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Abstract

We present a stability result for a class of non-trivial self-similarly vanishing solutions to a 3D Hele-Shaw moving boundary problem with surface tension and single-point suction. These solutions are domains that bifurcate from the trivial spherical solution. The moving domains have a geometric centre located at the suction point and they are axially symmetric. We show stability with respect to perturbations that preserve these properties.

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1 Introduction

The Hele-Shaw problem (in its various versions) is one of the most intensively studied free boundary problems [11] and plays a paradigmatic role for understanding other more complicated problems.

In [14] we exhibited the existence of families of self-similar solutions to the Hele-Shaw problem with suction, regularised by surface tension. These solution exist up to extinction time. The paradigmatic character of the Hele-Shaw problem shows up here once more, as similar families of solutions appear e.g. in the study of tumour models [8] and viscous drops in electric fields [7].

For such families of solutions, investigation of their stability is a natural and important issue. The present paper is aimed at answering this question in the context of [14].

In the problem of Hele-Shaw flow with suction at the origin and with surface tension one seeks for a given domain $\Omega(0)$ a family of moving domains $t \mapsto \Omega(t)$ in $\mathbb{R}^3$ and corresponding functions $p(\cdot,t):\Omega(t)\to\mathbb{R}$ that satisfy

$$\Delta p = \mu\delta \qquad \text{in } \Omega(t), \tag{1.1}$$
$$p = -\gamma\kappa \quad \text{on } \Gamma(t), \tag{1.2}$$
$$v_n = -\frac{\partial p}{\partial n} \quad \text{on } \Gamma(t). \tag{1.3}$$

Here v_n is the normal velocity of the moving boundary $t\mapsto\Gamma(t)$ and $\kappa:\Gamma(t)\to\mathbb{R}$ is the mean curvature of the boundary (taken negative for convex domains). The parameter $\mu>0$ is the suction speed, $\gamma>0$ is a positive constant and δ is the delta distribution.

Besides liquid flow in a Hele-Shaw cell [4], the model also describes porous media flow [5, 6]. In recent years, related models of tumour growth have attracted a lot of interest, see e.g. [1] and [8] and the references given there.

In the simplest case, the initial domain is the unit ball $\mathbb{B}^3$. Then, by radial symmetry, $\Omega(t)=\alpha(t)\mathbb{B}^3$, where

$$\alpha(t)=\sqrt[3]{1-\frac{3\mu t}{4\pi}},$$

for $t\in[0,\frac{4\pi}{3\mu})$. We refer to this particular case as the trivial solution.

The following results have recently been proved.

- **Nonlinear stability of the trivial solution** [18]
  Suppose $\mu/\gamma<32\pi/5$. Consider an initial domain $\Omega(0)$ that is star-shaped and close to $\mathbb{B}^3$ in the $\mathcal{C}^{4,\alpha}$-topology. Suppose that its volume is equal to $4\pi/3$ and its geometric centre is located at the origin. Then there exists a family of domains $t\mapsto\Omega(t)$ that solves (1.1)-(1.3). This family extincts at $t=\frac{4\pi}{3\mu}$ as an asymptotically spherical point.

- **Existence of non-trivial self-similarly vanishing solutions** [14]
  On the other hand, there exist non-trivial solutions of the type $t\mapsto\Omega(t)=\alpha(t)\Omega(0)$, solving (1.1)-(1.3), for certain initial domains $\Omega(0)$. These solutions bifurcate from the trivial solution $t\mapsto\alpha(t)\mathbb{B}^3$. Here μ/γ is the bifurcation parameter. The moving domains are invariant with respect to rotations around the z-axis. Furthermore, the geometric centre of the domains is located at the origin and the initial volume is equal to $4\pi/3$. We refer to these solutions as the bifurcation solutions.

Nonlinear stability of the trivial solution has also been proved for this problem in two dimensions [17]. In this case star-shaped perturbations extinct as asymptotically circular points for any value of μ.

In this paper we prove that some of the bifurcation solutions are stable under perturbations that respect axial symmetry, position of geometric centre and initial volume.

This paper is organised as follows. In Section 2 we rescale the moving domain by $\alpha(t)$ and introduce an evolution equation (2.1) in which both the trivial solution and the bifurcation solutions are represented by stationary solutions. Section 3 recalls our bifurcation result from [14] and provides additional information that characterises the local bifurcation picture. For μ slightly larger than a certain value μ_2 we find disk-like shapes and for μ slightly smaller than μ_2 we have cigar-like shapes, see Figure 1. Our main stability result is given in Section 4. It states that the branch of bifurcation solutions consisting of disk-like shapes is nonlinearly stable (near the bifurcation point). This relies on analysing the linearisation of the evolution operator around the bifurcation solution and applying the results of Crandall and Rabinowitz [2].

2 The evolution equation

We start by recalling some constructions from [18], to give an equation describing the evolution of our domain. Let $\left(\mathbb{H}^s(\mathbb{S}^2), (\cdot,\cdot)_s\right)$ be the Sobolev space of functions on the unit sphere of real order s.

Any continuous function $f : \mathbb{S}^2 \to (-1,\infty)$ describes a domain Ω_f in the following way:

$$\Omega_f := \left\{ x \in \mathbb{R}^3 \setminus \{0\} : |x| < 1 + f\left(\frac{x}{|x|}\right) \right\} \cup \{0\}.$$

For any domain $\Omega(t)$ that moves according (1.1)-(1.3) we introduce a continuous function $R(\cdot,t) : \mathbb{S}^2 \to (-1,\infty)$ such that $\Omega(t) = \Omega_{R(\cdot,t)}$. Note that we need to restrict ourselves to star-shaped domains. Rescaling the family of domains $t \mapsto \Omega_{R(\cdot,t)}$ by a factor $\alpha(t)^{-1}$ we obtain another family of domains $t \mapsto \Omega_{r(\cdot,t)} := \alpha(t)^{-1}\Omega_{R(\cdot,t)}$, where the function $r(\cdot,t) : \mathbb{S}^2 \to (-1,\infty)$ satisfies

$$r(x,t) = \frac{1 + R(x,t)}{\alpha(t)} - 1.$$

In this way, $r \equiv 0$ corresponds to the trivial solution and any stationary solution corresponds to a self-similar solution $\Omega(t) = \alpha(t)\Omega(0)$ of the original problem. Often, if no confusion is possible, we omit the x-argument or the t-argument in $r(x,t)$. The evolution of domains that are initially small perturbations of the unit ball, represented by small $r(0)$, are solutions to the equation

$$\frac{\partial r}{\partial t} = \frac{1}{\alpha(t)^3}(\gamma \mathcal{F}_1(r) - \mu \mathcal{F}_2(r)). \tag{2.1}$$

Here $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-3}(\mathbb{S}^2)$ and $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^2)$ are analytic operators on a small neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^2)$ for $s > 5$. The definition of these operators is given in Appendix A. For a derivation of (2.1) we refer to [18].

3 Bifurcation solutions

Let $(Y_k)_{k=0}^{\infty}$ be the zonal harmonics. These are the spherical harmonics that are rotationally symmetric around the z-axis. Let $\mathbb{H}^s_\times(\mathbb{S}^2)$ be the subspace of $\mathbb{H}^s(\mathbb{S}^2)$ consisting of functions that are invariant with respect to rotations around the z-axis and let X_k be the orthogonal complement of $\langle Y_k \rangle$ in $\mathbb{H}^s_\times(\mathbb{S}^2)$. Since the mappings $\mathcal{F}_1$ and $\mathcal{F}_2$ respect rotational symmetries it is possible to introduce $\mathcal{F}_{\times,\mu} : \mathcal{U}_\times \to \mathbb{H}^{s-3}_\times(\mathbb{S}^2)$ on $\mathcal{U}_\times := \mathcal{U} \cap \mathbb{H}^s_\times(\mathbb{S}^2)$ by

$$\mathcal{F}_{\times,\mu} := (\gamma \mathcal{F}_1 - \mu \mathcal{F}_2)|_{\mathcal{U}_\times}.$$

From now on we regard γ as fixed and μ will be the bifurcation parameter. Introduce the numbers

$$\zeta_l := 4\pi\frac{l^3+l^2-2l}{l+3}, \quad \mu_l = \gamma\zeta_l, \quad l = 2,3,4,\ldots.$$

We recall the main result of [14]: For each integer $k \geq 2$ there exist a positive δ_k and a curve $(\rho_k, m_k) : (-\delta_k, \delta_k) \to \mathbb{H}^s_\times(\mathbb{S}^2) \times \mathbb{R}$ with $\rho_k(0) = 0$ and $m_k(0) = \mu_k$ such that for $\sigma \in (-\delta_k, \delta_k)$

$$\mathcal{F}_{\times, m_k(\sigma)}(\rho_k(\sigma)) = 0. \tag{3.1}$$

Furthermore, there exist $\mathcal{C}^1$-functions $f_k : (-\delta_k, \delta_k) \to X_k$ with $f_k(0) = 0$ such that for $\sigma \in (-\delta_k, \delta_k)$ we have

$$\rho_k(\sigma) = \sigma Y_k + \sigma f_k(\sigma). \tag{3.2}$$

In other words, the functions $\rho_k(\sigma) \in \mathbb{H}^s_\times(\mathbb{S}^2)$ are stationary solutions of (2.1) that bifurcate from the trivial stationary solution $r \equiv 0$ in the direction of Y_k. Observe, moreover, that due to the smoothness of $\mathcal{F}_{\times,\cdot}$ as a function of (r, μ), the curve (ρ_k, m_k) is smooth as well.

Lemma 3.1. *We have $m_k'(0) < 0$ for k even and $m_k'(0) = 0$ for k odd.*

Proof. This is proved in Appendix B. □

4 Stability of bifurcation solutions

Let $\mathcal{F}'_{\times,\mu}(0)$ be the Fréchet derivative of $\mathcal{F}_{\times,\mu}$ at zero. The zonal harmonics form a complete $\mathbb{L}_2(\mathbb{S}^2)$-orthonormal set of eigenfunctions of $\mathcal{F}'_{\times,\mu}(0) : \mathbb{H}^s_\times(\mathbb{S}^2) \to \mathbb{H}^{s-3}_\times(\mathbb{S}^2)$ with

$$\mathcal{F}'_{\times,\mu}(0)[Y_l] = g_{l,\mu} Y_l, \tag{4.1}$$

where $g_{l,\mu}$ are the numbers

$$g_{l,\mu} := -\gamma(l^3 + l^2 - 2l) + \mu\frac{l+3}{4\pi},$$

[18]. For any value of μ both $g_{0,\mu}$ and $g_{1,\mu}$ are positive. In order to derive stability of the bifurcation solutions, we need to exclude these positive eigenvalues. We assume that $r_0 \in \mathfrak{M}_\times$ where

$$\mathfrak{M}_\times := \left\{ r \in \mathbb{H}^s_\times(\mathbb{S}^2) : \int_{\Omega_r} dx = \frac{4\pi}{3}, \int_{\Omega_r} x_3 dx = 0 \right\}.$$

This is the manifold of functions in $\mathbb{H}^s_\times(\mathbb{S}^2)$ for which the corresponding domains Ω_r have the volume of the unit ball and a geometric centre that is located at the origin. It is invariant under the nonlinear evolution, i.e. if r solves (2.1) and $r(0) \in \mathfrak{M}_\times$ then $r(t) \in \mathfrak{M}_\times$ for all t ([18] Lemma 5.5). Now we define

$$\mathbb{H}^s_{\times,1}(\mathbb{S}^2) := \{ r \in \mathbb{H}^s_\times(\mathbb{S}^2) : (r, Y_0)_0 = (r, Y_1)_0 = 0 \}$$

and $X_{k,1}$ as the orthoplement of $\langle Y_k \rangle$ in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$. On a sufficiently small neighbourhood $\mathcal{U}_\times$ of zero in $\mathbb{H}^s_\times(\mathbb{S}^2)$ we introduce the operator $\phi : \mathcal{U}_\times \to \mathbb{R}^2 \times \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ by

$$\phi(r) := (f(r), \mathcal{P}r)^T,$$

where $\mathcal{P}$ is $\mathbb{L}_2(\mathbb{S}^2)$-orthogonal projection on $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ and

$$f(r) := \left(\int_{\Omega_r} dx - \frac{4\pi}{3}, \int_{\Omega_r} x_3 dx \right).$$

The mapping ϕ is an analytic bijection between a neighbourhood of zero in $\mathbb{H}^s_\times(\mathbb{S}^2)$ and a neighbourhood of zero in $\mathbb{R}^2 \times \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$, see [18]. On a neighbourhood $\mathcal{U}_{\times,1}$ of zero in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ we define the analytic bijection $\psi : \mathcal{U}_{\times,1} \to \mathfrak{M}_\times$ by

$$\psi(r) = \phi^{-1}(0, 0, r)$$

and we define $\tilde{\mathcal{F}}_{\times,\mu} : \mathcal{U}_{\times,1} \to \mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ by

$$\tilde{\mathcal{F}}_{\times,\mu} := \mathcal{P} \circ \mathcal{F}_{\times,\mu} \circ \psi.$$

It is not hard to see that $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ is the tangent space of $\mathfrak{M}_\times$ at zero and $\psi'(0) : \mathbb{H}^s_{\times,1}(\mathbb{S}^2) \to \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ is the identity [18]. Consequently for $h \in \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$

$$\tilde{\mathcal{F}}'_{\times,\mu}(0)[h] = \mathcal{P}\mathcal{F}'_{\times,\mu}(0)[\psi'(0)[h]] = \mathcal{F}'_{\times,\mu}(0)[h]$$

and for the spectra of the operators $\tilde{\mathcal{F}}'_{\times,\mu}(0)$ and $\mathcal{F}'_{\times,\mu}(0)$ we have

$$\mathrm{sp}(\tilde{\mathcal{F}}'_{\times,\mu}(0)) = \mathrm{sp}(\mathcal{F}'_{\times,\mu}(0)) \setminus \{g_{0,\mu}, g_{1,\mu}\}.$$

Fix now $k \geq 2$. Introducing $\tilde{\rho}_k(\sigma) := \mathcal{P}\rho_k(\sigma)$ one gets from (3.1)

$$\tilde{\mathcal{F}}_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma)) = 0.$$

From [14] it is known that

$$\ker \tilde{\mathcal{F}}'_{\times,\mu_k}(0) = \left(R(\tilde{\mathcal{F}}'_{\times,\mu_k}(0)) \right)^\perp = \langle Y_k \rangle.$$

Here the orthoplement is taken in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$. From [14] Theorem 3.1 we know that in a neighbourhood of $(0, \mu_k)$ in $\mathbb{H}^s_\times(\mathbb{S}^2) \times \mathbb{R}$ any zero of $(r, \mu) \mapsto \mathcal{F}_{\times,\mu}(r)$ is either of the form $(0, \mu)$ or $(\rho_k(\sigma), m_k(\sigma))$. As a consequence, in a neighbourhood of $(0, \mu_k)$ in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2) \times \mathbb{R}$, any zero of $(r, \mu) \mapsto \tilde{\mathcal{F}}_{\times,\mu}(r)$ is either of the form $(0, \mu)$ or $(\tilde{\rho}_k(\sigma), m_k(\sigma))$. Combining these results one applies [2] Corollary 1.13 to find a an interval I_k, a $\delta_k > 0$ containing μ_k and continuously differentiable functions $\xi_k : I_k \to \mathbb{R}$, $\eta_k : (-\delta_k, \delta_k) \to \mathbb{R}$, $u_k : I_k \to \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$, and $w_k : (-\delta_k, \delta_k) \to \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ such that for $\varsigma \in I_k$ and $\sigma \in (-\delta_k, \delta_k)$

$$\tilde{\mathcal{F}}'_{\times,\varsigma}(0)[u_k(\varsigma)] = \xi_k(\varsigma) u_k(\varsigma)$$

and

$$\tilde{\mathcal{F}}'_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma))[w_k(\sigma)] = \eta_k(\sigma) w_k(\sigma).$$

Moreover, $\xi_k(\mu_k) = \eta_k(0) = 0$, $u_k(\mu_k) = w_k(0) = Y_k$, $u_k(\varsigma) - Y_k \in X_{k,1}$ for $\varsigma \in I_k$ and $w_k(\sigma) - Y_k \in X_{k,1}$ for $\sigma \in (-\delta_k, \delta_k)$. However, from (4.1) and continuity of ξ_k it follows that u_k is constant:

$$\forall \varsigma \in I_k : \xi_k(\varsigma) = g_{k,\varsigma},\ u_k(\varsigma) = Y_k.$$

Consequently for all $k \geq 2$ and $\varsigma \in I_k$

$$\xi_k'(\varsigma) = \frac{k+3}{4\pi} > 0. \tag{4.2}$$

Now we restrict our attention to $k \in 2\mathbb{N}$. Because of Lemma 3.1, $m_k(\sigma)$ is nonzero for small nonzero $|\sigma|$. From [2] Theorem 1.16 and (4.2) we get for small σ

$$\operatorname{sgn}(\eta_k(\sigma)) = -\operatorname{sgn}(\sigma m_k'(\sigma)\xi_k'(\mu_k)) = -\operatorname{sgn}(\sigma m_k'(\sigma)).$$

Lemma 3.1 yields

$$\operatorname{sgn}(\eta_k(\sigma)) = \operatorname{sgn}\sigma. \tag{4.3}$$

We conclude that for small positive σ, $\eta_k(\sigma)$ is a positive eigenvalue of $\tilde{\mathcal{F}}'_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma))$ and for small negative σ, $\eta_k(\sigma)$ is a negative eigenvalue of $\tilde{\mathcal{F}}'_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma))$. Now we consider the curve of bifurcation solutions for $k = 2$. Observe that zero is the largest element of $\operatorname{sp}(\tilde{\mathcal{F}}'_{\times,m_2(0)}(0))$. Now we prove that for small negative σ, $\operatorname{sp}(\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma)))$ is situated on the left of the imaginary axis and show stability of the disk-like shapes, see Figure 1.

Lemma 4.1. *There exists a positive δ such that for $\sigma \in (-\delta, 0)$ the operator $\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma))$ generates an analytic contraction semigroup on $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ with domain of definition $\mathbb{H}^{s}_{\times,1}(\mathbb{S}^2)$.*

Proof. To shorten notation we introduce $F_\sigma := \tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma))$, $Y := \mathbb{H}^{s}_{\times,1}(\mathbb{S}^2)$ and $X := \mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ with norms $\|\cdot\|_Y$ and $\|\cdot\|_X$. Let $\mathcal{H}(Y, X)$ be the set of operators that generate analytic semigroups on X with dense domain of definition $Y = \mathcal{D}(F_0)$ and let $\rho(F_\sigma)$ be the resolvent set of F_σ.

From [18] it is known that $F_0 \in \mathcal{H}(Y, X)$. Therefore there are $M > 0$, $\omega \in \mathbb{R}$ and $\vartheta \in (\frac{\pi}{2}, \pi)$ such that $S := \{z \in \mathbb{C} : |\arg(z - \omega) < \vartheta\} \subseteq \rho(F_0)$ and for $\lambda \in S$ we have

$$\|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|}.$$

Here $\mathcal{R}(\lambda, F_0) : X \to X$ is the resolvent of F_0 and $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)})$ is the space of bounded operators on X. Endow Y with the graph norm

$$\|x\|_{\mathcal{D}(F_0)} := \|x\|_X + \|F_0 x\|_X,$$

which is equivalent to the norm $\|\cdot\|_Y$.

Fix $\kappa \in (0, 1)$. Note at first that the mapping $\sigma \mapsto F_\sigma$ is continuous with values in $\mathcal{L}(\mathcal{D}(F_0), X)$. Therefore, $F_\sigma \in \mathcal{H}(Y, X)$ and

$$\|F_\sigma - F_0\|_{\mathcal{L}(\mathcal{D}(F_0),X)} < \frac{\kappa}{2(1+M)}, \tag{4.4}$$

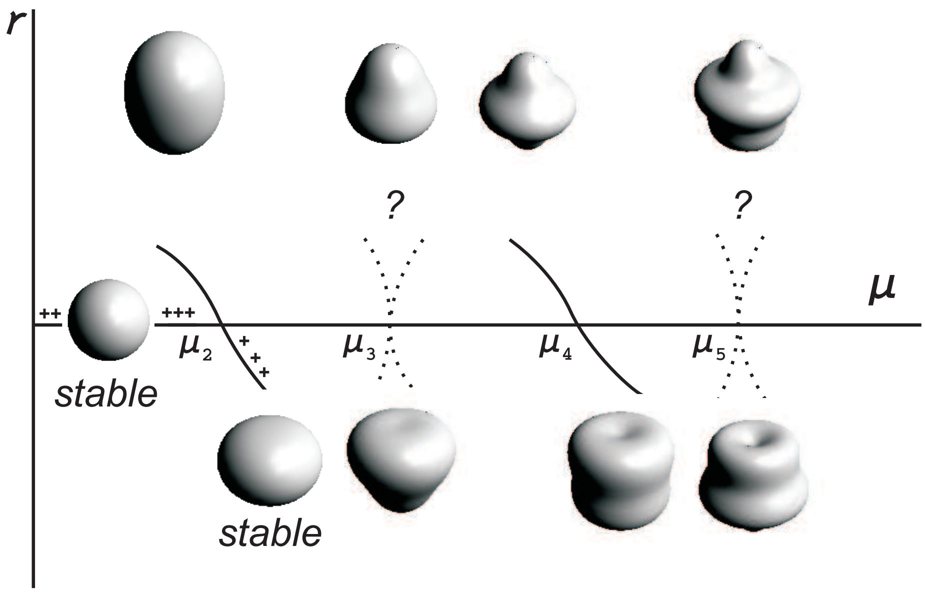

Figure 1: A sketch of the curves of nontrivial stationary domains bifurcating from the unit ball. We also see domains of the type $\Omega_{\pm\epsilon Y_k}$. These approximate our bifurcation solutions. The solutions approximated by $\Omega_{-\epsilon Y_2}$ for positive ϵ are stable.

Due to the additional symmetry of the problem with respect to the reflection $(x, y, z) \mapsto (x, y, -z)$, if (ρ, μ) is a solution to the bifurcation problem then so is $(\hat{\rho}, \mu)$ where $\hat{\rho}$ is defined by $\Omega_{\hat{\rho}} = -\Omega_{\rho}$. For even k, uniqueness and Lemma 3.1 imply $\rho = \hat{\rho}$ on the bifurcation branches. If k is odd, then the branch consists of pairs of different solutions ρ and $\hat{\rho}$ for the same value of μ, and a pitchfork bifurcation occurs. However, we do not know whether $\mu < \mu_k$ or $\mu > \mu_k$ on these branches.

for $|\sigma|$ sufficiently small.

For $\lambda \in S$

$$\begin{aligned}\|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X,\mathcal{D}(F_0))} &\leq \|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} + \|F_0\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} \\ &\leq \|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} + \|(\lambda\mathcal{I} - F_0)\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} + |\lambda|\|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} \\ &\leq 1 + \frac{(1+|\lambda|)M}{|\lambda - \omega|}.\end{aligned}$$

Combining this and (4.4) one sees that there exists a $\Lambda > 0$ such that if $\lambda \in S_\Lambda := \{z \in S : |z| > \Lambda\}$ then

$$\|(F_\sigma - F_0)\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} < \kappa. \tag{4.5}$$

For $\lambda \in S_\Lambda \subseteq \rho(F_0)$ and $f \in X$ we consider the problem

$$\lambda u - F_\sigma u = f, \tag{4.6}$$

that is to be solved for $u \in \mathcal{D}(F_0)$. Introducing

$$v := \lambda u - F_0 u$$

we get the following problem that is equivalent to (4.6):

$$v = (F_\sigma - F_0)\mathcal{R}(\lambda, F_0)v + f. \tag{4.7}$$

From (4.5) it follows that the mapping

$$\mathcal{Z} : v \mapsto (F_\sigma - F_0)\mathcal{R}(\lambda, F_0)v + f$$

defines a contraction on X. The Banach Contraction Theorem yields that there exists a unique $v' \in X$ such that $\mathcal{Z}(v') = v'$. As a consequence,

$$u' := \mathcal{R}(\lambda, F_0)v' \tag{4.8}$$

solves (4.6) uniquely. Because of (4.5) and (4.7) we have $\|v'\|_X \leq \kappa\|v'\|_X + \|f\|_X$ and get

$$\|u'\|_X \leq C_\lambda\|v'\|_X \leq \frac{C_\lambda}{1-\kappa}\|f\|_X,$$

for some C_λ depending only on λ. Consequently, $S_\Lambda \subseteq \rho(F_\sigma)$. It follows from the continuity of $\sigma \mapsto F_\sigma$ and the perturbation result given in [2], Lemma 1.3, that for sufficiently small $\delta, \zeta > 0$ we have

$$\mathrm{sp}(F_\sigma) \cap \{|z| < \zeta\} = \{\eta_2(\sigma)\}. \tag{4.9}$$

if $|\sigma| < \delta$. Define the compact sets

$$K := \left\{z \in \mathbb{C} \setminus S_\Lambda^c : \mathrm{Re}\, z \geq \frac{1}{2}g_{3,\mu_2}\right\}, \quad K_\zeta := \{z \in K : |z| \geq \zeta\}.$$

As $(\lambda, \sigma) \mapsto \lambda \mathcal{I} - F_\sigma$ is continuous and $\mathrm{sp}(F_0) \cap K_\zeta = \emptyset$ it follows from a standard compactness argument that

$$\mathrm{sp}(F_\sigma) \cap K_\zeta = \emptyset \tag{4.10}$$

for all σ sufficiently close to zero.

From (4.9), (4.10), and (4.3) we conclude that for σ negative and near zero, $\mathrm{sp}(F_\sigma)$ is in the open left half-space of the complex plane. This completes the proof. $\square$

From Lemma 4.1 we derive a stability result. First we make the evolution equation (2.1) autonomous by introducing a new time variable τ such that

$$\frac{d\tau}{dt} = \frac{1}{\alpha(t)^3}.$$

It follows that

$$\tau = \tau(t) = -\frac{4\pi}{3\mu} \ln \left(1 - \frac{3\mu t}{4\pi} \right), \tag{4.11}$$

and

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\times,\mu}(r).$$

Note that $t = \frac{4\pi}{3\mu}$ corresponds to $\tau = \infty$.

Now we state our main result in which we regard r as a function of τ.

Theorem 4.2. *(Stability of bifurcation solutions) Suppose that $s > 5$. There exists a positive δ_1, such that if $\sigma \in (-\delta_1, 0)$ and $\lambda \in (0, -\eta_2(\sigma))$, then there exists a δ_2 and an $M > 0$ such that the following statement holds. If $r_0 \in \mathfrak{M}_\times$ and $\|r_0 - \rho_2(\sigma)\|_s < \delta_2$, then there exists a solution $r \in \mathcal{C}([0,\infty), \mathbb{H}^s(\mathbb{S}^2)) \cap \mathcal{C}^1([0,\infty), \mathbb{H}^{s-3}(\mathbb{S}^2))$ to*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\times, m_2(\sigma)}(r), \quad r(0) = r_0.$$

Furthermore, for $\tau \in [0,\infty)$ we have $r(\tau) \in \mathfrak{M}_\times$ and

$$\|r(\tau) - \rho_2(\sigma)\|_s \le M e^{-\lambda\tau} \|r_0 - \rho_2(\sigma)\|_s.$$

Proof. First we show solvability of the problem

$$\frac{\partial \tilde{r}}{\partial \tau} = \tilde{\mathcal{F}}_{\times, m_2(\sigma)}(\tilde{r}), \tag{4.12}$$

with $\tilde{r}(0) = \mathcal{P} r_0$ near $\tilde{\rho}_2(\sigma)$. Since $\mathcal{I} + \tilde{\mathcal{F}}'_{\times, m_2(0)}(0)$ is an isomorphism between $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ and $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ (see [18]), $\mathcal{I} + \tilde{\mathcal{F}}'_{\times, m_2(\sigma)}(\tilde{\rho}_2(\sigma))$ is an isomorphism between $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ and $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ as well for σ near zero. This means that the graph norm of $\tilde{\mathcal{F}}'_{\times, m_2(\sigma)}(\tilde{\rho}_2(\sigma))$ is equivalent to $\|\cdot\|_s$. Since $\tilde{\mathcal{F}}_{\times, m_2(\sigma)}$ is analytic [18] it follows from Lemma 4.1 and [12] Theorem 9.1.2 that there exists a solution to (4.12) that satisfies

$$\|\tilde{r}(\tau) - \tilde{\rho}_2(\sigma)\|_s \le \tilde{M} e^{-\lambda\tau} \|\mathcal{P} r_0 - \tilde{\rho}_2(\sigma)\|_s,$$

for $\tilde{M}$ independent of $\mathcal{P} r_0$. This estimate follows from the fact that $\mathrm{sp}(\tilde{\mathcal{F}}'_{\times, m_2(\sigma)}(\tilde{\rho}_2(\sigma)))$ is on the left of the line $\mathrm{Re}\, z = -\lambda$. Now, $r = \psi(\tilde{r})$ solves the original problem and since ψ is analytic we have the estimate as well. $\square$

In view of (4.11), this exponential decay in τ translates into algebraic decay in t:

$$\|r(t)-\rho_2(\sigma)\|_s \leq M\left(1-\frac{3\mu t}{4\pi}\right)^{\lambda\frac{4\pi}{3\mu}}\|r_0-\rho_2(\sigma)\|_s.$$

A The structure of the evolution operators

Assume that $s>5$. Let r be small in $\mathbb{H}^s(\mathbb{S}^2)$, and let Γ_r be the boundary of Ω_r. Introduce

- $\tilde{z}(r,\cdot):\mathbb{S}^2\to\Gamma_r$ by
$$\tilde{z}(r,x)=(1+r(x))\,x,$$
- $n(r,\cdot)$ as the function that maps an element $x\in\mathbb{S}^2$ to the exterior unit normal vector on Γ_r at the point $\tilde{z}(r,x)$,
- $\kappa(r,\cdot)$ as the function that maps an element $x\in\mathbb{S}^2$ to the mean curvature of Γ_r at $\tilde{z}(r,x)$.

Let $\mathrm{Tr}\in\mathcal{L}(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^3),\mathbb{H}^s(\mathbb{S}^2))$ denote the trace operator. There exists an extension operator $E\in\mathcal{L}(\mathbb{H}^s(\mathbb{S}^2),\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^3))$ such that $\mathrm{Tr}\circ E$ is the identity (cf. [15] Theorem 6.108).

Introduce $z:\mathbb{H}^s(\mathbb{S}^2)\to\left(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^3)\right)^3$ by

$$z(r,x):=(1+(Er)(x))\,x.$$

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^2)$ we define the operators

- $\mathcal{A}:\mathcal{U}\to\mathcal{L}(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^3),\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^3))$ and $\mathcal{Q}:\mathcal{U}\to\mathcal{L}\left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^3),\left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^3)\right)^3\right)$ by
$$\mathcal{A}(r)u:=\left(\Delta\left(u\circ z(r)^{-1}\right)\right)\circ z(r),$$
$$\mathcal{Q}(r)u:=\left(\nabla\left(u\circ z(r)^{-1}\right)\right)\circ z(r).$$
- $\mathcal{S}:\mathcal{U}\to\mathcal{L}(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^3),\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^3)\times\mathbb{H}^{s-2}(\mathbb{S}^2))$ by
$$\mathcal{S}(r)u:=\begin{pmatrix}\mathcal{A}(r)u\\ \mathrm{Tr}u\end{pmatrix}.$$
- $\mathcal{E}:\mathcal{U}\to\mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^2),\mathbb{H}^{s-3}(\mathbb{S}^2))$ by
$$\mathcal{E}(r)u:=\frac{\mathrm{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\begin{bmatrix}0\\u\end{bmatrix}\right]\right)\cdot n(r)}{n(r)\cdot\mathrm{id}}.$$
- $\varphi:\mathcal{U}\to\mathbb{H}^s(\mathbb{S}^2)$ by
$$\varphi(r):=\frac{1}{4\pi(1+r)}-\frac{1}{4\pi}.$$

- $l : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^2)$ by
$$l(r) := \frac{1}{4\pi(1+r)^2} - \frac{1+r}{4\pi}.$$

These operators are well-defined (see [18]). The third-order operator $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-3}(\mathbb{S}^2)$ and first-order operator $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^2)$ are defined by

$$\mathcal{F}_1(r) = \mathcal{E}(r)\kappa(r),$$

$$\mathcal{F}_2(r) = \mathcal{E}(r)\varphi(r) + l(r).$$

B The sign of $m_k'(0)$

We start by parameterising $\mathbb{S}^2$ by spherical coordinates:

$$(\theta, \phi) \mapsto \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.$$

In this appendix, $m_k'(0)$ will be calculated in terms of I_k and J_k defined by

$$I_k := \int_{\mathbb{S}^2} (Y_k)^3 d\sigma, \quad J_k := \int_{\mathbb{S}^2} Y_k \left(\frac{\partial Y_k}{\partial \theta}\right)^2 d\sigma.$$

We note that
$$I_k = \begin{cases} (4\pi)^{-1/2}(2k+1)^{3/2}\frac{k!^3(3k/2)!^2}{(3k+1)!(k/2)!^6} & k \text{ even}, \\ 0 & k \text{ odd}, \end{cases} \tag{B.1}$$

see e.g. [13], Eqns. (C.16) and (C.23). Moreover, integration by parts shows

$$J_k = -J_k - \int_{\mathbb{S}^2} Y_k^2 \Delta_0 Y_k \, d\sigma = -J_k + k(k+1)I_k,$$

where Δ_0 is the Laplace-Beltrami operator on $\mathbb{S}^2$, hence

$$J_k = \frac{1}{2}k(k+1)I_k. \tag{B.2}$$

Thus, I_k and J_k are positive for k even and zero for k odd.

Lemma 3.1 will follow if we combine this fact with Lemmas B.1 and B.6.

We use $(\cdot,\cdot)_0$ to denote the $\mathbb{L}_2(\mathbb{S}^2)$-inner product and introduce the Dirichlet-to-Neumann operator $\mathcal{N} : \mathbb{H}^s(\mathbb{S}^2) \to \mathbb{H}^{s-1}(\mathbb{S}^2)$ by

$$\mathcal{N}u := \frac{\partial}{\partial n}\mathcal{S}(0)^{-1}\begin{pmatrix} 0 \\ u \end{pmatrix}.$$

It is known (see [18]) that
$$\mathcal{N}Y_k = kY_k,$$

for all $k \in \mathbb{N}_0$. In [18] is has been calculated that

$$\begin{aligned}
\mathcal{F}_1'(0) &= -\mathcal{N}^3 - \mathcal{N}^2 + 2\mathcal{N}, \\
\mathcal{F}_2'(0) &= -\frac{1}{4\pi}\left(\mathcal{N} + 3\mathcal{I}\right), && \text{(B.3)} \\
\mathcal{E}(0) &= \mathcal{N}, \\
\kappa'(0) &= -\mathcal{N}^2 - \mathcal{N} + 2\mathcal{I}, && \text{(B.4)} \\
\varphi'(0) &= -\frac{1}{4\pi}\mathcal{I}, && \text{(B.5)} \\
l'(0) &= -\frac{3}{4\pi}\mathcal{I},
\end{aligned}$$

where $\mathcal{I}$ is the identity operator.

Lemma B.1. *We have*

$$m_k'(0) = -\frac{2\pi}{k+3}(\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k], Y_k)_0.$$

Proof. Differentiating the expression

$$\gamma\mathcal{F}_1(\rho_k(\sigma)) - m_k(\sigma)\mathcal{F}_2(\rho_k(\sigma)) = 0$$

twice with respect to σ gives

$$\begin{aligned}
0 = {} & \gamma\mathcal{F}_1''(\rho_k(\sigma))[\rho_k'(\sigma), \rho_k'(\sigma)] - m_k(\sigma)\mathcal{F}_2''(\rho_k(\sigma))[\rho_k'(\sigma), \rho_k'(\sigma)] \\
& + \gamma\mathcal{F}_1'(\rho_k(\sigma))[\rho_k''(\sigma)] - m_k(\sigma)\mathcal{F}_2'(\rho_k(\sigma))[\rho_k''(\sigma)] \\
& - 2m_k'(\sigma)\mathcal{F}_2'(\rho_k(\sigma))[\rho_k'(\sigma)] - m_k''(\sigma)\mathcal{F}_2(\rho_k(\sigma)).
\end{aligned}$$

Setting $\sigma = 0$ and using $\rho_k(0) = 0$, $\mathcal{F}_2(0) = 0$, and $\rho_k'(0) = Y_k$ one obtains

$$\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k] + \mathcal{F}'_{\times,\mu_k}(0)[\rho_k''(0)] = 2m_k'(0)\mathcal{F}_2'(0)[Y_k]. \tag{B.6}$$

From (B.3) one has

$$(\mathcal{F}_2'(0)[Y_k], Y_k)_0 = -\frac{1}{4\pi}(k+3).$$

Moreover, since $\rho_k''(0) = 2f_k'(0) \in \langle Y_k \rangle^\perp$ (see (3.2)) and $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ respect the decomposition $\mathbb{H}^s_\times(\mathbb{S}^2) = \langle Y_k \rangle \oplus X_k$, we get

$$(\mathcal{F}'_{\times,\mu_k}(0)[\rho_k''(0)], Y_k) = 0.$$

Taking the inner product with Y_k on both sides of (B.6) and applying the last two equations yields the result. $\square$

Now we introduce the vector spherical harmonics $\vec{V}_k, \vec{W}_k : \mathbb{S}^2 \to \mathbb{R}^3$ conform [9] and [10] in the following way:

$$\vec{V}_k := -\sqrt{\frac{k+1}{2k+1}} Y_k e_\rho + \frac{1}{\sqrt{(k+1)(2k+1)}} \frac{\partial Y_k}{\partial \theta} e_\theta$$

$$\vec{W}_k := \sqrt{\frac{k}{2k+1}} Y_k e_\rho + \frac{1}{\sqrt{k(2k+1)}} \frac{\partial Y_k}{\partial \theta} e_\theta, \tag{B.7}$$

for $k \in \mathbb{N}_0$. Here e_ρ and e_θ and are the usual unit vectors corresponding to spherical coordinates.

From [9] or [10] we have the following formulas:

$$Y_k e_\rho = -\sqrt{\frac{k+1}{2k+1}} \vec{V}_k + \sqrt{\frac{k}{2k+1}} \vec{W}_k, \tag{B.8}$$

$$\nabla_0 Y_k = k\sqrt{\frac{k+1}{2k+1}} \vec{V}_k + (k+1)\sqrt{\frac{k}{2k+1}} \vec{W}_k = \frac{\partial Y_k}{\partial \theta} e_\theta, \tag{B.9}$$

$$\nabla(\rho^k Y_k) = \rho^{k-1} \sqrt{k(2k+1)} \vec{W}_k, \tag{B.10}$$

where the surface gradient ∇_0 maps a function $y : \mathbb{S}^2 \to \mathbb{R}$ to $\nabla E y - \frac{\partial E y}{\partial \rho} e_\rho$. Here and in the sequel the expression $\rho^k Y_k$ should be interpreted as the function that maps an element of $\mathbb{B}^3$ characterised by spherical coordinates to $\rho^k Y_k(\theta, \phi)$. Note that (B.10) can be obtained combining (B.8) and (B.9).

Lemma B.2. *We have*

$$\mathcal{A}'(0)[y] = -\Delta((Ey)\,\mathrm{id}\cdot\nabla) + ((Ey)\,\mathrm{id}\cdot\nabla)\Delta,$$

$$\mathcal{Q}'(0)[y] = -(\nabla E y)(\mathrm{id}\cdot\nabla) - (Ey)\nabla.$$

Proof. One obtains the first identity taking the Fréchet derivative of the identity

$$\mathcal{A}(r)\,(u \circ z(r)) = (\Delta u) \circ z(r)$$

for any suitable function u at $r = 0$ and using $\mathcal{A}(0) = \Delta$, $z(0) = \mathrm{id}$ and $z'(0)[y] = (Ey)\,\mathrm{id}$. In a similar way one finds

$$\mathcal{Q}'(0)[y] = -\nabla((Ey)\,\mathrm{id}\cdot\nabla) + (Ey)\,\mathrm{id}\cdot H,$$

where H stands for the Hessian. Therefore

$$\begin{aligned}
\mathcal{Q}'(0)[y] &= -(\nabla E y)(\mathrm{id}\cdot\nabla) - (Ey)\nabla(\mathrm{id}\cdot\nabla) + (Ey)\,\mathrm{id}\cdot H \\
&= -(\nabla E y)(\mathrm{id}\cdot\nabla) - (Ey)\nabla - (Ey)\,\mathrm{id}\cdot H + (Ey)\,\mathrm{id}\cdot H \\
&= -(\nabla E y)(\mathrm{id}\cdot\nabla) - (Ey)\nabla.
\end{aligned}$$

□

Lemma B.3. *We have*

$$(\mathcal{E}'(0)[Y_k]\kappa'(0)[Y_k], Y_k)_0 = (k^2+k-2)(kI_k+J_k).$$

Proof. From (B.4) we have

$$\kappa'(0)[Y_k] = -(k^2+k-2)Y_k,$$

so the lemma will be proved by showing

$$(\mathcal{E}'(0)[Y_k]Y_k, Y_k)_0 = -(kI_k+J_k). \tag{B.11}$$

Using $|n(r)| = 1$ and the consequence $n'(0)[Y_k]\cdot \mathrm{id} = 0$ we get

$$\mathcal{E}'(0)[Y_k]Y_k = A + B + C,$$

where A, B and C are given by

$$\begin{aligned}
A &= \nabla\mathcal{S}(0)^{-1}(0, Y_k)^\top \cdot n'(0)[Y_k],\\
B &= \mathcal{Q}'(0)[Y_k]\mathcal{S}(0)^{-1}(0, Y_k)^\top \cdot n(0),\\
C &= -\nabla\mathcal{S}(0)^{-1}\mathcal{S}'(0)[Y_k]\mathcal{S}(0)^{-1}(0, Y_k)^\top \cdot n(0),
\end{aligned}$$

with the trace operators suppressed for the sake of brevity.

Introduce $U : \mathbb{B}^3 \to \mathbb{R}$ by

$$U := \mathcal{S}(0)^{-1}(0\ Y_k)^\top = \rho^k Y_k.$$

For later use we note that by (B.10) and (B.7)

$$\nabla U = \rho^{k-1}\sqrt{k(2k+1)}\vec{W}_k, \qquad \mathrm{id}\cdot\nabla U = k\rho^k Y_k. \tag{B.12}$$

It is known that

$$n'(0)[Y_k] = -\nabla Y_k = -\frac{\partial Y_k}{\partial\theta}e_\theta. \tag{B.13}$$

(See e.g. the proof of Lemma 2.5 in [16]) Consequently, using (B.7) and (B.12)

$$A = \mathrm{Tr}\nabla U\cdot n'(0)[Y_k] = -\sqrt{k(2k+1)}\mathrm{Tr}(\rho^{k-1}\vec{W}_k)\cdot\frac{\partial Y_k}{\partial\theta}e_\theta = -\left(\frac{\partial Y_k}{\partial\theta}\right)^2.$$

Further, Lemma B.2 and (B.12) yield

$$\mathcal{Q}'(0)[Y_k]U = -\nabla(EY_k)k\rho^k Y_k - EY_k\left[\rho^{k-1}\sqrt{k(2k+1)}\vec{W}_k\right]$$

and by (B.7)

$$B = \mathrm{Tr}(\mathcal{Q}'(0)[Y_k]U)\cdot e_\rho = -k\left(\frac{\partial}{\partial n}(EY_k) + Y_k\right)Y_k.$$

To determine C, we use Lemma B.2 and the fact that U is harmonic. Setting

$$\Phi := (EY_k)\,\mathrm{id}\cdot\nabla U = k(EY_k)\rho^k Y_k$$

we find

$$
\begin{aligned}
C &= -\frac{\partial}{\partial n}\mathcal{S}(0)^{-1}(\mathcal{A}'(0)[Y_k]U,\ 0)^\top \\
&= \frac{\partial}{\partial n}\mathcal{S}(0)^{-1}(\Delta\Phi,\ 0)^\top = \frac{\partial}{\partial n}\big(\mathcal{S}(0)^{-1}(\Delta\Phi,\ \mathrm{Tr}\Phi)^\top - \mathcal{S}(0)^{-1}(0,\ \mathrm{Tr}\Phi)^\top\big) \\
&= \frac{\partial\Phi}{\partial n} - \mathcal{N}(\mathrm{Tr}\Phi) = k\frac{\partial}{\partial n}(EY_k)Y_k + k^2Y_k^2 - k\mathcal{N}(Y_k^2)
\end{aligned}
$$

Adding the results, taking the inner product with Y_k and using the symmetry of $\mathcal{N}$ we get (B.11). □

Lemma B.4. *We have*

$$
(\mathcal{N}\kappa''(0)[Y_k, Y_k], Y_k)_0 = 4k(k^2+k-1)I_k.
$$

Proof. For each $r \in \mathbb{H}^s(\mathbb{S}^2)$ introduce

$$
\mathcal{G}(r) := \sum_{i,j=1}^{2}\left(\frac{\partial\tilde{z}(r)}{\partial\omega_i}\right)^T\left(\frac{\partial\tilde{z}(r)}{\partial\omega_j}\right)e_i\otimes e_j = (1+r)^2\mathcal{G}_0 + \frac{\partial r}{\partial\omega}\otimes\frac{\partial r}{\partial\omega}
$$

and

$$
g(r) := \det\mathcal{G}(r),
$$

where $\omega_1 = \theta, \omega_2 = \phi$,

$$
\mathcal{G}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}
$$

and

$$
\frac{\partial r}{\partial\omega}\otimes\frac{\partial r}{\partial\omega} = \begin{pmatrix} \left(\frac{\partial r}{\partial\theta}\right)^2 & \frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\phi} \\ \frac{\partial r}{\partial\theta}\frac{\partial r}{\partial\phi} & \left(\frac{\partial r}{\partial\phi}\right)^2 \end{pmatrix}.
$$

The Laplace-Beltrami operator on Γ_r can be expressed in terms of $\mathcal{G}(r)$ as

$$
\mathcal{B}(r) := \sum_{i,j}\frac{1}{\sqrt{g(r)}}\frac{\partial}{\partial\omega_i}\left[\sqrt{g(r)}g^{ij}(r)\frac{\partial}{\partial\omega_j}\right],
$$

where $g^{ij}(r)$ are the components of the inverse of $\mathcal{G}(r)$. For the Laplace-Beltrami operator on $\mathbb{S}^2$ we have

$$
\Delta_0 = \mathcal{B}(0) = -\mathcal{N}^2 - \mathcal{N}. \tag{B.14}
$$

We have the following expansions around $r=0$:

$$
\begin{aligned}
\mathcal{G}(r) &= \mathcal{G}_0\left(\mathcal{I}+2r\mathcal{I}+r^2\mathcal{I}+\mathcal{G}_0^{-1}\frac{\partial r}{\partial\omega}\otimes\frac{\partial r}{\partial\omega}\right),\\
\mathcal{G}(r)^{-1} &= \mathcal{G}_0^{-1}\left(\mathcal{I}-2r\mathcal{I}+3r^2\mathcal{I}-\frac{\partial r}{\partial\omega}\otimes\frac{\partial r}{\partial\omega}\mathcal{G}_0^{-1}+\mathcal{O}(r^3)\right),\\
g(r) &= \sin^2\theta\left(1+4r+6r^2+\left(\frac{\partial r}{\partial\theta}\right)^2+\csc^2\theta\left(\frac{\partial r}{\partial\phi}\right)^2+\mathcal{O}(r^3)\right),\\
\sqrt{g(r)} &= \sin\theta\left(1+2r+r^2+\frac{1}{2}\left(\frac{\partial r}{\partial\theta}\right)^2+\frac{1}{2}\csc^2\theta\left(\frac{\partial r}{\partial\phi}\right)^2+\mathcal{O}(r^3)\right),\\
X(r) &:= \sqrt{g(r)}\mathcal{G}(r)^{-1}\\
&= \sin\theta\mathcal{G}_0^{-1}\left(\mathcal{I}-\frac{\partial r}{\partial\omega}\otimes\frac{\partial r}{\partial\omega}\mathcal{G}_0^{-1}+\frac{1}{2}\left(\frac{\partial r}{\partial\theta}\right)^2\mathcal{I}+\frac{1}{2}\csc^2\theta\left(\frac{\partial r}{\partial\phi}\right)^2\mathcal{I}+\mathcal{O}(r^3)\right),\\
Z(r) &:= \frac{1}{\sqrt{g(r)}}=\csc\theta\left(1-2r+3r^2-\frac{1}{2}\left(\frac{\partial r}{\partial\theta}\right)^2-\frac{1}{2}\csc^2\theta\left(\frac{\partial r}{\partial\phi}\right)^2+\mathcal{O}(r^3)\right).
\end{aligned}
$$

From these expansions it follows that for any $h\in\mathbb{H}^s(\mathbb{S}^2)$

$$
\begin{aligned}
\mathcal{G}'(0)[h] &= 2h\mathcal{G}_0,\\
\mathcal{G}''(0)[h,h] &= 2h^2\mathcal{G}_0+2\frac{\partial h}{\partial\omega}\otimes\frac{\partial h}{\partial\omega},\\
X'(0)[h] &= 0.
\end{aligned}
$$

Since zonal harmonics do not depend on the azimuthal coordinate ϕ we have

$$X''(0)[Y_k,Y_k]=\left(\frac{\partial Y_k}{\partial\theta}\right)^2\begin{pmatrix}-\sin\theta & 0\\ 0 & \csc\theta\end{pmatrix}, \tag{B.15}$$

$$Z''(0)[Y_k,Y_k]=\csc\theta\left(6(Y_k)^2-\left(\frac{\partial Y_k}{\partial\theta}\right)^2\right), \tag{B.16}$$

$$\mathcal{B}'(0)[Y_k]=-2Y_k\Delta_0, \tag{B.17}$$

$$\begin{aligned}\mathcal{B}''(0)[Y_k,Y_k]=&\left(6(Y_k)^2-\left(\frac{\partial Y_k}{\partial\theta}\right)^2\right)\Delta_0+\csc\theta\frac{\partial}{\partial\theta}\left[-\sin\theta\left(\frac{\partial Y_k}{\partial\theta}\right)^2\frac{\partial}{\partial\theta}\right]\\
&+\csc^2\theta\left(\frac{\partial Y_k}{\partial\theta}\right)^2\frac{\partial^2}{\partial\phi^2}.\end{aligned} \tag{B.18}$$

From the identity

$$\kappa(r)=(\mathcal{B}(r)\tilde{z}(r))\cdot n(r)$$

(see [3] Thm. 1 in Section 2.5) it follows that

$$\kappa''(0)[Y_k, Y_k] = A + B + C + D + E + F,$$

with

$$\begin{aligned}
A &= (\mathcal{B}''(0)[Y_k, Y_k]\tilde{z}(0)) \cdot n(0), \\
B &= (\Delta_0 \tilde{z}''(0)[Y_k, Y_k]) \cdot n(0), \\
C &= (\Delta_0 \tilde{z}(0)) \cdot n''(0)[Y_k, Y_k], \\
D &= 2(\mathcal{B}'(0)[Y_k]\tilde{z}'(0)[Y_k]) \cdot n(0), \\
E &= 2(\mathcal{B}'(0)[Y_k]\tilde{z}(0)) \cdot n'(0)[Y_k], \\
F &= 2(\Delta_0 \tilde{z}'(0)[Y_k]) \cdot n'(0)[Y_k].
\end{aligned}$$

Since $\tilde{z}(0) = n(0) = \mathrm{id}$ and $\frac{\partial\,\mathrm{id}}{\partial\theta} \perp \mathrm{id}$, $\frac{\partial\,\mathrm{id}}{\partial\phi} \perp \mathrm{id}$ we obtain

$$\begin{aligned}
A &= \left(6(Y_k)^2 - \left(\frac{\partial Y_k}{\partial\theta}\right)^2\right)(\Delta_0\,\mathrm{id}) \cdot \mathrm{id} - \left(\frac{\partial Y_k}{\partial\theta}\right)^2 \frac{\partial^2\,\mathrm{id}}{\partial\theta^2} \cdot \mathrm{id} + \csc^2\theta \left(\frac{\partial Y_k}{\partial\theta}\right)^2 \frac{\partial^2\,\mathrm{id}}{\partial\phi^2} \cdot \mathrm{id} \\
&= \left(6(Y_k)^2 - \left(\frac{\partial Y_k}{\partial\theta}\right)^2\right)(\Delta_0\,\mathrm{id}) \cdot \mathrm{id} = -12(Y_k)^2 + 2\left(\frac{\partial Y_k}{\partial\theta}\right)^2.
\end{aligned}$$

The last step follows from (B.14) and the identity $\mathcal{N}\,\mathrm{id} = \mathrm{id}$. We have $B \equiv 0$, because $z''(0) \equiv 0$. Taking the second Fréchet derivative at zero of the expression

$$n(r) \cdot n(r) = 1$$

we obtain from (B.13)

$$n''(0)[h, h] \cdot \mathrm{id} = -n'(0)[h] \cdot n'(0)[h] = -\left(\frac{\partial h}{\partial\theta}\right)^2.$$

Therefore,

$$C = 2\left(\frac{\partial Y_k}{\partial\theta}\right)^2.$$

In the following identities, that follow from [10] (B-6 a) and (B-6 c), Δ_0 has to be applied to every component of the vector fields separately.

$$\Delta_0 \vec{V}_k = -(k+1)(k+2)\vec{V}_k, \quad \Delta_0 \vec{W}_k = -k(k-1)\vec{W}_k. \tag{B.19}$$

From (B.8), (B.17), and (B.19) we get

$$\begin{aligned}
D &= 2\mathcal{B}'(0)[Y_k](Y_k\,\mathrm{id})\cdot \mathrm{id} = -4Y_k\Delta_0(Y_k\,\mathrm{id})\cdot \mathrm{id}\\
&= -4Y_k\Delta_0\left[-\sqrt{\frac{k+1}{2k+1}}\vec{V}_k + \sqrt{\frac{k}{2k+1}}\vec{W}_k\right]\cdot \mathrm{id}\\
&= 4\left[\frac{(k+1)^2(k+2)}{2k+1} + \frac{k^2(k-1)}{2k+1}\right](Y_k)^2\\
&= 4(k^2+k+2)(Y_k)^2.
\end{aligned}$$

Furthermore,

$$E = -4Y_k(\Delta_0\,\mathrm{id})\cdot n'(0)[Y_k] = 8Y_k\,\mathrm{id}\cdot n'(0)[Y_k] = 0.$$

Finally

$$\begin{aligned}
F &= 2(\Delta_0(Y_k\,\mathrm{id}))\cdot -\nabla_0 Y_k\\
&= 2\Delta_0\left[-\sqrt{\frac{k+1}{2k+1}}\vec{V}_k + \sqrt{\frac{k}{2k+1}}\vec{W}_k\right]\cdot -\nabla_0 Y_k\\
&= \left[2(k+1)(k+2)\sqrt{\frac{k+1}{2k+1}}\vec{V}_k - 2k(k-1)\sqrt{\frac{k}{2k+1}}\vec{W}_k\right]\cdot -\frac{\partial Y_k}{\partial\theta}e_\theta\\
&= \left[-2\frac{(k+1)(k+2)}{2k+1} + 2\frac{k(k-1)}{2k+1}\right]\left(\frac{\partial Y_k}{\partial\theta}\right)^2 = -4\left(\frac{\partial Y_k}{\partial\theta}\right)^2.
\end{aligned}$$

The lemma follows by adding the results. □

Lemma B.5. *We have*

$$(\mathcal{E}'(0)[Y_k]\varphi'(0)[Y_k], Y_k)_0 = \frac{1}{4\pi}(kI_k + J_k),$$

$$(\mathcal{N}\varphi''(0)[Y_k, Y_k], Y_k)_0 = \frac{k}{2\pi}I_k.$$

Proof. The first statement follows from (B.5) and (B.11). The second statement follows from the definition of φ. □

Lemma B.6. *We have*

$$\begin{aligned}
&(\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k], Y_k)_0\\
&= \frac{2k(k^3+7k^2+6k-6)}{k+3}I_k + \frac{6(k^2+k-2)}{k+3}J_k.
\end{aligned}$$

Proof. We have

$$\mathcal{F}''_{\times,\mu_k}(0) = \gamma\left(\mathcal{F}''_1(0) - \zeta_k \mathcal{F}''_2(0)\right).$$

Observe that $\varphi(0) = 0$ and as $\mathcal{E}(r)$ vanishes on constants, $\mathcal{E}''(0)[h,h]\kappa(0) = 0$. Therefore

$$\mathcal{F}''_1(0)[h,h] = 2\mathcal{E}'(0)[h]\kappa'(0)[h] + \mathcal{E}(0)\kappa''(0)[h,h],$$

$$\mathcal{F}''_2(0)[h,h] = 2\mathcal{E}'(0)[h]\varphi'(0)[h] + \mathcal{E}(0)\varphi''(0)[h,h] + l''(0)[h,h].$$

From the definition of l we find $l''(0)[h,h] = \frac{3}{2\pi}h^2$. Combining the previous lemmas and the fact that $\mathcal{E}(0) = \mathcal{N}$ we obtain

$$\begin{aligned}
&(\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k], Y_k)_0 \\
&= \gamma\big(2(\mathcal{E}'(0)[Y_k]\kappa'(0)[Y_k], Y_k)_0 - 2\zeta_k(\mathcal{E}'(0)[Y_k]\varphi'(0)[Y_k], Y_k)_0\big) \\
&\quad +\gamma\big((\mathcal{N}\kappa''(0)[Y_k, Y_k], Y_k)_0 - \zeta_k(\mathcal{N}\varphi''(0)[Y_k, Y_k], Y_k)_0 - \zeta_k(l''(0)[Y_k, Y_k], Y_k)_0\big) \\
&= \gamma\left(2\left(k^2 + k - 2 - \zeta_k\frac{1}{4\pi}\right)(kI_k + J_k) + 4k(k^2 + k - 1)I_k - \zeta_k\frac{k}{2\pi}I_k - \zeta_k\frac{3}{2\pi}I_k\right) \\
&= \gamma\left(\frac{6(k^2 + k - 2)}{k + 3}(kI_k + J_k) + 4k(k^2 + k - 1)I_k - 2k(k^2 + k - 2)I_k\right) \\
&= \gamma\left(\frac{6(k^2 + k - 2)}{k + 3}(kI_k + J_k) + 2k^2(k + 1)I_k\right).
\end{aligned}$$

□

Lemma 3.1 follows from (B.1), (B.2), Lemma B.1 and Lemma B.6.

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