For the record, a Construction of an Optimal Non-linear Binary Error-Correcting Code of Length~10 and Minimum Distance~6

DOI:
10.13140/RG.2.2.12705.79203

Document status and date:
Published: 24/01/2024

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

For the record, a Construction of an Optimal Non-linear Binary Error-Correcting Code of Length 10 and Minimum Distance 6

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24 January 2024

According to [1], there exists a (not necesaarily linear) binary error-correcting code of length 10 and minimum distance 6, consisting of 6 codewords. I could not immediately find a concrete example. In [2], this is given as Exercise 114 in Chapter 2 "Bounds on the size of codes":

> By Theorem 2.1.2, $A_2(9,5) = A_2(10,6)$.
>
> (a) Compute upper bounds on both $A_2(9,5)$ and $A_2(10,6)$ using the Sphere Packing Bound, the Plotkin Bound, and the Elias Bound. When computing the Elias Bound make sure all possible values of w have been checked.
>
> (b) What is the best upper bound for $A_2(9,5) = A_2(10,6)$?
>
> (c) Find a binary code of length 10 and minimum distance 6 meeting the bound in part (b). Hint: This can be constructed using the zero vector with the remaining codewords having weight 6. (Note: This verifies the entry in Table 2.1.)

Note that Table 2.1 in [2] is improved by [1]. In both, the entry for $A_2(10,6) = 6$.

Construction

The hint suggests that one finds 5 codewords of weight 6 that have minimum distance 6. By also including the all-0 codeword, one then gets a code of 6 codewords and minimum distance 6.

Imagine one could find 5 codewords of length 5, weight 3, and minimum distance 3. Then doubling each bit would yield 5 codewords of length 10 and weight 6 at distance 6. However, [1] and [2] list $A_2(5,3) = A_2(6,4) = 4$, So, that is not going to work.

Consider the 5 cyclic shifts of 11000. It is easy to verify that this yields 5 codewords of length 5, weight 2, and minimum distance 2. In fact, because of the cyclic nature, there are only 2 pairs to consider for the distance:

- $\begin{matrix} 11000 \\ 01100 \end{matrix}$, at distance 2
- $\begin{matrix} 11000 \\ 00110 \end{matrix}$, at distance 4

Now double each 1-bit and replace the 0-bits by respectively 00, 01, and 10. This yields 5 codewords of length 10 and weight 6. The 2 pairs for the distance distribution are then:

- $\begin{matrix} 11\ 11\ 00\ 01\ 10 \\ 10\ 11\ 11\ 00\ 01 \end{matrix}$, at distance $1 + 0 + 2 + 1 + 2 = 6$
- $\begin{matrix} 11\ 11\ 00\ 01\ 10 \\ 01\ 10\ 11\ 11\ 00 \end{matrix}$, at distance $1 + 1 + 2 + 1 + 1 = 6$

Thus, we found as 6 codewords of length 10:

$$\begin{matrix} 0000000000 \\ 1111000110 \\ 1100011011 \\ 0001101111 \\ 0110111100 \\ 1011110001 \end{matrix}$$

All pairwise distances equal 6. Note that this code is not linear, since its size is not a power of 2. In fact, $1111000110 + 1100011011 = 0011011101$, which does not occur in the code.

References

[1] Andries Brouwer. *Table of general binary codes*. 2018-11-27. `https://www.win.tue.nl/~aeb/codes/binary.html`

[2] Huffman, W. Carry and Pless, Vera. *Fundamentals of Error-Correcting Codes*. Cambridge University Press, 2003. `http://www.cambridge.org/9780521782807`