Capacitated Network Bargaining Games

Capacitated Network Bargaining Games: Stability and Structure

Laura Sanità[1] and Lucy Verberk[2](✉)

[1] Bocconi University, Milan, Italy
laura.sanita@unibocconi.it
[2] Eindhoven University of Technology, Eindhoven, The Netherlands
l.p.a.verberk@tue.nl

Abstract. Capacitated network bargaining games are popular combinatorial games that involve the structure of matchings in graphs. We show that it is always possible to stabilize unweighted instances of this problem (that is, ensure that they admit a stable outcome) via capacity-reduction and edge-removal operations, without decreasing the total value that the players can get.
Furthermore, for general weighted instances, we show that computing a minimum amount of vertex-capacity to reduce to make an instance stable is a polynomial-time solvable problem. We then exploit this to give approximation results for the NP-hard problem of stabilizing a graph via edge-removal operations.
Our work extends and generalizes previous results in the literature that dealt with an uncapacitated version of the problem, using several new arguments. In particular, while previous results mainly used combinatorial techniques, we here rely on polyhedral arguments and, more specifically, on the notion of *circuits* of a polytope.

Keywords: Matching · Network Bargaining Games · Circuits.

1 Introduction

This paper focuses on *stabilization* problems for *capacitated* Network Bargaining Games (NBG).

NBG were introduced by Kleinberg and Tardos [13] as an extension of Nash's 2-player bargaining solution [16]. Here we are given a graph $G = (V, E)$ with edge weights $w \in \mathbb{R}^E_{\geq 0}$, where the vertices represent players and the edges the deals that the players can make. Each player can enter in *at most one* deal with one of their neighbours, and together they agree on how to split the value of the corresponding edge. An outcome is then associated with a matching M of G representing the deals, and an allocation vector $y \in \mathbb{R}^V_{\geq 0}$ with $y_u + y_v = w_{uv}$ if $uv \in M$, and $y_v = 0$ if v is not matched. An outcome (M, y) is called *stable* if no player has an incentive to break the current agreements to enter in a deal with a different neighbour, which formally translates in the condition $y_u \geq \max_{v:uv \in E\setminus M} \{w_{uv} - y_v\}$, for each player u.

Bateni et al [3] introduced a more realistic *capacitated* setting, where players are allowed to enter in more than one deal. This generalization can be modeled by adding vertex capacities $c \in \mathbb{Z}^V_{\geq 0}$ to the input. An outcome to the capacitated NBG is now given by a c-matching M and a vector $a \in \mathbb{R}^{2E}_{\geq 0}$ that satisfies $a_{uv} + a_{vu} = w_{uv}$ if $uv \in M$, and $a_{uv} = a_{vu} = 0$ otherwise. The concept of stable outcome naturally generalizes (see [3] for a formal definition).

A key property of (capacitated) NBG is that instances admitting a stable outcome have a very nice *LP characterization*, as shown by [13,3]. Specifically, given an instance (G, w, c), there exists a stable outcome for the corresponding game on G if and only if the value of a maximum weight c-matching $\nu^c(G)$ equals the value of maximum maximum weight *fractional* c-matching $\nu^c_f(G)$, defined as

$$\nu^c_f(G) := \max \left\{ w^\top x : \sum_{u:uv \in E} x_{uv} \leq c_v \; \forall v \in V, \; 0 \leq x \leq 1 \right\}. \tag{1}$$

In other words, instances admitting stable outcomes are the ones for which the standard LP relaxation of the maximum-weight matching problem has an optimal integral solution. A graph G for which $\nu^c(G) = \nu^c_f(G)$ is called *stable*.

It can be easily seen from this characterization that there are instances which do not admit stable solutions, for example odd cycles. For this reason, several researchers in the past years investigated so-called *stabilization* problems, where the goal is to turn a given graph into a stable one, via some graph operations (see [4,15,5,7,1,6,12,14,10]). Two common ways to stabilize graphs are edge- and vertex-removal operations, which have a natural NBG interpretation: they ensure a stable outcome by blocking interactions and blocking players, respectively.

In the uncapacitated setting, stabilization via edge- and vertex-removal operations for NBG is quite well understood. In particular, (i) stabilizing a graph by removing a minimum number of edges (called the *edge-stabilizer problem*) is NP-hard, and even hard-to-approximate with a constant factor, but admits an $O(\Delta)$-approximation, with Δ being the maximum degree of a vertex in the graph [11,5,14]. Differently, (ii) stabilizing a graph by removing a minimum number of vertices (called the *vertex-stabilizer problem*) is solvable in polynomial-time [1,12,14]. Moreover, (iii) both problems for unweighted graphs exhibit a very nice structural property: optimal solution do not decrease the cardinality of a maximum matching, meaning that there is always a way to stabilize the graph without decreasing the total value that the players can get [5,1]. The study of stabilization problems in the capacitated setting was recently initiated in [10], where (among other things) it is shown that the vertex-stabilizer problem becomes NP-hard. However, to complete the picture regarding stabilization problems in capacitated NBG, a few questions remain:

(i) Can one efficiently stabilize graphs by reducing the capacity of vertices, instead of removing them?

(ii) Are there non-trivial approximation algorithms for the edge-stabilizer problem in capacitated NBG instances?

(iii) For capacitated NBG with unit-weights, can one still hope to stabilize a graph without decreasing the total value that the players can get?

Our results and techniques. In this paper, we give an affirmative answer to the above three questions.

Our work started by realizing that the hardness proved in [10] for the vertex-stabilizer problem crucially relies on the fact that vertices have different capacity values and get removed *completely*. Another (still natural) way to generalize the vertex-stabilizer problem from the uncapacitated setting, is to *decrease* the capacity of the vertices (that is, reduce the number of potential deals that a player can be engaged in). We prove that computing a minimum amount of vertex capacity to reduce to make an instance stable (which we call the *capacity-stabilizer problem*) is a polynomial-time solvable problem (results in section 4). Interestingly, our solution reduces the capacity of each vertex by at most one. This has a nice network bargaining interpretation: there is always an optimal and at the same time *fair* way to stabilize instances, as no player will have its capacity dramatically reduced compared to others.

In addition to answering question (i) in a positive way, the algorithm to solve the capacity-stabilizer problem becomes instrumental when dealing with edge-stabilizers. In fact, we crucially exploit it to extend the $O(\Delta)$-approximation for NBG to the capacitated settings, answering question (ii) (results in section 5).

Eventually, we manage to show that the key structural property of minimum stabilizers for instances with unit-weights still holds. Namely, that optimal solutions to both the capacity-stabilizer and the edge-stabilizer problem do not decrease the total value that the players can get (results in sections 4 and 5).

Besides extending the previous known results about NBG to the capacitated setting, what we find interesting are the new arguments we rely on in our proofs. Previous results mainly used combinatorial techniques for both the structural result in (iii) and the main algorithmic ingredient behind (i) and (ii) (which is the fact that the minimum number of odd cycles in the support of an optimal fractional solution to (1) provides a *lower bound* on the size of a stabilizer). Such arguments do not seem to extend easily in presence of capacities. We here instead rely on polyhedral arguments and, in particular, on the notion of *circuits* of a polytope, which are a key concept in optimization (see sections 2 and 3). Interestingly, our polyhedral view point allows us not only to deal more broadly with capacitated instances, but also to simplify some of the cardinal arguments previously used in the literature: in particular, our lower bound proof is (more general and) much simpler than the corresponding one in [14] for the uncapacitated setting.

2 Preliminaries

Problem definition. Let S be a multi-set of vertices V. We denote by $G[c_S - 1]$ the graph G with the capacity of all vertices in S reduced by one. Note that if a vertex appears e.g. twice in S, its capacity is reduced by two. We define a *capacity-stabilizer* as a multi-set S of vertices, such that $G[c_S - 1]$ is stable. Since S is a multi-set, the amount of capacity reduced equals the size of S. We define an *edge-stabilizer* as a set $F \subseteq E$, such that $G \setminus F := (V, E \setminus F)$ is stable.

Capacity-stabilizer problem: given $G = (V, E)$ with edge weights $w \in \mathbb{R}^E_{\geq 0}$ and vertex capacities $c \in \mathbb{Z}^V_{\geq 0}$, find a capacity-stabilizer of minimum cardinality.

Edge-stabilizer problem: given $G = (V, E)$ with edge weights $w \in \mathbb{R}^E_{\geq 0}$ and vertex capacities $c \in \mathbb{Z}^V_{\geq 0}$, find an edge-stabilizer of minimum cardinality.

Notation. We use (G, w, c) to refer to a graph with edge weights and vertex capacities, and $[(G, w, c), M]$ to refer to a graph with a given c-matching M. We say that $[(G, w, c), M]$ is stable if G is stable and M is a maximum-weight c-matching in G.

Let S be a multi-set of vertices. We use c^{S-1} to refer to the capacities in $G[c_S - 1]$. For any $s \in S$, with $S \setminus s$ we mean removing s just once from S.

For a vertex v, we let $\delta(v)$ be the set of edges of G incident into it. For $F \subseteq E$, we denote by d_v^F the degree of v in G with respect to the edges in F. For $f \in \mathbb{R}^E$, we define $f(F) := \sum_{e \in F} f_e$. Given a c-matching M, we say that $v \in V$ is *exposed* if $d_v^M = 0$, *unsaturated* if $d_v^M < c_v$ and *saturated* if $d_v^M = c_v$. We also use these terms for *fractional c-matchings* x, e.g., v is saturated if $x(\delta(v)) = c_v$. We let $\triangle$ denote the symmetric difference operator.

We denote a (uv-)walk W by listing its edges and endpoints sequentially, i.e., by $W = (u; e_1, \ldots, e_k; v)$. We say a walk is closed if $u = v$. A trail is a walk in which edges do not repeat. A path is a trail in which internal vertices do not repeat. A cycle is a path which starts and ends at the same vertex. Note that the edge set of a walk can be a multi-set.

Duality. The dual of (1) is given by

$$\tau_f^c(G) := \min\left\{c^\top y + 1^\top z : y_u + y_v + z_{uv} \geq w_{uv} \ \forall uv \in E, y \geq 0, z \geq 0\right\}. \quad (2)$$

A feasible solution (y, z) of (2) is called a *fractional vertex cover*. By LP theory, we have $\nu^c(G) \leq \nu_f^c(G) = \tau_f^c(G)$. The complementary slackness conditions of $\nu_f^c(G)$ and $\tau_f^c(G)$ are as follows.

$$(x_{uv} = 0 \vee y_u + y_v + z_{uv} = w_{uv}) \wedge (y_v = 0 \vee x(\delta(v)) = c_v) \wedge (z_{uv} = 0 \vee x_{uv} = 1) \quad (3)$$

2.1 Augmenting Structures

Definition 1. *We say that a walk W is M-alternating (w.r.t. a matching M) if it alternates edges in M and edges not in M. We say W is M-augmenting if it is M-alternating and $w(W \setminus M) > w(W \cap M)$. An M-alternating uv-walk W is* proper *if $W \triangle M$ is a c-matching.*

Definition 2. *Given an M-alternating walk $W = (u; e_1, \ldots, e_k; v)$ and an $\varepsilon > 0$, the* ε-augmentation *of W is the vector $x^{M/W}(\varepsilon) \in \mathbb{R}^E$ given by*

$$x_e^{M/W}(\varepsilon) = \begin{cases} 1 - \kappa(e)\varepsilon & \text{if } e \in M, \\ \kappa(e)\varepsilon & \text{if } e \notin M, \end{cases}$$

where $\kappa(e) = |\{i \in [k] : e_i = e\}|$. We say that W is feasible *if there exists an $\varepsilon > 0$ such that the corresponding ε-augmentation of W is a fractional c-matching.*

To get a better understanding of proper and feasible, we state for different kinds of walks what proper and feasible mean. *(i) Non-closed walks:* an M-alternating walk $W = (u; e_1, \ldots, e_k; v)$, where $u \neq v$, is proper and feasible if either $e_1 \in M$ or $d_u^M \leq c_u - 1$, and if either $e_k \in M$ or $d_v^M \leq c_v - 1$. *(ii) Even-length closed walks:* an M-alternating walk $W = (v; e_1, \ldots, e_k; v)$, such that k is even, is always proper and feasible. *(iii) Odd-length closed walks:* an M-alternating walk $W = (v; e_1, \ldots, e_k; v)$, such that k is odd, is proper if either $e_1, e_k \in M$ or $d_v^M \leq c_v - 2$, and feasible if either $e_1, e_k \in M$ or $d_v^M \leq c_v - 1$.

Theorem 1 (theorem 1 in [10]). *A c-matching M in (G, w, c) is maximum-weight if and only if G does not contain a proper M-augmenting trail.*

Theorem 2 (theorem 3 in [10]). $[(G, w, c), M]$ *is stable if and only if G does not contain a feasible M-augmenting walk.*

2.2 Basic Fractional c-Matchings

The polytope of fractional c-matchings in G is $\mathcal{P}_{\text{FCM}}(G)$, formally defined as

$$\mathcal{P}_{\text{FCM}}(G) := \left\{ x \in \mathbb{R}^E : x(\delta(v)) \leq c_v \ \forall v \in V, 0 \leq x \leq 1 \right\}.$$

We write $\mathcal{P}_{\text{FCM}}$ if G is clear from the context, or irrelevant. We refer to the vertices of $\mathcal{P}_{\text{FCM}}$ as *basic* fractional c-matchings. The next result is well known, but we provide a proof in appendix A for completeness.

Theorem 3. *A fractional c-matching x is basic if and only if its components are equal to 0, $\frac{1}{2}$ or 1, and the edges with $x_e = \frac{1}{2}$ induce vertex-disjoint odd cycles with saturated vertices.*

We partition the support of a basic fractional c-matching x into the odd cycles induced by $x_e = \frac{1}{2}$-edges: $\mathscr{C}_x = \{C_1, \ldots, C_q\}$, and matched edges: $\mathscr{M}_x = \{e \in E : x_e = 1\}$.

Definition 3. Alternate rounding $C = (v; e_1, \ldots, e_{2k+1}; v) \in \mathscr{C}_x$ exposing v, *means replacing x_e by $\hat{x}_e = 0$ for all $e \in \{e_1, e_3, \ldots, e_{2k+1}\}$ and by $\hat{x}_e = 1$ for all $e \in \{e_2, e_4, \ldots, e_{2k}\}$. Similarly, we define* alternate rounding $C \in \mathscr{C}_x$ covering v.

Let $\mathcal{X}$ be the set of basic maximum-weight fractional c-matchings in G. Define $\gamma(G) := \min_{x \in \mathcal{X}} |\mathscr{C}_x|$, as the minimum number of half-integral odd cycles in the support of any basic maximum-weight fractional c-matching in G. As already noted by [14] for the $c = 1$ case, we have

Proposition 1. *A graph (G, w, c) is stable if and only if $\gamma(G) = 0$.*

[14] proposes an algorithm to obtain a maximum-weight basic fractional matching with minimum number of odd cycles ($|\mathscr{C}_x| = \gamma(G)$). Their algorithm can be generalized to c-matchings. We just state the result here, all details can be found in appendix B.

Theorem 4. *A basic maximum-weight fractional c-matching x with $|\mathscr{C}_x| = \gamma(G)$ can be computed in polynomial time.*

2.3 Circuits of the Fractional c-Matching Polytope

Let e be an edge of a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, Bx \leq d\}$. The *edge direction* of e is $v - w$ for any two distinct points v and w on e. The *circuits* of a polyhedron are all potential edge directions that can appear for any choice of b and d [9, theorem 1.8]. Let $\mathcal{C}(\mathcal{P})$ denote the set of circuits with co-prime integer components of $\mathcal{P}$.

For a characterization of the circuits of the fractional c-matching polytope we rely on [8], which defined five classes of graphs $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5)$, listed below.

(i) Let $\mathcal{E}_1$ denote the set of all subgraphs $F \subseteq G$ such that F is an even cycle.
(ii) Let $\mathcal{E}_2$ denote the set of all subgraphs $F \subseteq G$ such that F is an odd cycle.
(iii) Let $\mathcal{E}_3$ denote the set of all subgraphs $F \subseteq G$ such that F is a path.
(iv) Let $\mathcal{E}_4$ denote the set of all subgraphs $F \subseteq G$ such that $F = C \cup P$, where C and P are an odd cycle and a non-empty path, respectively, that intersect only at an endpoint of P.
(v) Let $\mathcal{E}_5$ denote the set of all subgraphs $F \subseteq G$ such that $F = C_1 \cup P \cup C_2$, where C_1 and C_2 are odd cycles, and P is a path satisfying the following: if P is non-empty, then C_1 and C_2 are vertex-disjoint and P intersects each C_i exactly at its endpoints; if P is empty then C_1 and C_2 intersect at only one vertex.

A set of circuits can be associated to the subgraphs in these classes, by defining:

$$
\begin{aligned}
\mathcal{C}_1 = \bigcup_{F \in \mathcal{E}_1} \Big\{ g \in \{-1,1\}^E : \quad & g(e) \neq 0 && \text{iff } e \in E(F) \\
& g(\delta(v)) = 0 && \forall v \in V(F) \quad \Big\}, \\
\mathcal{C}_2 = \bigcup_{F \in \mathcal{E}_2} \Big\{ g \in \{-1,1\}^E : \quad & g(e) \neq 0 && \text{iff } e \in E(F) \\
& g(\delta(w)) \neq 0 && \text{for one } w \in V(F) \\
& g(\delta(v)) = 0 && \forall v \in V(F) \setminus \{w\} \quad \Big\}, \\
\mathcal{C}_3 = \bigcup_{F \in \mathcal{E}_3} \Big\{ g \in \{-1,1\}^E : \quad & g(e) \neq 0 && \text{iff } e \in E(F) \\
& g(\delta(v)) = 0 && \forall v : |\delta(v)| = 2 \quad \Big\}, \\
\mathcal{C}_4 = \bigcup_{F = (C \cup P) \in \mathcal{E}_4} \{ g \in \mathbb{Z}^E : \quad & g(e) \neq 0 && \text{iff } e \in E(F) \\
& g(\delta(v)) = 0 && \forall v : |\delta(v)| \geq 2 \\
& g(e) \in \{-1,1\} && \forall e \in E(C) \\
& g(e) \in \{-2,2\} && \forall e \in E(P) \quad \}, \\
\mathcal{C}_5 = \bigcup_{F = (C_1 \cup P \cup C_2) \in \mathcal{E}_5} \{ g \in \mathbb{Z}^E : \quad & g(e) \neq 0 && \text{iff } e \in E(F) \\
& g(\delta(v)) = 0 && \forall v \in V(F) \\
& g(e) \in \{-1,1\} && \forall e \in E(C_1 \cup C_2) \\
& g(e) \in \{-2,2\} && \forall e \in E(P) \quad \}.
\end{aligned}
$$

The authors of [8] showed that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$ is precisely the set of circuits of the *fractional matching polytope*, that is $\mathcal{P}_{\text{FCM}}$ with $c = 1$ and without the (redundant) constraints $x \leq 1$. Since the circuits are independent on the right hand side (note that the constraints $x \leq 1$ are parallel to $x \geq 0$), the same set of circuits apply to $\mathcal{C}(\mathcal{P}_{\text{FCM}})$. Hence

Proposition 2. *$\mathcal{C}(\mathcal{P}_{FCM}) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$.*

3 Key Polyhedral Tools

Consider the following general setting. Let $\mathcal{P}$ be any polytope, $a^\top x \leq b$ be an inequality of the description of $\mathcal{P}$, and $\delta \in \mathbb{R}_{>0}$. Let $\overline{x}$ be an optimal solution of the LP: $\max\left\{c^\top x : x \in \mathcal{P}, a^\top x \leq b - \delta\right\}$, and assume that $\overline{x}$ is a non-optimal vertex of the LP $\max\left\{c^\top x : x \in \mathcal{P}\right\}$. Furthermore, assume that there is no vertex $\tilde{x}$ of $\mathcal{P}$ satisfying $b - \delta < a^\top \tilde{x} < b$.

Theorem 5. *It is possible to move to an optimal solution of* $\max\left\{c^\top x : x \in \mathcal{P}\right\}$ *from $\overline{x}$ in one step over the edges of $\mathcal{P}$ (i.e., there is an optimal vertex of $\mathcal{P}$ adjacent to $\overline{x}$).*

Proof. Let x^* be the optimal solution of $\max\left\{c^\top x : x \in \mathcal{P}\right\}$ that is the *closest* vertex to $\overline{x}$ on $\mathcal{P}$ (that is, such that we need a minimum number of steps over the edges of $\mathcal{P}$ to reach x^* from $\overline{x}$). Note that $a^\top \overline{x} = b - \delta$ and $a^\top x^* = b$, otherwise $\overline{x} + \lambda(x^* - \overline{x})$, for some small $\lambda > 0$, and x^*, respectively, contradict the optimality of $\overline{x}$. We need to show that $\overline{x}$ and x^* are adjacent on $\mathcal{P}$.

Let $\mathcal{P}' = \left\{x \in \mathcal{P} : a^\top x \geq b - \delta\right\}$. Then $\overline{x}, x^* \in \mathcal{P}'$. Note that $\overline{x}$ and x^* are adjacent on $\mathcal{P}$ if and only if they are adjacent on $\mathcal{P}'$. So for the remainder of the proof we restrict ourselves to $\mathcal{P}'$.

For the sake of contradiction assume that $\overline{x}$ and x^* are not adjacent on $\mathcal{P}'$. Then, the line segment of all their convex combinations: $\lambda\overline{x} + (1-\lambda)x^*$ for $0 \leq \lambda \leq 1$, is not an edge of $\mathcal{P}'$. Hence, any point $\lambda'\overline{x} + (1-\lambda')x^*$ for a fixed $0 < \lambda' < 1$ is also a convex combination of other vertices of $\mathcal{P}'$: $\lambda'\overline{x} + (1-\lambda')x^* = \sum_i \alpha_i \hat{x}_i + \sum_j \beta_j \tilde{x}_j$, where $\alpha_i \geq 0$ for all i, $\beta_j \geq 0$ for all j, $\sum_i \alpha_i + \sum_j \beta_j = 1$, $\hat{x}_i$ is a vertex of $\mathcal{P}'$ with $a^\top \hat{x}_i = b - \delta$ for all i, and $\tilde{x}_j$ is a vertex of $\mathcal{P}'$ with $a^\top \tilde{x}_j = b$ for all j. If we multiply both sides by a we get

$$\begin{aligned} & a^\top\left(\lambda'\overline{x} + (1-\lambda')x^*\right) = a^\top\left(\textstyle\sum_i \alpha_i \hat{x}_i + \sum_j \beta_j \tilde{x}_j\right), \\ \Longleftrightarrow \quad & \lambda'(b-\delta) + (1-\lambda')b = \textstyle\sum_i \alpha_i (b-\delta) + \sum_j \beta_j b, \\ \Longleftrightarrow \quad & b - \lambda'\delta = \left(\textstyle\sum_i \alpha_i + \sum_j \beta_j\right) b - \sum_i \alpha_i \delta, \end{aligned}$$

hence $\lambda' = \sum_i \alpha_i$, and consequently $1 - \lambda' = \sum_j \beta_j$. We can also multiply both sides by c. Here we use that $\overline{x}$ is an optimal solution of $\max\{c^\top x : x \in \mathcal{P}, a^\top x \leq b - \delta\}$, and that x^* is an optimal solution of $\max\left\{c^\top x : x \in \mathcal{P}\right\}$.

$$\begin{aligned} c^\top\left(\lambda'\overline{x} + (1-\lambda')x^*\right) &= c^\top\left(\textstyle\sum_i \alpha_i \hat{x}_i + \sum_j \beta_j \tilde{x}_j\right) = \textstyle\sum_i \alpha_i c^\top \hat{x}_i + \sum_j \beta_j c^\top \tilde{x}_j \\ &\leq \textstyle\sum_i \alpha_i c^\top \overline{x} + \sum_j \beta_j c^\top x^* = \lambda' c^\top \overline{x} + (1-\lambda') c^\top x^* \end{aligned}$$

So we must have equality throughout. In particular, $c^\top \tilde{x}_j = c^\top x^*$, i.e., all $\tilde{x}_j$ are optimal solutions to $\max\left\{c^\top x : x \in \mathcal{P}\right\}$. Note that all $\tilde{x}_j$'s are also vertices of $\mathcal{P}$. We show that we can choose some $\tilde{x}_j$ to be adjacent to $\overline{x}$ on $\mathcal{P}'$, and hence also on $\mathcal{P}$, contradicting that x^* is the optimal solution closest to x.

Let x' be a vertex of $\mathcal{P}'$ that is adjacent to $\overline{x}$, such that $ax' = b$ (such an x' must exist). Consider the line segment between x' and $\lambda'\overline{x} + (1-\lambda')x^*$:

$\mu x' + (1-\mu)\left(\lambda'\overline{x} + (1-\lambda')x^*\right)$ for $0 \le \mu \le 1$. For $\mu < 0$, this line segment extends beyond $\lambda'\overline{x} + (1-\lambda')x^*$. If this is still in $\mathcal{P}'$, we can write $\lambda'\overline{x} + (1-\lambda')x^*$ as a convex combination of x' and some other $\hat{x}_i$'s and $\tilde{x}_j$'s. Reaching our desired contradiction. Otherwise, $\lambda'\overline{x} + (1-\lambda')x^*$ must be at the boundary, a face, of $\mathcal{P}'$. Because $\lambda'\overline{x} + (1-\lambda')x^*$ is in this face, the whole line segment $\lambda\overline{x} + (1-\lambda)x^*$ for $0 \le \lambda \le 1$ must be in this face. We can then repeat the argument, replacing $\mathcal{P}'$ by this face. Since this face has strictly smaller dimension than $\mathcal{P}'$, we either find a contradiction in one of the iterations, or we reach a face of dimension one, i.e., an edge of $\mathcal{P}'$. Since this edge contains the whole line segment $\lambda\overline{x} + (1-\lambda)x^*$ for $0 \le \lambda \le 1$, the line segment is the edge, a contradiction. □

We make use of this theorem for $\mathcal{P}_{\mathrm{FCM}}$ in two settings: to analyze what happens when we reduce the capacity of a vertex, and when we remove an edge, in capacitated NBG instances. For the first setting, we have:

Theorem 6. *Let $\overline{x}$ be a maximum-weight fractional c-matching in $G[c_v - 1]$ for some $v \in V$. If $\overline{x}$ is basic in G, but not maximum-weight in G, then it is possible to move to a basic maximum-weight fractional c-matching in G in one step over the edges of $\mathcal{P}_{FCM}(G)$.*

Proof. It readily follows from theorem 5 by letting $\mathcal{P} = \mathcal{P}_{\mathrm{FCM}}(G)$, $a^\top x \le b$ be $x(\delta(v)) \le c_v$, $\delta = 1$, and w be the objective function. □

In the second setting, we need to do a bit of extra work.

Theorem 7. *Let $\overline{x}$ be a maximum-weight fractional c-matching in $G \setminus e$ for some $e \in E$. If $\overline{x}$ is basic in G, but not maximum-weight in G, then it is possible to move to a basic maximum-weight fractional c-matching in G in at most two steps over the edges of $\mathcal{P}_{FCM}(G)$. If two steps are needed, the first one moves to a vertex with $x_e = \frac{1}{2}$, and the second one moves to a vertex with $x_e = 1$.*

Proof. Case 1: $x_e \in \{0,1\}$ for all vertices of $\mathcal{P}_{FCM}(G)$. It follows directly from theorem 5 that only one step is needed, by letting $\mathcal{P} = \mathcal{P}_{\mathrm{FCM}}(G)$, $a^\top x \le b$ be $x_e \le 1$, $\delta = 1$, and w the objective function.

Case 2: there are vertices of $\mathcal{P}_{FCM}(G)$ that satisfy $x_e = \frac{1}{2}$. In this case we split the problem in two. Let $\mathcal{P}^{\le} = \left\{x \in \mathcal{P}_{\mathrm{FCM}}(G) : x_e \le \frac{1}{2}\right\}$ and $\mathcal{P}^{\ge} = \left\{x \in \mathcal{P}_{\mathrm{FCM}}(G) : x_e \ge \frac{1}{2}\right\}$. Let x be a maximum-weight fractional c-matching in $G \setminus e$, such that x is basic in G, but x does not have maximum-weight in G. Note that x also does not have maximum-weight over $\mathcal{P}^{\le}$. In addition, since x is a vertex of $\mathcal{P}_{\mathrm{FCM}}(G)$, and feasible in $\mathcal{P}^{\le}$, it is also a vertex of $\mathcal{P}^{\le}$.

Let $\mathcal{P} = \mathcal{P}^{\le}$, $ax \le b$ be $x_e \le \frac{1}{2}$, $\delta = \frac{1}{2}$, and w be the objective function. We can then apply theorem 5: it is possible to move to an optimal solution $\hat{x}$ of $\max\left\{w^\top x : x \in \mathcal{P}^{\le}\right\}$ from x in one step over the edges of $\mathcal{P}^{\le}$. Note that $\hat{x}_e = \frac{1}{2}$. If $\hat{x}$ is a vertex of $\mathcal{P}_{\mathrm{FCM}}(G)$, then the edge of $\mathcal{P}^{\le}$ that is used, is also an edge of $\mathcal{P}_{\mathrm{FCM}}(G)$. Otherwise, the edge of $\mathcal{P}^{\le}$ that is used, is only part of an edge of $\mathcal{P}_{\mathrm{FCM}}(G)$. If $\hat{x}$ is a basic maximum-weight fractional c-matching in G, we are done. So suppose that that is not the case.

Subcase 2a: $\hat{x}$ is a vertex of $\mathcal{P}_{FCM}(G)$. Since $\hat{x}$ is feasible in $\mathcal{P}^{\geq}$, it is also a vertex in $\mathcal{P}^{\geq}$. By assumption, $\hat{x}$ is optimal over $\mathcal{P}^{\leq}$, so also over $\mathcal{P}^{\geq}$ with the additional constraint $x_e \leq \frac{1}{2}$, but not optimal over $\mathcal{P}_{\text{FCM}}(G)$, so also not over $\mathcal{P}^{\geq}$. Let $\mathcal{P} = \mathcal{P}^{\geq}$, $ax \leq b$ be $x_e \leq 1$, $\delta = \frac{1}{2}$, and w the objective function. We can then apply theorem 5: it is possible to move to an optimal solution x^* of $\max\left\{w^\top x : x \in \mathcal{P}^{\geq}\right\}$ from $\hat{x}$ in one step over the edges of $\mathcal{P}^{\geq}$. Note that $x^*_e = 1$. Then, x^* is also an optimal solution of $\max\left\{w^\top x : x \in \mathcal{P}_{\text{FCM}}(G)\right\}$, and a vertex of $\mathcal{P}_{\text{FCM}}(G)$. Since $\hat{x}$ and x^* are both vertices of $\mathcal{P}_{\text{FCM}}(G)$, the edge of $\mathcal{P}^{\geq}$ that is used, is also an edge of $\mathcal{P}_{\text{FCM}}(G)$.

All in all, we get: starting from x, it is possible to move to a basic maximum-weight fractional c-matching in G in two steps over the edges of $\mathcal{P}_{\text{FCM}}(G)$, such that $x_e = \frac{1}{2}$ after the first step, and $x_e = 1$ after the second step.

Subcase 2b: $\hat{x}$ is not a vertex of $\mathcal{P}_{FCM}(G)$. Since we moved from x to $\hat{x}$ over only part of an edge of $\mathcal{P}_{\text{FCM}}(G)$, it must be that $\mathcal{P}^{\leq}$ and $\mathcal{P}^{\geq}$ split this edge in two, and the splitting point, $\hat{x}$, is a vertex of both polytopes. By assumption, $\hat{x}$ is optimal over $\mathcal{P}^{\leq}$, so also over $\mathcal{P}^{\geq}$ with the additional constraint $x_e \leq \frac{1}{2}$. Since we reached $\hat{x}$ by moving over just part of an edge of $\mathcal{P}_{\text{FCM}}(G)$, and this increased the weight, moving further along this edge will increase the weight even further. Hence, $\hat{x}$ is not optimal over $\mathcal{P}^{\geq}$. Let $\mathcal{P} = \mathcal{P}^{\geq}$, $ax \leq b$ be $x_e \leq 1$, $\delta = \frac{1}{2}$, and w the objective function. We can again apply theorem 5: it is possible to move to an optimal solution x^* of $\max\left\{w^\top x : x \in \mathcal{P}^{\geq}\right\}$ from $\hat{x}$ in one step over the edges of $\mathcal{P}^{\geq}$. Note that $x^*_e = 1$. Then, x^* is also an optimal solution of $\max\left\{w^\top x : x \in \mathcal{P}_{\text{FCM}}(G)\right\}$, and a vertex of $\mathcal{P}_{\text{FCM}}(G)$. Since x^* is a vertex of $\mathcal{P}_{\text{FCM}}(G)$, but $\hat{x}$ is not, the edge of $\mathcal{P}^{\geq}$ that is used, is only part of an edge of $\mathcal{P}_{\text{FCM}}(G)$. In particular, it must be the remainder of the edge over which we moved in the first step.

All in all, we get: starting from x, it is possible to move to a basic maximum-weight fractional c-matching in G in one step over the edges of $\mathcal{P}_{\text{FCM}}(G)$. □

Consider $\mathcal{P}_{\text{FCM}}$. We introduce a non-negative slack-variable for each inequality of the form $x(\delta(v)) \leq c_v$. We get a polytope that naturally corresponds to the set of fractional perfect c-matchings on a modified graph $\overline{G} = (V, E \cup L)$, obtained from G by adding a loop edge $vv \in L$ for each vertex $v \in V$. We define

$$\mathcal{P}_{\text{FPCM}}(\overline{G}) := \left\{x \in \mathbb{R}^{E \cup L} : x(\delta(v)) = c_v \ \forall v \in V, x \geq 0, x_e \leq 1 \ \forall e \in E\right\},$$

as the polytope of fractional perfect c-matchings in $\overline{G}$. The following theorem is based on methods used in [17, section III-G].

Theorem 8. *If x and y are adjacent vertices of $\mathcal{P}_{FCM}(G)$, then $y = x + \alpha g$, where $g \in \mathcal{C}(\mathcal{P}_{FCM})$ and $\alpha \in \left\{\frac{1}{2}, 1\right\}$. Furthermore,*

- *if $\alpha = 1$, then $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ and $|\mathscr{C}_y| = |\mathscr{C}_x|$.*
- *if $\alpha = \frac{1}{2}$, then $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_5$, and*
  - *if $g \in \mathcal{C}_1$, then $|\mathscr{C}_y| = |\mathscr{C}_x|$.*
  - *if $g \in \mathcal{C}_2 \cup \mathcal{C}_4$, then $|\mathscr{C}_y| = |\mathscr{C}_x| \pm 1$, and the odd cycle in g belongs to either $\mathscr{C}_x$ or $\mathscr{C}_y$.*

- *if $g \in \mathcal{C}_5$, then $|\mathscr{C}_y| = |\mathscr{C}_x| \pm \{0,2\}$, and the odd cycles in g both belong to $\mathscr{C}_x$, or both to $\mathscr{C}_y$, or exactly one belongs to $\mathscr{C}_x$ and the other to $\mathscr{C}_y$.*

Proof. Let g be the edge direction of the edge between x and y, scaled in such a way that the components of g are co-prime. Then, clearly, $y = x + \alpha g$ for some $\alpha \neq 0$. Without loss of generality, we can assume that $\alpha > 0$, since $-g$ is also an edge direction of the same edge. All edge directions are circuits, hence $g \in \mathcal{C}(\mathcal{P}_{\text{FCM}})$, and in particular, $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$, by proposition 2. That means that the components of g have a magnitude of at least 1. Then it follows from $0 \leq x \leq 1$, that $\alpha \leq 1$. Likewise, for the circuits with components that have a magnitude of 2, it follows that $\alpha \leq \frac{1}{2}$. Finally, since x and y are vertices, i.e. they are basic, their components are half-integral, which implies $\alpha \in \left\{\frac{1}{2}, 1\right\}$.

Case 1: $\alpha = 1$. As discussed, for circuits with components that have a magnitude of 2, $\alpha \leq \frac{1}{2}$. Hence, in this case, $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Furthermore, all components of αg are integral, which means that half-integral edges, and in particular the half-integral odd cycles, are not affected. It follows that $|\mathscr{C}_y| = |\mathscr{C}_x|$.

Case 2: $\alpha = \frac{1}{2}$. Circuits in $\mathcal{C}_3$ correspond with paths. For either endpoint of this path, applying the circuit results in $\pm\alpha \cdot 1 = \pm\frac{1}{2}$ on a single edge incident with the vertex. Since x and y are both basic, this is not possible: $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_5$.

We extend x and y to fractional perfect c-matchings $\overline{x}$ and $\overline{y}$ in $\overline{G}$. This extension is uniquely obtained by setting $\overline{x}_{vv} = c_v - x(\delta(v))$ for each $vv \in L$, likewise for $\overline{y}$. Since x and y are adjacent vertices of $\mathcal{P}_{\text{FCM}}(G)$, it follows that $\overline{x}$ and $\overline{y}$ are adjacent vertices of $\mathcal{P}_{\text{FPCM}}(\overline{G})$. We extend g to $\overline{g}$ such that $\overline{y} = \overline{x} + \frac{1}{2}\overline{g}$.

Let $E^1 = \{e \in E : \overline{x}_e = \overline{y_e} = 1\}$, and $\mathcal{G}$ be the graph induced by the supports of $\overline{x}$ and $\overline{y}$, minus the edges in E^1. We claim that there is exactly one component of $\mathcal{G}$ that contains an edge e with $\overline{x}_e \neq \overline{y}_e$. Clearly there is at least one, since $x \neq y$ and hence $\overline{x} \neq \overline{y}$. Actually, $\overline{x} \neq \overline{y}$ only on the support of $\overline{g}$. Since $\overline{x}_e \neq \overline{y}_e$ for every edge e in the support of $\overline{g}$, we have that at least one of $\overline{x}_e$ and $\overline{y}_e$ is not zero, and at least one is not one. Hence, all those edges are in $\mathcal{G}$, and in particular they all are in the same component, since $\overline{g}$ is connected.

Let $\mathcal{K}$ be the component of $\mathcal{G}$ that contains an edge e with $\overline{x}_e \neq \overline{y}_e$, and let k be the number of vertices in this component. Let K be a subgraph of $\overline{G}$ induced by the vertices in $\mathcal{K}$, minus the edges in E^1, and change the capacities accordingly: for each vertex $v \in V(K)$, reduce the capacity of v by $\left|\delta(v) \cap E^1\right|$. Let $\overline{x}|_K$ be obtained from $\overline{x}$ by restricting to the edges in K, likewise for $\overline{y}|_K$. Note that $\overline{x}|_K$ and $\overline{y}|_K$ are perfect c-matchings in K with the adjusted capacity. In particular, they are adjacent vertices of $\mathcal{P}_{\text{FPCM}}(K)$, since $\overline{x}$ and $\overline{y}$ are adjacent vertices of $\mathcal{P}_{\text{FPCM}}(\overline{G})$.

Let A be the incidence matrix of K. Since the columns associated to the loop edges form an identity matrix, the rank of A is k. Since $\overline{x}|_K$ and $\overline{y}|_K$ are adjacent vertices, there must be $|E(K)| - 1$ linearly independent constraints that are tight for both of them. Since the rank of A is k, and we removed the edges for which the "≤ 1" constraint is tight for both $\overline{x}$ and $\overline{y}$, this implies that there are at least $|E(K)| - 1 - k$ edges for which the "≥ 0" constraint is tight for both of them. Consequently, there are at most $k + 1$ edges in the union of the supports of $\overline{x}|_K$ and $\overline{y}|_K$. Note that the graph induced by the supports of $\overline{x}|_K$ and $\overline{y}|_K$

is $\mathcal{K}$. With $k+1$ edges on k connected vertices, we have a spanning tree, plus two additional (possibly loop) edges: it is easy to realize then that there can be at most two odd cycles in $\mathcal{K}$.

Subcase 2a: $g \in \mathcal{C}_1$. The support of $\overline{g}$ is an even cycle, say C. If $\overline{x}|_K = \frac{1}{2}$ for all edges on C, then $\overline{x}|_K$, and therefore also x, contains a half-integral even cycle, which contradicts that x is basic. Similarly, if $\overline{x}|_K$ is integral for all edges on C, $\overline{y}|_K = \frac{1}{2}$ for all edges on C, contradicting that y is basic. Hence, $\overline{x}|_K$ has edges on C with integral value, and also edges with value $\frac{1}{2}$. The half-integral edges imply that $\overline{x}|_K$ has at least one half-integral odd cycle. The integral edges become half-integral in $\overline{y}|_K$, which means $\overline{y}|_K$ also has at least one half-integral odd cycle, different from the one of $\overline{x}|_K$. These odd cycles are distinct, and both in $\mathcal{K}$, and we have already shown that $\mathcal{K}$ contains at most two odd cycles. Hence, $\left|\mathscr{C}_{\overline{y}|_K}\right| = \left|\mathscr{C}_{\overline{x}|_K}\right|$.

Subcase 2b: $g \in \mathcal{C}_2 \cup \mathcal{C}_4$. The support of $\overline{g}$ is a (possibly empty) path, an odd cycle and a loop edge, of which only the odd cycle can influence the half-integral odd cycles in $\overline{x}|_K$ and $\overline{y}|_K$. If this odd cycle in the support of $\overline{g}$ is a half-integral odd cycle in $\overline{x}|_K/\overline{y}|_K$, then note that $\overline{y}|_K$ contains exactly one less/more half-integral odd cycle than $\overline{x}|_K$. Otherwise, both $\overline{x}|_K$ and $\overline{y}|_K$ have half-integral and integral edges on the odd cycle of g. That means that $\overline{x}|_K$ and $\overline{y}|_K$ both have at least one odd cycle in the component, different from the odd cycle in g and different from each other. But then there are at least three odd cycles in the component, a contradiction. Hence, the odd cycle in g belongs to either $\mathscr{C}_{\overline{x}|_K}$ or $\mathscr{C}_{\overline{y}|_K}$, and $\left|\mathscr{C}_{\overline{y}|_K}\right| = \left|\mathscr{C}_{\overline{x}|_K}\right| \pm 1$.

Subcase 2c: $g \in \mathcal{C}_5$. The support of $\overline{g}$ is two odd cycles connected by a (possibly empty) path. Since $\mathcal{K}$ contains at most two odd cycles, and $\overline{g}$ already contains two odd cycles, these are the only odd cycles. Similar to the previous subcase, for both the odd cycles separately we can argue that not both $\overline{x}|_K$ and $\overline{y}|_K$ can have half-integral and integral edges on the odd cycle, i.e., each odd cycle belongs to either $\mathscr{C}_{\overline{x}|_K}$ or $\mathscr{C}_{\overline{y}|_K}$. There are three options: both odd cycles belong to $\mathscr{C}_{\overline{x}|_K}$, or both to $\mathscr{C}_{\overline{y}|_K}$, or one to $\mathscr{C}_{\overline{x}|_K}$ and one to $\mathscr{C}_{\overline{y}|_K}$. It follows that $\left|\mathscr{C}_{\overline{y}|_K}\right| = \left|\mathscr{C}_{\overline{x}|_K}\right| \pm \{0,2\}$.

Since $\overline{x}$ equals $\overline{y}$ outside of K, our conclusions carry over from $\overline{x}|_K$ and $\overline{y}|_K$ to $\overline{x}$ and $\overline{y}$. In addition, removing loop edges, i.e., going back from $\overline{x}, \overline{y}$ to x, y, does not affect half-integral odd cycles. Hence, our conclusions also hold for x and y: If $g \in \mathcal{C}_1$, then $|\mathscr{C}_y| = |\mathscr{C}_x|$. If $g \in \mathcal{C}_2 \cup \mathcal{C}_4$, then $|\mathscr{C}_y| = |\mathscr{C}_x| \pm 1$ and the odd cycle in g belongs to either $\mathscr{C}_x$ or $\mathscr{C}_y$. If $g \in \mathcal{C}_5$, then $|\mathscr{C}_y| = |\mathscr{C}_x| \pm \{0,2\}$, and the odd cycles in g both belong to $\mathscr{C}_x$, or both to $\mathscr{C}_y$, or exactly one to $\mathscr{C}_x$ and the other to $\mathscr{C}_y$. □

4 Capacity-Stabilizer

We first exploit the above polyhedral results to prove that a lower bound on the size of a capacity-stabilizer is given by the minimum number of half-integral odd cycles in the support of any basic maximum-weight fractional c-matching.

Lemma 1. *For every capacity-stabilizer S, $|S| \geq \gamma(G)$.*

Proof. To prove the lemma, by proposition 1, it is enough to show that reducing the capacity of any vertex by one decreases the number of odd cycles by at most one. Therefore, from now on, we concentrate on proving the following statement:

$$\text{for all } v \in V,\ \gamma(G[c_v - 1]) \geq \gamma(G) - 1. \qquad (\star)$$

Let x be a basic maximum-weight fractional c-matching in $G[c_v - 1]$ with $\gamma(G[c_v - 1])$ odd cycles. Let (y, z) be an optimal fractional vertex cover in $G[c_v - 1]$, satisfying complementary slackness (see equation (3)) with x. Note that increasing the capacity does not influence the feasibility of x and (y, z), hence they are a fractional c-matching and vertex cover in G, respectively.

If x has maximum-weight in G, then $\gamma(G) \leq |\mathscr{C}_x| = \gamma(G[c_v - 1])$, and hence $(\star)$ holds.

Assume now that x does not have maximum-weight in G. Then, x and (y, z) cannot satisfy complementary slackness in G. The change from $G[c_v - 1]$ to G only influences the complementary slackness condition $y_v = 0 \vee x(\delta(v)) = c_v - 1$, so we must have $y_v > 0$ and $x(\delta(v)) = c_v - 1 < c_v$. We distinguish two cases.

Case 1: $v \in V(C)$ for some odd cycle $C \in \mathscr{C}_x$. Create a new fractional c-matching $\hat{x}$ by alternate rounding C covering v. One can check that $\hat{x}$ and (y, z) satisfy complementary slackness in G. Consequently, $\hat{x}$ is a maximum-weight fractional c-matching in G, and given that x is basic, so is $\hat{x}$. Then, $\gamma(G) \leq |\mathscr{C}_{\hat{x}}| = |\mathscr{C}_x| - 1 = \gamma(G[c_v - 1]) - 1$, and hence $(\star)$ holds.

Case 2: $v \notin V(\mathscr{C}_x)$. Since v is not part of any odd cycle, by theorem 3, x is basic in G. Then, by theorem 6, we can move to a basic maximum-weight fractional c-matching x^* in G in one step over the edges of $\mathcal{P}_{\text{FCM}}(G)$. By theorem 8, $x^* = x + \alpha g$ for $\alpha \in \left\{\frac{1}{2}, 1\right\}$ and $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. Because $w^\top x^* > w^\top x$, x^* cannot be feasible in $G[c_v - 1]$, so we must have $x^*(\delta(v)) = c_v = x(\delta(v)) + 1$. Consequently, $g \in \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. Then, by theorem 8, we have $|\mathscr{C}_{x^*}| = |\mathscr{C}_x| \pm \{0, 1\}$. Therefore $|\mathscr{C}_{x^*}| \leq |\mathscr{C}_x| + 1$, and consequently, $\gamma(G) \leq |\mathscr{C}_{x^*}| \leq |\mathscr{C}_x| + 1 = \gamma(G[c_v - 1]) + 1$, yielding $(\star)$. □

We can now state the polynomial-time algorithm to solve the stabilization problem via capacity reduction.

Algorithm 1: stabilization by capacity reduction

initialize $S \leftarrow \emptyset$
compute a basic maximum-weight fractional c-matching x in G with $\gamma(G)$ odd cycles $C_1, \ldots, C_{\gamma(G)}$, and a minimum fractional vertex cover (y, z) in G
for $i = 1$ *to* $\gamma(G)$ **do**
  $S \leftarrow S + \arg\min_{v \in V(C_i)} y_v$
return $G[c_S - 1]$

Theorem 9. *Algorithm 1 is a polynomial-time algorithm that computes a minimum capacity-stabilizer S for G. Moreover:*

(a) The solution S reduces the capacity of each vertex by at most one unit.
(b) The solution S preserves the weight of a maximum-weight matching by a factor of $\frac{2}{3}$, i.e., $\nu^c(G[c_S - 1]) \geq \frac{2}{3}\nu^c(G)$.

Proof. Let $S = \{v_1, \ldots, v_{\gamma(G)}\}$ be the set of vertices whose capacity is reduced in algorithm 1. Let $\hat{x}$ be obtained from x by alternate rounding C_i exposing v_i, for all $i \in \{1, \ldots, \gamma(G)\}$. Clearly, $\hat{x}$ is a fractional c-matching in $G[c_S - 1]$. In addition, (y, z) is still a fractional vertex cover in $G[c_S - 1]$. One can check that they satisfy complementary slackness with respect to $G[c_S - 1]$. Hence, they are optimal in $G[c_S - 1]$. Note that $\hat{x}$ is an integral matching. Hence, $G[c_S - 1]$ is stable. Moreover, $|S| = \gamma(G)$, which is minimum by lemma 1.

Since all cycles in $\mathscr{C}_x$ are vertex-disjoint, the set S is *not* a multi-set. Hence, (a) holds. To see (b), note that since $v_i = \arg\min_{v \in V(C_i)} y_v$, we have

$$y_{v_i} \leq \frac{y(V(C_i))}{|C_i|} \leq \tfrac{1}{3} y(V(C_i)).$$

Then, using stability of $G[c_S - 1]$ and optimality of (y, z) in $G[c_S - 1]$, we find

$$\begin{aligned}\nu^c(G[c_S - 1]) = \tau_f^c(G[c_S - 1]) = \left(c^{S-1}\right)^\top y + 1^\top z = c^\top y - \sum_{i=1}^{\gamma(G)} y_{v_i} + 1^\top z \\ \geq c^\top y - \sum_{i=1}^{\gamma(G)} \tfrac{1}{3} y(V(C_i)) + 1^\top z \geq \tfrac{2}{3}\left(c^\top y + 1^\top z\right) = \tfrac{2}{3}\tau_f^c(G) \geq \tfrac{2}{3}\nu^c(G).\end{aligned}$$

In the above chain of inequalities, we use the fact that $c_v - \frac{1}{3} \geq \frac{2}{3} c_v$ for all $v \in V(\mathscr{C}_x)$. This is trivially true since for all these vertices $c_v \geq 1$. □

The previous theorem shows that there always exists a capacity-stabilizer of minimum size that preserves the total value that the players can get by a factor of $\frac{2}{3}$. We note that, for arbitrary weighted graphs, this factor is asymptotically best possible, as shown by [14] already in the unit-capacity case. However, for unit-capacities *and* unit-weights, the authors in [1] proved a stronger statement: namely, that inclusion-wise minimal stabilizers completely preserve the total value that the players can get (i.e., up to a factor 1). Using our polyhedral tools, we can show that this statement still holds in the capacitated setting (and note that it is satisfied by the solution provided by our algorithm).

Theorem 10. *In $(G, 1, c)$: for any inclusion-wise minimal capacity-stabilizer S, we have $\nu^c(G[c_S - 1]) = \nu^c(G)$.*

Proof. Let M be a maximum-cardinality c-matching in $G[c_S - 1]$.

Claim. M is maximum in $G[c_{S \setminus v} - 1]$, for any $v \in S$.

Proof. For the sake of contradiction, suppose that $|M| < \nu^c(G[c_{S \setminus v} - 1])$.

Since S is a stabilizer, $G[c_S - 1]$ is stable, and hence there exists a fractional vertex cover (y, z) that satisfies complementary slackness with M in $G[c_S - 1]$.

Increasing the capacity of a vertex does not change feasibility of (y, z), hence, (y, z) is a fractional vertex cover in $G[c_{S\setminus v} - 1]$. Observe that, since the edges have unit-weight, we can assume w.l.o.g. that $y \leq 1$. Then

$$\tau_f^c(G[c_{S\setminus v} - 1]) \leq (c^{S-1})^\top y + y_v + 1^\top z = |M| + y_v \leq |M| + 1 \leq \nu^c(G[c_{S\setminus v} - 1]),$$

i.e., $G[c_{S\setminus v} - 1]$ is stable, contradicting the minimality of S. □

Claim. M is maximum in $G[c_{S\setminus\{u,v\}} - 1]$, for any $u, v \in S$.

Proof. For the sake of contradiction, suppose that $|M| < \nu^c(G[c_{S\setminus\{u,v\}} - 1])$.

Let x be the indicator vector of M, then by theorem 3, x is basic. Since S is inclusion-wise minimal, $G[c_{S\setminus v} - 1]$ is not stable, and thus x is not a maximum fractional c-matching in $G[c_{S\setminus v} - 1]$. We can apply theorem 6 to x, $G[c_S - 1]$ and $G[c_{S\setminus v} - 1]$, and conclude that there exists a basic maximum-weight fractional c-matching $\hat{x}$ in $G[c_{S\setminus v} - 1]$, which is adjacent to x on $\mathcal{P}_{\text{FCM}}$. By theorem 8, $\hat{x} = x + \alpha g$, where $\alpha \in \left\{\frac{1}{2}, 1\right\}$ and $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. Since M is maximum in $G[c_{S\setminus v} - 1]$ by the previous claim, $\hat{x}$ cannot be integral, so $\alpha = \frac{1}{2}$, and consequently by theorem 8, $g \notin \mathcal{C}_3$. Furthermore, the circuits in $\mathcal{C}_1 \cup \mathcal{C}_5$ are not augmenting in cardinality, so $g \in \mathcal{C}_2 \cup \mathcal{C}_4$, and necessarily v must be the only vertex with $g(\delta(v)) \neq 0$. Observing now that $|M|$ and $\nu^c(\cdot)$ are integral, we find that $\hat{x}$ is not a maximum fractional c-matching in $G[c_{S\setminus\{u,v\}} - 1]$, since

$$1^\top \hat{x} = 1^\top x + \tfrac{1}{2} = |M| + \tfrac{1}{2} < \nu^c(G[c_{S\setminus\{u,v\}} - 1]) \leq \nu_f^c(G[c_{S\setminus\{u,v\}} - 1]),$$

Applying theorem 6 again, we get that there exists a basic maximum-weight fractional c-matching x^* in $G[c_{S\setminus\{u,v\}} - 1]$, which is adjacent to $\hat{x}$ on $\mathcal{P}_{\text{FCM}}$. By theorem 8, $x^* = \hat{x} + \beta h$, where $\beta \in \left\{\frac{1}{2}, 1\right\}$ and $h \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. As before, the circuits in $\mathcal{C}_1 \cup \mathcal{C}_5$ are not augmenting in cardinality, so $h \in \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. Since S is an inclusion-wise minimal stabilizer, $G[c_{S\setminus\{u,v\}} - 1]$ is not stable, consequently $1^\top x^* > \nu^c(G[c_{S\setminus\{u,v\}} - 1])$. Using integrality of $\nu^c(\cdot)$ and half-integrality of x^* and $\hat{x}$, this implies $1^\top x^* \geq 1^\top \hat{x} + 1$. So we must have $\beta = 1$, hence $h \in \mathcal{C}_2 \cup \mathcal{C}_3$ by theorem 8, and $1^\top x^* = 1^\top \hat{x} + 1$. Furthermore, u must be one of the (possibly two) vertices with $h(\delta(u)) > 0$.

Assume first that $h \in \mathcal{C}_2$. Then, u is the only vertex with $h(\delta(u)) > 0$. However, as components of βh have a magnitude of one, it means that u is not saturated in $G[c_{S\setminus v} - 1]$. Therefore, $\hat{x} + \frac{1}{2}h$ is a fractional c-matching in $G[c_{S\setminus v} - 1]$ with higher objective function value than $\hat{x}$, contradicting its optimality.

We are left with $h \in \mathcal{C}_3$. It follows that the support of h is a path P_h with endpoints u and some vertex $t \neq u$. Note that, necessarily, $t \neq v$, as v is saturated in $\hat{x}$ while t is not. In particular, neither u nor t are saturated in $G[c_{S\setminus u} - 1]$. However, we know that $x + \beta h$ cannot be a fractional c-matching in $G[c_{S\setminus u} - 1]$, because M is of maximum cardinality in $G[c_{S\setminus u} - 1]$ by the previous claim. Since $x + \beta h$ does not violate any capacity bound in $G[c_{S\setminus u} - 1]$, the reason why $x + \beta h$ is not a fractional c-matching must the fact that either $0 \leq x + \beta h$ or $x + \beta h \leq 1$ does *not* hold. Since instead $0 \leq x + \alpha g + \beta h \leq 1$ holds, it follows that the supports of h and g must share some edge. Note that since all components of βh have a magnitude of one, the support of h cannot overlap with the cycle in the support of g. Let ℓ be the last vertex on the ut-path P_h that is an endpoint of a shared edge between the supports of h and g. By construction, the subpath P_1 from ℓ to t in P_h is then an M-alternating path. Let P_g denote the edges in the support of g, and let P_2 be the path from v to ℓ in P_g. Note that P_2 is also an M-alternating path. Then, one observes that either $P_2 \cup P_1$ is a proper M-augmenting tv-path in $G[c_{S\setminus v} - 1]$ (contradicting the previous claim), or $P_1 \cup (P_g \setminus P_2)$ is a circuit that we can apply to (fractionally) increase the cardinality of x in $G[c_S - 1]$, contradicting the stability of $G[c_S - 1]$. □

Suppose for the sake of contradiction that $|M| < \nu^c(G)$. Then there exists a proper M-augmenting st-trail T in G, by theorem 1 (note that, possibly, $s = t$). Since M is maximum in $G[c_S - 1]$, T cannot be proper in $G[c_S - 1]$, by theorem 1. Therefore, $|S \cap \{s,t\}| \geq 1$. We distinguish two cases.

Case 1: $|S \cap \{s,t\}| = 1$. Without loss of generality, let s be the vertex whose capacity gets reduced by S. If $s \neq t$, then $c_s^{S\setminus s-1} = c_s^{S-1} + 1$ and $c_t^{S\setminus s-1} = c_t$. If $s = t$ then $c_s^{S\setminus s-1} = c_s$. In both cases, T is a proper M-augmenting trail in $G[c_{S\setminus s} - 1]$, contradicting the first claim.

Case 2: $|S \cap \{s,t\}| = 2$. If $s \neq t$, then $c_s^{S\setminus\{s,t\}-1} = c_s^{S-1} + 1$ and $c_t^{S\setminus\{s,t\}-1} = c_t^{S-1} + 1$. If $s = t$ then $c_s^{S\setminus\{s,s\}-1} = c_s^{S-1} + 2$. In both cases, T is a proper M-augmenting trail in $G[c_{S\setminus\{s,t\}} - 1]$, contradicting the second claim. □

5 Edge-Stabilizer

In this section we state our results for the edge-stabilizer problem.

First, we show that we can generalize a lower bound on the size of an edge-stabilizer, provided in the uncapacitated setting.

Lemma 2. *For every edge-stabilizer F, $|F| \geq \frac{1}{2}\gamma(G)$.*

Proof. To prove the lemma, by proposition 1, it is enough to show that removing one edge decreases the number of odd cycles by at most two. Therefore, from now on, we concentrate on proving the following statement:

$$\text{for all } e \in E,\ \gamma(G \setminus e) \geq \gamma(G) - 2. \qquad (\diamond)$$

Let x be a basic maximum-weight fractional c-matching in $G \setminus e$ with $\gamma(G \setminus e)$ odd cycles. Extend x to G by setting $x_e = 0$. Then x is a basic fractional c-matching in G.

If x has maximum-weight in G, then $\gamma(G) \leq |\mathscr{C}_x| = \gamma(G \setminus e)$, and hence ($\diamond$) holds.

If x does not have maximum-weight in G, we can apply theorem 7: we can move to a basic maximum-weight fractional c-matching x^* in G in at most two steps over the edges of $\mathcal{P}_{\mathrm{FCM}}(G)$, and if two steps are needed, the first one moves to a vertex with $x_e = \frac{1}{2}$, and the second one to a vertex with $x_e = 1$.

Suppose only one step was needed. By theorem 8, $x^* = x + \alpha g$ for $\alpha \in \left\{\frac{1}{2}, 1\right\}$ and $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. Then, by theorem 8, we have $|\mathscr{C}_{x^*}| = |\mathscr{C}_x| \pm \{0, 1, 2\}$. So definitely, $|\mathscr{C}_{x^*}| \leq |\mathscr{C}_x| + 2$, and consequently, $\gamma(G) \leq |\mathscr{C}_{x^*}| \leq |\mathscr{C}_x| + 2 = \gamma(G \setminus e) + 2$, yielding ($\diamond$).

Suppose two steps were needed. Let $\hat{x}$ be the vertex reached after the first step. By theorem 8, $\hat{x} = x + \alpha g$ and $x^* = \hat{x} + \beta h$ for $\alpha, \beta \in \left\{\frac{1}{2}, 1\right\}$ and $g, h \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. Now note that we must have $x_e = 0$, $\hat{x}_e = \frac{1}{2}$ and $x^*_e = 1$. So g creates at least one cycle, and h breaks at least one cycle. This gives the following options for g:

- $g \in \mathcal{C}_1 \cup \mathcal{C}_5$ both breaks and creates a cycle,
- $g \in \mathcal{C}_2 \cup \mathcal{C}_4$ creates one cycle,
- $g \in \mathcal{C}_5$ creates two cycles.

Hence, $|\mathscr{C}_{\hat{x}}| = |\mathscr{C}_x| + \{0, 1, 2\}$. Similarly for h:

- $h \in \mathcal{C}_1 \cup \mathcal{C}_5$ both breaks and creates a cycle,
- $h \in \mathcal{C}_2 \cup \mathcal{C}_4$ breaks one cycle,
- $h \in \mathcal{C}_5$ that breaks two cycles.

Hence, $|\mathscr{C}_{x^*}| = |\mathscr{C}_{\hat{x}}| - \{0, 1, 2\}$. So definitely, $|\mathscr{C}_{x^*}| \leq |\mathscr{C}_x| + 2$, and consequently, as before, $\gamma(G) \leq \gamma(G \setminus e) + 2$, yielding ($\diamond$). □

The authors of [14] provided an example that shows this bound is tight, already in the uncapacitated setting. However, in the unweighted, capacitated setting we can get a stronger bound. Repeating the proof above, but replacing the "-2" in ($\diamond$) by "-1" and using that circuits in $\mathcal{C}_1 \cup \mathcal{C}_5$ are not augmenting in cardinality, we can proof:

Lemma 3. *In $(G, 1, c)$: for every edge-stabilizer F, $|F| \geq \gamma(G)$.*

The authors of [14] also give a $O(\Delta)$-approximation algorithm, based on their algorithm for the vertex-stabilizer problem (instead of removing the vertices, all edges incident to those vertices are removed). Similarly, for capacitated instances we use algorithm 1, and instead of reducing the capacity of the vertices, we remove all edges incident to those vertices, except the edges e with $x_e = 1$.

Theorem 11. *The edge-stabilizer problem admits an efficient $O(\Delta)$-approximation algorithm.*

Proof. Let $S = \left\{v_1, \ldots, v_{\gamma(G)}\right\}$ be the set of vertices found by algorithm 1. Set $F = \{\delta(v) \setminus \mathscr{M}_x : v \in S\}$. The size of F is at most $\Delta\gamma(G)$. We claim that $G \setminus F$ is stable, hence this gives us an $O(\Delta)$-approximation, by lemma 2.

Let $\hat{x}$ be obtained from x by alternate rounding C_i exposing v_i, for all $i \in \{1, \ldots, \gamma(G)\}$. Note that $\hat{x}$ is a basic fractional c-matching in $G \setminus F$ with $|\mathscr{C}_{\hat{x}}| = 0$. Let $(\hat{y}, \hat{z})$ be obtained from (y, z) as follows:

$$\hat{y}_v = \begin{cases} y_v & \text{if } v \notin S, \\ 0 & \text{if } v \in S, \end{cases} \qquad \hat{z}_{uv} = \begin{cases} z_{uv} + y_u + y_v & \text{if } u, v \in S, uv \in E \setminus F, \\ z_{uv} + y_u & \text{if } u \in S, v \notin S, uv \in E \setminus F, \\ z_{uv} + y_v & \text{if } u \notin S, v \in S, uv \in E \setminus F, \\ z_{uv} & \text{if } u, v \notin S, uv \in E \setminus F. \end{cases}$$

One can check that $\hat{x}$ and $(\hat{y}, \hat{z})$ satisfy complementary slackness in $G \setminus F$. Consequently, $\hat{x}$ is a basic maximum-weight fractional c-matching in $G \setminus F$ with $|\mathscr{C}_{\hat{x}}| = 0$, and so $G \setminus F$ is stable. $\square$

If we restrict ourselves to unweighted instances, like for capacity-stabilizers, we can show that any *inclusion-wise minimal* edge-stabilizer preserves the size of a maximum-cardinality c-matching.

Theorem 12. *In $(G, 1, c)$: for any inclusion-wise minimal edge-stabilizer F: $\nu^c(G \setminus F) = \nu^c(G)$.*

Proof. Let M be a maximum-cardinality c-matching in G such that the overlap with F is minimum, i.e., $|M \cap F|$ is minimum. Suppose for the sake of contradiction that $|M \cap F| > 0$.

Consider the graph $G \setminus (F \setminus M)$. Since M is avoided, M is a maximum-cardinality c-matching in $G \setminus (F \setminus M)$. Let x be the indicator vector of M, then by theorem 3, x is basic. Since F is inclusion-wise minimal, $G \setminus (F \setminus M)$ is not stable, and thus x is not a maximum fractional c-matching in $G \setminus (F \setminus M)$. So there must be a vertex x^* of $\mathcal{P}_{\text{FCM}}$, adjacent to x, with $1^\top x^* > 1^\top x$. By theorem 8, $x^* = x + \alpha g$, where $\alpha \in \left\{\frac{1}{2}, 1\right\}$ and $g \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. Since M is maximum, x^* cannot be integral, so $\alpha = \frac{1}{2}$, and consequently by theorem 8, $g \notin \mathcal{C}_3$. Furthermore, the circuits in $\mathcal{C}_1 \cup \mathcal{C}_5$ are not augmenting in cardinality, so $g \in \mathcal{C}_2 \cup \mathcal{C}_4$. Let $v \in V$ be the only vertex with $g(\delta(v)) \neq 0$. The circuits in $\mathcal{C}_2 \cup \mathcal{C}_4$ are only augmenting in cardinality if $g(\delta(v)) > 0$. Consequently, $x^*(\delta(v)) = x(\delta(v)) + 1 = d_v^M + 1$. Clearly, we have $x^*(\delta(v)) \leq c_v$, and so, $d_v^M < c_v$.

The support of g consists of a (possibly empty) path P_g and an odd cycle C_g, intersecting at only one vertex, such that the sign of g_e alternates. Since it is feasible to apply g to x, it must be that if $sgn(g_e) = -$, then $x_e = 1$ $(e \in M)$, and if $sgn(g_e) = +$, then $x_e = 0$ $(e \notin M)$. Since $g(\delta(v)) > 0$, the first and last edge of g satisfy $sgn(g_e) = +$. Recall that $d_v^M < c_v$. Consider the closed vv-walk $W = (P_g, C_g, P_g^{-1})$. Then W is a feasible M-augmenting walk in $G \setminus (F \setminus M)$.

If there exists some $e \in W \cap F$, then $e \in M$, since all other edges of F were removed. Then there exists an even-length M-alternating trail T: the part of W starting from v, to (and including) the edge e. Since $d_v^M < c_v$ and $e \in M$, T is proper. Then $M' = M \triangle T$ is a maximum-cardinality c-matching in G with $|M' \cap F| < |M \cap F|$, contradicting our assumption.

Hence, $W \cap F = \emptyset$, which means that W exist in $G \setminus F$. In addition, we have $d_v^{M \setminus F} \leq d_v^M < c_v$. Thus, W is a feasible $M \setminus F$-augmenting walk in $G \setminus F$. By theorem 2, $G \setminus F$ combined with $M \setminus F$ is not stable. However, $G \setminus F$ is stable, so it must be that $M \setminus F$ is not maximum in $G \setminus F$. Then, by theorem 1, there is a proper $M \setminus F$-augmenting st-trail T in $G \setminus F$. Note that since T is augmenting in cardinality, the first and last edge of T are not in $M \setminus F$. Since W cannot exists for a maximum-cardinality c-matching in $G \setminus F$, by theorem 2, there must be such a trail T that either makes W infeasible, or overlaps wit the edges of W. We consider such a T. Clearly, T is also an M-augmenting trail in G, but M is maximum in G, so by theorem 1, T is not proper for M in G.

Since T is proper for $M \setminus F$ in $G \setminus F$, we have, if $s \neq t$, $d_s^{M\setminus F} \leq c_s - 1$ and $d_t^{M\setminus F} \leq c_t - 1$, if instead $s = t$, then $d_s^{M\setminus F} \leq c_s - 2$. Since T is not proper for M in G, we have, if $s \neq t$, $d_s^M = c_s$ or $d_t^M = c_t$, if instead $s = t$, then $d_s^M \geq c_s - 1$. This gives us five cases:

1. $s \neq t$, $d_s^M = c_s$ and $d_t^M < c_t$,
2. $s \neq t$, $d_s^M < c_s$ and $d_t^M = c_t$,
3. $s \neq t$, $d_s^M = c_s$ and $d_t^M = c_t$,
4. $s = t$ and $d_s^M = c_s - 1$,
5. $s = t$ and $d_s^M = c_s$.

In the first case, since $d_s^M = c_s$ and $d_s^{M\setminus F} \leq c_s - 1$, there is at least one $e \in \delta(s) \cap M \cap F$. Since $e \in F$, we have $e \notin T$, which means $T \cup e$ is a trail. Note that $T \cup e$ is M-alternating, and has even length. Since $d_t^M < c_t$ and $e \in M$, $T \cup e$ is proper. (If $T \cup e$ is closed, it is still proper, because it has even-length.) Therefore, $M' = M \triangle (T \cup e)$ is a maximum-cardinality c-matching in G with $|M' \cap F| < |M \cap F|$, contradicting our assumption. Similar arguments can be made in the second and fourth case.

For the third and fifth case we take a look at the overlap of T and W. We know that $d_v^{M\setminus F} \leq d_v^M < c_v$. So, to make W infeasible in $G \setminus F$, T would need to increase the degree of v with respect to the matching. This is only possible if $v \in \{s, t\}$. However, s and t are both saturated by M (in both the third and fifth case) and v is not. Hence, T and W must overlap in at least one edge.

We create a new walk W' by combining W and T: follow W starting from v to the first edge e that is also on T, traverse e, then switch to T and follow T from here to one of its endpoints. Since T and W are both M-alternating and -augmenting, so is W'. The part of W from v to and including e is a trail, this part of W does not overlap with T, and T is a trail, hence W' is a trail. Since T exists in $G \setminus F$ and we found before that $W \cap F = \emptyset$, we have $W' \cap F = \emptyset$.

In both the third and fifth case, the degree of both s and t with respect to M is strictly larger than their degree with respect to $M \setminus F$. Hence, $\delta(s) \cap M \cap F$ and $\delta(t) \cap M \cap F$ are both nonempty. Without loss of generality assume W' ends at s. Let $e \in \delta(s) \cap M \cap F$. Since $e \in F$, we have $e \notin W'$, which means $W' \cup e$ is a trail. Note that $W' \cup e$ is M-alternating, and has even length. Since $d_v^M < c_v$ and $e \in M$, $W' \cup e$ is proper. Therefore, $M' = M \triangle (W' \cup e)$ is a maximum-cardinality c-matching in G with $|M' \cap F| < |M \cap F|$, contradicting our assumption. $\square$

We conclude this paper with some additional remarks. Note that, as opposed to the capacity-stabilizer case, when dealing with edge-removal operations it is always possible to stabilize a graph without decreasing the weight of a maximum-weight matching: for example, one could take any maximum-weight c-matching M in G and remove all edges in $E \setminus M$. The previous theorem shows that, for unweighted instances, this property comes essentially for free, as any edge-stabilizer of minimum cardinality will be weight-preserving. However, for general weighted instances, this is not the case and we can show that the size of a minimum weight-preserving edge-stabilizer and the size of a minimum edge-stabilizer, can differ by a very large factor, namely $\Omega(|V|)$, already for unit-capacities.

Theorem 13. *There exist graphs* $(G, w, 1)$ *where the sizes of a minimum edge-stabilizer and a minimum weight-preserving edge-stabilizer differ by* $\Omega(|V|)$.

Proof. Let G and M be the graph and matching given in figure 1, respectively. Any matching on G can contain (i) at most one edge of each 3-cycle, which all

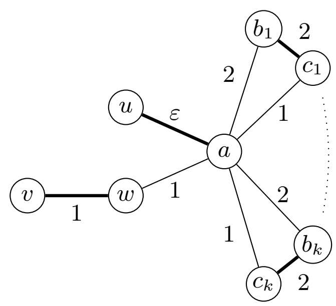

Fig. 1: Let $k \geq 3$ be an integer, and $0 < \varepsilon < 0.5$. The figure shows a graph $(G, w, 1)$ with k 3-cycles of the form (a, b_i, c_i) for $i = 1, \ldots, k$. Edge weights are given next to the edges. The bold lines indicate a matching M.

have a weight of at most 2, (ii) at most one of the edges incident to w, which both have weight 1, and (iii) the edge ua of weight ε. This gives us $\nu^c(G) \leq 2k+1+\varepsilon$. In particular, M is the only matching in G that attains this weight. So, M is the unique maximum-weight matching in G. There are k feasible M-augmenting walks: (u, a, b_i, c_i, a, u) for $i = 1, \ldots, k$, hence G is not stable, by theorem 2.

We can stabilize G by removing only two edges: ua and vw. We verify the stability by giving a matching and a vertex-cover of the same value. Let the matching be $wa \cup \{b_i c_i : i \in \{1, \ldots, k\}\}$. The weight of this matching is $2k+1$. Note that the weight decreased by ε. We set the vertex-cover y equal to 0 for u, v, w and equal to 1 for a and all k pairs b_i, c_i. The value of this vertex-cover is $2k+1$. So indeed, the graph is stable. This shows that the size of a minimum edge-stabilizer is at most two.

As mentioned, M is the unique maximum-weight matching in G. Consequently, any weight-preserving edge-stabilizer has to avoid M. Because the k feasible M-augmenting walks are edge-disjoint with respect to the edges from $E \setminus M$, any weight-preserving edge-stabilizer has to remove at least k edges.

We have $|V| = 3k+3$, or equivalently, $k = |V|/3 - 1$. The difference in size of a minimum edge-stabilizer and a minimum weight-preserving edge-stabilizer in this graph is at least $k-2$. So the difference in sizes is $\Omega(k) = \Omega(|V|)$. □

Theorem 9 of [14] shows that there is no constant factor approximation for the minimum edge-stabilizer problem in $(G, w, 1)$, unless P = NP. By adding one additional step, their proof can also be used for the minimum weight-preserving edge-stabilizer problem. We here state the result and prove the additional step, but refer to [14] for all details of the proof.

Theorem 14. *In* $(G, w, 1)$*: there is no constant factor approximation for the minimum weight-preserving edge-stabilizer problem, unless* $P = NP$.

Proof. Claim 8 in [14] shows that if G has an independent set of size at least k, then G^* has an edge-stabilizer of size at most k. The edge-stabilizer they construct is actually weight-preserving, which is the additional step we need.

Claim 9 in [14] shows that if G does not have an independent set of size at least k, then every edge-stabilizer of G^* has size at least $(\rho+1)k$. This claim covers *every* edge-stabilizer, which includes the weight-preserving edge-stabilizers.

Since both claims also hold for weight-preserving edge-stabilizers, the result follows.

Now for the additional step. They construct a matching M in $G^* \setminus F$ with weight $(m+4k)\rho k+n+k$ and a fractional vertex cover of the same value, to show that $G^* \setminus F$ is stable. This implies that M has maximum-weight in $G^* \setminus F$. We show that $\nu(G^*) = (m+4k)\rho k+n+k$, proving that M also has maximum-weight in G^*, and hence that F is weight-preserving.

Claim. If M is a maximum-weight matching in G^*, then no edge of the form $v_i u^l_{ij}$ is in M.

Proof. For the sake of contradiction, suppose $v_i u^l_{ij} \in M$ for some i, j, l. This implies that $v_i v'_i, u^l_{ij} u^l_{ji} \notin M$. Then, either $u^l_{ji} v_j \in M$ and thus $v_j v'_j \notin M$, or u^l_{ji} is M-exposed. In the first case, $(v'_i, v_i, u^l_{ij}, u^l_{ji}, v_j, v'_j)$ is an M-augmenting path, and in the second case, $(v'_i, v_i, u^l_{ij}, u^l_{ji})$ is an M-augmenting path. Both cases contradict that M has maximum-weight. □

As a result of this claim, any maximum-weight matching will contain all edges of the form $u^l_{ij} u^l_{ji}$. Additionally, in such a matching v_i will be matched to some b_j, or to v'_i.

Claim. If M is a maximum-weight matching in G^*, every b_i is matched to some v_j in M.

Proof. For the sake of contradiction, suppose there is an $i \in [k]$ such that $b_i v_j \notin M$ for every $j \in [n]$. Then, either b_i is matched to some vertex in H_i, or b_i is M-exposed. Since $k \leq n$, by the pigeonhole principle, there exists an $l \in [n]$ such that $v_l b_j \notin M$ for every $j \in [k]$. By the previous claim, and because M is a maximum-weight matching, $v_l v'_l \in M$. If b_i is matched to some vertex in H_i, there must be some other M-exposed vertex in H_i, because H_i has an odd number of vertices. Let a_i, c_i be such that $b_i a_i \in M$ and c_i is M-exposed. Then $(v'_l, v_l, b_i, a_i, c_i)$ is an M-augmenting path. If b_i is M-exposed, (v'_l, v_l, b_i) is an M-augmenting path. Both cases contradict that M has maximum-weight. □

Let M be any maximum-weight matching in G^*, i.e., $w(M) = \nu(G^*)$. By the previous claim, M must contain all k edges of the form $b_i v_j$, with a weight of 2 each. Since M is a maximum-weight matching, and $H_i \setminus \{b_i\}$ is a complete graph on an even number of vertices, it will be perfectly matched in M, resulting in a weight of $4\rho k$ for each H_i. The remaining $n-k$ vertices v_i that are not matched to some b_j in M, are matched to v_i' in M, with a weight of 1 each. All edges of the form $u_{ij}^l u_{ji}^l$ are in M, there are $m\rho k$ such edges and they have weight 1. Finally, each of the k remaining vertices v_i' cannot be matched in M, since its only adjacent vertex, v_i, is already matched to some b_j in M. This gives a total weight of $w(M) = 2k + 4\rho k^2 + n - k + m\rho k = (m + 4k)\rho k + n + k$. □

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A Basic Fractional c-Matchings

Proof of theorem 3. (⇒) Every extreme point of $\mathcal{P}_{\text{FCM}}$ is half-integral, as proven e.g. by theorem 21 of [2]. Let x be an extreme point, and let H be a connected component of the graph induced by the edges with half-integral value in x. First, note that H contains no even cycle, and no inclusion-wise maximal path with distinct endpoints: Otherwise, let D be an even cycle or an inclusion-wise maximal path with distinct endpoints in H. Let $g \in \mathcal{C}_1 \cup \mathcal{C}_3$ be the circuit associated to it. Then, $x + \varepsilon g$ and $x - \varepsilon g$ are both fractional c-matchings, for a small value of ε. However, as x is a convex combination of $x + \varepsilon g$ and $x - \varepsilon g$, this contradicts that x is an extreme point. Let T be any spanning tree of H. First, assume there exist two distinct edges $f_1, f_2 \in E(H) \setminus E(T)$. Then, adding f_1 (resp. f_2) to T creates an odd cycle D_1 (resp. D_2). These cycles cannot intersect in an edge, otherwise their support would contain an even cycle. Hence, they must be edge disjoint. The cycles can also not intersect in more than one vertex, otherwise their support again contains an even cycle. So, they either intersect at one vertex, or are connected via a path in T. In either case, one can associated to these edges a circuit $g \in \mathcal{C}_5$. Then, $x + \varepsilon g$ and $x - \varepsilon g$ are both fractional c-matchings, for a small value of ε, reaching a contradiction again. These arguments show that there is a unique edge $f \in E(H) \setminus E(T)$. Since H cannot contain inclusion-wise maximal paths, it contains at most one vertex of degree 1. If H contains exactly one such vertex u, then u and the odd cycle (created by adding f to T) are connected via a path. One can associate a circuit $g \in \mathcal{C}_4$ to the edges in this cycle and path. Again, considering $x + \varepsilon g$ and $x - \varepsilon g$ results in a contradiction. So, H does not have any vertex of degree 1, which means that the endpoints of f must be the leaves of T. Hence, T is a path and H is a cycle. Necessarily, H must be odd. Finally, no vertex in H can be unsaturated: as otherwise we can associate a circuit $g \in \mathcal{C}_2$ to H where the unsaturated vertex u is the only vertex with $g(\delta(u)) \neq 0$. Once again, we reach a contradiction by considering $x + \varepsilon g$ and $x - \varepsilon g$. In conclusion, if $x_e \notin \{0, 1\}$, it must equal $\frac{1}{2}$ and e must be part of an odd cycle. Furthermore, these odd cycles are vertex-disjoint and all vertices part of an odd cycle are saturated.

(⇐) Consider a vector w with $w_e = 1$ for all edges e in the support of x, and $w_e = -1$ for all other edges. Then, x is the unique optimal solution when maximizing the function w over $\mathcal{P}_{\text{FCM}}$. Hence x is an extreme point. □

B Maximum-Weight Basic Fractional c-Matching with Minimum Number of Odd cycles

[14] proposes an algorithm to obtain a maximum-weight basic fractional matching with minimum number of odd cycles ($|\mathscr{C}_x| = \gamma(G)$). We generalize their algorithm to c-matchings.

Given a graph (G, w, c) and fractional vertex cover (y, z), we say that an edge uv is *tight* if $y_u + y_v + z_{uv} = w_{uv}$. Similarly, in case of unit-capacities, we say that an edge uv is *tight* if $y_u + y_v = w_{uv}$. We say a walk is tight if all its edges are tight. By complementing on $F \subseteq E$, we mean replacing x_e by $1 - x_e$ for all $e \in F$. Without loss of generality, we assume that c_v is at most the degree of $v \in V$.

If we have a maximum-weight fractional c-matching with more than the minimum number of half-integral odd cycles, the next theorem tells us about existence of certain structures in the graph.

Theorem 15. *Let x be a basic maximum-weight fractional c-matching and (y, z) a minimum fractional vertex cover, in G. If $|\mathscr{C}_x| > \gamma(G)$, then there exists*

(i) a vertex $v \in V(C)$ for some odd cycle $C \in \mathscr{C}_x$ such that $y_v = 0$; or
(ii) a tight $\mathscr{M}_x$-alternating trail P which connects two odd cycles $C_i, C_j \in \mathscr{C}_x$ such that $z_e = 0$ for all $e \in P$, and all interior vertices of P are not in $V(\mathscr{C}_x)$; or
(iii) a tight and proper $\mathscr{M}_x$-alternating trail P which connects an odd cycle $C \in \mathscr{C}_x$ and a vertex $v \notin V(\mathscr{C}_x)$ such that $y_v = 0$, $z_e = 0$ for all $e \in P$, and all interior vertices of P are not in $V(\mathscr{C}_x)$.

Furthermore, alternate rounding the odd cycles and complementing on the path produces a basic maximum-weight fractional c-matching $\hat{x}$ such that $\mathscr{C}_{\hat{x}} \subset \mathscr{C}_x$.

Proof. We start by proving the second part of the theorem, namely that alternate rounding and complementing produces a basic maximum-weight fractional c-matching with fewer odd cycles. Note that x and (y, z) satisfy complementary slackness.

Case (i): let $\hat{x}$ be the basic fractional c-matching obtained by alternate rounding C exposing v. One can check that $\hat{x}$ satisfies complementary slackness with (y, z). Consequently, $\hat{x}$ is optimal. Furthermore, $\mathscr{C}_{\hat{x}} = \mathscr{C}_x \setminus C$.

Case (ii): let $u = V(P) \cap V(C_i)$ and $v = V(P) \cap V(C_j)$ be the endpoints of P. Let $\hat{x}$ be the basic fractional c-matching obtained by alternate rounding C_i and C_j at u and v, respectively and complementing on P. If P connects to C_i with an edge from $\mathscr{M}_x$, then alternate rounding C_i covers u, otherwise it exposes u. This is to make sure $\hat{x}(\delta(u)) = x(\delta(u))$. Similarly for C_j and v. One can check that $\hat{x}$ satisfies complementary slackness with (y, z). Consequently, $\hat{x}$ is optimal. Furthermore, $\mathscr{C}_{\hat{x}} = \mathscr{C}_x \setminus \{C_i, C_j\}$.

Case (iii): let $u = V(P) \cap V(C)$. Let $\hat{x}$ be the basic fractional c-matching obtained by alternate rounding C_i at u and complementing on P. If P connects to C_i with an edge from $\mathscr{M}_x$, then alternate rounding C_i covers u, otherwise it exposes u. Note that $\hat{x}(\delta(v)) \leq c_v$, because P is proper. One can check that $\hat{x}$ satisfies complementary slackness with (y, z). Consequently, $\hat{x}$ is optimal. Furthermore, $\mathscr{C}_{\hat{x}} = \mathscr{C}_x \setminus C$.

Note that if there exists a structure almost fitting the description in (ii) or (iii), with the only difference that there are interior vertices of P in $V(\mathscr{C}_x)$, then there also exists a structure exactly fitting the description in (ii) or (iii). In particular, this structure can be obtained by taking a suitable subtrail of P.

Finally, we prove the first part of the theorem. By the previous paragraph, for cases (ii) and (iii), we do not have to show that all interior vertices of P

are not in $V(\mathscr{C}_x)$. Let $\hat{x}$ be a basic maximum-weight fractional c-matching such that $|\mathscr{C}_{\hat{x}}| = \gamma(G)$. Using x, $\hat{x}$ and (y, z), we construct an auxiliary unit-capacity graph, with corresponding fractional matchings and vertex cover, as follows.

1. For each edge $uv \in E$ such that $z_{uv} > 0$, remove it, and correspondingly decrease c_u and c_v by one.
2. Initialize $V' = \emptyset$ and $E' = \emptyset$.
3. For each $v \in V$, set $V(v) = \{v_1, \ldots, v_{c_v}\}$ and $J_x(v) = J_{\hat{x}}(v) = \{1, \ldots, c_v\}$, add $V(v)$ to V', and set $y'_{v_i} = y_v$ for all $v_i \in V(v)$.
4. For each $uv \in E(\mathscr{C}_x)$, add u_1v_1 to E', set $w'_{u_1v_1} = w_{uv}$ and $x'_{u_1v_1} = \frac{1}{2}$.
5. For each $v \in V(\mathscr{C}_x)$, remove 1 from $J_x(v)$.
6. For each $uv \in \mathscr{M}_x$, arbitrarily choose some $i \in J_x(u)$ and $j \in J_x(v)$, and remove them from $J_x(u)$ and $J_x(v)$, respectively. Add u_iv_j to E', set $w'_{u_iv_j} = w_{uv}$ and $x'_{u_iv_j} = 1$.
7. For each $uv \in E(\mathscr{C}_{\hat{x}})$, add u_1v_1 to E' (if not yet present), set $w'_{u_1v_1} = w_{uv}$ and $\hat{x}'_{u_1v_1} = \frac{1}{2}$.
8. For each $v \in V(\mathscr{C}_{\hat{x}})$, remove 1 from $J_{\hat{x}}(v)$.
9. For each $uv \in \mathscr{M}_{\hat{x}}$, arbitrarily choose some $i \in J_{\hat{x}}(u)$ and $j \in J_{\hat{x}}(v)$, and remove them from $J_{\hat{x}}(u)$ and $J_{\hat{x}}(v)$, respectively. Add u_iv_j to E' (if not yet present), set $w'_{u_iv_j} = w_{uv}$ and $\hat{x}'_{u_iv_j} = 1$.
10. Set $G' = (V', E')$, and return $(G', w', \mathbf{1})$, x', $\hat{x}'$ and y'.

One can verify that x' and $\hat{x}'$ are basic fractional matchings, and that y' is a fractional vertex cover, and that x' and $\hat{x}'$ both satisfy complementary slackness with y'. It follows that x' and $\hat{x}'$ are basic maximum-weight fractional matchings, and that y' is a minimum fractional vertex cover.

We have $|\mathscr{C}_{x'}| = |\mathscr{C}_x| > \gamma(G) = |\mathscr{C}_{\hat{x}}| = |\mathscr{C}_{\hat{x}'}| \geq \gamma(G')$. Now we can apply theorem 5 from [14], it tells us that one of the following exists:

(a) a vertex $v \in V(C)$ for some odd cycle $C \in \mathscr{C}_{x'}$ such that $y'_v = 0$; or
(b) a tight $\mathscr{M}_{x'}$-alternating path P which connects two odd cycles $C_i, C_j \in \mathscr{C}_{x'}$; or
(c) a tight and proper $\mathscr{M}_{x'}$-alternating path P which connects an odd cycle $C \in \mathscr{C}_{x'}$ and a vertex $v \notin V(\mathscr{C}_{x'})$ such that $y'_v = 0$.

In case (a), vertex v corresponds to a vertex in G, and in particular to a vertex in $V(\mathscr{C}_x)$, that also has y-value zero, and thus we have found a structure of the form given in (i).

In case (b), the tight $\mathscr{M}_{x'}$-alternating path P, corresponds with a tight and $z = 0$ $\mathscr{M}_x$-alternating walk P^* in G ($z = 0$ because $z \neq 0$ edges where not added to G'). Note that this remains true when we select just part of P^*. Let C_i^* and C_j^* be the odd odd cycles in $\mathscr{C}_x$ corresponding to C_i and C_j, respectively. If P^* intersects with C_i^* and C_j^* in other vertices than its endpoints, then select a subwalk of P^* that only intersects with C_i^* and C_j^* at its endpoints. Finally, we can translate the walk to a trail by leaving out the parts between two occurrences of the same edge, e.g., $(\boldsymbol{e_1}, e_2, \boldsymbol{e_3}, e_4, \boldsymbol{e_5}, e_2, \boldsymbol{e_6}) \rightarrow (\boldsymbol{e_1}, e_2, \boldsymbol{e_6})$, where bold edges represent $\mathscr{M}_x$, note that the result will always be $\mathscr{M}_x$-alternating. So, given (b) is the case in G', it is possible to find a structure of the form given in (ii) in G.

In case (c), vertex v corresponds to a vertex v^* in G that also has y-value zero. If $v^* \in V(\mathscr{C}_x)$, case (i) holds, so for the remainder assume $v^* \notin V(\mathscr{C}_x)$. Again, P corresponds to a tight and $z = 0$ $\mathscr{M}_x$-alternating walk P^* in G. Let C^* be the odd cycle in $\mathscr{C}_x$ corresponding to C. Select the subwalk of P^* between v^* and the first vertex that is on C^*. This (non-closed) walk is proper, because one endpoint is in an odd cycle, which means it is unsaturated by $\mathscr{M}_x$, and the other endpoint corresponds with v, and we know that P is proper. Finally, we translate the walk to a trail, like in the previous case. So, given (c) is the case in G', it is possible to find a structure of the form given in (i) or (iii) in G. ☐

We will reduce the number of odd cycles in a basic maximum-weight fractional c-matching by looking for the structures described in theorem 15. We do this with the help of an auxiliary unit-weight graph. We are given a graph (G, w, c) and a basic maximum-weight fractional c-matching x on G. Fix a minimum fractional vertex cover (y, z). Then, let the auxiliary unit-weight graph G' be obtained by applying the following operations to G.

(a) Delete all non-tight edges.
(b) Delete all edges uv such that $z_{uv} > 0$, and reduce the capacity of u and v by one, if the capacity of a vertex is reduced to zero, remove the vertex.
(c) Add a vertex a.
(d) For every vertex $v \in V$ with $x(\delta(v)) > 0$ and $y_v = 0$, add the edge va.
(e) For every vertex $v \in V$ with $x(\delta(v)) < c_v$ and $y_v = 0$, add the vertex v_i and the edges vv_i, $v_i a$, for all $i = 1, \ldots, c_v - x(\delta(v))$.
(f) Shrink every $C_i \in \mathscr{C}_x$ into a pseudonode i and add the loop edge ii.

Consider the edge set

$$M' = \{e \in \mathscr{M}_x : z_e = 0\} \cup \left\{vv_1, \ldots, vv_{c_v - x(\delta(v))} : v \in V, x(\delta(v)) < c_v, y_v = 0\right\}.$$

It is easy to see that M' is a c-matching in G', and that it saturates all vertices except the pseudonodes and the vertex a. Note: for $F = \{vv\}$, $d_v^F = 2$.

Lemma 4. *M' is a maximum-cardinality c-matching in G' if and only if $|\mathscr{C}_x| = \gamma(G)$.*

Proof. ($\Rightarrow$) Suppose $|\mathscr{C}_x| > \gamma(G)$. Applying theorem 15 yields three cases. In case (i), there exists a vertex $v \in V(C_i)$ for some $C_i \in \mathscr{C}_x$ such that $y_v = 0$. Then the edge ia is in G', but not in M'. Furthermore, i and a are unsaturated by M', hence ia is a proper M'-augmenting path.

In case (ii), there exists a tight $\mathscr{M}_x$-alternating trail P which connects two odd cycles $C_i, C_j \in \mathscr{C}_x$ such that $z_e = 0$ for all $e \in P$, and all interior vertices on P are not in $V(\mathscr{C}_x)$. In G', P exists, and is an M'-alternating trail between the pseudonodes i and j, such that no interior vertices of P are pseudonodes. If P ends at i using an edge from M', we add ii to P. Likewise for j. This way P starts and ends with an edge not in M', and its endpoints are unsaturated. Hence, P is M'-augmenting and proper.

In case (iii), there exists a tight and proper $\mathscr{M}_x$-alternating trail P which connects an odd cycle $C_i \in \mathscr{C}_x$ and a vertex $v \notin V(\mathscr{C}_x)$ such that $y_v = 0$, $z_e = 0$ for all $e \in P$, and all interior vertices on P are not in $V(\mathscr{C}_x)$. In G', P exists, and is an M'-alternating trail between the pseudonode i and vertex v, such that no interior vertices of P are pseudonodes. If P ends at i using an edge from M', we add ii to P. If P ends at v using an edge from M', then the edge va is in G', and $P + va$ is a proper (i and a are unsaturated) M'-augmenting trail. If P ends at v using an edge not in M', then because P is proper (for $\mathscr{M}_x$), v is unsaturated. Then the edges vv_i and v_ia are in G', for any $i \in \{1, \ldots, c_v - x(\delta(v))\}$, and $P + vv_i + v_ia$ is a proper (i and a are unsaturated) M'-augmenting trail.

In all cases we find that there exists a proper M'-augmenting trail in G', hence by theorem 1, M' is not a maximum-cardinality c-matching in G'.

($\Leftarrow$) Suppose M' is not a maximum-cardinality c-matching in G'. Then, by theorem 1, there exists a proper M'-augmenting trail P in G'. Without loss of generality we assume that all interior vertices on P are not pseudonodes. Otherwise, we can select a sub-trail of P that satisfies this property. If this sub-trail ends with an edge from M' at a pseudonode, add the corresponding loop edge, then this trail is also proper and M'-augmenting. Since P is M'-augmenting in cardinality, it starts and ends with an edge not in M'. In addition, since P is proper, it starts and ends at an unsaturated vertex.

Suppose both of P's endpoints are pseudonodes i and j. If it ends with a loop edge at i or j, we remove this loop edge from P. Then, P corresponds with a tight $\mathscr{M}_x$-alternating trail which connects $C_i, C_j \in \mathscr{C}_x$, such that $z_e = 0$ for all $e \in P$, and all interior vertices on P are not in $V(\mathscr{C}_x)$ (case (ii) of theorem 15).

Since both of P's endpoints must be unsaturated, the only other option is that the endpoints are a pseudonode i and the vertex a. Again, if ii is on P, we can just remove it from P. If $ia \in P$, then there exists a vertex $v \in V(C_i)$ such that $y_v = 0$ (case (i) of theorem 15). Note that $x(\delta(v)) = c_v$ for all $v \in V(\mathscr{C}_x)$, so there are never edges of the form ii_j, i_ja. If $va \in P$ for some $v \in V$, then $y_v = 0$ and v is not a pseudonode. Because P is M'-alternating, $P - va$ ends with an edge from M' at v. In this case, $P - va$ corresponds with a tight and proper $\mathscr{M}_x$-alternating trail which connects $C_i \in \mathscr{C}_x$ and $v \notin V(\mathscr{C}_x)$, such that $y_v = 0$, $z_e = 0$ for all $e \in P$, and all interior vertices on P are not in $V(\mathscr{C}_x)$ (case (iii) of theorem 15). Otherwise, if $v_ja \in P$ for some $v \in V$ and $j \in \{1, \ldots, c_v - x(\delta(v))\}$, then $y_v = 0$, v is unsaturated, and v is not a pseudonode. In this case, $P - vv_j - v_ja$ corresponds with a tight and proper $\mathscr{M}_x$-alternating trail which connects $C_i \in \mathscr{C}_x$ and $v \notin V(\mathscr{C}_x)$, such that $y_v = 0$, $z_e = 0$ for all $e \in P$, and all interior vertices on P are not in $V(\mathscr{C}_x)$ (case (iii) of theorem 15).

In both cases, by theorem 15, $|\mathscr{C}_x| > \gamma(G)$. □

So, instead of looking in G for the structures described in theorem 15, we can also look for M'-augmenting trails in G'. To make this easier, we consider yet another auxiliary graph, now with unit-capacities. Let the unit-weight, unit-capacity auxiliary graph G'' and matching M'' in G'' be obtained by applying the following operations to G' and M'.

(a) For each edge $uv \in E(G')$ add the vertices e_u and e_v. Then replace the edge by the three edges ue_u, e_ue_v and e_vv. If $uv \in M'$, add ue_u and e_vv to M'', otherwise, add e_ue_v to M''.
(b) For each vertex $v \in V(G')$, replace v by c_v copies: $v_1, \dots, v_{c_v}$.
(c) For each edge of the form ve_v, replace it by c_v copies: $v_1e_v, \dots, v_{c_v}e_v$. If $ve_v \in M''$, replace it by v_ie_v, for some M''-exposed copy v_i of v.

Lemma 5. *$\nu(G'') = \nu^c(G') + |E(G')|$, and $|M'| = \nu^c(G')$ if and only if $|M''| = \nu(G'')$.*

Proof. Suppose M' is a maximum-cardinality matching in G', i.e., $|M'| = \nu^c(G')$. Then $|M''| = |M'| + |E(G')|$, and hence $\nu(G'') \geq \nu^c(G') + |E(G')|$.

Suppose M'' is a maximum-cardinality matching in G'', i.e., $|M''| = \nu(G'')$. Without loss of generality, we can assume that for each $e = uv \in E(G')$, either u_ie_u and e_vv_j are in M'' for some i and j, or e_ue_v is in M'', because if just u_ie_u or e_vv_j is in M'', for some i or j, we can replace it by e_ue_v. Now let $e = uv$ be in M' if u_ie_u and e_vv_j are in M'' for some i and j. Then $|M'| = |M''| - |E(G')|$, and hence $\nu^c(G') \geq \nu(G'') - |E(G')|$. The result follows. □

So, instead of looking for M'-augmenting trails in G', we can look for M''-augmenting paths in G''. Suppose we have an M''-augmenting path P'' in G''. We find the corresponding M'-augmenting trail in G' as follows. If P'' contains two consecutive edges of the form v_ie_v, e_vv_j, for some vertex copies v_i, v_j of v, remove them. This disconnects P'', because it jumps form v_i to v_j, but they both correspond to v, so the discontinuity will be fixed when mapped to G'. Since P'' is augmenting in cardinality, its endpoints have to be M''-exposed. Consequently, vertices of the form e_v cannot be endpoints. We know now that P'' can be partitioned in groups of three consecutive edges that look like u_ie_u, e_ue_v, e_vv_j. Then, when we map to an M'-alternating walk P' in G', we map groups of edges to their corresponding edge uv. Since the endpoints of P'' are M''-exposed, the corresponding endpoints of P' are M'-unsaturated, i.e., P' is M'-augmenting and proper. Finally, P' is a trail, since if it would repeat an edge uv, P'' would need to repeat e_ue_v, contradicting that P is a path.

Finally, we are ready to prove theorem 4, using algorithm 2.

Proof of theorem 4. Lines 1 to 3 can be done in polynomial time. There are at most $O(|V|)$ odd cycles in $\mathscr{C}_x$, and hence at most $O(|V|^2)$ pseudonodes in G''. At every iteration, we eliminate at least one odd cycle from $\mathscr{C}_x$, or we eliminate a tree T from G'', which includes a pseudonode. When we eliminate an odd cycle, all trees T are reset, which means that per odd cycle, we might remove $O(|V|^2)$ trees. And since there are at most $O(|V|)$ odd cycles, there are at most $O(|V|^3)$ iterations. In addition, all operations done within the while-loop can be done in polynomial time. Hence, algorithm 2 terminates in polynomial time.

Next, we prove correctness. By construction of G' and M', the only M'-unsaturated vertices are pseudonodes and the new vertex a. Consequently, the only M''-exposed vertices in G'' are exactly one copy of each pseudonode and the vertex a. When the algorithm terminates, it means that the only M''-exposed

Algorithm 2: Minimize number of odd cycles in a basic maximum-weight fractional c-matching

```
compute a basic maximum-weight fractional c-matching x in G
compute a minimum fractional vertex cover (y, z) in G
construct G', M', G'' and M'' as described above
while there is an M''-exposed pseudonode r in G'' do
    grow an M''-alternating tree T rooted at r using Edmonds' algorithm
    if an M''-augmenting rs-path P'' is found in G'' then
        let P' be the corresponding M'-augmenting trail in G' (found as
          described above)
        if interior vertices of P' are pseudonodes then
            select a sub-trail of P' such that no interior vertices are
              pseudonodes, like mentioned in the proof of lemma 4
        let P be the corresponding 𝓜_x-alternating trail in G
        if s is a pseudonode then
            alternate round x on C_r, C_s and complement x on P
        else
            alternate round x on C_r and complement x on P
        recreate G', M', G'' and M'' (so also add the removed trees again)
    else
        G'' ← G'' \ V(T)
return x
```

vertices are pseudonodes that are part of removed trees T and a. From the moment these trees are found, nothing changed in the graph, except the removal of the trees. Which means all removed trees are still frustrated upon termination of the algorithm. In addition, in the remaining graph there is just a single M''-exposed vertex (a), which means no M''-augmenting path can exist. This implies that the last matching M'' created by the algorithm is maximum. By lemmas 4 and 5, it follows that $|\mathscr{C}_x| = \gamma(G)$. □