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A switching method for constructing cospectral gain graphs

Aida Abiad a,b,c,*, Francesco Belardo d, Antonina P. Khramova a

a Department of Mathematics and Computer Science, Eindhoven University of Technology, the Netherlands
b Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
c Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium
d Department of Mathematics and Applications R. Caccioppoli, University of Naples Federico II, Italy

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A gain graph over a group G, also referred to as G-gain graph, is a graph where an element of a group G, called gain, is assigned to each oriented edge, in such a way that the inverse element is associated with the opposite orientation. Gain graphs can be regarded as a generalization of signed graphs, among others. In this work, we show a new switching method to construct cospectral gain graphs. Some previous methods known for graph cospectrality follow as a corollary of our results.

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1. Introduction

Gain graphs, which can be regarded as a generalization of signed graphs [7] where the group is \([-1, 1]\), have been extensively studied, see the survey paper [19]. Gain graphs can be also considered as particular cases of biased graphs [18] and are related to voltage graphs [11]. Among the key notions concerning signed and gain graphs, there are switching equivalence and balance, see for instance [12,18].

The spectrum of complex unit gain graphs and signed graphs with respect to the identical representation, both of which are special cases of gain graphs and the more general G-spectrum, has been considered in the literature, see [19]. Reff [14] introduced complex unit gain graphs and investigated their spectral properties by extending some fundamental results from spectral graph theory. For signed graphs, characterizations of some families have been done with respect to the identical representation, such as [1,2,6]. Other cases related to gain graphs have been considered, including signed directed graphs [5] and quaternion unit gain graphs [16]. For a general group G, it has been shown that any G-gain graph with a cycle as its underlying graph is determined by its G-spectrum [9].

Less is known regarding cospectrality of gain graphs. The well-known switching method by Godsil and McKay [10] (GM-switching), and the more recent switching by Wang, Qiu and Hu [13,15] (WQH-switching), are operations on graphs that do not change the spectrum of the adjacency matrix. The GM-switching has been extended to signed graphs [3] and complex unit graphs [4], and more recently, also to gain graphs over an arbitrary group [8]. The WQH-switching has also been extended, in its simpler form, to signed graphs [3] and complex unit graphs [4] under the name Modified Godsil-McKay switching. In this paper we show that both results from [3,4] hold in a more general form. In particular, we present a new method to construct G-gain graphs over an arbitrary group G that are cospectral with respect to the G-spectrum, which is...
independent of the choice of the particular representation of $G$. We will also investigate how this new switching behaves when considering a particular representation of $G$. Our results provide new insights into the spectral theory of gain graphs with respect to $G$-cospectrality, which to the best of our knowledge, has only been investigated by Cavaleri, Donno and Spessato [8,9].

2. Preliminaries

We use the notation and the definitions from [8, Section 2].

For an abstract group $G$, its neutral element is denoted by $1_G$, except for when $1_G$ coincides with 1 of the field of complex numbers $\mathbb{C}$, in which case the index $G$ is omitted.

Let $\mathbb{T}$ denote the group of complex units, or the group of elements $\{z \in \mathbb{C} : |z| = 1\}$. The group $\mathbb{T}_n$ is a group of $n$-th roots of 1 \in \mathbb{C}$, so $\mathbb{T}_n = \{z \in \mathbb{C} : z^n = 1\}$. For example, $\mathbb{T}_2 = \{-1, 1\}$, or $\mathbb{T}_4 = \{i, -i, -1, 1\}$.

$M_{n,m}(\mathbb{F})$ denotes a set of matrices of size $n \times m$ with entries in $\mathbb{F}$, typically $\mathbb{F} \in \{\mathbb{C}, \mathbb{G}\}$, where $\mathbb{C}G = \{\sum_{x \in \mathbb{G}} c_x x : c_x \in \mathbb{C}\}$ is the group algebra of $\mathbb{G}$ endowed with the product

\[
\left(\sum_{x \in \mathbb{G}} f_x x\right)\left(\sum_{y \in \mathbb{G}} h_y y\right) = \sum_{x, y \in \mathbb{G}} f_x h_y xy,
\]

and with the involution

\[
\left(\sum_{x \in \mathbb{G}} f_x x\right)^* = \sum_{x \in \mathbb{G}} f_x x^\ast
\]

for $f_x, h_y \in \mathbb{C}$. We assume $M_0(\mathbb{F}) := M_{0,0}(\mathbb{F})$. Throughout the paper, the identity matrix of size $n$ is denoted by $I_{M_n(\mathbb{F})}$, while $f_{M_n(\mathbb{F})}$ is a matrix in $M_n(\mathbb{F})$ such that all its entries are $1_G$ in case $\mathbb{F} = \mathbb{C}G$ or 1 in case $\mathbb{F} = \mathbb{C}$.

A gain graph over a group $G$, also referred to as $G$-gain graph, is a pair $(\Gamma, \psi)$, where $\Gamma = (V, A)$ is an underlying directed graph such that any arc $(u, w) \in A$ has a reverse $(w, u) \in A$, and $\psi : A \rightarrow G$ is a gain function such that for any arc $(u, w) \in A$ we have $\psi(w, u) = \psi(u, w)^{-1}$. In particular, a $\mathbb{T}_2$-gain graph is a graph with each arc labeled by 1 or $-1$ (or, alternatively, a sign $+$ or $-$), which is also called a signed graph. The concept of gain graphs has been first introduced by Zaslavsky [18]. For a regularly updated bibliography on signed and gain graphs we refer to [19].

For a $G$-gain graph $(\Gamma, \psi)$ on $n$ vertices $V = \{v_1, v_2, \ldots, v_n\}$ we define its adjacency matrix $A_{(\Gamma, \psi)}$ with entries

\[
(A_{(\Gamma, \psi)})_{i,j} = \begin{cases} \psi(v_i, v_j), & v_i \sim v_j; \\ 0, & \text{otherwise.} \end{cases}
\]

This way, $A_{(\Gamma, \psi)} \in M_n(\mathbb{C}G)$.

Similar to how ordinary graphs are usually considered up to isomorphism, gain graphs are typically considered up to switching isomorphism [17]. Two gain functions $\psi_1$ and $\psi_2$ on the same underlying graph $\Gamma = (V, A)$ are switching equivalent if there exists a function $f : V \rightarrow G$ such that for any pair of adjacent vertices $v, w$ we have

\[
\psi_2(v, w) = f(v)^{-1} \psi_1(v, w) f(w).
\]

Two gain graphs $(\Gamma_1, \psi_1)$ and $(\Gamma_2, \psi_2)$ with vertex sets $V_1$ and $V_2$ respectively are then switching isomorphic if there exists a graph isomorphism $\phi : V_1 \rightarrow V_2$ and the gain function $\psi_1$ is switching equivalent to $\psi_2 \circ \phi$ defined by $(\psi_2 \circ \phi)(v, w) = \psi_2(\phi(v), \phi(w))$ for any arc $(v, w)$ in $\Gamma_1$.

It is known that the switching equivalence relation can be expressed in terms of adjacency matrices of the gain graphs on $n$ vertices. Namely, $\psi_1$ and $\psi_2$ are switching equivalent if and only if there exists a diagonal matrix $\Lambda \in M_n(\mathbb{C}G)$ such that $A_{(\Gamma_1, \psi_1)} = \Lambda^* A_{(\Gamma_2, \psi_2)} \Lambda$ [7, Theorem 4.1]. Here and elsewhere, $M^*$ is obtained from matrix $M \in M_n(\mathbb{C}G)$ by taking the transpose of $M$ and applying the involution $f \mapsto f^*$ to each entry: $(M^*)_{i,j} = (M_{j,i})^*$. It follows that the gain graphs $(\Gamma_1, \psi_1)$ and $(\Gamma_2, \psi_2)$ are switching isomorphic if and only if there exists a diagonal matrix $\Lambda \in M_n(\mathbb{C}G)$ with $\Lambda_{i,i} \in G$ for $i \in \{1, 2, \ldots, n\}$ and a permutation matrix $P \in M_n(\mathbb{C}G)$ with entries 0 and 1 such that $(\Lambda \circ P) A_{(\Gamma_1, \psi_1)} = (\Lambda \circ P) A_{(\Gamma_2, \psi_2)}$.

\[
A_{(\Gamma_2, \psi_2)} = (P \Lambda)^* A_{(\Gamma_1, \psi_1)} (P \Lambda).
\]

In case when $G \subseteq \mathbb{C}$, it is natural to define two $G$-gain graphs to be cospectral if the multisets of eigenvalues of their adjacency matrices (both in $M_n(\mathbb{C})$) coincide since the characteristic polynomial and the eigenvalues of a complex-valued matrix are well understood. This is the definition used in [4,14] where $\mathbb{T}_n$-gain graphs were considered. In a general case when $G$ is not necessarily a group of complex numbers, we define $G$-cospectrality through the equality of the sequences of trace powers (the spectral moments) of the two adjacency matrices up to conjugacy.

To be more precise, let $Tr : M_n(\mathbb{C}G) \rightarrow \mathbb{C}G$ be a function that maps matrix $A$ to $\sum_{i=1}^n A_{i,i}$. Also, let $[g]$ be a conjugacy class of $g \in G$ and let $\mathbb{C}_{\text{Class}}[G]$ be a set of finitely supported class functions, i.e. functions $f : G \rightarrow \mathbb{C}$ such that $f(g_1) = f(g_2)$ whenever $g_1, g_2$ are in the same conjugacy class $[g]$. We consider a natural map $\mu : \mathbb{C}G \rightarrow \mathbb{C}_{\text{Class}}[G]$ defined by
\[
\mu \left( \sum_{x \in G} a_x x \right) (g) = \sum_{x \in [G]} a_x.
\]

Then two matrices \(A, B \in M_n(\mathbb{C}G)\) are \(G\)-cospectral if \(\mu(Tr(A^h)) = \mu(Tr(B^h))\) for any positive integer \(h\), and two \(G\)-gain graphs \((\Gamma_1, \psi_1)\) and \((\Gamma_2, \psi_2)\) are \(G\)-cospectral if their adjacency matrices are \(G\)-cospectral. While two switching isomorphic graphs are always \(G\)-cospectral, the converse is not true. Hence, it is possible that two \(G\)-cospectral \(G\)-gain graphs are switching non-isomorphic. We are mainly interested in constructing pairs of \(G\)-cospectral non-isomorphic \(G\)-gain graphs.

Note that, similarly to ordinary graphs, for a gain graph \((\Gamma, \psi)\) with vertex set \(V = \{v_1, v_2, \ldots, v_n\}\) and a positive integer \(h\) the entry \((A_{\Gamma, \psi})^h_{i,j}\) corresponds to the sum of gains of all walks of length \(h\) from \(v_i\) to \(v_j\) [7, Lemma 4.1]. As such, the definition of \(G\)-cospectrality is a generalization of the cospectrality notion of gain graphs in the case of complex-valued adjacency matrices.

An instrumental tool in asserting \(G\)-cospectrality of gain graphs constructed in Section 3 is the following lemma.

**Lemma 1.** [8, Lemma 2.6] Let \((\Gamma_1, \psi_1)\) and \((\Gamma_2, \psi_2)\) be two \(G\)-gain graphs with adjacency matrices \(A_1, A_2 \in M_n(\mathbb{C}G)\), respectively. Let \(Q, R \in M_n(\mathbb{C}G)\) be such that every entry of \(R\) is a complex multiple of \(1_G\) and \(QR = RQ = I_{M_n(\mathbb{C}G)}\). If \(A_2 = QA_1R\) then \((\Gamma_1, \psi_1)\) and \((\Gamma_2, \psi_2)\) are \(G\)-cospectral.

A representation of a group \(G\) of degree \(k\) is a group homomorphism \(\pi : G \rightarrow GL_k(\mathbb{C})\), where \(GL_k(\mathbb{C})\) is a set of all invertible matrices in \(M_k(\mathbb{C})\). An easy example is for a symmetric group on \(k\) elements \(S_k\). A homomorphism \(\pi : S_k \rightarrow GL_k(\mathbb{C})\) that maps a permutation in \(S_k\) to a respective permutation matrix of size \(k\) is a representation. Moreover, it is a unitary representation, i.e. a representation such that \(\pi(g) \in U_k(\mathbb{C})\) for any \(g \in G\), where \(U_k(\mathbb{C}) = \{M \in GL_k(\mathbb{C}) : M^{-1} = M^*\}\) is a set of unitary matrices of size \(k\) over \(\mathbb{C}\). It is a well-known fact that any finite group \(G\) can be isomorphically embedded into a symmetric group \(S_k\), so a unitary representation always exists for any finite group. Another example is a trivial representation \(\pi_0 : G \rightarrow \mathbb{C}\) that maps any \(g \in G\) to \(1\) which is also unitary. In case \(G \subseteq \mathbb{C}\) one can also consider an identical representation \(\pi_{id} : G \rightarrow GL_1(\mathbb{C})\) such that \(\pi_{id}(g) = g\) for any \(g \in G\).

Along with a group homomorphism \(\pi : G \rightarrow GL_k(\mathbb{C})\), \(\pi\) will also sometimes denote its natural linear extension to \(\mathbb{C}G\), which is an algebra homomorphism. Namely,

\[
\pi : \mathbb{C}G \rightarrow M_k(\mathbb{C}) \text{ s.t.} \quad \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g \pi(g).
\]

This can be even further extended to \(\pi : M_{n,m}(\mathbb{C}G) \rightarrow M_{mk,mk}(\mathbb{C})\), where \(A \in M_{n,m}(\mathbb{C}G)\) is mapped to a matrix \(\pi(A)\) obtained by replacing each entry \(A_{i,j} \in \mathbb{C}G\) with a matrix \(\pi(A_{i,j}) \in M_k(\mathbb{C})\).

Let \((\Gamma, \psi)\) be a \(G\)-gain graph, and let \(\pi\) be a unitary representation of \(G\). We say that \(\pi(A_{\Gamma, \psi})\) is the represented adjacency matrix of \((\Gamma, \psi)\) with respect to \(\pi\). According to [7, Proposition 3.4], \(\pi(A_{\Gamma, \psi})\) is Hermitian since \(\pi\) is unitary, and its (real) spectrum is called the \(\pi\)-spectrum of \((\Gamma, \psi)\). Finally, two \(G\)-gain graphs \((\Gamma_1, \psi_1)\) and \((\Gamma_2, \psi_2)\) are \(\pi\)-cospectral if they have the same \(\pi\)-spectrum. Note that two \(\pi\)-cospectral \(G\)-gain graphs do not need to be switching isomorphic. However, if two \(G\)-gain graphs are \(G\)-cospectral, then they are also \(\pi\)-cospectral, and the converse is not true in general. We refer the reader to [9, Figure 1] for a diagram of the relationships between the notions of \(G\)-cospectrality, \(\pi\)-cospectrality, switching isomorphism, and switching equivalence for \(G\)-gain graphs.

It follows from [9, Theorem 4.14] that, for a finite group \(G\), two \(G\)-gain graphs \((\Gamma_1, \psi_1)\) and \((\Gamma_2, \psi_2)\) are \(G\)-cospectral if any only if they are \(\pi\)-cospectral for every unitary representation \(\pi\) of \(G\), including the trivial representation \(\pi_0\). This way, \(G\)-cospectrality is a more general notion than \(\pi\)-cospectrality.

### 3. A switching to construct \(G\)-cospectral gain graphs

The main goal of this section is to show a new method to obtain pairs of \(G\)-cospectral gain graphs. We are inspired by the Godsil-McKay switching [10], the Wang-Qiu-Hu switching [13,15] and its generalizations [3,4].

Let \((\Gamma, \psi)\) be a \(G\)-gain graph on \(n\) vertices, and suppose that

\[
\alpha = \{C_0, C_1, C_2, \ldots, C_{2k-1}, C_{2k}\}
\]

is a partition of the vertex set of \(V_\Gamma\). With respect to \(\alpha\) for every vertex \(v\), we define

\[
\Psi_\gamma(v) := \sum_{w \in C_i, w \sim v} \psi(v, w).
\]

**Definition 2.** A partition \(\alpha\) is a \(G\)-WQH partition if the following conditions hold:

- \(|C_0| = n_0\) and \(|C_i| = |C_{i+1}| = n_i\) for any odd \(i < 2k\);
for $i, j \in \{1, 2, \ldots, 2k\}$ and $v, v' \in C_i$ we have $\Psi_j(v) = \Psi_j(v')$;
- for odd $i, j < 2k$ and $v \in C_i, \ v' \in C_{i+1}$ we have $\Psi_j(v) = \Psi_{j+1}(v')$ and $\Psi_{j+1}(v) = \Psi_j(v')$;
- for every $v \in C_0$ and an odd $i < 2k$ we have either
  (a) $\Psi_i(v) = \Psi_{i+1}(v)$, or
  (b) $\Psi_i(v) = |C_i| g_1$ and $\Psi_{i+1}(v) = |C_{i+1}| g_2$ for some distinct $g_1, g_2 \in G \cup \{0\} \subset C_G$.

Alternatively, in a $G$-WQH partition, the total gain over all edges from $v \in C_i$ to vertices in $C_j$ does not depend on the choice of $v$. Additionally, for odd $i, j < 2k$ the pairs of subsets $C_i \cup C_{i+1}$ and $C_j \cup C_{j+1}$ are subject to the following relation: the total gain over all edges from $v \in C_i$ to vertices in $C_j \cup C_{j+1}$ must be the same as the total gain over all edges from $v' \in C_{i+1}$ to vertices in $C_j \cup C_{j+1}$. Moreover, for a vertex $v \in C_0$ and an odd $i < 2k$, either the total gain summed over all edges to $C_i$ is the same as the total gain summed over all edges to $C_{i+1}$, or such $v$ must be adjacent to all vertices of $C_i$ with the same gain $g_1$ and to all vertices of $C_{i+1}$ with the same gain $g_2$, where either gain may be zero, implying non-adjacency. The requirement for $g_1$ and $g_2$ to be distinct is stated to prevent overlap between the two cases, although we note that the following definitions and proofs do not conflict if $g_1 = g_2$.

Example 3. Let $(\Gamma, \psi)$ be a $T_4$-gain graph on 13 vertices $v_0, v_1, \ldots, v_{12}$ with adjacency matrix:

$$
A(\Gamma, \psi) = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

We consider the partition $\alpha$ into subsets

$C_0 = \{v_0\}, C_1 = \{v_1, v_2, v_3\}, C_2 = \{v_4, v_5, v_6\}, C_3 = \{v_7, v_8, v_9\}, C_4 = \{v_{10}, v_{11}, v_{12}\}$

see the left of Fig. 1. In the matrix, we use single lines to illustrate the partition into these five blocks, and doubled lines to highlight the two pairs of switching cycles $C_1 \cup C_2$ and $C_3 \cup C_4$. The partition $\alpha$ is G-WQH partition. Indeed, we have $|C_1| = |C_2|$ and $|C_3| = |C_4|$. Moreover, it is easily verified that for any $i, j \in \{1, 2, 3, 4\}$ and any vertex $w_i \in C_i$ we have $\Psi_i(w_i) = 0$, so the second and third conditions in Definition 2 hold up. Finally, for $v_0 \in C_0$ and the subsets $C_3$ and $C_4$ we have $\Psi_3(v_0) = \Psi_4(v_0) = 1$, which satisfies (a) of the fourth condition, and for the subsets $C_1$ and $C_2$ we have $\Psi_1(v_0) = 3 = 3 \cdot 1$ and $\Psi_2(v_0) = 0 = 3 \cdot 0$, which satisfies (b) of that condition for $g_1 = 1 \in T_4$ and $g_2 = 0$.

Definition 4. For a G-WQH partition $\alpha$ and a G-gain graph $(\Gamma, \psi)$, we define a gain graph $(\Gamma^\alpha, \psi^\alpha)$ as follows (assuming $\psi(v, w) = 0$ iff $v \sim w$):

- for $v, w \in C_1 \cup \cdots \cup C_{2k}$, we have $\psi^\alpha(v, w) = \psi(v, w)$ (that is, the adjacency and the gains between pairs of vertices in $\bigcup_{i=1}^{2k} C_i$ are the same as in $(\Gamma, \psi)$);
- for $v \in C_0$ and an odd $i < 2k$ such that $\Psi_i(v) = \Psi_{i+1}(v)$ and $w \in C_i \cup C_{i+1}$, we have $\psi^\alpha(v, w) = \psi(v, w)$;
- for $v \in C_0$ and an odd $i < 2k$ such that $\Psi_i(v) = |C_i| g_1$ and $\Psi_{i+1}(v) = |C_{i+1}| g_2$ for some $g_1, g_2 \in G$ we have $\psi^\alpha(v, w) = g_2$ if $w \in C_i$ and $\psi^\alpha(v, w) = g_1$ if $w \in C_{i+1}$.

The G-WQH partition is said to be nontrivial if $(\Gamma, \psi)$ and $(\Gamma^\alpha, \psi^\alpha)$ are not switching isomorphic.

Finally, for a G-WQH partition $\alpha$, let $Q_\alpha \in M_n(C_G)$ be a block-diagonal matrix $Q_\alpha = \text{diag}(I_{M_0(C_G)}, Q_1, \ldots, Q_k)$, where for each $i = 1, 2, \ldots, k$

$$Q_i = \begin{pmatrix}
I_{M_i(C_G)} & \frac{1}{n_i} \sum_{j=1}^{2k} M_i(C_G)
\frac{1}{n_i} \sum_{j=1}^{2k} M_i(C_G)
\frac{1}{n_i} \sum_{j=1}^{2k} M_i(C_G)
\end{pmatrix}$$

where $n_i$ is the size of $C_i$ and $C_{i+1}$ for any odd $i < 2k$. 

Observe that the nonzero entries of $Q_\alpha$ are all real multiples of $1_G$ and $Q_\alpha^* = Q_\alpha$.

**Lemma 5.** Let $A$ be a $2n_l \times 2n_j$ block matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

where each block has size $n_l \times n_j$ and such that the blocks $A_{1,1}$ and $A_{2,2}$ (as well as $A_{1,2}$ and $A_{2,1}$) have constant row sum of the same value and constant column sum of the same value. Then $Q_{n_l}A Q_{n_j} = A$.

**Proof.** Suppose $r_1$ and $c_1$ are, respectively, row sum and column sum constants for $A_{1,1}$ and $A_{2,2}$, while $r_2$ and $c_2$ are such constants for $A_{1,2}$ and $A_{2,1}$. Observe that

$$J_{M_n(G)}A_{l,m} = c_1 J_{M_{n_l,n_j}(G)} \quad \text{for} \quad l \in \{1, 2\};$$
$$J_{M_n(G)}A_{l,m} = c_2 J_{M_{n_l,n_j}(G)} \quad \text{for} \quad l \in \{1, 2\};$$
$$J_{M_n(G)}J_{M_{n_l,n_j}(G)} = n_l J_{M_{n_l,n_j}(G)} \quad \text{for} \quad l \in \{1, 2\}.$$ 

Using the above equalities, it is straightforward to verify that $Q_{n_l}A Q_{n_j} = A$ by performing the block matrix multiplication. \(\square\)

**Theorem 6.** Let $(\Gamma, \psi)$ be a $G$-gain graph and let $\alpha$ be a $G$-WQH partition with associated matrix $Q_\alpha \in M_n(G)$. Then

$$A(\Gamma^\alpha, \psi^\alpha) = Q_\alpha A(\Gamma, \psi) Q_\alpha^*.$$

In particular, $(\Gamma, \psi)$ and $(\Gamma^\alpha, \psi^\alpha)$ are $G$-cospectral.

**Proof.** With a suitable labeling of vertices of $\Gamma$ we can write

$$A(\Gamma, \psi) = \begin{pmatrix} C_{0,0} & C_{0,1} & \cdots & C_{0,k} \\ C_{0,1} & C_{1,1} & \cdots & C_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,k} & C_{1,k} & \cdots & C_{k,k} \end{pmatrix}.$$ 

Here, for all $i, j \in \{1, 2, \ldots, k\}$ the block $C_{i,j} \in M_{2n_{2j-1,2n_{2j-1}}(G)}$ defines the adjacencies and gains from vertices in $C_{2i-1} \cup C_{2i}$ to vertices in $C_{2j-1} \cup C_{2j}$. The block $C_{0,j} \in M_{n_0,2n_{2j-1}}(G)$ defines the adjacencies and gains from vertices in $C_0$ to vertices in $C_{2j-1} \cup C_{2j}$. Later we will also use a split of this block $C_{0,j} = \left( C_{0,j}^{(1)}, C_{0,j}^{(2)} \right)$, where $C_{0,j}^{(1)}$ defines adjacencies between
the vertices of $C_0$ and $C_{2j-1}$, while $C^{(2)}_{0,j}$ does so for $C_0$ and $C_{2j}$. Finally, the block $C_{0,0} \in M_{n_0}(\mathbb{C}G)$ defines adjacencies and gains within the vertex subset $C_0$.

The following is true for $A(\Gamma, \psi)$ by the definition of $G$-WQH partition (Definition 2):

- for $i \in \{0, 1, \ldots, k\}$ we have $C^*_i = C_{i,i}$;  
- for $i, j \in \{0, 1, \ldots, k\}$ the matrix $C_{i,j}$ satisfies the conditions of Lemma 5.

By block matrix multiplication and from Lemma 5 we have:

$$Q_\alpha A(\Gamma, \psi) Q_\alpha = \begin{pmatrix} C_{0,0} & C_{0,1}Q_{n_1} & \cdots & C_{0,k}Q_{n_k} \\ Q_{n_1}C_{0,1} & C_{1,1} & \cdots & C_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n_k}C_{0,k} & C^*_1 & \cdots & C^*_k \\ \end{pmatrix}.$$  

Let $v \in C_0$ and $j \in \{1, 2, \ldots, k\}$, and let $x$ be the row of $C_{0,j}$ which corresponds to $v$ (the following observations are just as valid for the column $x^*$ of $C^*_0,j$ corresponding to $v$). With the use of the split $C_{0,j} = (C_{0,j}^{(1)}, C_{0,j}^{(2)})$ we can write down (assuming $J := J_{n_0,j}(\mathbb{C}G)$)

$$C_{0,j} = \left( \begin{array}{cc} C_{0,j}^{(1)} - \frac{1}{n_j} C_{0,j}^{(1)} J \frac{1}{n_j} C_{0,j}^{(2)} J \\
\frac{1}{n_j} C_{0,j}^{(1)} J + C_{0,j}^{(2)} - \frac{1}{n_j} C_{0,j}^{(2)} J \end{array} \right).$$

There are two possible cases. First, if $\Psi_{2j-1}(v) = \Psi_{2j}(v)$, then the row corresponding to $v$ in $A(\Gamma, \psi)$ is also $x$ as the contribution of $\frac{1}{n_j} C_{0,j}^{(1)} J$ and $\frac{1}{n_j} C_{0,j}^{(2)} J$ terms amounts to zero in this case. On the other hand, suppose $\Psi_{2j-1}(v) = n_{2j-1}g_1$ and $\Psi_{2j}(v) = n_{2j-1}g_2$ for some $g_1, g_2 \in G \cup \{0\}$, or, in other words, $x$ takes the form $(g_1, \ldots, g_1, g_2, \ldots, g_2)$, a row of $n_{2j-1}$ entries $g_1$ followed by $n_{2j-1}$ entries $g_2$. Then the row of $C_{0,j}Q_{n_j}$ corresponding to the vertex $v$ takes the form $(g_2, \ldots, g_2, g_1, \ldots, g_1)$. Either case is consistent with the described construction of the graph $(\Gamma^\alpha, \psi^\alpha)$, and so the matrix $Q_\alpha A(\Gamma, \psi) Q_\alpha$ is indeed equal to the adjacency matrix $A(\Gamma^\alpha, \psi^\alpha)$. \square

We now apply Theorem 6 to the $G$-gain graph from Example 3.

Example 7. Let $(\Gamma, \psi)$ be a $\mathbb{T}\delta$-gain graph on 13 vertices with the partition $\alpha$ as in Example 3. It was already shown that $\alpha$ is a $G$-WQH partition. Moreover, the partition is nontrivial. Indeed, the graph $(\Gamma^\alpha, \psi^\alpha)$ is obtained from $(\Gamma, \psi)$ by removing edges $\{v_0, v_1\}, i \in \{1, 2, 3\}$ and adding new edges $\{v_0, v_i\}, i \in \{4, 5, 6\}$ all with gain 1, see Fig. 1. This graph has 5 isolated vertices unlike $(\Gamma, \psi)$, and hence cannot be switching isomorphic to it. It also cannot be constructed using the GM switching for $G$-cospectral graphs from [8] since there is no suitable $G$-GM partition of $V_G$.

4. A switching with respect to a representation: $\pi$ -cospectral gain graphs

In this section, we present a method to obtain pairs of $\pi$ -cospectral gain graphs for some unitary representation $\pi$ of a group $G$.

Let $(\Gamma, \psi)$ be a $G$-gain graph on $n$ vertices, and let $\pi$ be a unitary representation $\pi : G \rightarrow U_k(\mathbb{C})$ ($\pi$ also stands for a linear extension $\pi : \mathbb{C}G \rightarrow M_k(\mathbb{C})$). Recall the function

$$\Psi_\pi(v) := \sum_{w \in C_i, w \sim v} \psi(v, w)$$

defined with respect to a partition $\alpha = [C_0, C_1, \ldots, C_{2k}]$. In this section, we will also use a function $\Psi_{\pi,i+1}(v) = \Psi_\pi(v) + \Psi_{\pi,i+1}(v)$ for an odd $i < 2k$.

Remark 8. If $\pi_0(v)$ is the trivial representation mapping each element of $G$ to 1, then $\pi_0(\psi_\pi(v))$ is the number of vertices of $C_i$ which are adjacent to $v$.

Definition 9. A partition $\alpha = [C_0, C_1, \ldots, C_{2k}]$ of the vertex set of $\Gamma$ is a $\pi$ -WQH partition if the following conditions hold:

- $|C_i| = n_0$ and $|C_i| = |C_{i+1}| = n_i$ for an odd $i < 2k$;
- for $i, j \in \{1, 2, \ldots, 2k\}$ and $v, v' \in C_i$ we have $\pi(\Psi_{\pi,j}(v)) = \pi(\Psi_{\pi,j}(v'))$;
- for odd $i, j < 2k$ and $v \in C_i, v' \in C_{i+1}$ we have $\pi(\Psi_{\pi,j}(v)) = \pi(\Psi_{\pi,j+1}(v'))$ and $\pi(\Psi_{\pi,j+1}(v)) = \pi(\Psi_{\pi,j}(v'))$;
- for every $v \in C_0$ and an odd $i < 2k$ we have either
  (a) $\Psi_{\pi,j}(v) = \Psi_{\pi,i+1}(v)$, or
  (b) $\Psi_{\pi,i}(v) = |C_i|g_1$ and $\Psi_{\pi,i+1}(v) = |C_{i+1}|g_2$ for some distinct $g_1, g_2 \in G \cup \{0\}$.  

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The above definition is closely related to Definition 2. For a \( \pi \)-WQH partition \( \alpha \), the graph \((\Gamma^\alpha, \psi^\alpha)\) is constructed exactly as described in Definition 4.

**Example 10.** This example builds on [8, Example 4.3]. Consider an \( S_4 \)-gain graph \((\Gamma, \psi)\) depicted in the left of Fig. 2; the \( \psi \)-image of any unlabeled edge is \( 1_{S_4} \), and we use edges labeled \((12)(34)\), \((12)\), and \((34)\) to depict an arc and its reverse both labeled \((12)(34)\), \((12)\), and \((34)\), respectively. Let \( \omega \) be a partition \([C_0, C_1, C_2, C_3, C_4]\), where \( C_0 = \{v_1\}, C_1 = \{v_2, v_3, v_4, v_5\}, C_2 = \{v_6, v_7, v_8, v_9\}, C_3 = \{v_{10}, v_{11}, v_{12}, v_{13}\}, \) and \( C_4 = \{v_{14}, v_{15}, v_{16}, v_{17}\} \). It is not a \( G \)-WQH partition, since \( \Psi_1(v) = 1_{S_4} + (12)(34) \neq (12) + (34) = \Psi_2(v) \) for \( v \in C_1 \) and \( v' \in C_2 \). However, by choosing a unitary representation \( \pi \) that sends a permutation from \( S_4 \) to a respective \( 4 \times 4 \) permutation matrix, we obtain

\[
\pi(1_{S_4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi((12)(34)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

and so \( \pi(\Psi_1(v)) = \pi(\Psi_2(v')) \). Similarly, for \( v \in C_3 \) and \( v' \in C_4 \) we have \( \pi(\Psi_3(v)) = \pi(\Psi_4(v')) \). As for the other conditions, we have \([C_1] = [C_2]\) and \([C_3] = [C_4]\), as well as \( \Psi_i(v_j) = 0 \) for any distinct \( i, j \in \{1, 2, 3, 4\} \), so the first three conditions in Definition 9 are satisfied. For the final condition, we have \( \Psi_1(v_1) = 4 \times 1_{S_4} \) and \( \Psi_2(v_1) = 4 \times 0 \) which satisfies the (a) part, and \( \Psi_3(v_1) = \Psi_4(v_1) = 1_{S_4} \) satisfying part (b).

**Theorem 11.** Let \((\Gamma, \psi)\) be a \( G \)-gain graph, let \( \pi \) be a unitary representation of \( G \), and let \( \omega \) be a \( \pi \)-WQH partition. Then

\[
\pi(A_{\Gamma\omega, \psi^\omega}) = \pi(Q_\omega)\pi(A_{\Gamma, \psi})\pi(Q_\omega),
\]

and in particular, \((\Gamma, \psi)\) and \((\Gamma^\omega, \psi^\omega)\) are \( \pi \)-cospectral.

**Proof.** Observe that (the extension to \( CG \) of) \( \pi \) is a homomorphism, so \( \pi(Q_\omega)\pi(A_{\Gamma, \psi})\pi(Q_\omega) = \pi(Q_\omega)A_{\Gamma, \psi}Q_\omega \). Also, by the argument similar to one presented in the proof of Theorem 6, the first \( n_0 \) rows and columns of matrix \( Q_\omega A_{\Gamma, \psi}Q_\omega \) relate to the adjacencies between vertices from \( C_0 \) and \( C_i \) for \( i \in \{0, 1, \ldots, 2k\} \) in \((\Gamma^\omega, \psi^\omega)\). Hence all that is left to prove is that the \( \pi \)-image of \((n-n_0) \times (n-n_0)\) principal submatrix of \( Q_\omega A_{\Gamma, \psi}Q_\omega \) obtained by removing first \( n_0 \) rows and columns is the same as the \( \pi \)-image of a similarly constructed submatrix of \( A_{\Gamma^\omega, \psi^\omega} \). We will do so by considering individual blocks of this matrix.

We fix a pair of odd (not necessarily distinct) integers \( i, j < 2k \) and consider pairs of subsets \( C_i \cup C_{i+1} \) and \( C_j \cup C_{j+1} \) with vertices \( v_1, v_2, \ldots, v_{2n_i} \) and \( w_1, w_2, \ldots, w_{2n_j} \), respectively. Let \( A' \) denote a \( 2n_i \times 2n_j \) principal submatrix of \( A_{\Gamma^\omega, \psi^\omega} \) obtained by taking rows corresponding to vertices \( v_1, \ldots, v_{2n_i} \) and columns corresponding to vertices \( w_1, w_2, \ldots, w_{2n_j} \). The labeling of the vertices is such that an element \( A'_{i,j} \) is \( \psi(v_i, w_j) \) if \( v_i \) and \( w_j \) are adjacent or \( A'_{i,j} = 0 \) otherwise. In this case, after the conjugation \( Q_\omega A_{\Gamma, \psi}Q_\omega \) this block takes the form \( Q_\omega A' Q_\omega \).

The matrix \( A' \) can be represented as a block matrix

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.
\]

Let \( S_1 \) be the sum of all elements in block \( A_{11} \) (same as the sum of elements in block \( A_{21} \)), and \( S_2 \) be the sum of all elements in block \( A_{12} \) (\( A_{21} \)). Let \( D \) be a matrix of four \( n_i \times n_j \) blocks

\[
D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},
\]

where

\[
D_{11} = \begin{pmatrix} \frac{S_1}{n_i} + \frac{S_1}{n_j} - \psi(v_1) - \psi(w_1) & \frac{S_1}{n_i} - \psi(w_1) & \cdots & \frac{S_1}{n_i} - \psi(w_{n_j}) \\ \frac{S_1}{n_i} - \psi(v_2) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{S_1}{n_i} - \psi(v_{n_i}) & \cdots & 0 & 0 \\ \frac{S_1}{n_j} - \psi(v_{n_i+1}) & \cdots & 0 & \cdots & 0 \\ \frac{S_2}{n_i} - \psi(v_{n_i+1}) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{S_2}{n_i} - \psi(v_{2n_i}) & \cdots & 0 & \cdots & 0 \\ \frac{S_2}{n_j} - \psi(v_{n_i+1}) & \cdots & 0 & \cdots & 0 \end{pmatrix},
\]

\[
D_{12} = \begin{pmatrix} \frac{S_1}{n_i} + \frac{S_1}{n_j} - \psi(v_{n_i+1}) - \psi(w_{n_i+2}) & \frac{S_1}{n_i} - \psi(w_{n_i+2}) & \cdots & \frac{S_1}{n_i} - \psi(w_{2n_j}) \\ \frac{S_2}{n_i} - \psi(v_{n_i+1}) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{S_2}{n_i} - \psi(v_{2n_i}) & \cdots & 0 & 0 \\ \frac{S_2}{n_j} - \psi(v_{n_i+1}) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{S_2}{n_j} - \psi(v_{2n_i}) & \cdots & 0 & \cdots & 0 \\ \frac{S_2}{n_j} - \psi(v_{n_i+1}) & \cdots & 0 & \cdots & 0 \end{pmatrix}.
\]
Observe that $A′ + D$ has a constant row and column sum of the same value, as well as its two principal submatrices obtained by removing first (last) $n_i$ rows and $n_j$ columns. Indeed, using the fact that $\Psi_{i,j+1}(v)$ is the sum of the row of $A'$ that corresponds to the vertex $v$ and $\Psi_{j,j+1}^*(w)$ is the sum of the respective column, it is easy to confirm that the sum of any row or column of $A′ + D$ is equal to $\frac{S_1}{n_i}$ or $\frac{S_2}{n_j}$ respectively, while the four blocks have row sum $\frac{S_3}{n_i}$ and column sum $\frac{S_4}{n_j}$ where $t \in \{1, 2\}$ depending on the block (analogous to the proof of [8, Theorem 4.4]). This means we can apply Lemma 5 to show that $Q_{n_i}(A′ + D)Q_{n_j} = A′ + D$.

In addition, we have $\pi(D) = 0$. This follows from the following equalities that hold for any $v \in C_i$, $v' \in C_{i+1}$, $w \in C_j$, and $w' \in C_{j+1}$ due to the definition of $\pi$-WQH partition:

$$\pi(S_1) = \pi(n_i\Psi_i(v)) = \pi(n_i\Psi_i^*(w)) = \pi(n_j\Psi_{i+1}(v')) = \pi(n_j\Psi_{i+1}^*(w')),$$

$$\pi(S_2) = \pi(n_i\Psi_{i+1}(v)) = \pi(n_j\Psi_{i+1}^*(w)) = \pi(n_i\Psi_i(v')) = \pi(n_j\Psi_{i+1}^*(w')).$$

From this it is easily derived that the $\pi$-image in any element of $D$ is 0.

By combining $Q_{n_i}(A′ + D)Q_{n_j} = A′ + D$ and $\pi(D) = 0$, we obtain

$$\pi(Q_{n_i}(A′ + D)Q_{n_j}) = \pi(Q_{n_j}(A′ + D) - D)Q_{n_j},$$

$$\pi(Q_{n_i}(A′ + D)Q_{n_j}) = \pi(Q_{n_i}(A′ + D)Q_{n_j}) - \pi(Q_{n_j}) \cdot 0 \cdot \pi(Q_{n_j}),$$

$$\pi(A′ + D) = \pi(A′) + 0 = \pi(A′). \ \square$$

**Example 12.** Consider the $S_4$-gain graph $(\Gamma, \psi)$ and a partition $\alpha$ from Example 10. It was already shown that $\alpha$ is a $\pi$-WQH partition where $\pi$ is a unitary representation that maps an element of $S_4$ to a respective permutation matrix of size 4. The graph $(\Gamma^\alpha, \psi^\alpha)$ (on the right in Fig. 2) is not switching isomorphic to $(\Gamma, \psi)$. Additionally, there is no partition that both satisfies the conditions for $\pi$-GM partition for some $\pi$ (defined in [8]) and produces a graph that is switching isomorphic to $(\Gamma^\alpha, \psi^\alpha)$.

5. Concluding remarks

The generalization of the spectral graph theory to gain graphs is far from trivial. Switching methods for gain graphs were previously described in the literature for some particular cases. In [3, Section 4], WQH-switching (referred to as Modified
Godsil–McKay switching) was generalized to signed graphs. In [4, Section 3], the same switching was described for complex unit gain graphs. In both cases, adaptations of WQH-switching were used to construct cospectral graphs with respect to the identical representation. For a general group \( G \subset \mathbb{C} \), an approach based on group representations was introduced in [7] in order to discuss the spectrum of a \( G \)-gain graph with respect to any representation \( \pi \). In [9], the notion of \( G \)-cospectrality independent of the choice of \( \pi \) was first introduced, and in [8], generalizations of GM-switching with respect to both \( G \)– and \( \pi \)-cospectrality were described. To the authors’ knowledge, only the work by Cavaleri, Donno and Spessato [8,9] contributed before to the investigation of the cospectrality in the context of gain graphs over a general group \( G \).

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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