangular lattice. This follows directly from the Wexler-Raz identities (6) and (8). We choose N in Eq. (10) such that the norm of the difference of the dual window $\gamma(t)$ and the optimum dual window $\gamma_{opt}(t)$ remains constant.

For this Gaussian window $g(t)$ it is shown that the optimal choice is $\alpha = \beta$ for the rectangular lattice [9]. It is not difficult to show that the optimal choice is $\alpha = \sqrt{3}\beta$ for the quincunx lattice.

In Fig. 2 we have depicted the dual windows of $g(t)$ and the optimum dual windows $\gamma_{opt}(t)$ for several choices of α and β for the quincunx lattice. In Fig. 3 we have depicted the same for the rectangular lattice.

From this example we see that the number of elements in $\{g^{(mk)}\}$ used in the estimate depends on the oversampling rate. The smaller the number of elements contained in the dual function, the snugger the frame.

Furthermore, we can conclude from this example that the dual windows for the quincunx lattice for different values of oversampling are better in the sense that the dual windows resemble better the optimum dual windows in the L_2 sense for this Gaussian window $g(t)$.

Note that in this example the oversampling rates are rational; however, with this method the oversampling may be irrational, as well.

VI. Conclusions

We presented a fast computation technique for approximating the dual window of the Gabor transform in the case of the quincunx lattice. The dual window is generated by the biorthogonal function that is spanned by a collection of functions that depend on the window. The number of elements used in the estimate depends on the oversampling rate. The higher the oversampling rate, the fewer elements in the dual window. The estimation of the frame bounds is avoided with this method.

The dual windows of a Gaussian window for the quincunx lattice for different values of oversampling are better in the sense that the dual windows resemble better the optimum dual windows in the L_2 sense than the dual windows in the case of a rectangular lattice.

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