

## An invitation to quantum tomography

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# An invitation to quantum tomography

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## 1 Introduction

It is curious that it took more than eighty years from its discovery till it was possible to experimentally determine and visualize the most fundamental object in quantum mechanics, the wave function. The forward route from quantum state to probability distribution of measurement results has been the basic stuff of quantum mechanics textbooks for decennia. That the corresponding mathematical inverse problem had a solution, provided (speaking metaphorically) that the quantum state has been probed from a sufficiently rich set of directions, had also been known for many years. However it was only in 1993, with [14], that it became feasible to actually carry out the corresponding measurements on one particular quantum system—in that case, the state of one mode of electromagnetic radiation (a pulse of laser light at a given frequency). The resulting pictures have since made it to the front covers of journals like *Nature* and *Science*, and experimentalists use the technique to establish that they have succeeded in creating non-classical forms of laser light such as squeezed light and Schrödinger cats. The experimental technique we are referring to here is called quantum homodyne tomography: the word homodyne referring to a comparison between the light being measured with a reference light beam at the same frequency. We will explain the word tomography in a moment.

The quantum state can be represented mathematically in many different but equivalent ways, all of them linear transformations on one another. One favourite is as the Wigner function  $W$ : a real function of two variables, integrating to plus one over the whole plane, but not necessarily nonnegative. It can be thought of as a “generalised joint probability density” of the electric and magnetic fields,  $q$  and  $p$ . However one cannot measure both fields at the same time and in quantum mechanics it makes no sense to talk about the values of both electric and magnetic fields simultaneously. It does, however, make sense to talk about the value of any linear combination of the two fields, say  $\cos(\phi)q + \sin(\phi)p$  (in fact, one can think of  $\phi$  as simply representing time). And one way to think about the statistical problem is as follows: the unknown parameter is a joint probability density  $W$  of two variables  $Q$  and  $P$ . The data consists of independent samples from the distribution of  $(X, \Phi) = (\cos(\Phi)Q + \sin(\Phi)P, \Phi)$ , where

$\Phi$  is chosen independently of  $(Q, P)$ , and uniformly in the interval  $[0, \pi]$ . Write down the mathematical model expressing the joint density of  $(X, \Phi)$  in terms of that of  $(Q, P)$ . Now just allow that latter joint density,  $W$ , to take negative as well as positive values (subject to certain restrictions which we will mention later). And that is the statistical problem of this paper.

This is indeed a classical tomography problem: we take observations from all possible one-dimensional projections of a two-dimensional probability density. The non-classical feature is that though all these one-dimensional projections are indeed bona-fide probability densities, the underlying two-dimensional “joint density” need not itself be a bona-fide joint density, but can have small patches of “negative probability density”.

Though the parameter to be estimated may look weird from some points of view (for instance, when one looks at it “as a probability density”), it is mathematically very nice from other points of view. For instance, one can also represent it by a matrix of (a kind of) Fourier coefficients: one speaks then of the “density matrix”  $\rho$ . This is an infinite dimensional matrix of complex numbers, but it is a positive and selfadjoint matrix with trace one. The diagonal elements are real numbers summing to one, and forming the probability distribution of the number of photons found in the light beam (if one could do that measurement). Conversely, any such matrix  $\rho$  corresponds to a physically possible Wigner function  $W$ , so we have here a concise mathematical characterization of precisely which “generalized joint probability densities” can occur.

The initial reconstructions were done by borrowing analytic techniques from classical tomography—the data was binned and smoothed, the inverse Radon transform carried out, followed by some Fourier transformations. At each of a number of steps, there are numerical discretization and truncation errors. The histogram of the data will not lie in the range of the forward transformation (from quantum state to density of the data). Thus the result of blindly applying an inverse will not be a bona-fide Wigner function or density matrix. Moreover the various numerical approximations all involve arbitrary choices of smoothing, binning or truncation parameters. Consequently the final picture can look just how the experimenter would like it to look and there is no way to statistically evaluate the reliability of the result. On the other hand the various numerical approximations tend to destroy the interesting “quantum” features the experimenter is looking for, so this method lost in popularity after the initial enthusiasm.

So far there has been almost no attention paid to this problem by statisticians, which is a shame, since on the one hand it is one of the most important statistical problems coming up in modern physics, and on the other hand it is “just” a classical nonparametric statistical inverse problem. The unknown parameter is some object  $\rho$ , or if you prefer  $W$ , lying in an infinite dimensional linear space (the space of density matrices, or the space of Wigner functions; these are just two concrete representations in which the experimenter has particular interest). The data has a probability distribution which is a linear transform of the parameter. Considered as an analytical problem, we have an ill-posed inverse problem, but one which has a lot of beautiful mathematical structure

and about which a lot is known (for instance, close connection to the Radon transform). Moreover it has features in common with nonparametric missing data problems (the projections from bivariate to univariate, for instance, and there are more connections we will mention later) and with nonparametric density and regression estimation. Thus we think that the time is ripe for this problem to be “cracked” by mathematical and computational statisticians. In this paper we will present some first steps in that direction.

Our main results will be consistency theorems for two estimators. Both estimators are based on approximating the infinite dimensional parameter  $\rho$  by a finite dimensional parameter, in fact, thinking of  $\rho$  as an infinite dimensional matrix, we simply truncate it to an  $N \times N$  matrix where the truncation level  $N$  will be allowed to grow with the number of observations  $n$ . The first estimator employs some analytical inverse formulas expressing the elements of  $\rho$  as mean values of certain functions of the observations  $(X, \Phi)$ . Simply replace the theoretical means by empirical averages and one has unbiased estimators of the elements of  $\rho$ , with moreover finite variance. If one applies this technique without truncation the estimate of the matrix  $\rho$  as a whole will typically not satisfy the nonnegativity constraints. The resulting estimator will not be consistent either, with respect to natural distance measures. But provided the truncation level grows with  $n$  slowly enough, the truncated estimate will satisfy the constraints, and provided it grows fast enough, the overall estimator will be consistent.

There are many unbiased estimators of the matrix elements of  $\rho$  and the choice we make is based on analytic tractability, not on any optimality criteria.

The second estimator we study further exploits the same idea, in a more canonical way: we study the sieved maximum likelihood estimator based on the same truncation to a finite dimensional problem. The truncation level  $N$  needs to depend on sample size  $n$  to balance bias and variance. We prove consistency of the sieved mle under an appropriate choice of  $N(n)$  by applying a general theorem of [18]. In order to verify the conditions we need to bound certain metric entropy integrals (with bracketing) which express the size (infinite-dimensionality) of the statistical model under study.

This turns out to be feasible, and indeed to have an elegant solution, by exploiting features of the mapping from parameters (density matrices) to distributions of the data. Various distances between probability distributions possess analogues as distances between density matrices, the mapping from parameter to data turns out to be a contraction, so we can bound metric entropies for the statistical model for the data with quantum metric entropies for the class of density matrices. And the latter can be calculated quite conveniently.

Our results form just a first attempt at studying the statistical properties of estimators which are already being used by experimental physicists, but they show that the basic problem is both rich in interesting features and tractable to analysis. The main result so far is a consistency theorem for a sieved maximum likelihood estimator, which depends on an assumption of the rate at which a truncated density matrix approximates the true one. It seems that the assumption is satisfied for the kinds of states which are met with in practice. However,

further work is needed here to describe in physically interpretable terms, when the estimator works. Secondly, we need to obtain rates of consistency and to further optimize the construction of the estimator. Thirdly, one should explore the properties of penalized maximum likelihood, and if possible make it adaptive to the rate of approximation of the truncated model, so that the truncation level  $N(n)$  is determined from the data.

We largely restrict attention to an ideal case of the problem where there is no further noise in the measurements. In practice, the observations have added to them Gaussian disturbances of known variance. There are some indications that when the variance is larger than a threshold of  $1/2$ , reconstruction becomes impossible or at least, qualitatively much more difficult. This needs to be researched from the point of view of optimal rates of convergence. The threshold should not be an absolute barrier for sieved or penalized maximum likelihood, though it may well have qualitative impact on how well this works.

We also only considered one particular though quite convenient way of sieving the model, i.e., one particular class of finite dimensional approximations. There are many other possibilities and some of them might allow easier analysis and easier computation. For instance, instead of truncating the matrix  $\rho$  in a given basis, one could truncate in an arbitrary basis, so that the finite dimensional approximations would correspond to specifying  $N$  arbitrary state vectors (eigenvectors) and a probability distribution over these “pure states”. Now the problem has become a missing data problem, where the “full data” would assign to each observation also the label of the pure state from which it came. In the full data problem we need to reconstruct not a matrix but a set of vectors, together with an ordinary probability distribution over the set, so the “full data” problem is statistically speaking a much easier problem than the missing data problem. One could imagine that the EM algorithm, or Bayesian reconstruction methods, could exploit this structure.

We concentrated on estimation of  $\rho$  but it would also be interesting to obtain results on estimation of  $W$ . The analogy with density estimation could suggest new statistical approaches here. Finally, it is most important to add to the estimated parameter, estimates of its accuracy. This is absolutely vital for applications, but so far no valid approach is available.

The quantum mathematical physics of this problem is identical to that of the quantum simple harmonic oscillator, where  $q$  and  $p$  stand for position and momentum of a particle, oscillating inside a quadratic potential well. In the next section we describe this mathematics using the terminology of position and momentum.

Section 3 is devoted to the ad hoc estimator based on truncation of  $\rho$ , and Section 4 to the sieved maximum likelihood estimator. That section finishes with some concluding remarks to the whole paper.

## 2 Quantum systems and measurements

In classical mechanics the state of macroscopic systems like billiard balls, pendulums or stellar systems is described by its “coordinates” in a phase space, each coordinate corresponding to an attribute which we can measure such as position and momentum. Therefore the functions on the phase space are called observables. When there exists uncertainty about the exact point in the phase space, or we deal with a statistical ensemble, the state is a probability distribution and the observables become random variables. Quantum mechanics also deals with observables such as position and momentum of a particle, spin of an electron, number of photons in a cavity but breaks from classical mechanics in that these are no longer functions but *selfadjoint operators* on a complex Hilbert space  $\mathcal{H}$  with inner product  $\langle \cdot, \cdot \rangle$  which is linear in the right slot and anti-linear in the left one. For example, the components in different directions of the spin of an electron are certain selfadjoint operators on  $\mathbb{C}^2$ , or hermitian  $2 \times 2$  matrices which do not commute with each other. Another quantum system with which we will deal in this paper is the quantum particle. Its basic observables position and momentum, are two unbounded selfadjoint operators  $\mathbf{Q}$  and  $\mathbf{P}$  respectively acting on the complex Hilbert space  $\mathcal{H} = L^2(\mathbb{R})$  as

$$\begin{aligned} (\mathbf{Q}\psi_1)(x) &= x\psi_1(x), \\ (\mathbf{P}\psi_2)(x) &= -i\frac{d\psi_2(x)}{dx}, \end{aligned}$$

for  $\psi_1, \psi_2$ , vectors in their respective domains. The operators satisfy Heisenberg’s *canonical commutation relations*  $\mathbf{QP} - \mathbf{PQ} = i\mathbf{1}$ . We note that the algebra generated by all bounded functions of  $\mathbf{Q}$  and  $\mathbf{P}$  is dense in the space of bounded operators  $\mathcal{B}(\mathcal{H})$  with respect to the weak operator topology, defined by the seminorms  $|\langle \psi, \cdot \psi \rangle|$  for all  $\psi \in \mathcal{H}$ . For this reason  $\mathcal{B}(\mathcal{H})$  is usually considered the algebra of observables of the system.

The state of the quantum system is given by a *density matrix*  $\rho$ , i.e. a positive trace-class operator with  $\text{Tr}(\rho) = 1$ . This is analogue to the probability distribution on the phase space, the expectation of an observable  $\mathbf{X} \in \mathcal{B}(\mathcal{H})$  being given by  $\mathbb{E}_\rho(\mathbf{X}) := \text{Tr}(\rho\mathbf{X})$ . The states form a convex subset  $\mathcal{S}(\mathcal{H})$  of the space of trace-class operators on  $\mathcal{H}$ , the latter being denoted  $\mathcal{T}_1(\mathcal{H})$ . The lower script 1 refers to the norm

$$\|\tau\|_1 := \text{Tr}(|\tau|), \tag{2.1}$$

with respect to which  $\mathcal{T}_1(\mathcal{H})$  is a Banach space. If  $\tau$  is selfadjoint then it can be represented as an infinite diagonal matrix (in a certain basis of the space  $\mathcal{H}$ ) with elements  $\tau_i$ , thus  $\|\tau\|_1 = \sum_i |\tau_i|$ . Any state can be written, in general non-uniquely, as convex combination of *pure* or vector states which have expectations of the form

$$\mathbb{E}_\psi(\mathbf{X}) := \text{Tr}(\mathbf{P}_\psi\mathbf{X}) = \langle \psi, \mathbf{X}\psi \rangle \tag{2.2}$$

where  $\mathbf{P}_\psi$  is the orthogonal projection on the space  $\mathbb{C}\psi$ . There exists a duality relation

$$\mathcal{T}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$$

which is the non-commutative analogue of  $\ell_1^* = \ell_\infty$ .

But how do we actually measure an observable of the system? This is in general a difficult question from the practical point of view, as we will see in this paper only certain observables can be measured with the present technology. But we can describe how the probability distribution of the results will look if we perform the measurement. Any selfadjoint operators  $\mathbf{X}$  has a spectral decomposition or “diagonalization”

$$\mathbf{X} = \sum_{i \in \sigma(\mathbf{X})} x_i \mathbf{P}_i$$

where the sum is taken over the spectrum of  $\mathbf{X}$  and  $\mathbf{P}_i$  is the projection associated to the eigenvalue  $x_i$ . The sum should be replaced by an integral for operators with continuous spectrum. If the system is in the state  $\rho$  then the probability of obtaining the value  $x_i$  is

$$p_\rho(i) = \text{Tr}(\rho \mathbf{P}_i). \quad (2.3)$$

which depends only on the spectral projections, the eigenvalues  $x_i$  being just labels of the results. More realistic measurements are modeled by *positive operator valued measures* (POVM) which are maps  $\mathbf{M}$  from the  $\sigma$ -algebra of a measure space  $(\Omega_{\mathbf{M}}, \Sigma_{\mathbf{M}})$  into  $\mathcal{B}(\mathcal{H})$  with the following properties:  $\mathbf{M}(A) = \mathbf{M}(A)^* \geq 0$  for any  $A \in \Sigma_{\mathbf{M}}$ ,  $\mathbf{M}(\cup_i A_i) = \sum_i \mathbf{M}(A_i)$  for a countable number of arbitrary disjoint  $A_i \in \Sigma_{\mathbf{M}}$ , and  $\mathbf{M}(\Omega_{\mathbf{M}}) = \mathbf{1}$ . Similarly to the projection valued case, the probability distribution of the results is

$$P_\rho^{(\mathbf{M})}(A) = \text{Tr}(\rho \mathbf{M}(A)).$$

An important feature of the map  $\rho \mapsto P_\rho^{(\mathbf{M})}$  is that it is *contractive* in appropriate norms. The total variation distance between two probability distributions on  $(\Omega_{\mathbf{M}}, \Sigma_{\mathbf{M}})$  is defined by

$$d_{\text{tv}}(P_1, P_2) := \sup_{|F| \leq 1} \left| \int F(x) P_1(dx) - \int F(x) P_2(dx) \right| \quad (2.4)$$

Then

$$\begin{aligned} d_{\text{tv}}(P_\rho^{(\mathbf{M})}, P_{\rho'}^{(\mathbf{M})}) &= \sup_{|F| \leq 1} \left| \int F(x) P_\rho^{(\mathbf{M})}(dx) - \int F(x) P_{\rho'}^{(\mathbf{M})}(dx) \right| \\ &= \sup_{|F| \leq 1} \left| \text{Tr}((\rho - \rho') \int F(x) \mathbf{M}(dx)) \right| \leq \|\rho - \rho'\|_1, \end{aligned}$$

where in the last step we have used the fact that  $\int F(x) \mathbf{M}(dx) \leq \mathbf{1}$  and then we applied the inequality  $|\text{Tr}(\tau \mathbf{Y})| \leq \|\tau\|_1 \|\mathbf{Y}\|$  for all  $\tau$  trace-class and  $\mathbf{Y}$  bounded, in which the reader recognizes its classical counterpart  $|\int fg| \leq \|f\|_1 \|g\|_\infty$ .

Notice that we are merely concerned here with the distribution of the results and do not specify the state of the system after the measurement. The “no

quantum cloning theorem” shows that measurements on a single system cannot completely reveal its state, in other words if the state is left unchanged after measurement then the results do not give us any information on the state.

We can now formulate our problem in the following way: we have at our disposal a large number of systems identically prepared in an unknown state  $\rho$ , on each one of them we can perform a certain measurement, and we want to construct an estimator of  $\rho$  based on the measurement results. Suppose for simplicity that we make the same measurement  $\mathbf{M}$  on all particles, then the results are i.i.d. random variables  $X_1, X_2, \dots$  on  $(\Omega_{\mathbf{M}}, \Sigma_{\mathbf{M}})$  with distribution  $p_{\rho}^{\mathbf{M}}$ . We will be interested in identifiable models, meaning that the map  $\mathbf{T}_{\mathbf{M}} : \rho \mapsto p_{\rho}^{\mathbf{M}}$  is one-to-one. For further details on quantum statistical inference we refer to the review [2] and the classical textbook [9].

### 3 Quantum homodyne tomography

Let us return to the quantum particle described by the observables  $\mathbf{Q}$  and  $\mathbf{P}$  satisfying the canonical commutation relations  $\mathbf{QP} - \mathbf{PQ} = \mathbf{1}$ . The problem of measuring observables other than position and momentum has been elusive until ten years ago when pioneering experiments in quantum optics conducted by Raymer’s group [14] lead to a powerful measurement technique called *homodyne tomography*. The quantum system to be measured is laser light with a fixed frequency whose observables are the field amplitudes satisfying commutation relation identical to those which characterize the quantum particle. Their linear combinations  $\mathbf{X}_{\phi} = \cos \phi \mathbf{Q} + \sin \phi \mathbf{P}$  are called *quadratures*, and homodyne tomography is about measuring the quadratures for an *arbitrary* phase  $\phi \in [0, \pi]$ . The experimental setup consists of an additional laser of high intensity called local oscillator (LO), which is combined with the mode of unknown state through a fifty-fifty beam splitter, and two photon detectors each one measuring one of the emerging beams. Then a rescaled difference of the measurement results turns out to have the same probability distribution as that of the quadrature  $\mathbf{X}_{\phi}$  in the limit of infinite intensity LO. It can be shown that the probability distribution  $P_{\rho}(\cdot, \phi)$  on  $\mathbb{R}$  has density  $p_{\rho}(\cdot, \phi)$  with respect to the Lebesgue measure and generating function

$$\mathbb{E}(e^{itX} | \phi) = \text{Tr}(\rho e^{it\mathbf{X}_{\phi}}). \quad (3.1)$$

The phase  $\phi$  is controlled by the experimenter by adjusting a parameter of the local oscillator, and it will be assumed to be chosen randomly uniformly distributed over the interval  $[0, \pi]$ . Then the joint probability distribution for the pair consisting in measurement results and phases  $(X, \Phi)$  has density  $p_{\rho}(x, \phi)$  with respect to the measure  $dx \times \frac{d\phi}{\pi}$  on  $\mathbb{R} \times [0, \pi]$ . A natural way of representing the state  $\rho$  is by writing down its matrix elements  $\rho_{i,j} := \langle \psi_i, \rho \psi_j \rangle$  in an orthonormal basis of the Hilbert space  $L_2(\mathbb{R})$ , for example

$$\psi_i(x) := \frac{H_i(x)}{(\sqrt{\pi} 2^i i!)^{1/2}} e^{-x^2/2}, \quad (3.2)$$



where  $H_i$  are Hermite polynomials. This basis has a special relevance in quantum optics,  $\psi_i(x)$  being pure states of exactly  $i$  photons. Here is the concrete formula for  $p_\rho$  in terms of  $\rho_{j,k}$ :

$$p_\rho(x, \phi) = \sum_{j,k=0}^{\infty} \rho_{j,k} \psi_k(x) \psi_j(x) e^{-i(j-k)\phi}. \quad (3.3)$$

An important feature of this homodyne detection scheme is the invertibility of the map  $\mathbf{T} : \rho \rightarrow p_\rho(\cdot, \cdot)$ , making it theoretically possible to infer the state of the system from the knowledge of the distribution of results. This was not possible had we measured only a finite number of quadratures! But what is the connection of this method with the more familiar computerized tomography used in the hospitals? Well, physicists like to represent the state of a quantum system by a certain function on  $\mathbb{R}^2$  called the *Wigner function*  $W(q, p)$  which is much like a joint probability distribution for  $\mathbf{P}$  and  $\mathbf{Q}$  in the sense that its marginals are the probability distributions for measuring  $\mathbf{Q}$  and respectively  $\mathbf{P}$ . Of course the two observables cannot be measured simultaneously so we cannot speak of a joint distribution, in fact the Wigner function need not be positive but many interesting features of the quantum state can be visualized in this way. It turns out that  $p_\rho(x, \phi)$  is the *Radon transformation* of the Wigner function

$$p_\rho(q, \phi) = \int_{-\infty}^{\infty} W(q \cos \phi + p \sin \phi, q \sin \phi - p \cos \phi) dp.$$

The Radon transformation and its inverse play a distinguished role in computerized tomography. Here one reconstructs a “shape”, for example the spatial distribution of the absorption coefficient for X-ray in a cross-section of the human body, by recording the transmitted radiation along an axis perpendicular to the beam and repeating this with the apparatus rotated at different angles. In our case the Wigner function is the unknown function while the probability density  $p_\rho$  represents the transmitted angle-dependent signal. The term optical homodyne tomography was coined in 1993 [14] when the first Wigner function was reconstructed from experimental data using the homodyne scheme. The Fourier transform of the Wigner function has the following expression

$$\widetilde{W}(u, v) = \text{Tr}(\rho e^{-iu\mathbf{Q} - iv\mathbf{P}}),$$

and we note that if  $\mathbf{Q}$  and  $\mathbf{P}$  were commuting operators then  $W(q, p)$  would indeed be the joint probability distribution of outcomes of their measurement. Finally, from  $\widetilde{W}(u, v)$  we can obtain the matrix elements of the state  $\rho$  with respect to a fixed orthonormal basis by integrating with certain kernel functions [10]. In practice this procedure has its drawbacks because it involves “filtering” the data as in usual tomography which as argued in [7] amounts to tampering with the state that is, making it more “classical”.

In [7] D’Ariano et al. presented a technique which provides the matrix elements without calculating the Wigner function as an intermediary step. The

method has been further analyzed in [3, 6, 12]. The key formula shows that any operator  $\tau \in \mathcal{T}_1(\mathcal{H})$  can be expressed as a linear superposition of functions of the observables  $\mathbf{X}_\phi$ :

$$\tau = \frac{1}{4} \int_{-\infty}^{\infty} dr |r| \int_0^\pi \frac{d\phi}{\pi} \text{Tr}(\tau e^{ir\mathbf{X}_\phi}) e^{-ir\mathbf{X}_\phi}. \quad (3.4)$$

which is an application of the general theory of *quantum tomography* developed by D'Ariano and his collaborators [8, 4]. By applying this formula to the state  $\rho$  and using (3.1) we get

$$\rho = \int_{-\infty}^{\infty} dx \int_0^\pi \frac{d\phi}{\pi} p_\rho(x, \phi) K(x - \mathbf{X}_\phi),$$

where  $K$  is the generalized function given by

$$K(x) = -\frac{1}{2} \mathcal{P} \frac{1}{x^2} = -\lim_{\epsilon \rightarrow 0^+} \text{Re} \frac{1}{(x + i\epsilon)^2}. \quad (3.5)$$

In order to obtain a mathematically sound expression we take the matrix elements on both side

$$\rho_{k,j} = \int_{-\infty}^{\infty} dx \int_0^\pi \frac{d\phi}{\pi} p_\rho(x, \phi) f_{k,j}(x) e^{-i(j-k)\phi}, \quad (3.6)$$

with  $f_{k,j}$  bounded functions which in the quantum tomography literature are called *pattern functions*. A first concrete expression using Laguerre polynomials was found in [5], and was followed by a more transparent one [11]

$$f_{k,k+d}(x, \phi) = e^{-id\phi} \frac{d}{dx} (\psi_k(x) \varphi_{d+k}(x)), \quad (3.7)$$

in terms of the basis vectors  $\psi_k$  and a certain un-normalizable solution of the Schrödinger equation

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right] \varphi_j = \omega_j \varphi_j, \quad (3.8)$$

Equation (3.6) suggests the *unbiased estimator* of  $\rho$  based on the first  $n$  i.i.d. results  $(X_l, \Phi_l)$  whose matrix elements are [6, 12, 11]:

$$\hat{\rho}_{k,j}^{(n)} = \frac{1}{n} \sum_{l=1}^n f_{k,j}(X_l, \Phi_l). \quad (3.9)$$

By the strong law of large numbers the individual matrix elements of this estimator converge to the matrix elements of the true parameter  $\rho$  and has the advantage that it can be computed in real time. The disadvantages are that the matrix  $\hat{\rho}^{(n)}$  as a whole need not be positive, normalized or even trace-class, and one has no control on the convergence  $\hat{\rho}^{(n)} \rightarrow \rho$  in any relevant distance such as for example  $\|\cdot\|_1$  due to the infinite number of matrix elements and

ranges of the pattern functions  $f_{i,i+d}$  increasing with  $i$  and  $d$ . We can avoid this problem by choosing  $\hat{\rho}^{(n)}$  to be an effectively finite dimensional selfadjoint matrix of dimension  $N(n)$  growing with  $n$  that is,  $\hat{\rho}_{i,i+d}^{(n)} = 0$  for  $i + d > N(n)$ , and  $\hat{\rho}_{i,i+d}^{(n)}$  given by (3.9) for  $i + d \leq N(n)$ . We apply now Hoeffding's inequality for the matrix elements,

$$\mathbb{P}(|\hat{\rho}_{i,i+d}^{(n)} - \rho_{i,i+d}| \geq a) \leq \exp\left(\frac{-na^2}{\|f_{i,i+d}\|_\infty^2}\right), \quad (3.10)$$

and let  $\rho^{(n)}$  denote the restriction of the true density matrix to  $N(n)$  dimensional subspace on which  $\hat{\rho}^{(n)}$  is non-trivial. We will look at the  $\|\cdot\|_2$ -distance defined in general by

$$\|\tau - \tau'\|_2^2 := \text{Tr}(|\tau - \tau'|^2) = \sum_{j,k \geq 0} |\tau_{j,k} - \tau'_{j,k}|^2.$$

Then from (3.10) we obtain

$$\mathbb{P}(\|\hat{\rho}^{(n)} - \rho^{(n)}\|_2 \geq a) \leq N(n)^2 \exp\left(\frac{-na^2}{\sum_{k,j=0}^{N(n)} \|f_{k,j}\|_\infty^2}\right). \quad (3.11)$$

**Lemma 3.1** *The following holds:*

$$\sum_{k \geq j \geq 0}^N \|f_{k,j}\|_\infty^2 = O(N^{7/3}). \quad (3.12)$$

*Proof.* We refer to the paper [11] for a more detailed analysis of the functions  $\psi_k, \varphi_j$  and we mention here only some qualitative features. Let  $\epsilon > 0$  be fixed. The Plancherel-Rotarch formulas [15] give asymptotic formulas for  $\psi_k$  and  $\varphi_k$  in three regions of  $\mathbb{R}$ : the ‘‘classical region’’  $|x| \leq \epsilon\sqrt{2k+1}$  where both have an oscillatory behavior and have absolute values bounded by the envelope function  $\sqrt{2/\pi}(1-x^2)^{-1/4}$ , the ‘‘classically forbidden region’’  $|x| \geq \epsilon\sqrt{2k+1}$  in which  $\psi_k$  decays as  $x^k e^{-x^2/2}$  while  $\varphi_k$  grows as  $x^{-k-1} e^{x^2/2}$ , and the ‘‘transition region’’ with width  $\epsilon k^{-1/6}$  centered around the turning point  $\sqrt{2k+1}$  in which

$$\psi_k(x) = 2^{1/4} k^{-1/12} \text{Ai}\left(\sqrt{2} k^{1/6} (x - \sqrt{2k+1})\right) \quad (3.13)$$

and similarly for  $\varphi_k$  with the Airy function [1] Ai replaced by Bi.

The range of the pattern functions  $f_{k,j}$  increases slowly with the distance to the diagonal  $j - k$ , thus the main contribution in (3.12) is brought by terms which lie away from the diagonal. Let  $C$  be a fixed constant, then for the pattern function  $f_{k,j}$  situated in the upper corner  $j \geq Ck$ , the maximum is attained in the overlap of the classical region for  $\varphi_j$  with the transition region of  $\psi_k$ , and can be estimated by using the Plancherel-Rotarch formulas

$$\|f_{k,j}\|_\infty = O\left(\frac{j^{1/4}}{k^{1/12}}\right). \quad (3.14)$$

We sum now over the upper corner to obtain asymptotic behavior of the sum (3.12). □

In particular we have the following necessary condition for the  $\|\cdot\|_2^2$ -consistency:

$$n^{-1}N(n)^{7/3} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.15)$$

**Theorem 3.2** *Let  $(\epsilon_n, N(n))$  be such that  $\epsilon_n \rightarrow 0$ ,  $N(n) \rightarrow \infty$  and*

$$\frac{n\epsilon_n^2}{N(n)^{7/3}} - 2 \log N(n) \rightarrow \infty. \quad (3.16)$$

*Then*

$$\|\hat{\rho}^{(n)} - \rho\|_2^2 = \|\rho^{(n)} - \rho\|_2^2 + O_{\mathbb{P}}(\epsilon_n^2). \quad (3.17)$$

*Moreover if*

$$\sum_{n=1}^{\infty} \exp\left(-\frac{n\epsilon_n^2}{N(n)^{7/3}} + 2 \log N(n)\right) < \infty \quad (3.18)$$

*then  $\|\hat{\rho}^{(n)} - \rho\|_2 \rightarrow 0$  almost surely.*

*Proof.* The first statement follows directly from (3.11) and the fact that  $\|\hat{\rho}^{(n)} - \rho\|_2^2 = \|\rho^{(n)} - \rho\|_2^2 + \|\hat{\rho}^{(n)} - \rho^{(n)}\|_2^2$ . The almost sure convergence follows from the first Borel-Cantelli lemma. □

The homodyne tomography as presented in the beginning of this section does not take into account various losses (mode mismatching, failure of detectors) in the detection process which modify the distribution of results in a real measurement compared with the idealized case. Fortunately, an analysis of such losses [10] shows that they can be quantified by a single *efficiency* coefficient  $0 < \eta < 1$  and the change in probability distributions amounts replacing  $X_i$  by

$$X'_i := \sqrt{\eta}X_i + \sqrt{(1-\eta)/2}Y_i \quad (3.19)$$

with  $Y_i$  a sequence of i.i.d. standard Gaussian independent of all  $X_j$ . The efficiency-corrected probability density is then

$$p_{\rho}(y, \phi; \eta) = (\pi(1-\eta))^{-1/2} \int_{-\infty}^{\infty} p(x, \phi) \exp\left[-\frac{\eta}{1-\eta}(x - \eta^{-1/2}y)^2\right] dx. \quad (3.20)$$

The problem is again the inference of the parameter  $\rho$  from  $(X'_1, \Phi_1), (X'_2, \Phi_2)$ . One could follow two routes: use a deconvolution technique for the variable  $X$  to obtain  $p_{\rho}$  and then apply the previous kernel estimator for  $\rho$ , or find new pattern functions  $f_{k,j}(x; \eta)$  such that

$$\rho_{k,j} = \int_{-\infty}^{\infty} dx \int_0^{\pi} \frac{d\phi}{\pi} p_{\rho}(x, \phi; \eta) f_{k,j}(x; \eta). \quad (3.21)$$

Such functions are analyzed in [3, 6] where it is argued that the the method has a fundamental limitation for  $\eta \leq 1/2$  in which case the pattern functions are unbounded, while for  $\eta > 1/2$  numerical calculations show that their range grows exponentially fast with both indices  $j, k$ . However there exists no proof of the conjecture which is implicitly made in the literature that it is impossible to estimate  $\rho$  consistently for  $\eta \leq 1/2$ . A third route is to first estimate an intermediary state  $\rho^{(\text{meas})}$  as in the  $\eta = 1$  case, and then to obtain  $\rho$  from  $\rho^{(\text{meas})}$  by inverting a *Bernoulli transformation* [10]:

$$p_\rho(\cdot, \cdot; \eta) \xrightarrow{f_{k,j}} \rho^{(\text{meas})} \xrightarrow{\text{inverse Bernoulli}} \rho. \quad (3.22)$$

To understand the (inverse) Bernoulli transformation let us consider first the diagonal elements  $\{p_k = \rho_{k,k}, k = 0, 1..\}$  and  $\{q_j = \rho_{j,j}^{(\text{meas})}, j = 0, 1..\}$  which are both probability distributions over  $\mathbb{N}$  and represent the statistics of the number of photon in the two states. Let  $b_k^{k+p} = \binom{k+p}{k} \eta^k (1-\eta)^p$  be the binomial distribution. Then

$$q_j = \sum_{k=j}^{\infty} b_j^k(\eta) p_k \quad (3.23)$$

which is interpreted as result of the ‘‘absorption’’ process by which each photon is allowed to pass with probability  $\eta$  and absorbed with probability  $1 - \eta$ . The general formula is

$$\rho_{j,k}^{(\text{meas})} = \sum_{p=0}^{\infty} \left[ b_j^{j+p}(\eta) b_k^{k+p}(\eta) \right]^{1/2} \rho_{j+p, k+p}, \quad (3.24)$$

and its inverse is obtained by replacing  $\eta$  with  $\eta^{-1}$ ! For  $\eta \leq 1/2$  the power series  $(1 - \eta^{-1})^k$  appearing in the inverse transformation diverges, reflecting the obstruction for obtaining bounded pattern functions  $f_{k,j}(x; \eta)$ .

## 4 Sieve maximum likelihood estimation

In this section we will develop a maximum likelihood approach to the estimation of the state  $\rho$ . Let us remind the reader of the terms of the problem: we are given a sequence  $(X_1, \Phi_1), (X_2, \Phi_2) \dots$  of i.i.d. random variables with values in  $\mathbb{R} \times [0, \pi]$  with probability density  $p_\rho$  with respect to the Lebesgue measure  $dx \times \frac{d\phi}{\pi}$  depending on the parameter  $\rho \in \mathcal{S}(\mathcal{H})$ . When taking into consideration the efficiency  $\eta < 1$  we have replace  $p_\rho$  by  $p_\rho(\cdot, \cdot; \eta)$ . We would like to find

$$\hat{\rho}^{(n)} = \hat{\rho}^{(n)}(X_1, \Phi_1, \dots, X_n, \Phi_n),$$

such that the  $\|\cdot\|_1$ -consistency holds:

$$\lim_{n \rightarrow \infty} \|\hat{\rho}^{(n)} - \rho\|_1 = 0, \quad \text{a.s.}$$

Let  $\hat{p}_n := p_{\hat{\rho}^{(n)}}$  be the corresponding probability density. We denote by

$$h(P_1, P_2) := \left( \int (\sqrt{p_1} - \sqrt{p_2})^2 d\mu \right)^{1/2}, \quad (4.1)$$

the Hellinger distance between two probability distributions on  $(\Omega, \Sigma, \mu)$  with densities  $p_1, p_2$  with respect to  $\mu$ . The following relations are well known

$$\frac{1}{2}d_{\text{tv}}(P_1, P_2) \leq h(P_1, P_2) \leq \sqrt{d_{\text{tv}}(P_1, P_2)}, \quad (4.2)$$

and combined with (2.5) give in the case of our measurement

$$h(P_\tau, P_{\tau'}) \leq \sqrt{\|\tau - \tau'\|_1} \quad (4.3)$$

for arbitrary states  $\tau, \tau' \in \mathcal{S}(\mathcal{H})$ . As a consequence, the Hellinger consistency

$$\lim_{n \rightarrow \infty} h(\hat{P}_n, P_\rho) \rightarrow 0, \quad \text{a.s.}, \quad (4.4)$$

is weaker than the  $\|\cdot\|_1$ -consistency.

The maximum likelihood estimator is usually defined as the parameter  $\tau$  which maximizes the log-likelihood  $\sum_{a=1}^n \log p_\tau(X_a, \Phi_a)$ . In this case the maximum is not achieved over the whole space and it seems more appropriate to restrict the attention to a subspace  $\mathcal{Q}(n)$  on which the maximum exists, whose size grows with the number of data and such that  $\cup_{n \geq 1} \mathcal{Q}(n)$  is dense in  $\mathcal{S}(\mathcal{H})$  in the norm topology. Such a method is called sieved maximum likelihood and we refer to [16, 18] for the general theory. The choice of the spaces  $\mathcal{Q}(n)$  should be tailored according to the problem one wants to solve, the class of states one is interested in, etc. We will use here the *number states sieves* for which  $\mathcal{Q}(n)$  consists of density matrices over the subspace spanned by the basis vectors  $\psi_0, \dots, \psi_{N(n)}$  defined in (3.2), with  $N(n)$  an increasing function of  $n$  which will be fixed later:

$$\mathcal{Q}(n) = \{\tau \in \mathcal{T}_1(\mathcal{H}) : \tau_{j,k} = 0 \text{ for all } j > N(n) \text{ or } k > N(n)\}. \quad (4.5)$$

The dimension of the space  $\mathcal{Q}(n)$  is  $N(n)^2$ . Let

$$\hat{\rho}^{(n)} := \arg \max_{\tau \in \mathcal{Q}(n)} \sum_{a=1}^n \log p_\tau(X_a, \Phi_a), \quad (4.6)$$

and notice that by compactness arguments the maximum always exists.

We define the convex map

$$\mathbf{T} : \mathcal{S}(\mathcal{H}) \ni \tau \mapsto p_\tau \in L_1(\mathbb{R} \times [0, \pi], dx \times \frac{d\phi}{\pi}), \quad (4.7)$$

whose image  $\mathcal{P}$  is the class of probability densities of the form (3.3), but for the moment we lack a more intrinsic characterization of its elements. The image of

the sieve  $\mathcal{Q}(n)$  is the convex hull  $\mathcal{P}(n)$  of densities of the form

$$p_\psi(x, \phi) = \left| \sum_{k=0}^{N(n)} \alpha_k e^{ik\phi} \psi_k(x) \right|^2, \quad (4.8)$$

with  $\psi = \sum_{k=0}^{N(n)} \alpha_k \psi_k$  a unit vector.

In order to obtain results on consistency of estimators, it is essential to bound the “size” of the sieve by entropy numbers which we define here for with respect to the  $\|\cdot\|_1$ -distance.

**Definition 4.1** *Let  $\mathcal{F}$  be a class of probability densities. Let  $N_{B,1}(\delta, \mathcal{F})$  be the smallest value of  $p \in \mathbb{N}$  for which there exist pairs of functions  $\{[f_j^L, f_j^U]\}$  with  $j = 1, \dots, p$  such that  $\|f_j^L - f_j^U\|_1 \leq \delta$  for all  $j$ , and such that for each  $f \in \mathcal{F}$  there is a  $j = j(f) \in \{1, \dots, p\}$  such that*

$$f_j^L \leq f \leq f_j^U.$$

*Then  $H_{B,1}(\delta, \mathcal{F}) = \log N_{B,1}(\delta, \mathcal{F})$  is called  $\delta$ -entropy with bracketing of  $\mathcal{F}$ .*

We note that this definition relies on the concept of positivity and distance between  $L_1$ -functions. But the same notions exist for the space of trace-class operators  $\mathcal{T}_1(\mathcal{H})$ , thus by replacing probability densities with density matrices and functions with selfadjoint trace class operators we obtain the definition of the  $\delta$ -entropy with bracketing  $H_{B,1}(\delta, \mathcal{Q})$  for some space of density matrices  $\mathcal{Q}$ .

**Proposition 4.2** *Let  $\mathcal{Q}(n)$  be the class of density matrices of dimension  $N(n)$ . Then*

$$H_{B,1}(\delta, \mathcal{Q}(n)) \leq CN(n)^2 \log \frac{N(n)}{\delta}. \quad (4.9)$$

*for some constant  $C$  independent of  $n$  and  $\delta$ .*

*Proof.*

Let  $\{\rho_j, j = 1, \dots, c(\delta, n)\}$  be a maximal set of density matrices in  $\mathcal{Q}(n)$  such that for any  $j \neq k$  we have  $\|\rho_j - \rho_k\|_1 > \frac{\delta}{2N(n)}$ . We define

$$\rho_j^U = \rho_j + \frac{\delta}{2N(n)} \mathbf{1}, \quad \rho_j^L = \rho_j - \frac{\delta}{2N(n)} \mathbf{1}.$$

Then for any  $\rho$  in the ball  $B_1(\rho_j, \frac{\delta}{2N(n)})$  we have  $\rho - \rho_i \leq \frac{\delta}{2N(n)} \mathbf{1}$ , thus

$$\rho_j^L \leq \rho \leq \rho_j^U,$$

and clearly  $\|\rho_j^L - \rho_j^U\|_1 = \delta$ . It remains to estimate the number of balls  $c(\delta, n)$ . From standard arguments on dimension we obtain

$$c(\delta, n) \left(\frac{\delta}{4N(n)}\right)^{N(n)^2} \leq \left(1 + \frac{\delta}{4N(n)}\right)^{N(n)^2} - \left(1 - \frac{\delta}{4N(n)}\right)^{N(n)^2}$$

where the difference on the right side represents the volume between two balls of radii  $1 - \frac{\delta}{4N(n)}$  and  $1 + \frac{\delta}{4N(n)}$ . As a rough estimation we obtain

$$c(\delta, n) \leq \left(1 + \frac{4N(n)}{\delta}\right)^{N(n)^2} \leq \left(5 \frac{N(n)}{\delta}\right)^{N(n)^2},$$

The bracketing entropy is at most  $\log c(\delta, n)$  and we obtain (4.9) with  $C = 1 + \log 5$ . □

**Corollary 4.3** *Let  $\mathcal{P}(n)^{1/2}$  be the class of  $L_2$ -functions  $\{\sqrt{p_\rho} : p_\rho \in \mathcal{P}(n)\}$  and  $H_B(\delta, \mathcal{P}(n)^{1/2})$  be the bracketing entropy with the  $\|\cdot\|_2$ -distance. Then*

$$H_{B,1}(\delta, \mathcal{P}(n)) \leq N(n)^2 \log \frac{N(n)}{\delta} \quad (4.10)$$

$$H_B(\delta, \mathcal{P}(n)^{1/2}) \leq N(n)^2 \log \frac{N(n)}{2\delta^2} \quad (4.11)$$

*Proof.*

Let  $\tilde{\mathbf{T}}$  be the linear extension to  $\mathcal{T}_1(\mathcal{H})$  of the map  $\mathbf{T}$ . Then  $\tilde{\mathbf{T}}$  is positivity preserving that is, for any  $\tau, \tau' \in \mathcal{T}_1(\mathcal{H})$  such that  $\tau \geq \tau'$  then  $p_\tau \geq p_{\tau'}$  where we extend the notation  $p_\tau = \tilde{\mathbf{T}}(\tau)$  to all trace-class operators. Let  $[\rho_j^U, \rho_j^L]$  be the  $\delta$ -bracketing matrices from the previous proposition. Then by the above observation  $[\tilde{\mathbf{T}}(\rho_j^U), \tilde{\mathbf{T}}(\rho_j^L)]$  is a set of brackets for  $\mathcal{P}(n) = \mathbf{T}(\mathcal{Q}(n))$ . From the monotonicity on the  $\|\cdot\|_1$  proved (2.5) we obtain  $\|\tilde{\mathbf{T}}(\rho_j^U) - \tilde{\mathbf{T}}(\rho_j^L)\|_1 \leq \delta$ .

For the second inequality we note that  $[\tilde{\mathbf{T}}(\rho_j^U)^{1/2}, (\tilde{\mathbf{T}}(\rho_j^L)_+)^{1/2}]$  is a set of brackets for  $\mathcal{P}(n)^{1/2}$  and then it can be shown that

$$\|\tilde{\mathbf{T}}(\rho_j^U)^{1/2} - (\tilde{\mathbf{T}}(\rho_j^L)_+)^{1/2}\|_2^2 \leq \frac{\delta}{2}.$$

□

We will concentrate now on the Hellinger consistency of the sieve maximum likelihood estimator  $\hat{P}_n$ . We will appeal to a theorem from [18], which is similar to other results in the literature on non-parametric  $M$ -estimation (see for example [16]). There are two competing factors which contribute to the convergence of  $h(\hat{P}_n, P_\rho)$ . The first is related with the approximation properties of the sieves with respect to the whole parameter space. Such a “distance” from  $\rho$  to the sieve  $\mathcal{Q}(n)$  can take different expressions, for example in terms of Kullback-Leibler divergence  $K(q, p) := \int p \log \frac{p}{q}$ ,

$$\delta_n(0+) := \inf_{\rho' \in \mathcal{Q}(n)} K(p_{\rho'}, p_\rho). \quad (4.12)$$

and

$$\tau_n = \lim_{k \rightarrow \infty} \int p_\rho (\log \frac{p_\rho}{p_k})^2 \quad (4.13)$$



where  $\{p_k, k = 1, 2, \dots\} \subset \mathcal{P}(n)$  is a sequence such that  $\lim_{k \rightarrow \infty} K(q_k, p_\rho) = \delta_n(0+)$ . Another natural rate which will be used later is

$$\gamma_n = \inf_{\rho' \in \mathcal{Q}(n)} \|\rho - \rho'\|_1. \quad (4.14)$$

Notice that all this numbers depend on the growth rate of the sieve  $N(n)$ . The second factor influencing the convergence of  $h(\hat{P}_n, P_\rho)$  is the size of the sieves which is expressed by the bracketing entropy. The non-parametric m.l. estimation theory shows that the following *entropy integral inequality* plays an important role in determining the rate of convergence

$$J_B(\delta, \mathcal{P}^{1/2}(n)) := \int_{\delta^2/2^8}^{\sqrt{2}\delta} H_B^{1/2}\left(\frac{u}{c_3}, \mathcal{P}(n)^{1/2}\right) du \leq c_4 \sqrt{n} \delta^2. \quad (4.15)$$

**Theorem 4.4** *There exist constants  $c_i, i = 1, \dots, 4$  such that if  $\delta_n$  is the smallest value satisfying (4.15) and we define*

$$\epsilon_n = \begin{cases} \delta_n, & \text{if } \delta_n(0+) < \frac{1}{4}c_1\delta_n \\ (4\delta_n(0+)/c_1)^{1/2}, & \text{otherwise} \end{cases}$$

then

$$\mathbf{P}\left(h(\hat{P}_n, P_\rho) \geq \epsilon_n\right) \leq 5e^{-c_2 n \epsilon_n^2} + \frac{4\tau_n}{c_1 n \epsilon_n^2}. \quad (4.16)$$

In calculating the entropy integral we take into account (4.11),

$$\begin{aligned} J_B(\delta, \mathcal{P}^{1/2}(n)) &= O\left[N(n) \int_{\delta^2}^{\delta} \left(\log \frac{N(n)^{1/2}}{u}\right)^{1/2} du\right] \\ &= O\left[N(n)^{3/2} \int_{N(n)^{1/2}/\delta}^{N(n)/\delta^2} w^{-2} (\log w)^{1/2} dw\right] \\ &= O\left[N(n)\delta \left(\log \frac{N(n)}{\delta}\right)^{1/2}\right]. \end{aligned} \quad (4.17)$$

From the entropy inequality we obtain the rate  $\delta_n$  satisfying

$$\frac{N(n)}{\delta_n} = O\left(\sqrt{\frac{n}{\log n}}\right). \quad (4.18)$$

**Theorem 4.5** *Suppose that the state  $\rho$  satisfies  $\tau_n = O(N(n)^{-\tau})$  for some  $\tau > 0$ . Let  $\hat{\rho}^{(n)}$  be the sieve MLE with  $N(n) = o\left(\left(\frac{n}{\log n}\right)^{1/2}\right)$  and  $N(n)^{-1} = o(n^{-\theta})$  for some  $\theta > 0$ . Then  $\hat{p}_n$  is Hellinger consistent, i.e.*

$$h(\hat{P}_n, P_\rho) \rightarrow 0 \quad \text{a.s.} \quad (4.19)$$

*Proof.* We apply Theorem 4.4 to our particular situation. We can choose a rate  $\delta_n \rightarrow 0$  satisfying (4.18) for our particular choice of  $N(n)$  and decreasing slower than  $1/\log n$ . Then

$$\sum_{n=1}^{\infty} \left( 5e^{-c_2 n \epsilon_n^2} + \frac{4\tau_n}{c_1 n (\epsilon_n)^2} \right) < \infty$$

because the lower bound for  $N(n)$  and the class assumption imply that  $\tau_n$  decreases faster than some power of  $n$ . A standard application of the first Borel-Cantelli lemma proves almost sure convergence of  $h(\hat{P}_n, P) \rightarrow 0$ .  $\square$

From the physical point of view, we are more interested in the convergence of the state estimator  $\hat{\rho}^{(n)}$  which is in principle a stronger requirement than Hellinger consistency. We will show however that the two are equivalent by applying a quantum analogue of the classical Scheffé's lemma [17] which says that if a sequence of probability densities converge pointwise almost everywhere to a probability density, then they also converge in  $\|\cdot\|_1$ . We will replace the  $L_1$  space by the space of trace-class operators  $\mathcal{T}_1(\mathcal{H})$ , and the pointwise convergence by *weak operator convergence* which is roughly  $\langle \psi, \mathbf{X}_n \psi \rangle \rightarrow \langle \psi, \mathbf{X} \psi \rangle$  for all  $\psi \in \mathcal{H}$ . In particular for density matrices it is sufficient to check the individual convergence of all matrix elements in a given basis. For the proof and other non-commutative convergence theorems we refer to [13].

**Theorem 4.6** *Let  $\rho_n$  be a sequence of density matrices converging weakly to another density matrix  $\rho$ . Then  $\|\rho_n - \rho\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .*

**Corollary 4.7** *The Hellinger consistency of  $\hat{P}_n$  is equivalent to the  $\|\cdot\|_1$ -consistency of  $\hat{\rho}^{(n)}$ . In particular, under the assumptions of Theorem 4.5 we have  $\|\hat{\rho}^{(n)} - \rho\|_1 \rightarrow 0$ , a.s..*

*Proof.* By Theorem 4.6 it is enough to prove almost sure convergence of each matrix element individually. But we have shown in (3.6) that  $\rho_{k,j}$  and  $\hat{\rho}_{k,j}^{(n)}$  can be expressed as the integral of  $p_\rho$  and respectively  $\hat{p}_n$  with bounded pattern functions  $f_{k,j}(x)e^{-i(j-k)\phi}$ .  $\square$

**Concluding Remarks.** There are many open questions related to quantum tomography and we would like to enumerate a few of them here.

The equivalence in last corollary holds as well for efficiency  $\eta > \frac{1}{2}$  as we only use the fact that the pattern functions are bounded, but seems to fail for  $\eta \leq \frac{1}{2}$  when the pattern functions are unbounded. Is  $\eta = \frac{1}{2}$  some kind of transition point between two convergence regimes?

Another problem which has not been treated here is that of rates of convergence for estimators. A possible way to obtain this is to find the rates  $\epsilon_n$  of convergence for  $h(\hat{P}_n, P_\rho)$  and then to use the modulus of continuity  $\omega_n(\epsilon)$  of the inverse map on the sieves

$$\mathbf{T}^{-1} : \mathcal{P}(n) \rightarrow \mathcal{Q}(n) \tag{4.20}$$

to obtain the rough rate  $\omega_n(\epsilon_n)$  for  $\|\hat{\rho}^{(n)} - \rho\|_1$ . This will lead to a slower increase of the sieve dimension  $N(n)$ . Is there a more direct approach to the estimation of the rates? Does the maximum likelihood estimator converge faster than the kernel estimator using pattern functions presented in section 3? Can we use penalization instead of arbitrarily choosing the dimension of the sieve?

On the practical side of the problem, finding the maximum of the likelihood function over a set of density matrix is non-trivial. The positivity and normalization constraints must be taken into account.

In the case  $\eta < 1$  we have to deconvolve the noise introduced by the detection imperfection. The analysis made for perfect detection should be made also in this case. It seems to us that the conjecture made by D'Ariano referring to the impossibility of reconstructing the state for  $\eta \leq \frac{1}{2}$  is not true in general, but it does pose a kind of restriction. One should identify the class of states for which the reconstruction is still possible.

Needless to say, the methods used here for quantum tomography can be applied in other problems of quantum estimation, such as for example estimating how certain devices transform the states of quantum systems.

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