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Travelling wave solutions for gravity fingering in porous media flows

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Abstract

We study an imbibition problem for porous media. When a wetted layer is above a dry medium, gravity leads to the propagation of the water downwards into the medium. In experiments, the occurrence of fingers was observed, a phenomenon that can be described with models that include hysteresis. In the present paper we describe a single finger in a moving frame and set up a free boundary problem to describe the shape and the motion of one finger that propagates with a constant speed. We show the existence of solutions to the travelling wave problem and investigate the system numerically.

MSC: 76S05; 35C07; 47J40

Keywords: Porous media; Travelling waves; Hysteresis

1. Introduction

Standard models for flow in unsaturated porous media fail in the description of a fundamental process, namely the imbibition into a dry medium with gravity as the driving force. While

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standard Richards models predict the formation of uniform imbibition fronts, the experimentally observed fingers [11,27] can only be described with a model that incorporates hysteresis.

Models for incompressible unsaturated porous media flow typically use the water pressure p and the water saturation s as primary variables. The Darcy law for the velocity together with the mass balance equation leads to

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + g e_z]) \,, \tag{1.1a}$$

we refer to [2,13,23,26] for the modelling. In the Richards equation (1.1a), the function $k : [0,1] \to \mathbb{R}$ is the permeability function which has to be determined from experiments, g is the gravitational acceleration, e_z is the normal vector pointing upwards. It is always assumed that s takes only values in $[0,1]$.

Equation (1.1a) must be accompanied by a relation between saturation s and pressure p. Models without hysteresis demand either the algebraic relation $p = p_c(s)$ for some given function $p_c : [0,1] \to \bar{\mathbb{R}}$, or they include the "$\tau$-correction" and demand, for some physical parameter $\tau > 0$, known as the dynamic capillary number, that $p = p_c(s) + \tau \partial_t s$; this latter model takes inertia in the material law into account, see [12]. If, additionally, hysteresis in an imbibition process shall be modelled, a possible simple law is

$$\partial_t s = \frac{1}{\tau}[p - p_c(s)]_+ \,, \tag{1.1b}$$

where $[\cdot]_+ := \max\{0, \cdot\}$ denotes the positive part. Our aim is a travelling wave analysis of equation (1.1). We recall that $p_c : (0,1) \to \mathbb{R}$ is a given imbibition capillary pressure function and $\tau > 0$ is a given constant.

Regarding the modelling we note that, if both imbibition and drainage should be modelled, one replaces (1.1b) by the model of [4],

$$\partial_t s = \frac{1}{\tau}[p - p_c(s)]_+ + \frac{1}{\tau}[p - p_d(s)]_- \,. \tag{1.2}$$

Here, $p_d : (0,1) \to \mathbb{R}$ is a drainage capillary pressure function with $p_d(s) \le p_c(s)$ for all $s \in (0,1)$, and $[\cdot]_- := \min\{0, \cdot\}$ is the negative part function. Equation (1.2) is a hysteresis model since, pointwise in space and time, all pressure values in the closed interval $[p_d(s), p_c(s)]$ are permitted for a fixed saturation s. The play-type hysteresis model with dynamic capillary pressure was analyzed in many works, we give an overview over the existing literature in the last part of the introduction. Since we are interested in an infiltration problem with $\partial_t s \ge 0$, we restrict ourselves to the case $p_d(s) = -\infty$ as in [9], i.e., we study (1.1b) instead of (1.2).

Numerical results for the time dependent system (1.1a)–(1.1b) are shown in Fig. 1, originally published in [15,22]. The figure illustrates a gravity driven imbibition process into an originally dry medium. Several fingers evolve in the process. It is observed that each finger travels approximately with constant speed. This has also been verified experimentally [27]. The present work aims at the description of a single finger in a co-moving frame of coordinates.

Travelling wave ansatz, domains and boundary conditions. Since we are interested in imbibition fronts in columns of porous media, we choose a cylindrical spatial domain Ω_∞. Restricting to two dimensions for convenience and denoting the width of the cylinder by $L > 0$, we consider

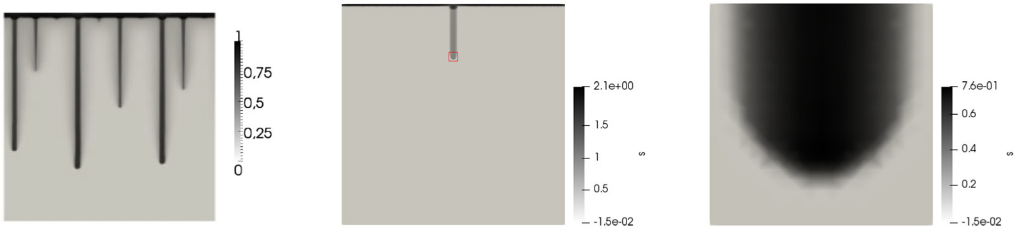

Fig. 1. Motivation for this contribution. Left: A snapshot of a solution to the time dependent system (1.1). Fingers are clearly visible; the solution is comparable to experimental observations [22]. Middle: With another choice of boundary values and p_c, a single finger is generated. A small squared region of size 2×2 around the finger-tip is marked [15]. Right: Enlargement of the marked region. We see the typical shape of the single finger in time-dependent calculations. The aim of this contribution is to analyze the travelling wave equations corresponding to (1.1) in order to obtain the shape of the single finger without a time-dependent calculation.

$\Omega_\infty := (0, L) \times \mathbb{R} \subset \mathbb{R}^2$. Points in $\mathbb{R}^2$ are denoted as $x = (y, z)$. We seek time-dependent solutions to (1.1) that move with a constant speed $c > 0$ in negative z-direction, i.e., downwards. This motivates the travelling wave coordinates

$$\tilde{z} = z + ct\,, \quad p(y, z, t) = p(y, \tilde{z})\,, \quad s(y, z, t) = s(y, \tilde{z})\,. \tag{1.3}$$

In the following, we omit the tilde symbol and write z instead of $\tilde{z}$. The new coordinates transform system (1.1) into

$$c\partial_z s = \nabla \cdot (k(s)[\nabla p + g e_z])\,, \tag{1.4a}$$

$$c\tau\partial_z s = [p - p_c(s)]_+\,. \tag{1.4b}$$

Even though the physical interpretation of a travelling wave solution requires the study of domains $\Omega_\infty = (0, L) \times \mathbb{R}$ that extend to $z \to \pm\infty$, we choose here to study problem (1.4) on the semi-infinite domain

$$\Omega := (0, L) \times \mathbb{R}_+ \text{ with bottom boundary } \Sigma := (0, L) \times \{0\} = \{(y, 0) : 0 < y < L\}\,.$$

Truncations of the domain are a common approach for numerical calculations and they facilitate the analysis. The problem is translation invariant; one should consider the bottom $\Sigma = \{z = 0\}$ as being far below the finger.

The boundary data are given by a prescribed saturation $s_0 > 0$ and a prescribed pressure p_0 at the bottom Σ of the domain, and by a prescribed total influx F_∞ on the top of the domain. More precisely, we assume that we are given $s_0 : [0, L] \to [0, 1]$, $p_0 : [0, L] \to \mathbb{R}$, and $F_\infty \in \mathbb{R}_+ = (0, \infty)$, and impose the boundary conditions

$$\int_0^L k(s(y, z))[\partial_z p(y, z) + g]\,dy \to F_\infty \qquad \text{as } z \to +\infty\,, \tag{1.5a}$$

$$s = s_0 \qquad \text{at } z = 0\,, \tag{1.5b}$$

$$p = p_0 \qquad \text{at } z = 0\,, \tag{1.5c}$$

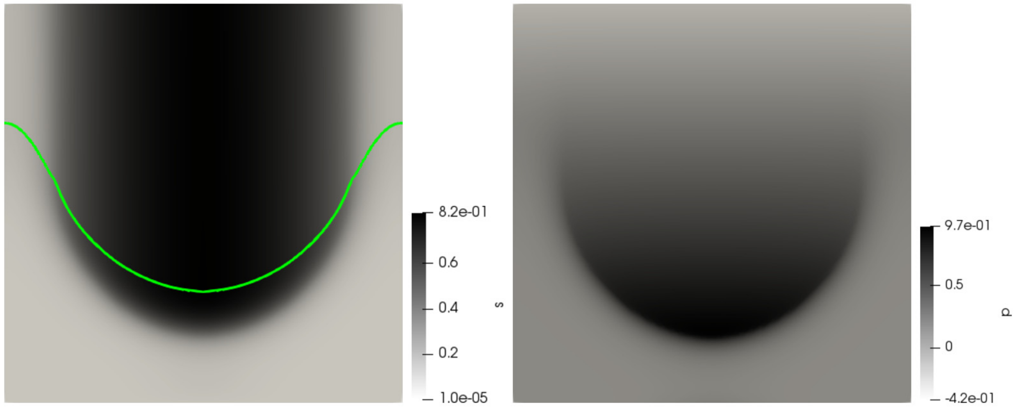

Fig. 2. A numerical solution of the free boundary travelling wave problem. The grey scale indicates the values of the saturation s (left) and the pressure p (right). The level line $\Gamma = \{x \mid p = p_c(s)\}$ is marked in the left image. The line Γ shows the free boundary: Below the line, the saturation is increasing, above the line, the saturation remains constant (increasing in vertical direction and, hence, increasing in time when interpreted as a time dependent solution).

and, along the lateral boundaries of Ω, a no-flux boundary condition: $\partial_y p(y, z) = 0$ for $y = 0$ and $y = L$, and arbitrary $z > 0$.

Before we start the analysis of (1.4)–(1.5), let us make one more comment regarding the relation to the original time dependent system (1.1). Problem (1.1) requires an initial condition for the saturation. In the simplest case, the initial saturation of the medium is constant and given by a number $s_* \in (0, 1)$; we would then demand $s \equiv s_*$ as the initial condition for (1.1). Our analysis suggests that, with some inflow, after a long time, fingers as in Fig. 1 appear. The shape of the fingers is determined by (1.4)–(1.5), the boundary data in (1.5) should then be chosen as $s_0 \equiv s_*$ and $p_0 \equiv p_c(s_*)$. This models that an imbibition process takes place at the lower boundary.

Main results. We perform an analysis of the travelling wave problem (1.4)–(1.5) on Ω. For the most part of this article, we prescribe the relaxation parameter τ, the frame speed c, and the boundary data s_0, p_0, and F_∞. Only in our last result, Theorem 4.7, we choose c in dependence of the other parameters in order to satisfy a physically adequate flux condition on the lower boundary, compare Remark 3.3.

The first part of our results concerns the system (1.4)–(1.5) on the bounded truncated domain $\Omega^H = (0, L) \times (0, H)$. We choose boundary conditions on the upper boundary appropriately and show that the system has a solution. The solution can be found with a variational principle, the analysis is given in Section 3.

The numerical part of this paper deals with this truncated problem. One result is the calculation of a finger solution, see Fig. 2. The numerical method and the results are described in Section 5.

The limit $H \to \infty$ for the solutions on the bounded domain is studied in Section 4. We find that every sequence of solutions (s_H, p_H) to truncated domain problems possesses a subsequence and a limit (s, p) which is a solution of the original problem (1.4). The limit process shows an interesting dichotomy: In one case, the flux boundary condition for $z \to \infty$ as in (1.5) remains satisfied (“large solution”). In the other case (“small solution”), only a corresponding inequality is satisfied.

The two cases are analyzed further. We find that “large solutions” are of the type that we would like to see in the fingering process: they possess a free boundary, the pressure p tends to $-\infty$ as $z \to \infty$, and the solution is “large” in the sense that the saturation exceeds a certain threshold. In the second case, the properties are reverted: The solution has a bounded pressure and it is “small” in the same sense as the solution was “large” in the other case. Interestingly, both types of solutions are found numerically, see Section 5.

Free boundary problem. Let us emphasize that we treat a free boundary problem. By (1.4b), one has to distinguish between the subdomain $\{x \in \Omega \mid \partial_z s(x) > 0\}$ (expected to be in the bottom) and the subdomain $\{x \in \Omega \mid \partial_z s(x) = 0\}$ (expected in the top part). In physical terms, this means that an imbibition process occurs near and below the finger-tip, whereas, in the region around the developed finger, the saturation does not change any more. With reference to the hysteresis relation, we note that the z-independent saturation implies that the pressure can take arbitrary values (below $\min p_c(s)$). Therefore, the pressure profile does not have to reflect the saturation profile and the fingers can remain stable in their upper part; no blurring by pressure differences occurs.

With Theorem 4.7 we provide the result that, for every F_∞ within appropriate bounds, there exists a wave speed c such that a physical flux condition at the lower boundary is satisfied.

Literature. The classical porous media equation is obtained by setting $\tau = 0$ and by replacing (1.1b) by the algebraic law $p = p_c(s)$, the resulting system is the Richards equation. This equation is widely used in the modelling of porous media flow. The Richards equation is analytically challenging when a degenerate permeability coefficient is considered, i.e., $k(0) = 0$. This degenerate behaviour is relevant in applications. Existence and uniqueness for this classical model is shown in [1], the uniqueness is made more general in [21]. The hysteresis model (1.1b) was introduced in [3,4,12]. It combines dynamic effects ($\tau > 0$) with a play-type hysteresis relation; the latter allows for an interval of pressure values p for a fixed saturation s. This is a desired feature in the modelling of porous media and important for the generation of fingers. For a review of the modelling, we refer to [26].

For the model (1.1), well-posedness results have been obtained in one space dimension in [4], and in higher dimension in [15]. Existence results were also derived in [22], but that paper, additionally, made the following connection clear: The effect of $\tau > 0$ is the generation of saturation overshoots, and a saturation overshoot in a travelling wave means that the system experiences a drainage process – it is this drainage process that makes the static hysteresis relevant and results in the possibility to create fingers in higher space dimension.

In [24], it was shown that the model does not define an L^1-contraction; in this sense, it can explain the fingering effect. The fingers were found numerically for unsaturated media in [15], for the two-phase flow in [14]. Other mechanisms for the creation of Fingers were also studied: Fingers were observed numerically in [5,7], where a free-energy based approach is used for modelling the capillary pressure. For a result with a degenerate p_c-curve, see [25]. Existence of solutions for an extension of the play-type model was shown in [17]. A uniqueness result was derived in [6].

Travelling waves have been investigated in several one-dimensional settings. A two-phase flow is studied in [18], the work includes static hysteresis and the τ-term (dynamic capillarity). The travelling wave form of the system is derived and analyzed as a Riemann problem, the authors construct entropy solutions. One of the most striking results regards the development of multiple plateaus in travelling wave solutions. In the older contribution [28], the different non-

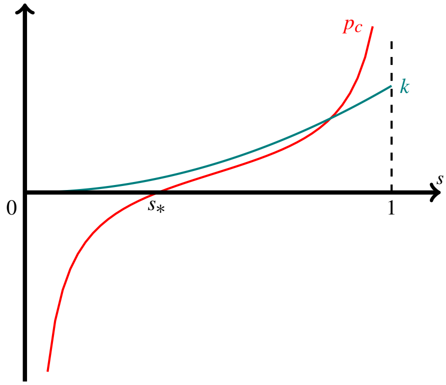

Fig. 3. Typical functions p_c and k.

equilibrium effects are studied separately and, also here, non-monotonic profiles are found. The combination of both effects was first studied in [20].

Another one-dimensional analysis was performed for pure imbibition (again in one space dimension) in [9]. The requirement of pure imbibition, $\partial_t s \geq 0$, allows to set $p_d(s) = -\infty$. This simplifies the analysis. The result is a very general statement about the existence of travelling waves whenever τ exceeds a critical value.

The present work can be regarded as an extension of the existence result of [9] to two space dimensions. Let us note that the methods are independent of the dimension and that, up to notation, the results remain valid, e.g., in three space dimensions. The dimension enters only in Sobolev embeddings that are used for regularity statements in the appendix.

2. Preliminaries

The coefficient functions k and p_c are fixed throughout this work. We make assumptions that are quite common and consistent with experiments, see [13]. For an illustration see Fig. 3.

Assumption 2.1. The functions $k : [0, 1] \to [0, \infty)$ and $p_c : (0, 1) \to \mathbb{R}$ satisfy:

(Ass-pc) The function p_c is differentiable and for some $\rho > 0$ holds $p_c' \geq \rho$ on $(0, 1)$. Upon normalization of the pressure, we can set $p_c(s_*) = 0$ for a given saturation value $s_* \in \mathbb{R}$. We assume $p_c(s) \to -\infty$ as $s \searrow 0$ and $p_c(s) \to \infty$ as $s \nearrow 1$.

(Ass-k) The function k is differentiable, $k|_{(0,1)} \in C^2$, and $k'(.), k''(.) > 0$ on $(0, 1)$.

We emphasize that typical concrete models for unsaturated flow in porous media satisfy the above assumptions.

The free boundary description. What qualitative behaviour can we expect for solutions of the travelling wave problem (1.4)–(1.5)? We expect that the pressure stabilizes, as $z \to +\infty$, to an affine function with $\nabla p \approx -g_F e_z$. If s (and hence $k(s)$) does not depend on z, then both sides of (1.4a) can vanish. This is what we expect for solutions in the upper part of the domain. We will be interested in solutions (p, s) that satisfy, for some $h \in \mathbb{R}_+$,

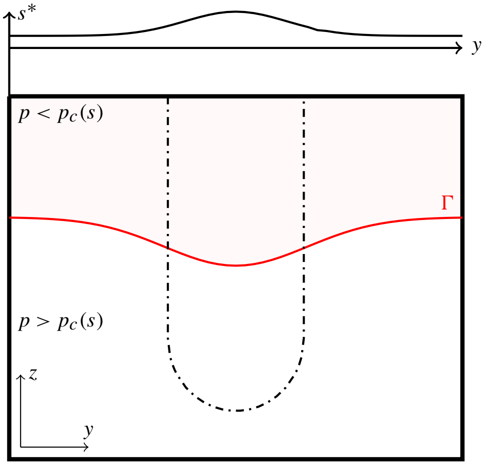

Fig. 4. When interpreted as a solution of the time-dependent problem, the finger moves with a constant speed downwards. The dashed line represents the boundary of the finger; one may think of an isoline of the saturation. The graph at the top part of the Figure indicates a profile of the limiting saturation s^* as defined in (2.3).

$$\partial_z s = 0 \text{ and } p \leq p_c(s) \text{ for all } (y, z) \text{ with } y \in (0, L) \text{ and } z > h\,. \tag{2.1}$$

For such a solution we can define a function $\Psi : [0, L] \to [0, \infty)$ as

$$\Psi(y) := \inf\{z_0 > 0 |\, \partial_z s(y, z) = 0 \text{ for all } z \geq z_0\}\,. \tag{2.2}$$

The graph of Ψ is a part of the free-boundary, $\{(y, \Psi(y))|\ y \in (0, L)\} \subset \Gamma$. For the rest of the paper, we define the function $s^* : [0, L] \to [0, 1]$ as

$$s^*(y) := \lim_{z\to\infty} s(y, z). \tag{2.3}$$

By positivity $\partial_z s(y, z) \geq 0$ and boundedness of s, the function s^* is well-defined for solutions (s, p) of (1.4). When a solution satisfies (2.1), there holds $s(y, z) = s^*(y)$ for all $z > h$.

We refer to Fig. 4 for an illustration. It is important not to confuse the free boundary Γ with the shape of the finger (the region of high saturation). We emphasize that the saturation profile remains unchanged (independent of z) above Γ; in particular, the finger extends to $z \to +\infty$.

Relations in the travelling wave formulation. A fundamental problem in travelling wave analysis is the determination of free parameters, in our case the wave speed c. The other parameters are fixed: $\tau, g > 0$ are physical constants, $L > 0$ a geometrical constant, and the boundary conditions fix $F_\infty > 0$ and $s_* > 0$. In the travelling wave formulation, $c \geq 0$ is a further unknown of the system. Nevertheless, for the most part of our analysis, we fix boundary values s_0 and p_0 and treat the problem with prescribed c. Only in our final result we determine c from an additional boundary condition for $z \to -\infty$.

Let us collect some properties of the real parameters.

Lemma 2.2 *(Wave speed and limiting pressure in the doubly infinite domain). Let* $(s, p) \in C^1(\Omega_\infty) \times C^2(\Omega_\infty)$ *be a classical solution to* (1.4) *on* Ω_∞ *with the boundary condition* (1.5a) *and the two conditions* $s \to s_*$ *and* $k(s)\nabla p \to 0$ *as* $z \to -\infty$*. Then, with* s^* *as in* (2.3)*, the wave speed satisfies*

$$c = (F_\infty - k(s_*)gL) \Bigg/ \left(\int_0^L (s^*(y) - s_*)\, dy \right) . \tag{2.4}$$

If the solution possesses a free boundary, i.e. (2.1) *holds for some* $h > 0$*, then*

$$g_F := g - \left(F_\infty \Bigg/ \int_0^L k(s^*(y))\, dy \right) \tag{2.5}$$

satisfies $g_F > 0$ *and there holds* $\nabla p(y,z) + g_F e_z \to 0$ *as* $z \to \infty$ *for every* $y \in (0, L)$*.*

Proof. Integrating (1.4a) over $(0, L) \times (-H, H)$ yields

$$c \int_0^L s(y,z)\, dy \Bigg|_{z=-H}^{H} = \int_0^L k(s(y,z))[\partial_z p(y,z) + g]\, dy \Bigg|_{z=-H}^{H} .$$

Sending $H \to \infty$ provides (2.4).

Relation (2.1) implies that $s(y,z) = s^*(y)$ holds for $z > h$. Therefore, the elliptic equation reduces to

$$\nabla \cdot (k(s^*)\nabla p) = 0 \text{ in } (0, L) \times (h, \infty) . \tag{2.6}$$

In particular, the flux quantity $\int_0^L k(s^*)\, \partial_z p(y,z)\, dy$ is independent of z for $z > h$. The boundary condition (1.5a) allows to evaluate this flux for $z \to \infty$; we find

$$\int_0^L k(s^*(y))\partial_z p(y,z)\, dy = F_\infty - g \int_0^L k(s^*(y))\, dy = -g_F \int_0^L k(s^*(y))\, dy . \tag{2.7}$$

This provides that, for $z > h$, the weighted average of $\partial_z p$ coincides with $-g_F$.

Solutions p of the elliptic equation (2.6) with homogeneous Neumann boundary conditions on unbounded domains have the property that ∇p stabilizes to a constant as $z \to \infty$ (a consequence of the strong maximum principle for $\partial_z p$). Relation (2.7) shows that this constant is $-g_F e_z$.

Let us assume for a contradiction $g_F < 0$. Then p is a growing function for $z \to \infty$. This is in contradiction with (1.4b), in which the left hand side vanishes for $z > h$ and $p_c(s)$ is independent of z for $z > h$.

Let us now assume $g_F = 0$ in order to exclude also this case. We use a maximum principle for p in the interior of the set $\{(y,z) | \partial_z s = 0\} = \{(y,z) | p \le p_c(s)\}$. The minimum of p is attained at the boundary. At the lower boundary of this set, there holds $p = p_c(s)$. This implies that the minimum is attained in a point of the form $(y,z) = (y, \Psi(y))$. We now use, for any $\varepsilon > 0$, the strong maximum principle: $p(y, \Psi(y) + \varepsilon) > p(y, \Psi(y)) = p_c(s(y, \Psi(y))) = p_c(s(y, \Psi(y) +$

ε)). This implies $p > p_c(s)$ in $(y, \Psi(y) + \varepsilon)$ and hence $\partial_z s(y, \Psi(y) + \varepsilon) > 0$, in contradiction to the construction of Ψ. $\square$

Notation. Let us collect the most relevant notations for this work. The unbounded domain is $\Omega = (0, L) \times \mathbb{R}_+$ with bottom boundary $\Sigma = (0, L) \times \{0\}$. For any height $H > 0$, the bounded domain is $\Omega^H = (0, L) \times (0, H)$, the bottom boundary is again $\Sigma = (0, L) \times \{0\}$, the top boundary is $\Sigma^H = (0, L) \times \{H\}$. We always impose homogeneous Neumann conditions at the lateral boundaries $\{0\} \times \mathbb{R}_+$ and $\{L\} \times \mathbb{R}_+$ of Ω, and at the lateral boundaries $\{0\} \times (0, H)$ and $\{L\} \times (0, H)$ of Ω^H.

The function sign : $\mathbb{R} \to \{0, 1\}$ is defined as $\operatorname{sign}(u) := 0$ for $u \le 0$, and $\operatorname{sign}(u) := 1$ otherwise. The positive part of a number or a function is defined as $[q]_+ = \max\{0, q\} = (q + |q|)/2$. The negative part of a number or a function is defined as $[q]_- = \min\{0, q\} = -[-q]_+$. The letter C denotes a generic positive constant and the value may change from one line to the next in calculations.

3. Existence result for bounded domains

Let $\tau > 0$, $s_* \in (0, 1)$, and two functions $p_0 \in H^{\frac{1}{2}}(\Sigma) \cap C^0(\bar{\Sigma})$ and $s_0 \in H^1(\Sigma)$ be given. We assume $s_* \le s_0 \le 1$ and $p_0 \ge p_c(s_0)$. For a height parameter $H > 0$ we introduce the following truncated problem.

Definition 3.1 *(Truncated domain travelling wave problem).* Let $c, F_\infty > 0$ be given. A pair $(s, p) \in H^1(\Omega^H) \times H^2(\Omega^H)$ on the domain $\Omega^H = (0, L) \times (0, H)$ with upper boundary Σ^H and lower boundary Σ is a truncated domain travelling wave solution (TW_H-solution) if there holds

$$c\partial_z s = \nabla \cdot (k(s)[\nabla p + g e_z]) \qquad \text{in } \Omega^H\,, \tag{3.1a}$$

$$c\tau \partial_z s = [p - p_c(s)]_+ \qquad \text{in } \Omega^H\,, \tag{3.1b}$$

$$s = s_0\,,\ \ p = p_0 \qquad \text{on } \Sigma\,, \tag{3.1c}$$

$$p \equiv p^* \in \mathbb{R} \qquad \text{on } \Sigma^H\,, \tag{3.1d}$$

$$\int\limits_{\Sigma^H} k(s)[\partial_z p + g] = F_\infty\,. \tag{3.1e}$$

We emphasize that the constant pressure value $p^* \in \mathbb{R}$ is a free parameter and part of the solution of the problem.

We note that for every TW_H-solution (s, p), the flux quantity

$$F_c(z) := \int\limits_0^L k(s(y, z))[\partial_z p(y, z) + g] - cs(y, z)\, dy \tag{3.2}$$

is independent of $z \in (0, H)$ by (3.1a). Evaluating this flux in the upper and in the lower boundary provides, by (3.1e),

$$\int_{\Sigma} k(s_0)\partial_z p + \int_{\Sigma} (k(s_0)g - cs_0) = F_\infty - \int_{\Sigma^H} cs\,. \tag{3.3}$$

Remark 3.2. Let us give a sloppy description of the consequences of (3.3) for small boundary data s_0. There is the possibility that $\partial_z p$ is large at Σ. This means that a sharp transition occurs near the lower boundary. In the opposite case (without boundary layer), the left hand side of (3.3) is small. In this case, a moderate flux $F_\infty > 0$ forces the system that s is not small at Σ^H. This is the desired behaviour for finger-like travelling wave solutions; they should connect a small saturation at $z = 0$ with a moderate or large saturation at $z = H$.

Remark 3.3 *(A condition for the wave speed c).* Let us highlight another consequence of the fact that F_c of (3.2) is independent of z. When (s, p) is a solution on the doubly unbounded domain Ω_∞ then we expect, in the limit $z \to -\infty$, that $s \to s_*$, $p \to p_c(s_*)$, and $\partial_z p \to 0$. In this situation, the constant flux quantity is necessarily $F_c = (gk(s_*) - cs_*)L$.

We use this observation in order to choose a closure condition for the case when the speed c is treated as an unknown: Even when we solve a Dirichlet problem in the truncated domain Ω with boundary conditions s_0 and p_0 at the lower boundary Σ, we will seek for c and solutions to the Dirichlet problem that satisfy the additional relation

$$F_c = \int_{\Sigma} (k(s_0)[\partial_z p + g] - cs_0) = (gk(s_*) - cs_*)L\,. \tag{3.4}$$

Theorem 4.7 yields that, given s_0, p_0, s_*, and F_∞, we find a speed c such that (3.4) is satisfied.

In the remainder of this section, we seek for TW_H-solutions (s, p). We use the space of functions

$$H^1_\sharp(\Omega^H) := \left\{ u \in W^{1,2}(\Omega^H) \,\middle|\, \mathrm{tr}(u) = 0 \text{ on } \Sigma\,,\ \exists u^* \in \mathbb{R} :\ u = u^* \text{ on } \Sigma^H \right\}. \tag{3.5}$$

The weak formulation of (3.1a) and (3.1e) is:

$$\int_{\Omega^H} c\partial_z s\,\phi + \int_{\Omega^H} k(s)[\nabla p + ge_z]\cdot\nabla\phi = \int_{\Sigma^H} F_\infty\,\phi \quad \text{for all } \phi \in H^1_\sharp(\Omega^H)\,. \tag{3.6}$$

Theorem 3.4 *(Existence of TW_H-solutions to prescribed data). Let $H, c, \tau, F_\infty > 0$ and $s_* \in (0, 1)$ be given, let $p_0 \in H^{\frac{1}{2}}(\Sigma) \cap C^0(\bar{\Sigma})$ and $s_0 \in H^1(\Sigma)$ satisfy*

$$s_* \le s_0 < 1\,, \quad \textit{and} \quad 0 < p_0 - p_c(s_0) \textit{ on } \Sigma\,.$$

Then there exists a TW_H-solution (s, p) with $s, \partial_z s \in L^2(\Omega^H)$, $p \in H^1(\Omega^H) \cap H^2_{\mathrm{loc}}(\Omega^H)$.

The assumptions $s_* > 0$ and $s_0 < 1$ are critical for the proof of the theorem. The assumptions are used to avoid the end-points of the saturation interval [0, 1] at all times in the construction. This is necessary since p_c is degenerate at these end-points.

Proof. We use an iteration over saturation fields.

Definition of the iteration. Let there be given a saturation field

$$s^{i-1} \in Y := \left\{ s \in L^2(\Omega^H) \,|\, s_* \le s \le 1 \right\} .$$

We define the coefficient functions $a := k(s^{i-1})$ and $b := p_c(s^{i-1})$ on Ω^H. We seek a solution p of

$$\frac{1}{\tau}[p-b]_+ = \nabla \cdot (a\,[\nabla p + g e_z]) \text{ in } \Omega^H \,, \tag{3.7}$$

with the boundary conditions $p = p_0$ on Σ and (3.1d)–(3.1e). This solution can be found with a variational method. We define the space of admissible functions as $X_{p_0} := \left\{ u \in H^1(\Omega^H) \,|\, u = p_0 \text{ on } \Sigma\,,\ \exists u^* \in \mathbb{R} :\ u = u^* \text{ on } \Sigma^H \right\}$ and minimize the functional

$$A : X_{p_0} \to \mathbb{R}\,, \quad A(p) := \int_{\Omega^H} \frac{1}{2\tau}[p-b]_+^2 + \frac{1}{2} a\,|\nabla p + g e_z|^2 - F_\infty \int_{\Sigma^H} p\,. \tag{3.8}$$

The functional is convex and coercive, which implies that a minimizer p exists. The Euler-Lagrange equation for p reads

$$\int_{\Omega^H} \frac{1}{\tau}[p-b]_+\,\varphi + a\,[\nabla p + g e_z] \cdot \nabla\varphi = F_\infty \int_{\Sigma^H} \varphi \qquad \forall \varphi \in H^1_\sharp(\Omega^H)\,.$$

Since arbitrary compactly supported test-functions φ can be inserted, equation (3.7) holds for p. The Euler-Lagrange equation additionally encodes the boundary condition $\int_{\Sigma^H} a\,(\partial_z p + g) = F_\infty$. Given $p^i = p$, we can solve the family of ordinary differential equations

$$c\tau\,\partial_z s = [p^i - p_c(s)]_+\,, \tag{3.9}$$

with initial data $s = s_0$ on $z = 0$; this system is related to (3.1b) together with the first equation in (3.1c). We denote the solution of this system by $s =: s^i$.

Fixed point of the iteration. We claim that, for some constant $C = C(H, c, \tau)$ independent of s^{i-1}, the pressure $p = p^i$ satisfies

$$\|p\|^2_{L^2(\Omega^H)} + \|\nabla p\|^2_{L^2(\Omega^H)} \le C\,. \tag{3.10}$$

In order to show this estimate, we first choose an H^1-extension $\hat{p}_0$ of the data p_0, vanishing at the upper boundary. We can now multiply equation (3.7) with $p - \hat{p}_0$ and integrate to obtain

$$\int_{\Omega^H} \frac{1}{\tau}[p-b]_+ \big([p-b] - \hat{p}_0 + b\big) + \int_{\Omega^H} (a\,[\nabla p + g e_z]) \cdot \nabla(p - \hat{p}_0) = \int_{\Sigma^H} F_\infty\, p\,.$$

One of the integrals on the left hand side is an upper bound for $k(s_*)\|\nabla p\|^2_{L^2(\Omega^H)}$, the other term with quadratic growth in p is on the left hand side and positive because of $[p-b]_+[p-b] \ge$

0. The remaining terms have linear growth in p and can therefore be estimated with Youngs inequality and with the Poincaré inequality.

The corresponding solutions $s^i = s$ of the ordinary differential equation satisfy $0 \le s \le 1$ by the growth assumption on p_c. In particular, there holds $s^i \in Y$. With $R := |\Omega^H|^{1/2} = |LH|^{1/2}$, we find that the above construction provides a map

$$\mathcal{T} : Y \supset B_R(0) \to B_R(0) \subset Y\,, \qquad s^{i-1} \mapsto s^i\,.$$

We claim that the map $\mathcal{T}$ is compact. We will show the compactness below with the characterization of compact subsets of $L^2(\Omega^H)$ by Kolmogorov-Riesz. An application of Schauder's fixed point theorem yields the existence of the desired solution s.

Let us turn to compactness of $\mathcal{T}$. We consider the family $p = p^i$ of solutions for $s = s^{i-1} \in B_R(0)$. This family of solutions is bounded in $H^1(\Omega^H)$, hence the finite differences $p(y,.) - p(y+\delta,.) \in L^2((0,H);\mathbb{R})$ are small for $\delta > 0$ small, independent of s. More precisely,

$$\int_0^{L-\delta}\int_0^H |p(y,z) - p(y+\delta,z)|^2\,dz\,dy \le \eta(\delta)\,,$$

with $\eta(\delta) \to 0$ as $\delta \to 0$, independent of s. We now consider two solutions of the ordinary differential equation (3.9), $s(y,.)$ and $s(y+h,.)$ to inputs $p(y,.)$ and $p(y+h,.)$. The solutions differ only as much as their right hand sides and their initial values differ. Because of our assumption $s_0 \in H^1(\Sigma)$, we therefore find also for the solutions

$$\int_0^{L-\delta}\int_0^H |s(y,z) - s(y+\delta,z)|^2\,dz\,dy \le C\eta(\delta)\,.$$

On the other hand, since $\partial_z s$ is bounded in $L^2(\Omega^H)$, the corresponding estimate $\int_0^L \int_0^{H-\delta} |s(y, z+\delta) - s(y,z)|^2\,dz\,dy \le C\eta(\delta)$ is clear. This shows compactness of the image set of s-fields. □

4. Unbounded domain solutions for $H \to \infty$

In this section we analyze the solutions (s_H, p_H) in the limit $H \to 0$. Again, for the larger part of this section, we keep $\tau > 0$, $s_* \in (0,1)$, F_∞, and $c > 0$ fixed; only in Theorem 4.7 we determine c from the other parameters. The main result of this section is the following: Let (s_H, p_H) denote the TW_H-solution as discussed in Theorem 3.4. Then, for $H \to \infty$, there holds $(s_H, p_H) \to (s, p)$ in an appropriate sense for some limit pair (s, p), which is defined on the unbounded domain Ω. The pair (s, p) is a travelling wave solution for the semi-infinite domain Ω.

It turns out that two different limiting solution types are possible. Type I is the "large solution". It is characterized by the following properties: 1) The solution is large in the sense that $\int_0^L gk(s(y,z_0))\,dy \ge F_\infty$ for some z_0. This means that a certain F_∞-dependent threshold is exceeded by the saturation variable. 2) The solution has a free boundary: For some $h > 0$ there holds $\partial_z s(y,z) = 0$ for every $z \ge h$. 3) The solution has an unbounded pressure, $p \to -\infty$ as $z \to \infty$.

Accordingly, Type II solutions are the “small solutions”. They have a bounded pressure and no free boundary.

To proceed with the analysis, we consider different assumptions.

Assumption 4.1. The following properties can be considered for the solution sequence (s_H, p_H) of (3.1), obtained in Theorem 3.4.

Bounds for parameters The limiting saturation $s^* \in (0,1)$, the wave speed c, and the flux F_∞ satisfy

$$gk'(s_*) < c < g(k(1) - k(s_*))/(1 - s_*)\,, \tag{4.1a}$$

$$gL[k(s_*) + k'(s_*)(1 - s_*)] < F_\infty < gLk(1)\,. \tag{4.1b}$$

Bound for the pressure For a real number $\bar{p} < \infty$ independent of H holds

$$p_H \le \bar{p} \quad \text{in } \Omega^H\,. \tag{4.2}$$

Local bound for the gradient There exists $C_P > 0$ such that, for every $H > 0$,

$$\|\nabla p_H\|_{L^\infty(\Omega^H)} \le C_P\,. \tag{4.3}$$

Regularity The saturation has the regularity properties

$$s_H,\ \partial_z s_H \in H^1(\Omega^H)\,. \tag{4.4}$$

The assumptions have a quite different character. Inequalities (4.1) are ranges for the physical parameters; we expect the existence of travelling waves in this parameter regime. The uniform upper bound of (4.2) is expected to hold, but it should be derived from the system of equations, which we did not succeed to do. The regularity estimate (4.3) and the local regularity (4.4) can be shown with the tools of elliptic regularity theory, see [10]. We formulate them here as assumptions, since the regularity theory is not the focus of this contribution.

We note that the relations (4.2)–(4.3) imply three further estimates:

$$\|s_H\|_{L^\infty(\Omega^H)} \le \bar{s} := {p_c}^{-1}(\bar{p}) < 1\,. \tag{4.5a}$$

In Lemma A.3 we prove that, for a constant $C_s = C_s(C_P, s_0, p_0)$,

$$\|\nabla s_H\|_{L^\infty(\Omega^H)} \le C_s\,. \tag{4.5b}$$

Since (4.5b) provides $\|\partial_z s_H\|_{L^\infty(\Omega^H)} < C_s$, one also has from (3.1b) that

$$p_H \le p_c(s_H) + c\tau C_s \quad \text{in } \Omega^H\,. \tag{4.5c}$$

Our main result on unbounded domains is the following.

Theorem 4.2 *(Limits of TW_H-solutions). Let $c, F_\infty, \tau > 0$, $s_* \in (0,1)$, and boundary data s_0, $p_0 \in C^1(\Sigma)$ with $s_* \leq s_0 < 1$ and $p_c(s_0) < p_0$ be given. Let all the properties of Assumption 4.1 be satisfied. For a sequence $H \to \infty$, let (s_H, p_H) be solutions to (3.1). Then, for a limiting pair (s,p), there holds $(s_H, p_H) \to (s,p)$ locally in $L^2(\Omega)$. The limits satisfy $s \in C_b^0(\Omega)$, $\partial_z s \in L^2(\Omega)$, $p \in H^2_{\mathrm{loc}}(\Omega) \cap H^1_{\mathrm{loc}}(\Omega \cup \Sigma)$, $(s,p) = (s_0, p_0)$ on Σ, and (1.4). The solution (s,p) is either of Type I or of Type II:*

- **Type I: "Large solution"** *The solution has a free-boundary: There exists $h \in \mathbb{R}_+$ such that $\partial_z s = 0$ for all $y \in (0,L)$ and $z \geq h$. The solution is large in the sense that, with $s^*(y) := \lim_{z\to\infty} s(y,z)$, there holds $g\int_0^L k(s^*(y))\,dy \geq F_\infty$, with strict inequality if p, s, and $\partial_z s$ are continuous. Furthermore, $p(y,z) \to -\infty$ as $z \to \infty$ in this case.*
- **Type II: "Small solution"** *The solution has a bounded pressure, there holds $p \in L^\infty(\Omega)$. Furthermore, $\nabla p \in L^2(\Omega)$. The solution is "small" in the sense that $g\int_0^L k(s^*(y))\,dy \leq F_\infty$.*

Type I solutions satisfy additionally the boundary condition (1.5a).

The theorem follows from Propositions 4.4 and 4.5. Before we can prove these results, we have to establish an a priori estimate, which is the basis for both propositions.

Lemma 4.3 *(A priori estimate for TW_H-solutions). Let $F_\infty, c, \tau, s_* > 0$ and $s_0, p_0 \in C^1(\Sigma)$ with $s_* \leq s_0(y) < 1$ and $p_c(s_0) < p_0$. For a sequence $0 < H \to \infty$, let (s_H, p_H) be solutions to (3.1). We assume that the solution sequence satisfies relations (4.3) and (4.4). We use the characteristic functions $\mathbb{1}_> := \mathbb{1}_{\{\partial_z s_H > 0\}}$ and $\mathbb{1}_0 := \mathbb{1}_{\{\partial_z s_H = 0\}}$ on Ω^H. There exists a constant $C_1 := C_1(c, \tau, s_0, p_0, C_P)$, independent of H, such that*

$$\int_{\Omega^H} \mathbb{1}_>\, {p_c}'(s_H)|\nabla s_H|^2 + \int_{\Omega^H} \mathbb{1}_0\, \frac{1}{{p_c}'(p_c^{-1}(p_H))}|\nabla p_H|^2 + c\tau \int_{\Sigma^H} |\nabla s_H|^2 \leq C_1\,. \tag{4.6a}$$

If, additionally, (4.2) is satisfied, there exists $C_2 := C_2(c, \tau, s_0, p_0, C_P, \bar{p})$ such that

$$\int_{\Omega^H} \mathbb{1}_> |\nabla p_H|^2 + \int_{\Omega^H} |\nabla(\partial_z s_H)|^2 \leq C_2\,. \tag{4.6b}$$

Proof. Within this proof, we write (s,p) instead of (s_H, p_H) to have shorter formulas. With $C > 0$ we refer to generic constants that may depend on $c, \tau, s_0, p_0, C_P, \bar{p}$, but not on H.

Step 1: Test function $K(s)$. We use $K : [0,1] \to [0,\infty)$, defined as $K(s) := \int_0^s k(\varrho)^{-1}\,d\varrho$. Equivalently, we may say that K is the primitive of k^{-1}, satisfying

$$K'(s) = \frac{1}{k(s)}\,, \qquad K(0) = 0\,.$$

Below, we will use additionally the primitive of K; we denote by $\tilde{K}$ the function that satisfies $\tilde{K}'(s) = K(s)$ and $\tilde{K}(0) = 0$.

We use $K(s)(y,z) = K(s(y,z))$ as a test function in (3.1a) and study

$$c \int_{\Omega^H} K(s)\partial_z s = \int_{\Omega^H} K(s)\nabla \cdot (k(s)[\nabla p + g e_z])\,.$$

Using an integration by parts, we may write this relation as

$$\begin{aligned} c \int_{\Omega^H} \partial_z \tilde{K}(s) &+ \int_{\Omega^H} k(s)[\nabla p + g e_z] \cdot \nabla K(s) \\ &= \int_{\Sigma^H} K(s)k(s)[\partial_z p + g] - \int_{\Sigma} K(s_0)k(s_0)[\partial_z p + g]\,. \end{aligned} \tag{4.7}$$

We have constructed K such that $\nabla K(s) = k(s)^{-1}\nabla s$. This gives a simple formula for the second integral. With another integration by parts and with $\mathbb{1} = \mathbb{1}_> + \mathbb{1}_0$ we find

$$\begin{aligned} c \int_{\Sigma^H} \tilde{K}(s) - c \int_{\Sigma} \tilde{K}(s_0) + \int_{\Omega^H} \mathbb{1}_> \nabla p \cdot \nabla s + \int_{\Omega^H} \mathbb{1}_0\, \partial_y p\, \partial_y s + \int_{\Omega^H} g e_z \cdot \nabla s \\ = \int_{\Sigma^H} K(s)\,k(s)\,\partial_z p - \int_{\Sigma} K(s_0)k(s_0)\partial_z p + g \int_{\Sigma^H} K(s)\,k(s) - g \int_{\Sigma} K(s_0)k(s_0)\,. \end{aligned} \tag{4.8}$$

We note that the last two integrals on the right hand side and the first two integrals on the left hand side are bounded. Since we assumed (4.3), actually the entire right hand side of (4.8) is bounded. The last integral of the left hand side can be integrated, which shows that also this term is bounded. We therefore find

$$\int_{\Omega^H} \mathbb{1}_> \nabla p \cdot \nabla s + \int_{\Omega^H} \mathbb{1}_0\, \partial_y p\, \partial_y s \le C\,. \tag{4.9}$$

We want to rewrite the first integral. With this aim, we observe that $c\tau\partial_z s = [p - p_c(s)]_+$ in Ω^H implies $c\tau\nabla\partial_z s = (\nabla p - {p_c}'(s)\nabla s)\mathbb{1}_>$ (we recall that we assumed $\partial_z s \in H^1(\Omega)$). This yields

$$\int_{\Omega^H} \mathbb{1}_> \nabla p \cdot \nabla s = c\tau \int_{\Omega^H} \nabla s \cdot \nabla\partial_z s + \int_{\Omega^H} \mathbb{1}_>\, p_c'(s)|\nabla s|^2\,. \tag{4.10}$$

The first term on the right hand side of (4.10) is

$$\begin{aligned} c\tau \int_{\Omega^H} \nabla s \cdot \nabla \partial_z s &= c\tau \int_{\Omega^H} \partial_z \left(\frac{1}{2}|\nabla s|^2\right) = \frac{c\tau}{2}\int_{\Sigma^H} |\nabla s|^2 - \frac{c\tau}{2}\int_{\Sigma} |\nabla s|^2 \\ &= \frac{c\tau}{2}\int_{\Sigma^H} |\nabla s|^2 - \frac{1}{2c\tau}\int_{\Sigma} [p_0 - p_c(s_0)]_+^2 - \frac{c\tau}{2}\int_{\Sigma} |\partial_y s_0|^2\,. \end{aligned} \tag{4.11}$$

At this point, we obtained from (4.9)

$$\int_{\Omega^H} \mathbb{1}_0\, \partial_y s\, \partial_y p + \int_{\Omega^H} \mathbb{1}_>\, p_c'(s)|\nabla s|^2 + \frac{c\tau}{2}\int_{\Sigma^H} |\nabla s|^2 \le C\,. \tag{4.12}$$

Step 2: Test function Φ. We next consider the new test function

$$\Phi := [K(s) - K({p_c}^{-1}(p))]_+ \in H^1(\Omega^H)\,.$$

Note that $\partial_z s > 0 \iff p > p_c(s) \iff {p_c}^{-1}(p) > s \iff K({p_c}^{-1}(p)) > K(s)$. This shows

$$\Phi = [K(s) - K({p_c}^{-1}(p))]\mathbb{1}_0\,.$$

Using Φ as a test function for (3.1a) and exploiting that $\Phi \neq 0$ only when $\partial_z s = 0$, we find

$$\int_{\Omega^H} \Phi\nabla\cdot(k(s)[\nabla p + g e_z]) = c\int_{\Omega^H} \Phi\partial_z s = 0\,. \tag{4.13}$$

Also on the left hand side, the term $\Phi\nabla\cdot[k(s)ge_z] = \Phi k'(s)g\partial_z s$ vanishes identically. Integration by parts in (4.13) yields, using $\Phi = 0$ on Σ,

$$\int_{\Omega^H} k(s)\nabla\Phi\cdot\nabla p = \int_{\Sigma^H} \Phi k(s)\partial_z p\,. \tag{4.14}$$

Because of $\nabla\Phi = \left(\frac{1}{k(s)}\nabla s - \frac{1}{k({p_c}^{-1}(p))}\frac{1}{p_c'({p_c}^{-1}(p))}\nabla p\right)\mathbb{1}_0$, we find

$$\int_{\Omega^H} \nabla s\cdot\nabla p\,\mathbb{1}_0 - \int_{\Omega^H} \frac{k(s)}{k({p_c}^{-1}(p))}\,\frac{|\nabla p|^2}{p_c'({p_c}^{-1}(p))}\,\mathbb{1}_0 = \int_{\Sigma^H} \Phi\, k(s)\partial_z p\,. \tag{4.15}$$

The first integral is $\int_{\Omega^H} \nabla s\cdot\nabla p\,\mathbb{1}_0 = \int_{\Omega^H} \partial_y s\,\partial_y p\,\mathbb{1}_0$, hence it coincides with the first term in (4.12). Since $k(s)\mathbb{1}_0 > k({p_c}^{-1}(p))\mathbb{1}_0$, from (4.12) we arrive at

$$\int_{\Omega^H} \frac{1}{p_c'({p_c}^{-1}(p))}|\nabla p|^2\,\mathbb{1}_0 + \int_{\Omega^H} \mathbb{1}_>\, {p_c}'(s)|\nabla s|^2 + \frac{c\tau}{2}\int_{\Sigma^H} |\nabla s|^2 \le C\,, \tag{4.16}$$

where we exploited once more (4.3). At this point, we have shown (4.6a).

Step 3: Test function $\partial_z s$. To show (4.6b), we use the test function $\partial_z s = \frac{1}{c\tau}[p - p_c(s)]_+ \in H^1(\Omega^H)$ in (3.1a). With an integration by parts we obtain

$$\begin{aligned}&\frac{1}{c\tau}\int_{\Omega^H} k(s)\nabla p\cdot\nabla[p - p_c(s)]_+ \\ &\quad = \int_{\Sigma^H} \partial_z s\, k(s)\partial_z p - \int_{\Sigma} \partial_z s\, k(s_0)\partial_z p + \int_{\Omega^H} (gk'(s) - c)|\partial_z s|^2\,.\end{aligned} \tag{4.17}$$

We observe that, by (4.3) and (4.5b), the first two integrals on the right hand side are bounded. Furthermore, the middle term of (4.16) shows that also the last integral is bounded.

Using the algebraic manipulation $2a(a-b) = a^2 - b^2 + (a-b)^2$, the left hand side of (4.17) is written as

$$\begin{aligned}\frac{1}{c\tau}\int_{\Omega^H} k(s)\nabla p\cdot\nabla[p-p_c(s)]_+ &= \frac{1}{c\tau}\int_{\Omega^H} k(s)\nabla p\cdot(\nabla p-\nabla p_c(s))\mathbb{1}_>\\ &= \frac{1}{2c\tau}\int_{\Omega^H} k(s)[|\nabla p|^2+|\nabla(p-p_c(s))|^2-|\nabla p_c(s)|^2]\mathbb{1}_>\\ &= \frac{1}{2c\tau}\int_{\Omega^H} k(s)[\mathbb{1}_>|\nabla p|^2+(c\tau)^2|\nabla(\partial_z s)|^2-\mathbb{1}_>(p_c{}'(s))^2|\nabla s|^2]\,.\end{aligned}$$

Inequality (4.16) along with (4.5a) shows that the negative term has a bounded integral. This shows (4.6b) and concludes the proof. □

To investigate the free-boundary structured solution described in Theorem 4.2 we define the function $h:\mathbb{R}_+\to\mathbb{R}_+$ with (2.1) in mind: For $H>0$ and (s_H, p_H) solving (3.1), $h = h(H)$ is defined as

$$h(H) := \inf\{z_0\in[0,H] : \partial_z s_H = 0 \text{ a.e. in } (0,L)\times(z_0,H)\}\,. \tag{4.18}$$

The height h marks a horizontal line such that, above that line, $\partial_z s$ vanishes. We note that $h\in[0,H]$ is well-defined and that $h=H$ is possible.

Proposition 4.4 *(Free-boundary solutions). We consider the situation of Theorem 4.2 with a sequence (s_H, p_H) of TW_H-solutions for $H\to\infty$. Additionally, we assume for the sequence $H\to\infty$ that the height*

$$h(H) \quad \textit{is bounded.} \tag{4.19}$$

Under this assumption, a free-boundary travelling wave solution (s,p) exists. More precisely, there exists a pair (s,p) with $s\in C^0_b(\Omega)$, $\partial_z s\in L^2(\Omega)$, $p\in H^2_{\mathrm{loc}}(\Omega)\cap H^1_{\mathrm{loc}}(\Omega\cup\Sigma)$, satisfying (1.4)–(1.5). The solution is of free boundary type in the sense that there exists $h^*>0$ such that $\partial_z s = 0$ for all $y\in(0,L)$ and $z\geq h^*$. The flux satisfies

$$F_\infty \leq g\int_0^L k(s^*(y))\,dy\,. \tag{4.20}$$

Under the additional regularity assumptions $s, \partial_z s, p\in C^0(\Omega)$, the strict inequality holds in (4.20).

Proof. Let $h^* > 0$ denote an upper bound of the function $h(H)$, i.e.

$$h(H) \leq h^* \text{ for all } H. \tag{4.21}$$

Step 1: An additional a priori estimate. We consider once more the function

$$s_H^*(y) = s_H(y, h^*) \text{ for all } y \in (0, L)\,. \tag{4.22}$$

Let $g_{F,H} \in \mathbb{R}$ be the number

$$g_{F,H} := g - \left(F_\infty \middle/ \int_0^L k(s_H^*(y))\,dy \right), \tag{4.23}$$

and let $\tilde{p}_H \in H^1(\Omega^H)$ be the function

$$\tilde{p}_H(y, z) := p_H(y, z) + g_{F,H}\, z \text{ for } (y, z) \in \Omega^H\,. \tag{4.24}$$

We note that these definitions reflect the observations of Lemma 2.2. We finally define $\varphi_H \in C^2([0, 1])$ as the function

$$\varphi_H(s) := cs - (g - g_{F,H})k(s)\,. \tag{4.25}$$

This allows to write (3.1a) in the form

$$\nabla \cdot [k(s_H)\nabla \tilde{p}_H] = \partial_z \varphi_H(s_H)\,. \tag{4.26}$$

We observe that, by (3.1e) and the choice of $g_{F,H}$ in (4.23),

$$\int_{\Sigma^H} k(s_H)\partial_z \tilde{p}_H = F_\infty + (g_{F,H} - g) \int_{\Sigma^H} k(s_H) = 0\,. \tag{4.27}$$

The test function $\tilde{p}_H$ in (4.26) provides the identity

$$\int_{\Omega^H} \tilde{p}_H\, \nabla \cdot [k(s_H)\nabla \tilde{p}_H] = \int_{\Omega^H} \tilde{p}_H\, \partial_z \varphi_H(s_H)\,. \tag{4.28}$$

The left hand side of (4.28) is calculated with an integration by parts, exploiting the fact that p_H on the upper boundary is constant, $p_H \equiv p_H^*$ on Σ_H. In the last line of the calculation we use (4.27).

$$\begin{aligned}
&\int_{\Omega^H} \tilde p_H \nabla\cdot[k(s_H)\nabla \tilde p_H] \\
&\quad = -\int_{\Omega^H} k(s_H)|\nabla \tilde p_H|^2 + \int_{\Sigma^H} \tilde p_H k(s_H)\partial_z \tilde p_H - \int_{\Sigma} \tilde p_H k(s_H)\partial_z \tilde p_H \\
&\quad = -\int_{\Omega^H} k(s_H)|\nabla \tilde p_H|^2 + (p_H^* + g_{F,H} H)\int_{\Sigma^H} k(s_H)\partial_z \tilde p_H - \int_{\Sigma} p_0\, k(s_0)[\partial_z p_H + g_{F,H}] \\
&\quad = -\int_{\Omega^H} k(s_H)|\nabla \tilde p_H|^2 - \int_{\Sigma} p_0\, k(s_0)[\partial_z p_H + g_{F,H}]\,.
\end{aligned}$$

The right hand side of (4.28) is treated with two integrations by parts,

$$\begin{aligned}
&\int_{\Omega^H} \tilde p_H\, \partial_z \varphi_H(s_H) \\
&\quad = -\int_{\Omega^H} \varphi_H(s_H)\partial_z \tilde p_H + \int_{\Sigma^H} \tilde p_H\, \varphi_H(s_H^*) - \int_{\Sigma} \tilde p_H\, \varphi_H(s_H) \\
&\quad = -\int_{\Omega^H} \varphi_H(s_H)\partial_z \tilde p_H + \left[\int_{\Omega^H} \partial_z \tilde p_H\, \varphi_H(s_H^*) + \int_0^L p_0\, \varphi_H(s_H^*)\right] - \int_{\Sigma} p_0\, \varphi_H(s_0) \\
&\quad = \int_{\Omega^H} (\varphi_H(s_H^*) - \varphi_H(s_H))\partial_z \tilde p_H + \int_0^L (\varphi_H(s_H^*) - \varphi_H(s_0))\, p_0\,.
\end{aligned}$$

Boundedness of many of the above terms can be concluded from the facts that $g_{F,H}$ is bounded, $\varphi_H \in C^1([0,1])$, and boundedness of $\partial_z p$ from (4.3). From (4.28) and Young's inequality we obtain

$$\begin{aligned}
\int_{\Omega^H} k(s_H)|\nabla \tilde p_H|^2 &\le C - \int_{\Omega^H} (\varphi_H(s_H^*) - \varphi_H(s_H))\partial_z \tilde p_H \\
&\le C + \int_{\Omega^H} \frac{1}{2k(s_H)}|\varphi_H(s_H^*) - \varphi_H(s_H)|^2 + \int_{\Omega^H} \frac{k(s_H)}{2}|\partial_z \tilde p_H|^2\,.
\end{aligned}$$

We have applied Young's inequality in such a way that the last term on the right hand side can be subtracted from both sides. Since $\varphi_H(s_H^*) - \varphi_H(s_H) = 0$ holds for $z \ge h^*$, the first integral on the right hand side is bounded. We conclude

$$\int_{\Omega^H} k(s_H)|\nabla \tilde p_H|^2 \le C(1 + h^*)\,.$$

Recalling additionally the estimates from Lemma 4.3, we have the following estimates for the solution sequence:

$$\int_{\Omega^H} [|\nabla \tilde{p}_H|^2 + |\partial_z s_H|^2 + |\nabla \partial_z s_H|^2] \leq C\,. \tag{4.29}$$

Step 2: Limit equations. It remains to exploit the bounds of (4.29) to construct the limit solution for $H \to \infty$. Since the sequence $g_{F,H}$ is bounded, we can choose a subsequence $\{H_i\}_{i\in\mathbb{N}}$ with $\lim H_i = \infty$ and $g_F \in \mathbb{R}$ such that $g_{F,H_i} \to g_F$. In the following, we only use this subsequence. The estimate (4.29) allows to choose a further subsequence and a pair (s, p) with $s \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ and $p \in H^1_{\mathrm{loc}}(\Omega)$ such that, for any bounded compact subset $\Omega' \subset \Omega$, there holds

$$s_H \to s \text{ and } \partial_z s_H \to \partial_z s \text{ strongly in } L^2(\Omega')\,, \tag{4.30a}$$

$$p_H \rightharpoonup p \text{ weakly in } H^1(\Omega') \text{ and } p_H \to p \text{ strongly in } L^2(\Omega')\,. \tag{4.30b}$$

These convergences imply that also the limit (s, p) satisfies (1.4) in Ω and the boundary conditions at the lower boundary. Furthermore, $\partial_z s_H \equiv 0$ for all H on $\{z \geq h^*\}$ implies $\partial_z s \equiv 0$ on $\{z \geq h^*\}$.

Step 3: Flux relations. Regarding the flux we use that the quantity

$$F_c^H(z) := \int_0^L k(s_H(y,z))[\partial_z p_H(y,z) + g] - c s_H(y,z)\, dy \tag{4.31}$$

is independent of $z \geq 0$ (compare F_c in (3.2)). Since the saturation s_H is independent of z for $z \geq h^*$, also the quantity

$$F^H(z) := \int_0^L k(s_H(y,z))[\partial_z p_H(y,z) + g]\, dy \tag{4.32}$$

is independent of z for $z \geq h^*$. Because of this independence and because of $F^H(H) = F_\infty$, we find, as $H \to \infty$, for every $z \geq h^*$,

$$F_\infty = F^H(z) \to \int_0^L k(s(y,z))[\partial_z p(y,z) + g]\, dy\,. \tag{4.33}$$

This shows that the boundary condition (1.5a) is satisfied by the limit functions.

We have found a free boundary solution on an unbounded domain. As in Lemma 2.2, there follows $g_F \geq 0$ and, under the regularity assumptions $s, \partial_z s, p \in C^0(\Omega)$, the strict inequality $g_F > 0$. This implies $F_\infty = (g - g_F)\int_0^L k(s^*(y))dy \leq g \int_0^L k(s^*(y))\, dy$, and hence (4.20). □

Proposition 4.5 *(Bounded pressure solutions). Let the situation be that of Theorem 4.2, with TW_H-solutions (s_H, p_H) along a sequence $H \to \infty$. We assume here that the sequence of heights $h(H)$ diverges,*

$$h(H) \to \infty \quad as \quad H \to \infty\,. \tag{4.34}$$

Then, a bounded pressure travelling wave solution (s, p) exists. More precisely, there exists a pair (s, p) with $s \in C^0_b(\Omega)$, $\partial_z s \in L^2(\Omega)$, $p \in H^2_{\mathrm{loc}}(\Omega) \cap H^1_{\mathrm{loc}}(\Omega \cup \Sigma)$ satisfying (1.4). For $C > 0$ there holds

$$\|p\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^2(\Omega)} + \|\partial_z s\|_{H^1(\Omega)} \le C\,.$$

The solution satisfies

$$g \int_0^L k(s^*(y))\,dy \le F_\infty\,. \tag{4.35}$$

We note that we do not obtain the flux condition (1.5).

Proof. In this proof, we only write $H \to \infty$ and $h \to \infty$ for the two sequences. We furthermore use $\Omega^h = [0, L] \times (0, h)$.

Step 1: L^∞-bound for the pressure. The upper bound for the pressure was assumed in (4.2), $p_H \le \bar{p}$ in Ω^H. Our aim in this step is to show a lower bound for the pressure.

On the lower boundary Σ there holds $p_H = p_0 \ge 0$. We claim that there is a lower bound also along the upper boundary Σ^h of Ω^h. Indeed, by definition of h in (4.18), there is a subset of non-vanishing measure in $(0, L) \times (h - 1, h)$ on which $\partial_z s_H > 0$ holds, i.e. $p_H > p_c(s_H) \ge 0$. The Lipschitz bound (4.3) implies that $p_H \ge -C_L$ holds on Σ^h for $C_L = C_P\sqrt{1 + L^2}$.

We can now exploit a maximum principle to obtain

$$-C_L \le p_H \le \bar{p} \qquad \text{a.e. in } \Omega^h\,. \tag{4.36}$$

The maximum principle is derived by using $[p_H + C_L]_-$ as a test function in (3.1a), which results in

$$\int_{\Omega^h} [p_H + C_L]_- \nabla \cdot [k(s_H)\nabla p_H] = \int_{\Omega^h} [p_H + C_L]_- (c - gk'(s_H))\partial_z s_H\,.$$

An integration by parts yields

$$\int_{\Omega^h} k(s_H)|\nabla[p_H + C_L]_-|^2 = \int_{\Sigma^h} [p_H + C_L]_- k(s_H)\partial_z p_H$$
$$- \int_{\Sigma} [p_H + C_L]_- k(s_H)\partial_z p_H + \int_{\Omega^h} [p_H + C_L]_- (c - gk'(s_H))\partial_z s_H\,.$$

As analyzed before, the boundary terms vanish because of $p_H + C_L \geq 0$ along Σ and along Σ^h. Regarding the last integral we note that in every point x with $\partial_z s(x) > 0$, there holds $p_H(x) \geq p_c(s_H(x)) \geq p_c(s_*) = 0$, and hence $[p_H + C_L]_- = 0$. This shows that all terms on the right hand side vanish. We obtain (4.36).

Step 2: A further a priori estimate. From the uniform pressure bound (4.36) we conclude that $p_c^{-1}(p_H)$ is bounded away from 1. With this information, the bound of (4.6a) provides, with a constant $C > 0$ independent of H, the inequality

$$\int\limits_{\Omega^h} \mathbb{1}_0 \, |\nabla p_H|^2 + \int\limits_{\Omega^H} \mathbb{1}_> \, p_c{}'(s_H) |\nabla s_H|^2 \leq C \, .$$

Similarly, (4.6b) implies

$$\int\limits_{\Omega^H} \mathbb{1}_> \, |\nabla p_H|^2 + \int\limits_{\Omega^H} |\nabla(\partial_z s_H)|^2 \leq C.$$

Combining both of these inequalities with (4.36), and recalling $\partial_z s_H = 0$ in $\Omega^H \setminus \Omega^h$, we obtain

$$\max_{\Omega^h} |p_H|^2 + \int\limits_{\Omega^h} |\nabla p_H|^2 + \int\limits_{\Omega^H} [|\partial_z s_H|^2 + |\nabla \partial_z s_H|^2] \leq C \, . \tag{4.37}$$

Step 3: Limit $H \to \infty$. Because of $h \to \infty$, we find a limiting pair (s, p) such that the local convergences of (4.30) hold for any compact subset Ω' of Ω. It is straightforward to verify that (s, p) solves (1.4). Moreover, (4.37) together with $h \to \infty$ implies the additional properties $\nabla p \in L^2(\Omega)$ (as a bounded solution to an elliptic equation) and $\partial_z s \in L^2(\Omega)$.

Regarding the limiting flux, we start from the relation $\int_{\Sigma^H} k(s_H)(\partial_z p_H + g) = F_\infty$. In order to calculate limits, we once more use the quantity $F_c^H(z)$ of (4.31), which is independent of z. The local strong convergence of s_H and the local weak convergence of ∇p_H yield, for almost every z, as $H \to \infty$,

$$\begin{aligned}
&\int\limits_0^L k(s(y,z))[\partial_z p(y,z) + g] - c s(y,z) \, dy =: F_c(z) \\
&\quad \leftarrow F_c^H(z) := \int\limits_0^L k(s_H(y,z))[\partial_z p_H(y,z) + g] - c s_H(y,z) \, dy \\
&\quad = F_c^H(H) = \int\limits_0^L k(s_H(y,H))[\partial_z p_H(y,H) + g] - c s_H(y,H) \, dy \\
&\quad = F_\infty - c \int\limits_0^L s_H(y,H) \, dy \, .
\end{aligned}$$

Because of $s_H(y,H) \geq s_H(y,z)$ for every z, and $s_H \to s$, there holds

$$\lim_{z\to\infty} \int_0^L s(y,z)\,dy \leq \lim_{H\to\infty} \int_0^L s_H(y,H)\,dy\,.$$

Taking in the above calculation both limits, $z \to \infty$ and $H \to \infty$, exploiting $\nabla p \in L^2(\Omega)$, we find

$$\int_0^L gk(s^*(y))\,dy = \lim_{z\to\infty} \int_0^L gk(s(y,z))\,dy \leq F_\infty\,.$$

This concludes the proof. □

Remark 4.6 *(Both solution types occur).* The one-dimensional travelling wave results in [9] indicate that both solution types exist for a given $s_* \in (0,1)$ and F_∞ satisfying (4.1). Type I (large) solutions occur in the one-dimensional model when τ is large. On the other hand, if $\|p_0 - p_c(s_*)\|_{L^\infty(\Sigma)}$ is small, then Type II (small) solutions are expected to occur for small τ values. Our numerical results confirm that both solution types occur.

We finally want to show that, for a given flux F_∞, it is possible to find a wave-speed c such that condition (3.4) is satisfied.

Theorem 4.7 *(Selecting a wave-speed c in dependence of F_∞ and s_*). Let $\tau > 0$, $s^* \in (0,1)$, and boundary data s_0, $p_0 \in C^1(\Sigma)$ be given, $p_c(s_*) \leq p_c(s_0) < p_0$ on Σ, furthermore F_∞ in the bounds of (4.1). We assume that, for all $c \in [c_1, c_2]$ with $c_1 := k'(s_*)g$ and $c_2 := g(k(1) - k(s_*))/(1 - s_*)$, a sequence (s_H, p_H) of solutions to (3.1) satisfying Assumption 4.1 exists. We consider the corresponding limit solutions (s,p) and their fluxes*

$$F_c = \int_\Sigma (k(s_0)[\partial_z p + g] - cs_0)\,, \tag{4.38}$$

and assume that F_c depends continuously on c. Then there exists a wave-speed $\bar{c} \in (c_1, c_2)$ such that the corresponding pair (s,p) satisfies (3.4), $F_c = (gk(s_*) - cs_*)L$.

Proof. We consider the continuous function $G : [c_1, c_2] \to \mathbb{R}$

$$G(c) := F_c - (gk(s_*) - cs_*)L\,. \tag{4.39}$$

We recall that G depends in an explicit way on c, but also implicitly, since s and p (and hence F_c) depend on c. The flux quantity F_c is independent of z, we choose to evaluate it at $z \to \infty$. We denote the limit of the first two terms as

$$F_0 := F_0(c) := \lim_{z\to\infty} \int_0^L k(s(y,z))[\partial_z p(y,z) + g]\,dy\,.$$

We observe that, by Theorem 4.2,

$$F_0 = \begin{cases} F_\infty & \text{if} \quad g \int_0^L k(s^*) > F_\infty\,, \\ g \int_0^L k(s^*) & \text{if} \quad g \int_0^L k(s^*) \le F_\infty\,. \end{cases}$$

In both cases holds $F_0 \le F_\infty$ and $F_0 \le g \int_0^L k(s^*)$. The function G can be written as

$$G(c) = F_0 - k(s_*)gL - c \int_0^L (s^*(y) - s_*)\, dy\,.$$

Showing $G(c) > 0$ as $c \to c_1$. If the solution is of Type II (small solution, second case in the above distinction), then

$$G(c) = \int_0^L (s^*(y) - s_*)\, dy \left(g\, \frac{\int_0^L (k(s^*(y)) - k(s_*))\, dy}{\int_0^L (s^*(y) - s_*)\, dy} - c \right).$$

We exploit that $s^* > s_0 \ge s_*$ implies, for every $y \in (0, L)$, that $k(s^*(y)) - k(s_*) > k'(s_*)(s^*(y) - s_*)$. This implies that, for c close to $c_1 = k'(s_*)g$, there holds $G(c) > 0$. On the other hand, if (s, p) is of Type I (large solutions), then

$$\begin{aligned} G(c) &= F_\infty - k(s_*)gL - c \int_0^L (s^*(y) - s_*)\, dy \\ &\ge F_\infty - k(s_*)gL - c(1 - s_*)L \\ &\ge gLk'(s_*)(1 - s_*) - c(1 - s_*)L + \varepsilon \\ &= (gk'(s_*) - c)(1 - s_*)L + \varepsilon\,, \end{aligned}$$

where we exploited the lower bound $gL[k(s_*) + k'(s_*)(1 - s_*)] + \varepsilon \le F_\infty$ for some $\varepsilon > 0$. We see that, also in this case, for c close to $c_1 = k'(s_*)g$, there holds $G(c) > 0$.

Showing $G(c) < 0$ as $c \to c_2$. Consider solutions of Type II (small solutions). For $\mu := (k(1) - k(s_*))/(1 - s_*)$, we show that in this case, there exists $\varepsilon > 0$ independent of $c \in [c_1, c_2]$ such that

$$\int_0^L (k(s^*) - k(s_*)) \le (\mu - \varepsilon) \int_0^L (s^* - s_*)\,. \tag{4.40}$$

Since $(k(s) - k(s_*))/(s - s_*)$ is a strictly increasing function for $s > s_*$, $\int_0^L (k(s^*) - k(s_*)) = \mu \int_0^L (s^* - s_*)$ if and only if $s^*(y) \in \{s_*, 1\}$ for all $y \in (0, L)$. From Jensen's inequality, one has

$$k\left(\frac{1}{L} \int_0^L s^* \right) \le \frac{1}{L} \int_0^L k(s^*) \le \frac{F_\infty}{gL} < k(1)\,, \tag{4.41}$$

implying $\frac{1}{L}\int_0^L s^* < 1$. Hence, the possibility $s* \equiv 1$ in $(0, L)$ is ruled out. Moreover, since $s^* > s_0 \geq s_*$, the possibility $s* \equiv s_*$ in $(0, L)$ is also ruled out. From Lemma A.3, $\|\nabla s\|_{L^\infty(\Omega)}$ is bounded. Hence s^* cannot take both the values s_* and 1 without transitioning through the intermediate values. Thus (4.40) holds.

If the solution is of Type II, then, for c close enough to $g\mu = g(k(1) - k(s_*))/(1 - s_*)$, we obtain from (4.40),

$$G(c) = F_0 - k(s_*)gL - c\int_0^L (s^* - s_*) = \int_0^L [g(k(s^*) - k(s_*)) - c(s^* - s_*)]$$

$$\leq \int_0^L (s^* - s_*)\,[g\mu - g\varepsilon - c] \leq 0\,.$$

If the solution is of Type I, then for $g_F > 0$ as defined in (2.5) (see also Proposition 4.4), one has

$$\begin{aligned} G(c) &= F_\infty - k(s_*)gL - c\int_0^L (s^* - s_*) \\ &= \int_0^L [g(k(s^*) - k(s_*)) - c(s^* - s_*)] - g_F\int_0^L k(s^*) \\ &\leq (g\mu - c)\int_0^L (s^* - s_*) - g_F\int_0^L k(s_0)\,. \end{aligned}$$

Consequently, $G(c) < 0$ for c close enough to $c_2 = g\mu$. Hence, there exists a zero $\bar{c}$ of $G(\cdot)$ in (c_1, c_2). This was the claim. □

5. Numerics

5.1. Numerical solution of system (3.1)

The primary numerical task is to solve system (3.1) for s and p, where the speed c and the total influx F_∞ are given. The existence of a solution was established in Theorem 3.4. We use an iterative method in order to deal with the nonlinearities. With a positive number $M > 0$, we use the iteration $(s^{i-1}, p^{i-1}) \mapsto (s^i, p^i)$ that is given by

$$Mp^i - \nabla\cdot[k(s^{i-1})(\nabla p^i + ge_z)] = Mp^{i-1} - \frac{1}{\tau}[p^{i-1} - p_c(s^{i-1})]_+\,, \tag{5.1a}$$

$$\partial_z s^i - \varepsilon\Delta s^i = \frac{1}{c\tau}[p^i - p_c(s^{i-1})]_+\,. \tag{5.1b}$$

The equations are solved in the rectangular computational domain Ω^H for some initial guess (s^0, p^0). They are supplemented by the boundary conditions (3.1c)–(3.1e) and no-flux conditions at the lateral boundaries.

We introduce an elliptic regularization in (5.1b) (which is otherwise first order in s), in order to stabilize the finite element approximation. Numerical experiments are run with a small number $\varepsilon > 0$. Depending on the precise value of ε, one has to choose a sufficiently high spatial resolution, such that the scheme is stable.

For $\varepsilon = 0$, a fixed point of the iteration scheme (5.1) provides a solution of (3.1). The set-up is such that the equations can be solved subsequently: One can solve the first equation for p^i, then the second equation for s^i. The iteration strategy is based on the L-scheme, see [16]. The iteration is expected to converge for $M \geq \tau^{-1}$, irrespective of the initial guess. Compare [19] for a modified L-scheme replacing M by a variable function defined at each iteration.

In order to discretize (5.1), we introduce a uniform triangulation Ω^H_{h} of the domain Ω^H and apply linear finite elements. In this sense, the discretization is based on the weak formulation in (3.5)–(3.6). The resulting scheme has been implemented in the adaptive finite element tool box AMDiS [29]. The linear equations arising from the discretization are treated with the direct solver UMFPACK, [8].

The physical parameters of the problem are chosen as in [15],

$$p_c(s) = s\,, \quad k(s) = \begin{cases} \kappa & \text{for} \quad s < a\,, \\ \kappa + (s-a)^2 & \text{for} \quad s \geq a\,, \end{cases} \tag{5.2}$$

and

$$g = 1\,, \quad \tau = 2\,, \quad \kappa = 0.001\,, \quad a = 0.32\,, \quad F_\infty = 0.056\,.$$

The domain is $\Omega^H = (-1, 1)^2$; up to a shift of the domain, this coincides with $L = 2$ and $H = 2$ in analytical results. The parameters for the numerical code are

$$M = 4\,, \quad \varepsilon = 0.0008\,.$$

The initial values for the iteration have been chosen as

$$p^0 = 4.5\,, \quad s^0 = 10^{-5}\,.$$

Regarding the lower boundary, we use the constant function $s_0 = 10^{-5}$ and the slightly perturbed pressure boundary condition

$$p_0(y) = p_c(s_0) + \delta e^{-(y/d)^2}\,.$$

The positive parameter $\delta = 0.078$ measures the amplitude of the perturbation and the scaling factor $d = 0.25$ measures the width of the perturbation.

Fig. 5 shows results for four different values of the speed c. We see a remarkable difference between the solution for $c = 0.04785$ and the solution for $c = 0.04786$. The abrupt change finds its counterpart in Theorem 4.2 (we recall that the theorem is treating unbounded domains while the numerical results are for a fixed bounded domain): The two images on the left show Type I

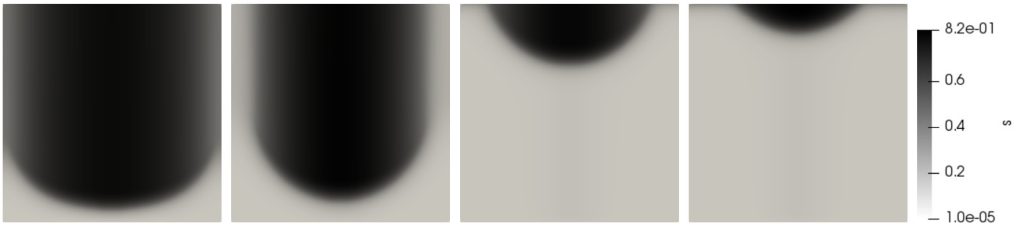

Fig. 5. The discrete solutions s_h of the iteration scheme for (from left to right) $c = 0.04$, $c = 0.04785$, $c = 0.04786$, $c = 0.05013166020$.

Table 1
Values for $G_1(c)$ for various c-values.

c	0.0476	0.0477
$G_1(c)$	$0.0476 + 0.0218 > 0$	$0.0477 - 0.3304 < 0$

solutions, i.e., “large solutions” with a free boundary. The two images on the right show “small solutions”.

In the above experiments, we have solved system (3.1) for different values of c. We now ask: What is the correct wave speed c in the sense of (3.4)? We use the following finite domain approximations: s_* can be neglected, hence, in particular, $k(s_*) = \kappa$. Furthermore, s_0 is constant and so small that also $k(s_0)$ can be replaced by κ. Condition (3.4) then reads $G_1(c) := c - (\kappa \int_\Sigma \partial_z p_h)/(L s_0) \overset{!}{=} 0$. We find the values as displayed in Table 1. We conclude that $G_1(\bar{c}_1) = 0$ is satisfied for some $\bar{c}_1 \in [0.0476, 0.0477]$. Up to the above finite domain approximations, we expect the travelling wave speed to be about 0.0477. This is remarkably close to the jump point, compare Fig. 5. We furthermore note that the value is not far from the value $c = 0.053$ that can be extracted from simulation results reported in [15].

5.2. Path-following algorithm to adjust c

So far, for each value of c, we started the iterative scheme (5.1) with constant functions s^0 and p^0 as initial guess. Since we are interested in solutions for a whole range of c-values, there is a very natural idea to speed up calculations: After having changed the value of c, instead of starting the iterative scheme from scratch, we start the iteration with the solution of the last value of c. Thereby, we increased c in every interaction step by 10^{-4} in some experiments, by 10^{-11} in others.

Interestingly, it turns out that this scheme produces results that are different from those reported in Section 5.1. Results are displayed in Fig. 6, and once more, we observe that, below a critical value for c, solutions are “large solutions”, above the critical value, we find “small solutions”. This feature is as in the sequence of Fig. 5, but the critical value of c is now different: It is about $\bar{c}_2 = 0.050$ and no longer about $\bar{c}_1 = 0.048$. For values of c below $\bar{c}_1$ and for values above $\bar{c}_2$, the results of the two schemes coincide.

We conclude with an evaluation of the integral condition in Theorem 4.2, where the criterion for a “large solution” was $g \int_0^L k(s^*(y))\, dy \geq F_\infty$ for the saturation values $s^*(y) := \lim_{z\to\infty} s(y, z)$ at infinity. With the approximation $s^* \approx s|_{\Sigma^H}$ and with (3.1e), the criterion for a “large solution” reads

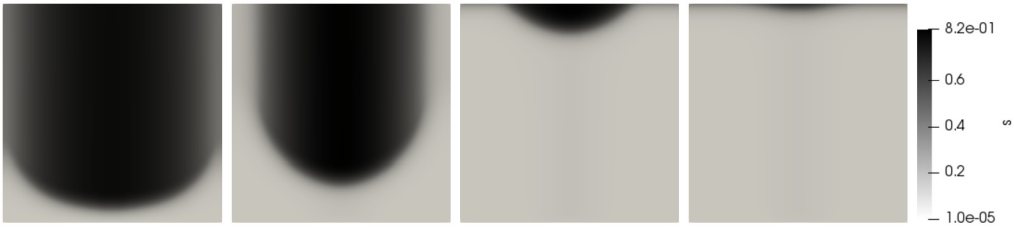

Fig. 6. Plots of the discrete solutions s_h of the path-following iteration scheme for (from left to right) $c = 0.04$, $c = 0.05013166020$, $c = 0.05013166023$, $c = 0.0625$.

Table 2
Values for $G_2(c)$ for various c-values.

c	0.04	0.05013166020	0.05013166023	0.0625
$G_2(c)$	−0.1948	−0.1630	−0.1492	0.0183

$$G_2(c) := \int_{\Sigma^H} k(s)\partial_z p \leq 0\,. \tag{5.3}$$

Our simulations yield the values in Table 2. We observe that the change of sign of G_2 occurs only after the point that the solution switched to the “small solution”.

Our observations may be interpreted as follows: For a range of values of c, there are two solutions of system (3.1). This is not in contradiction with our analysis, since Theorem 3.4 provides the existence, but not the uniqueness of solutions. A numerical scheme has the tendency to find the “stable” solution (“stable” has to be interpreted appropriately). In a path-following code as described here (in Section 5.2), due to numerical stabilization aspects, the code can follow one path beyond the point where it looses stability. We conjecture that this is what is visible in the observation $\bar{c}_2 > \bar{c}_1$.

6. Conclusions

We studied the travelling wave equations for a porous media imbibition problem with hysteresis. Denoting by c the unknown speed of the travelling wave, we treat a free boundary problem with an additional parameter. Our analysis shows that, after a domain truncation and for boundary conditions within physically reasonable limits: (i) For a prescribed speed c, travelling wave solutions exist. In the limit of infinite domains, different types of limit solutions can occur. (ii) A critical wave speed c can be selected by a flux condition. (iii) Numerical experiments provide solutions with the shape of a finger. We find values of c that are in good agreement with time-dependent calculations. Different numerical algorithms yield slightly different values for c, an effect that may be related to non-uniqueness of solutions.

Data availability

No data was used for the research described in the article.

Appendix A

The following result on solution sequences (s_H, p_H) does not rely on Assumption 4.1, but follows directly from the variational principle.

Lemma A.1 *(Large solution sequences have unbounded pressure). For a sequence* $0 < H \to \infty$, *let* (s_H, p_H) *be solutions to* (3.1). *We assume that, for some height parameter* $z_0 > 0$ *and some bound* $C_k > 0$, *every solution* s_H *satisfies the integral condition*

$$\int_0^L gk(s_H(y, z_0))\, dy \ge C_k > F_\infty\,. \tag{A.1}$$

In this situation, the sequence of pressure functions is unbounded,

$$\|p_H\|_{L^\infty} \to \infty\,. \tag{A.2}$$

In particular, it generates a "large" Type I solution.

Proof. For a contradiction argument we assume that, for some $\bar{p} > 0$, the pressure functions are bounded, $|p_H| \le \bar{p}$ on Ω^H. We recall that p_H is the minimizer for the functional A of (3.8), for given $s = s_H$. This provides a lower bound for A: For any function $\varphi \in X_{p_0}$, there holds, by Lemma A.2,

$$\begin{aligned} A(\varphi) \ge A(p_H) &\ge \int_{\Omega^H} \frac{1}{2} k(s_H)|\nabla p_H + g e_z|^2 - F_\infty L \bar{p} \\ &\ge \frac{1}{2} g^2 \left(\int_{\Omega^H} k(s_H) \right) - C_1(\bar{p}) - F_\infty L \bar{p}\,. \end{aligned}$$

Our aim is to find a contradiction, which we obtain by constructing a comparison function with lower energy. We choose a function $\tilde{p}_H$ that connects, in the domain $\{z \in (0, 1)\}$, the boundary data p_0 in a smooth way with $\tilde{p}_H \equiv 0$ for $z = 1$. For larger z, we set $\tilde{p}_H(y, z) = -g_F(z - 1)$, where the coefficient $g_F \in (0, g)$ is chosen below. We calculate for the energy

$$A(\tilde{p}_H) \le C_2 + \frac{1}{2}|g - g_F|^2 \left(\int_{\Omega^H} k(s_H) \right) + F_\infty H g_F\,.$$

Combining the two inequalities and using $\bar{C}_k := \left(g \int_0^L \int_{z_0}^H k(s_H) \right) / (H - z_0)$, we find

$$\frac{1}{2} g \bar{C}_k H \le C_3 + F_\infty H g_F + \frac{H \bar{C}_k}{2g} |g - g_F|^2\,. \tag{A.3}$$

Optimizing in g_F leads to the choice $g_F := g - q$ with $q := (gF_\infty)/\bar{C}_k < g$. In order to compare the prefactors of H on both sides we study

$$\begin{aligned}\frac{1}{2}\bar{C}_k g - \frac{\bar{C}_k}{2g}|g-g_F|^2 - F_\infty g_F &= \frac{1}{2}\bar{C}_k g - \frac{\bar{C}_k}{2g}|q|^2 - F_\infty(g-q)\\ = \frac{\bar{C}_k}{2g}\left(g^2-q^2-2q(g-q)\right) &= \frac{\bar{C}_k}{2g}\left(g^2+q^2-2qg\right) = \frac{\bar{C}_k}{2g}(g-q)^2 > 0\,.\end{aligned}$$

For large H, this yields a contradiction in (A.3). □

Lemma A.2 *(A Jensen type inequality). For $\Omega^H = (0,L)\times(0,H)$ with points $x=(y,z)$, $k:\Omega^H\to[0,k_0]$ monotonically increasing in z, and $u:\Omega^H\to\mathbb{R}$ with the uniform bound $\|u\|_{L^\infty}\le\bar{u}$, there exists a constant $C_1 = C_1(\bar{u},k_0)$, independent of H, such that*

$$\int_{\Omega^H} k\,|\nabla u + g e_z|^2 \ge -C_1 + g^2\int_{\Omega^H} k\,. \tag{A.4}$$

Proof. We use the averaging operator $M: L^2(\Omega^H)\to\mathbb{R}$, defined by

$$M(v) := \left(\int_{\Omega^H} kv\right)\Big/\left(\int_{\Omega^H} k\right).$$

This operator is linear and maps the constant function $v\equiv a\in\mathbb{R}$ to $M(v)=a$. We furthermore use the convex function $\psi:\mathbb{R}\to\mathbb{R}$, $\xi\mapsto|\xi+g|^2$. Jensen's inequality provides

$$M(\psi(\partial_z u)) \ge \psi(M(\partial_z u))\,.$$

In our setting and with $m := \int_{\Omega^H} k$, this yields

$$\int_{\Omega^H} k\,|\nabla u + g e_z|^2 \ge \int_{\Omega^H} k\,|\partial_z u + g|^2 = m\,M(\psi(\partial_z u)) \ge m\,\psi(M(\partial_z u))\,.$$

We calculate, using that k is increasing in z,

$$\begin{aligned}|M(\partial_z u)| &= \left|\frac{1}{m}\int_{\Omega^H} k\partial_z u\right| \le \frac{1}{m}\left|\int_0^L k\,u\Big|_0^H - \int_{\Omega^H}\partial_z k\,u\right|\\ &\le \frac{1}{m}\left(2k_0\bar{u} + \bar{u}\int_{\Omega^H}\partial_z k\right) \le \frac{3k_0\bar{u}}{m}\,.\end{aligned}$$

Inserting above we obtain

$$\int_{\Omega^H} k\,|\nabla u + g e_z|^2 \geq m\,\psi(M(\partial_z u)) = m\,|g + M(\partial_z u)|^2 \geq m g^2 - 6 g k_0 \bar{u}\,.$$

This shows the claim. □

Lemma A.3 *(Lipschitz continuity of s_H). Let $F_\infty, c, \tau, s_* > 0$ and $s_0, p_0 \in C^1(\Sigma)$ with $s_* \leq s_0 < 1$ and $p_0 \geq p_c(s_0)$ be fixed. For $H > 0$, let (s_H, p_H) be the TW_H-solution to (3.1) satisfying (4.3) and (4.4). Then, for $\rho = \min\{p_c{}'\} > 0$, there holds*

$$\|\partial_z s_H\|_{L^\infty(\Omega^H)}, \|\partial_y s_H\|_{L^\infty(\Omega^H)} \leq C_P/\rho + \|\partial_y s_0\|_{L^\infty(\Sigma)} + \tfrac{1}{c\tau}\|p_0 - p_c(s_0)\|_{L^\infty(\Sigma)} =: C_s\,. \quad \text{(A.5)}$$

Proof. To prove the lemma, we consider a regularization of the signum function, denoted as $\text{sign}_\varepsilon : \mathbb{R} \to [-1, 1]$. A possible choice is $\text{sign}_\varepsilon(\eta) := \eta/\varepsilon$ for $\eta \in [-\varepsilon, \varepsilon]$, $\text{sign}_\varepsilon(\eta) := -1$ for $\eta < -\varepsilon$ and $\text{sign}_\varepsilon(\eta) := 1$ for $\eta > \varepsilon$. We also introduce the primitive $H_\varepsilon(\eta) = \int_0^\eta \text{sign}_\varepsilon(\varrho)d\varrho$. We demand that, as $\varepsilon \to 0$, there holds $\text{sign}_\varepsilon(\eta) \to \text{sign}(\eta)$, $H_\varepsilon(\eta) \to |\eta|$, and $\eta\,\text{sign}_\varepsilon(\eta) \to |\eta|$.

We differentiate relation (3.1b), in the sense of distributions, with respect to x_j for $x_j = y$ and for $x_j = z$. The regularity assumption (4.4) on s_H allows to write

$$c\tau\partial_z\partial_{x_j}s_H + p_c{}'(s_H)\partial_{x_j}s_H\,\text{sign}(\partial_z s_H) = \partial_{x_j}p_H\,\text{sign}(\partial_z s_H)\,.$$

Multiplying both sides with $\text{sign}_\varepsilon(\partial_{x_j}s_H)$ yields

$$\begin{aligned} &c\tau\partial_z H_\varepsilon(\partial_{x_j}s_H) + p_c{}'(s_H)\,\text{sign}_\varepsilon(\partial_{x_j}s_H)\partial_{x_j}s_H\,\text{sign}(\partial_z s_H) \\ &\quad = \text{sign}(\partial_z s_H)\,\text{sign}_\varepsilon(\partial_{x_j}s_H)\partial_{x_j}p_H\,. \end{aligned} \quad \text{(A.6)}$$

Passing to the limit $\varepsilon \to 0$, we obtain for $x_j = z$ the relation

$$c\tau\partial_z|\partial_z s_H| + p_c{}'(s_H)|\partial_z s_H| \leq |\partial_z p_H| \leq C_P\,, \quad \text{(A.7)}$$

where we used (4.3) in the last inequality. We exploit that, for $z = 0$, there holds $\partial_z s_H = \frac{1}{c\tau}[p_0 - p_c(s_0)]_+$, and hence also $|\partial_z s_H| \leq \frac{1}{c\tau}\|p_0 - p_c(s_0)\|_{L^\infty(\Sigma)}$. Inequality (A.7) implies that $|\partial_z s_H|$ cannot exceed the value C_s of (A.5).

We now study $x_j = y$ in (A.6). In the limit $\varepsilon \to 0$, exploiting $\partial_z s_H \geq 0$, we find

$$c\tau\partial_z|\partial_y s_H| + p_c{}'(s_H)|\partial_y s_H|\,\text{sign}(\partial_z s_H) \leq |\partial_y p_H|\,\text{sign}(\partial_z s_H)\,. \quad \text{(A.8)}$$

With the uniform bound $|\nabla p_H| \leq C_P$ of (4.3) we can write

$$c\tau\partial_z|\partial_y s_H| \leq (C_P - \rho|\partial_y s_H|)\,\text{sign}(\partial_z s_H)\,. \quad \text{(A.9)}$$

For $z = 0$, there holds $\partial_y s_H = \partial_y s_0$. Inequality (A.9) implies that $|\partial_y s_H|$ cannot exceed the value C_s of (A.5). □

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