Solution to Problem 85-16* : A conjectured definite integral

An OC Curve Inequality

Problem 86-20*, *by* P. A. ROEDIGER *and* J. G. MARDO (U.S. Army Armament, Munitions and Chemical Command, Dover, NJ).

Let

$$\mathrm{OC}(n,c,q)=\sum_{i=0}^{c}\binom{n}{i}q^{n-i}(1-q)^{i}$$

where $n>c>0$ and $0<q<1$. Prove or disprove that

$$\mathrm{OC}(n,c,q)<\left[\mathrm{OC}(n,c,q^{1/m})\right]^{m}\quad \text{for all } m>1.$$

In the terminology of lot-by-lot sampling inspection by attributes, e.g., per MIL-STD-105D, the Operating Characteristic (OC) curve defines the probability of accepting a lot whose true fraction effective is q, when the criterion is to accept if and only if $(n-c)$ or more effectives are found in a random n-sample. When $m>1$ quality characteristics are distinguished, having effect rates q_i, lot quality is described by the profile $\tilde{Q}=(q_1,q_2,\cdots,q_m)$ and, generally, the accept/reject criteria are such that probability of acceptance has the form

$$\mathrm{PA}(\tilde{Q})=\prod_{i=1}^{m}\mathrm{OC}(n,c,q_i).$$

Since total lot quality q is the product of the q_i's, one is naturally interested in $\mathrm{PA}(\tilde{Q}\mid q)$, for a given q. The two sides of the proposed inequality can be shown to be optimal PA values, under this constraint. The difficulty is deciding which is the max and which is the min.

SOLUTIONS

A Conjectured Definite Integral

Problem 85-16*, *by* A. H. NUTTALL (Naval Underwater Systems Center, New London, CT).

It is conjectured that

$$\int_0^{\pi}\left\{\frac{\sin x}{x}\exp(x\cot x)\right\}^{\nu}dx=\frac{\pi\nu^{\nu}}{\Gamma(1+\nu)}\quad \text{for } \nu\geqq 0.$$

Prove or disprove.

The integral arose in a study of cross correlators. The conjectured result was discovered numerically first from the result for $\nu=\frac{1}{2}$ for which the computer output for the integral was recognized as $\sqrt{2\pi}$. The above result has been confirmed numerically to 15 decimal places for numerous values of ν in the range [0,150].

Solution by C. J. BOUWKAMP (Technische Hogeschool Eindhoven, Eindhoven, the Netherlands).

Nuttall's conjecture is true. Further, the integral is a special case of a general class of integral representations. Let $r(\beta)>0$ be continuously differentiable on $[0,\pi)$ with $r(\beta)\to\infty$ as $\beta\to\pi$; define

$$f(\nu,\beta):=\cos\{\nu(r\sin\beta-\beta)\}+(r'/r)\sin\{\nu(r\sin\beta-\beta)\}.$$

Then

$$\int_0^{\pi} r^{-\nu}\exp(\nu r\cos\beta)f(\nu,\beta)\,d\beta=\frac{\pi\nu^{\nu}}{\Gamma(\nu+1)}. \tag{1}$$

The special (and most simple) choice for r is $r(\beta)=\beta/\sin\beta$, which makes $f=1$. Then replacing β by x gives the required formula.

The proof of (1) goes via a Hankel-type integral,

$$\frac{i}{2}\int_{\infty}^{(0^+)}(-s)^{-\nu-1}e^{-\nu s}\,ds=\frac{\pi\nu^{\nu}}{\Gamma(\nu+1)},$$

the integration-path being parametrized through polar coordinates:

$$s=-r(\beta)\exp(i\beta),\qquad ds=-ir(1-ir'/r)\exp(i\beta)\,d\beta,$$

and assuming that the path is symmetric with respect to the real axis.

Also solved by D. J. BORDELON (Naval Underwater Systems Center, New London, CT), N. G. DE BRUIJN (Eindhoven University of Technology, the Netherlands), C. COSGROVE AND M. L. GLASSER (Clarkson University), W. E. HORNOR AND C. C. ROUSSEAU (Memphis State University), A. A. JAGERS (Technische Hogeschool Twente, Enschede, the Netherlands), D. S. JONES (University of Dundee, Scotland, U. K.), W. A. J. LUXEMBURG (California Institute of Technology), C. L. MALLOWS (AT&T Bell Laboratories, Murray Hill, NJ), O. G. RUEHR (Michigan Technological University), N. M. TEMME (Centrum voor Wiskunde en Informatica, Amsterdam, the Netherlands) *and* P. WAGNER (University of Innsbruck, Austria).

Most solvers employed contour integration in some fashion using a Hankel-type integral as above. Jones wrote

$$\lim_{x\to 0}\pi\left(\frac{2}{x}\right)^{\nu}J_{\nu}(\nu x)=\frac{\pi\nu^{\nu}}{\nu!}$$

where

$$J_{\nu}(\nu x)=\frac{1}{\pi}\int_0^{\pi}\exp\{-\nu F(\theta,x)\}\,d\theta,$$

and

$$F(\theta,x)=-\cot\theta(\theta^2-x^2\sin^2\theta)^{1/2}+\ln\{\theta+(\theta^2-x^2\sin^2\theta)^{1/2}\}/x\sin\theta,$$

Mallows specialized his previous result [1] which was proved independently [2]. Cosgrove and Glasser also used Mallow's result which they generalized and will describe elsewhere.

Hornor and Rousseau point out the connection between this problem and the analytic continuation of the series $\sum_{n=1}^{\infty}(n^{n-1}/n!)w^n$ which is a well-known Lagrange–Bürman expansion for the solution of $w=ze^{-z}$.

REFERENCES

[1] C. L. MALLOWS, *Problem* 6245, posed December, 1978, solution in Amer. Math. Monthly, 83 (1980), p. 584.

[2] R. EVANS, M. E. H. ISMAIL AND D. STANTON, *Coefficients in expansions of certain rational functions*, Canad. J. Math., 34 (1982), pp. 1011–1024.