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Vorticity scattering in shear flows at soft wall – hard wall transition

by

D.K. Singh
Vorticity scattering in shear flows at soft wall – hard wall transition

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An analytically exact solution of the problem of low Mach number incident vorticity scattering at a soft-to-hard wall transition in incompressible linear shear flow is obtained in terms of Fourier integrals by Wiener-Hopf method. The Harmonic vortical perturbations of inviscid linear shear flow travelling along a soft boundary are scattered at the soft - hard transition singularity that results in a far field which is qualitatively different for low shear and high shear cases. The local pressure field behaves as $r^{-\frac{1}{2}}$ and as a constant for low and high shear case respectively, similar to the hard-soft transition [1]. The subtleties of current Wiener-Hopf analysis poses challenges that are tackled with the help of pressure release wall limit ($Z = 0$) in case of which, the solution is analytically integrable and hence explicit.

The incompressible hydrodynamic inner solution is then matched asymptotically to a compressible acoustic outer field in order to determine the sound associated to the scattering process. The low shear case matches successfully and produce a soundfield pressure decaying as $r^{-\frac{1}{2}}$, including a $U_0$ relation for the radiated acoustic power, similar to [1]. The high shear case predicts a strong outerfield pressure that behaves as a constant and can not be matched to the acoustic outer field.

I. Introduction

The scattering of 2D vortical perturbations in an inviscid low Mach number linear shear flow with vanishing velocity at the wall passing over a hard to soft transition of this wall has been examined by [1,2]. The current work extends the analysis to a soft to hard transition, i.e. reversal of the boundary conditions of such wall. Because of the presence of the shear layer, the problem is non symmetric and demands an analysis from scratch. The soft hard transitions are as important as the hard soft transitions and are very common in aircraft engines and ventilation ducts and hence, it is important to analyse the scattering process associated with such discontinuities which is reported in current work. The similar nature of the problem allows to use some of the mathematical results published by [1,2] and are borrowed with permission. The Wiener-Hopf method in the incompressible limit poses some mathematical intricacies that are tackled with the help of the physical pressure release wall limit $Z = 0$. Under this limit, the Wiener-Hopf solution in terms of Fourier integrals is analytically integrable and hence explicit.


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thus provide rather a deep understanding of the problem that can be used to judge our finite impedance solution.

The 2D vorticity perturbations in an inviscid low Mach number linear shear flow with the pressure and velocity ratio at the wall equal to negative of impedance $-Z$ passing over a soft to hard transition are scattered in a local pressure which behaves differently depending upon whether the wake frequency $\omega$ being smaller (high shear case) or larger (low shear case) than the shear $\sigma = U'$. The solution confirms that soft -hard singularity behaves exactly as the hard soft singularity of [1][2] in both, low and high shear cases. The incompressible inner solution is obtain analytically exactly in terms of Fourier integrals and matched asymptotically to the compressible acoustic outer solution. The resulting sound field is found to behave similar to the hard-soft scattering soundfield.

The schematic of the considered problem is shown in Fig.1. We have low Mach number in the incompressible domain with linear shear $\sigma = U'$ and the flow velocity given by $U = \sigma y$. In 2D, the vortex stretching term is zero because there is no component of the flow velocity along the vorticity (into the plane) so we can assume that the incident field is produced by a mass source placed far upstream, although a non-conservative force field would give similar results to the plane. Consider that the total vorticity in the flow is

$$\nabla \times \mathbf{v} = 2 \pi \delta(x-x_0) \delta(y-y_0) \delta(z-z_0).$$

This has, under causal free field conditions (allowing only perturbations generated by the source) and $U_0 > 0$, the solution [4]

$$\hat{\chi} + \frac{U'(y)}{\rho_0 c_0^2} \hat{\sigma} = \frac{1}{2 \pi} \int_{\mathcal{G}} \left[ \frac{\hat{q}(x, y_0) \sigma_0}{\rho_0 U_0} H(x-x_0) e^{-ik_0(x-x_0)} \delta(y-y_0) \right] dx_0 dy_0, \quad k_0 = \frac{\omega}{U_0}. \tag{4}$$

Downstream the source we have just $H(x-x_0) = 1$. Utilising linearity we will consider a single $(x_0, y_0)$-component with unit amplitude and phase factor $e^{ik_0x_0} = 1$, in the incompressible limit, leading to the vortex sheet

$$\hat{\chi} = \frac{\sigma_0}{\rho_0 U_0} e^{-ik_0x} \delta(y-y_0).$$

With a simple shear flow given by $U(y) = \sigma y$ and uniform impedance boundary conditions along the wall $y = 0$, the corresponding velocity and pressure fields can be determined, far enough downstream the source, relatively easily. This field will act as the incident field for our scattering problem.

II. Model

We follow [1] and summarise the model as follows. Consider the two-dimensional incompressible inviscid problem of perturbations of a linearly sheared mean flow with time dependent
(\(e^{i\omega t}\)) vortex sheet along \(y = y_0\) in \(y > 0\) and a wall at \(y = 0\) which is soft (impedance) for \(x < 0\) and hard for \(x > 0\) with \(U(y) = \sigma y\); see figure 1. In this configuration we will have no contribution of a critical layer \(h_c\) or an instability like in [5].

As described above, we have a mass source placed at \(x = x_0 \rightarrow -\infty, y = y_0\) which produce the downstream travelling vorticity that decays exponentially away from the line \(y = y_0\) in the order \(\sim e^{-k_0|y-y_0|-ik_0x}\). When the convected vorticity field hits the soft-to-hard wall transition point \(x = 0\), it is scattered into a local pressure field that will radiate as sound into the far field.

The flow in the domain shown in figure 1 is governed by the linearised Euler equations with mixed boundary conditions (soft for \(x < 0\) and hard for \(x > 0\)), which makes the Wiener-Hopf technique [6, 7] a natural choice for obtaining the solution. Once we have obtained this (in the context of the acoustic field) inner solution, we can determine the source strength at the singularity \(x = 0\). In order to assess the produced sound, the incompressible inner solution will be matched with a compressible (acoustic) outer solution. In the incompressible limit, the sound speed approaches to infinity and hence the velocities are scaled with the mean flow velocity \(U_0\). In the compressible solution, the finite sound speed is used for such scaling.

### III. Mathematical formulation

The governing equation of mass and momentum conservation written in frequency domain are [3]

\[
\rho_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0,
\]

\[
\rho_0 \left( i\omega + U \frac{\partial}{\partial x} \right) u + \rho_0 \frac{\partial U}{\partial y} v + \frac{\partial p}{\partial x} = 0,
\]

\[
\rho_0 \left( i\omega + U \frac{\partial}{\partial x} \right) v + \frac{\partial p}{\partial y} = 0.
\]

Boundary conditions for finite impedance and hard wall at \(y = 0\) are

\[
v = 0 \quad \text{if } x > 0, \quad p = -Zv \quad \text{or} \quad i\omega p = \zeta p_y \quad \text{if } x < 0.
\]

Similarly, the boundary conditions for the pressure release and hard wall are

\[
v = 0 \quad \text{if } x > 0, \quad p = 0 \quad \text{if } x < 0.
\]

Apart from (7) and (8), we have an edge condition of vanishing energy flux from \((0, 0)\). The far field boundary conditions will be of vanishing velocity, but maybe not of vanishing pressure.
The incident field of the undulating vortex sheet at \( y = y_0 = U_0/\sigma \) is determined by the soft wall boundary condition at \( y = 0 \) and is given as

\[
\begin{align*}
    u_{in} &= U_0 e^{-ik_0 x} \left[ \lambda \text{sign}(y - y_0) e^{-k_0|y-y_0|} + e^{-k_0(y+y_0)} \right], \\
    v_{in} &= iU_0 e^{-ik_0 x} \left[ -\lambda e^{-k_0|y-y_0|} - e^{-k_0(y+y_0)} \right], \\
    p_{in} &= \frac{\sigma}{\omega} \rho_0 U_0^2 e^{-ik_0 x} \left[ -\lambda(1 + k_0|y-y_0|) e^{-k_0|y-y_0|} - (1 + k_0(y-y_0)) e^{-k_0(y+y_0)} \right],
\end{align*}
\]

with \( k_0 = \omega/U_0 \), and so \( k_0 y_0 = \omega/\sigma \). The effect of the soft wall is contained in dimensionless constant \( \lambda = \lambda_+ / \lambda_- \), where \( \lambda_+ = (\omega + \sigma + i k_0 \zeta) \) and \( \lambda_- = (\omega - \sigma - i k_0 \zeta) \) respectively. In the limit \( \zeta \to \infty \), \( \lambda = -1 \), hence the initial field converges to the hard wall initial field in [1].

In case of the pressure release wall, we have \( \lambda_0 = (\omega - \sigma)/(\omega + \sigma) \). Figure 2 shows the initial field for a typical case of low shear.

![Figure 2. The initial field \( p_{in} \) and \( v_{in} \) respectively. \( \omega = 5, \sigma = 4, U_0 = 5, k_0 = 1, y_0 = 1.25 \).](image)

The triple \((u_{in}, v_{in}, p_{in})\) satisfies the differential equations (6), continuity of \( p_{in} \) and \( v_{in} \) across \( y = y_0 \), and the soft-wall boundary condition \( p_{in} + Z v_{in} \) at \( y = 0 \). The scattered perturbations are due to the vanishing velocity \( v_{in} = 0 \) along \( y = 0, x > 0 \).

We split up the field in the incident part and the scattered part as follows

\[
    u = u_{in} + \overline{u}, \quad v = v_{in} + \overline{v}, \quad p = p_{in} + \overline{p}.
\]

After Fourier transformation in \( x \) (formally assuming the convergence of the integrals)

\[
    \overline{p}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(y, k) e^{-ikx} \, dk,
\]

(the same for \( \overline{u} \) and \( \overline{v} \)) we obtain the following set of equations

\[
    \rho_0(-ik \tilde{u} + \tilde{v}) = 0, \quad i \rho_0 \Omega \tilde{u} + \rho_0 \sigma \tilde{v} - \rho_0 i k \tilde{p} = 0, \quad i \rho_0 \Omega \tilde{v} + \tilde{p}' = 0,
\]

where \( \Omega = \omega - kU \). The system of equations has two independent solutions, namely \( \sim e^{\pm ky} \) [8,9]. The one, bounded for \( y \to \infty \), is then

\[
    \tilde{u}(y) = k A(k) e^{-|k|y}, \quad \tilde{v}(y) = -i |k| A(k) e^{-|k|y}, \quad \tilde{p}(y) = \rho_0 (\Omega - \text{sign}(\text{Re} k) \sigma) A(k) e^{-|k|y},
\]

with amplitude \( A(k) \) to be determined, and

\[
    |k| = \text{sign}(\text{Re} k) k = \sqrt{k^2},
\]

where \( \sqrt{\cdot} \) denotes the principal value square root, and \(|k|\) has thus branch cuts along the imaginary axis given by \((-i\infty, 0)\) and \((0, i\infty)\).
IV. Wiener-Hopf procedure

To facilitate the following Wiener-Hopf procedure, we introduce a small positive parameter ε and have an upper and a lower half plane, and a strip of overlap

\[ \mathbb{C}^+ = \{ k \in \mathbb{C} \mid \text{Im } k > -\varepsilon \}, \quad \mathbb{C}^- = \{ k \in \mathbb{C} \mid \text{Im } k < \varepsilon \}, \quad S = \{ k \in \mathbb{C} \mid -\varepsilon < \text{Im } k < \varepsilon \}, \]

The physical problem will be the limit \( \varepsilon \to 0 \) of a regularised problem with \( k_0 \) replaced by \( k_0 - i\varepsilon \) (an incident field \( \sim e^{-ik_0x} \) slightly decaying with \( x \)) and \( |k| \) replaced by the smoother function

\[ |k| = \sqrt{k^2 + \varepsilon^2} \]

with branch cuts \((-i\infty, -i\varepsilon) \cup (i\varepsilon, i\infty)\) avoiding strip \( S \). This way, we removed the branches of \( |k| \) away from the strip (cf. [10]).

Introduce the auxiliary functions \( F_-(k) \) and \( G_+(k) \) that are analytic in lower \( \text{Im}(k) < \varepsilon \) and upper \( \text{Im}(k) > -\varepsilon \) half of the complex plane respectively.

\[
F_-(k) = \int_{-\infty}^{0} \overline{v}(x,0) e^{ikx} \, dx, \quad G_+(k) = \int_{0}^{\infty} \left[ \overline{p}(x,0) + Z\overline{v}(x,0) \right] e^{ikx} \, dx
\]

Furthermore, we have

\[
G_+(k) = \int_{0}^{\infty} \left[ \overline{p}(x,0) + Z\overline{v}(x,0) \right] e^{ikx} \, dx = \int_{-\infty}^{\infty} \left[ \overline{p}(x,0) + Z\overline{v}(x,0) \right] e^{ikx} \, dx = -i|k|A(k) + (\lambda + 1)U_0 e^{-k_0y_0/(k - k_0)}
\]

with Wiener-Hopf kernel

\[
K(k) = 1 + \frac{a}{k} - \frac{b}{|k|}, \quad a = \frac{\sigma}{i\zeta}, \quad b = \frac{-\omega}{i\zeta}.
\]

With \( \varepsilon = 0 \), \( K(k) \) has 0, 1, or 2 zeros in the 1st, 2nd, or 4th quadrant, as shown in table[1] depending on the signs of \( \sigma - \omega \) and \( \text{Im } \zeta \), and assuming that \( \sigma, \omega, \Re \zeta > 0 \). As \( K(k) \) has a singularity in \( k = 0 \), which is inside strip \( S \), we follow [11] and consider the regularised version

\[
K(K) = 1 + \frac{a}{k - i\varepsilon} - \frac{b}{\sqrt{k^2 + \varepsilon^2}}.
\]

This \( K(k) \) has 3 zeros, which are for small \( \varepsilon \) approximated as shown in table[2]. So in general the zeros and singularities of \( K \) are not real and there is a neighbourhood of the real axis where \( K \) is analytic.
<table>
<thead>
<tr>
<th>case</th>
<th>$\sigma - \omega$</th>
<th>$\text{Im} \zeta$</th>
<th>$k_1 = -a + b$</th>
<th>$k_2 = -a - b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>$k = k_1 \in I$</td>
<td>no solution</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>-</td>
<td>no solution</td>
<td>$k = k_2 \in II$</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>+</td>
<td>no solution</td>
<td>no solution</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>$k = k_1 \in IV$</td>
<td>$k = k_2 \in II$</td>
</tr>
</tbody>
</table>

Table 1. Roots of non-regularised WH kernel $K(k)$ in (18)

<table>
<thead>
<tr>
<th>case</th>
<th>$\sigma - \omega$</th>
<th>$\text{Im} \zeta$</th>
<th>$k_1 \simeq -a + b + i \frac{a}{a+b}$</th>
<th>$k_2 \simeq -a - b + i \frac{a}{a+b}$</th>
<th>$k_3 \simeq -i \frac{a^2 + b^2}{a^2 - b^2} + \frac{\varepsilon^2}{(a^2 - b^2)^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>$k \simeq -a + b \in I$</td>
<td>no solution</td>
<td>no solution</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>-</td>
<td>no solution</td>
<td>$k \simeq -a - b \in II$</td>
<td>no solution</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>+</td>
<td>no solution</td>
<td>no solution</td>
<td>$k \simeq -i \frac{a^2 + b^2}{a^2 - b^2}$</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>$k \simeq -a + b \in IV$</td>
<td>$k \simeq -a - b \in II$</td>
<td>$k \simeq -i \frac{a^2 + b^2}{a^2 - b^2}$</td>
</tr>
</tbody>
</table>

Table 2. Roots of the regularised WH kernel $K(k)$ in (18)

Hence we arrive at the Wiener-Hopf equation

$$F_-(k) = \frac{G_+(k)}{\rho_0 \zeta} \frac{1}{K(k)} + (\lambda + 1) U_0 \frac{e^{-k_0 y_0}}{(k - k_0)}$$ \hspace{1cm} (19)

where $K(k)$ is to be solved in the standard way [6] by writing

$$K(k) = \frac{K_+(k)}{K_-(k)}$$ \hspace{1cm} (20)

where splitfunction $K_+$ is analytic in $\mathbb{C}^+$ and $K_-$ is analytic in $\mathbb{C}^-$. These splitfunctions are constructed in the usual way [11] as explained below.

Consider $k \in S$ inside a large rectangular contour $C \subset S$ between $k = -L - i\eta \varepsilon$ and $k = L + i\eta \varepsilon$, where $\eta$ is small enough, as shown in figure 9. In general $K$ has no zeros $k_{1,2,3}$ (if any) within $C$ and we assume a definition of $\log K(k)$ with branch cuts not crossing $S$. As it happens, with the present choice of the regularised $K$, this is achieved by taking the principal value logarithm. Then by Cauchy’s integral representation theorem,

$$\log K(k) = \lim_{L \to \infty} \frac{1}{2\pi i} \int_C \frac{\log K(\xi)}{\xi - k} \, d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log K(\xi - i\eta \varepsilon)}{\xi - i\eta \varepsilon - k} \, d\xi - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log K(\xi + i\eta \varepsilon)}{\xi + i\eta \varepsilon - k} \, d\xi$$ \hspace{1cm} (21)

where it may be noted that the integrals converge at infinity since

- $\text{Im}(k)$
- $\text{Re}(k)$
- $L$

Figure 3. Contour $C$
\[
\frac{\log K(\xi)}{\xi - k} = O(1/\xi^2) \quad (\xi \to \infty).
\]

Considered as a function of \( k \), the first integral can be analytically continued to \( \mathbb{C}^+ \), and the second integral can be analytically continued to \( \mathbb{C}^- \). So we can identify

\[
\log K_+(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \frac{K(\xi - i\eta\varepsilon)}{\xi - i\eta\varepsilon - k} \, d\xi, \quad k \in \mathbb{C}^+, \quad (22)
\]
\[
\log K_-(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \frac{K(\xi + i\eta\varepsilon)}{\xi + i\eta\varepsilon - k} \, d\xi, \quad k \in \mathbb{C}^- . \quad (23)
\]

If \( \varepsilon \to 0 \), the representations of \( K_+ \) and \( K_- \) become the same, in the sense that it becomes \( K_+ \) if \( k \in \mathbb{C}^+ \) and \( K_- \) if \( k \in \mathbb{C}^- \).

Although the splitfunctions for \( \varepsilon > 0 \) are only available numerically, it appears that for \( \varepsilon = 0 \) they can be given analytically exactly [1], by equation (28), albeit by using the somewhat unusual dilogarithm function. Furthermore, by extensive comparison with the numerical versions for very small but non-zero \( \varepsilon \), we could verify that the analytical splitfunctions as defined above are indeed the proper limit for \( \varepsilon \to 0 \). This remarkable result will be important later for the far field analysis of the physical solutions represented by a Fourier integrals. Altogether, we conclude that in \( S \),

\[
\frac{F_-(k)}{K_-(k)} - \frac{G_+(k)}{\rho_0\zeta K_+(k)} = (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \frac{1}{K_-(k)}
\]
\[
= (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \left[ \frac{1}{K_-(k)} - \frac{1}{K_-(k_0)} \right] + (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \frac{1}{K_-(k_0)}
\]

where we have isolated the pole \( k_0 \in \mathbb{C}^- \) from \( K_- \) and write

\[
\frac{F_-(k)}{K_-(k)} - (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \left[ \frac{1}{K_-(k)} - \frac{1}{K_-(k_0)} \right] = \frac{G_+(k)}{\rho_0\zeta K_+(k)} + (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \frac{1}{K_-(k_0)}
\]

The left and right side of (25) that are analytic in \( \mathbb{C}^+ \) and in \( \mathbb{C}^- \) respectively, are via their equivalence in \( S \) each other’s analytic continuations, and define an entire function \( E \)

\[
E(k) = \frac{F_-(k)}{K_-(k)} - (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \left[ \frac{1}{K_-(k)} - \frac{1}{K_-(k_0)} \right] = \frac{G_+(k)}{\rho_0\zeta K_+(k)} + (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \frac{1}{K_-(k_0)}.
\]

Following [D] we have \( E \equiv 0 \), hence we can write from (67) and (26)

\[
F_-(k) = (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{k - k_0} \left[ 1 - \frac{K_-(k)}{K_-(k_0)} \right],
\]
\[
G_+(k) = - (\lambda + 1)\rho_0\zeta U_0 \frac{e^{-k_0y_0}}{K_-(k_0)} \frac{K_+(k)}{(k - k_0)}
\]
\[
A(k) = -i (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{K_-(k_0)} \frac{K_-(k)}{|k|(k - k_0)}.
\]
\( A(k) \) obtained from (27) can be substituted back into (13). This gives, with the inverse Fourier transform from (11) added to the initial field, the formal solution of the problem

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_\text{in} - \frac{1}{2\pi} \int_{-\infty}^{\infty} i \text{sign}(k)(\lambda + 1) U_0 \frac{e^{-k\rho_0} K_{-}(k)}{K_{-}(k_0)} \frac{1}{k - k_0} e^{-|k|y} e^{-ikx} \, dk, \\
\mathbf{v} &= \mathbf{v}_\text{in} - \frac{1}{2\pi} \int_{-\infty}^{\infty} (\lambda + 1) U_0 \frac{e^{-k\rho_0} K_{-}(k)}{K_{-}(k_0)} \frac{1}{k - k_0} e^{-|k|y} e^{-ikx} \, dk, \\
\mathbf{p} &= \mathbf{p}_\text{in} - \frac{1}{2\pi} \int_{-\infty}^{\infty} i (\lambda + 1) \rho_0 U_0 \frac{e^{-k\rho_0} (\Omega - \text{sign}(\text{Re} k) \sigma) K_{-}(k)}{|k|(k - k_0)} e^{-|k|y} e^{-ikx} \, dk.
\end{align*}
\]

The pole \( k = k_0 \) is to be included when \( x > 0 \) and corresponds to the trailing vorticity. The other singularity at \( k = 0 \) is the one responsible for the farfield sound. The integrals in (28) can be evaluated numerically and depend on the \( K_{-}(k) \) function which is essentially different for the low and high shear cases. Please note that in the hard wall limit \( \lambda \to -1 \), the scattering field (28) will vanish as expected.

**V. Hydrodynamic solution**

The solution set \( u, v \) and \( p \) is the solution of incompressible inner problem of a larger compressible acoustic problem. Although a strict Matched Asymptotic Expansion analysis has not been laid out here in detail, we will refer to it as the inner solution. The outer limit of this inner solution \( r = \sqrt{x^2 + y^2} \to \infty \) is used to match it with the inner limit of the outer compressible solution. In order to evaluate the solutions, in the form of Fourier integrals (28), numerically or asymptotically in the far field, we need to know the behaviour of \( K_{-}(k) \) at \( k = 0 \). The following asymptotic behaviour of \( K_{-}(k \to 0) \) can be confirmed from 3.B and 3.C

\[
\begin{align*}
K_{-}(k) &\sim \frac{c_1}{\left(a - \text{sign} (\text{Re} k) b \right)^{\frac{1}{2}} k^{\frac{1}{2}} - i \delta} \text{ for } \sigma < \omega, \\
K_{-}(k) &\sim \frac{c_1}{\left(a - \text{sign} (\text{Re} k) b \right)^{-1} k^{-i \delta}} \text{ for } \sigma > \omega,
\end{align*}
\]

where \( c_1 \) is a complex constant and \( \delta = \frac{1}{2\pi} \log \left| \frac{\sigma + \omega}{\sigma - \omega} \right| \) is real positive.

**V.A. Solution of velocities \( u \) and \( v \)**

We see by combining (29) and (28) that in either case, the velocities \( u \) and \( v \) are integrable at \( k = 0 \). In the farfield, the \( u \) and \( v \) solution behaves like \( r^{\frac{1}{2} + i \delta} \) and \( r^{-1 + i \delta} \) for low and high shear cases respectively. Shown in figure 5 (top and middle) are the solutions (total = incident + scattered) of velocities for a typical representative case. Apparently, the high mean shear weaken the velocity field especially downstream of the edge. In the hard-soft, similar behaviour of the velocities was found hence, we see that the boundary reversal has no change on the behaviour of velocities except that the new \( \delta \) is negative of the previous one.

**V.B. Solution of pressure \( p \)**

As we noticed, the behaviour of the singularity at \( k = 0 \) is different for the cases \( \sigma < \omega \) and \( \sigma > \omega \). Hence, the far field solution in pressure is different for these cases. This splits our problem into 2 different cases in terms of radiated pressure. We will discuss them in separate sections.
V.B.1. Low shear

The low shear case corresponds with \( \sigma < \omega \), i.e. \( k_0y_0 > 1 \). The behaviour of \( K_-(k) \sim k^{\frac{1}{2} - i\delta} \) in the limit \( k \to 0 \) weakens the non-integrable singularity \( |k|^{-1} \) to an integrable singularity \( k^{-\frac{1}{2} - i\delta} \) of the integrand in (28). Hence the pressure solution can be obtained by direct integration along the contour shown in Fig. 4 like the velocities. For a typical case, this is shown in figure 5 (bottom left). It can be predicted, for example by invoking a version of Watson’s Lemma, even at this stage that a weaker singularity at \( k = 0 \) produces a weaker far field sound that decays like \( r^{-\frac{1}{2} + i\delta} \). Please note that we found similar behaviour in the case of hard-soft wall transition as well except that the new \( \delta \) is negative of the previous one.

V.B.2. High shear

The high shear case corresponds with \( \sigma > \omega \), i.e. \( k_0y_0 < 1 \). The behaviour of \( K_-(k) \sim k^{-i\delta} \) in the limit \( k \to 0 \) does not weaken the singularity in this case and the integral function behaves as \( \sim |k|^{-1-i\delta} \) as \( k \to 0 \) and hence diverges. The divergent behaviour at \( k = 0 \) in Fourier space suggests a strong far field at \( r = \sqrt{x^2 + y^2} \to \infty \) in the physical plane. The Fourier representation of pressure is too singular to interpret and hence should be regularised, using generalised functions, by splitting off the singular part and the part which is integrable. From (28) and (29), we have

\[
\bar{p}(x, y) = -i(\lambda + 1) \rho_0 U_0 \left[ \frac{e^{-k_0y_0}}{2\pi K_-(k_0)} \right] \int_{-\infty}^{0} \left( \frac{\Omega + \sigma}{(k - k_0)|k|K_+(k)} - \frac{c_1(\omega + \sigma)}{-(a + b)k_0|k|^{1-i\delta}} \right) e^{-ikx - |k|y} dk \\
- i(\lambda + 1) \rho_0 U_0 \left[ \frac{e^{-k_0y_0}}{2\pi K_-(k_0)} \right] \int_{0}^{\infty} \left( \frac{\Omega - \sigma}{(k - k_0)|k|K_+(k)} - \frac{c_1(\omega - \sigma)}{-(a - b)k_0|k|^{1-i\delta}} \right) e^{-ikx - |k|y} dk \\
- i(\lambda + 1) \rho_0 U_0 c_1 \left[ \frac{e^{-k_0y_0}}{2\pi K_-(k_0)} \right] \int_{-\infty}^{0} \left[ \frac{\zeta}{|k|^{1-i\delta}} e^{-ikx - |k|y} dk + \int_{0}^{\infty} \frac{-i\zeta}{|k|^{1-i\delta}} e^{-ikx - |k|y} dk \right] \\
(30)
\]

The split of the singularity renders the pressure integral of \( O(1) \) at \( k = 0 \) and hence integrable. In (30), the first 2 integrals have a finite limit at \( k = 0 \) and therefore can be evaluated along the integration contour 1 and 2 respectively, as shown in figure 4. The last integrals in (30) are those which carry the singularity and diverge at \( k = 0 \) which makes them difficult to interpret. They can be evaluated as generalised functions [12][13]. With Appendix B we have

\[
- i(\lambda + 1) \rho_0 U_0 c_1 \left[ \frac{e^{-k_0y_0}}{2\pi K_-(k_0)} \right] \int_{-\infty}^{0} \frac{\omega + \sigma}{|k|^{1-i\delta}} e^{-ikx - |k|y} dk + \int_{0}^{\infty} \frac{\omega - \sigma}{|k|^{1-i\delta}} e^{-ikx - |k|y} dk \\
= i^{1+i\delta} \Gamma(-i\delta) \frac{\zeta c_1}{2\pi} (\lambda + 1) \rho_0 U_0 \frac{e^{-k_0y_0}}{k_0 K_-(k_0)} (z^{i\delta} - \bar{z}^{i\delta}) \\
(31)
\]

where \( z = x + iy \). The results from (31) used with the first two integrals in (30) added to the initial field \( p_\text{in} \) gives the final solution of the inner pressure \( p \). Shown in figure 5 (bottom right) is the pressure for a typical case. The pressure field is clearly more intense for high shear than for low shear. This behaviour was found similar in the hard-soft transition case where the singularity at \( k = 0 \) was behaving like \( |k|^{-1+i\delta} \) and hence the farfield pressure was behaving as a constant in modulus \( \sim r^{-i\delta} \).

V.C. Far field of inner solution \( \bar{p}_\text{inner} \) – inside shear layer

In order to have an estimate of the far field radiated pressure, we need the asymptotic evaluation of the pressure integral (28) in the limit \( k \to 0 \) because small \( k \) in Fourier space relates to large
Figure 4. Integration contour

\[
r = \sqrt{x^2 + y^2} \sim \infty \text{ in the physical plane.}
\]

(a) **Low shear, } \sigma < \omega \):

From (28) and (29), we have in the limit \( k \to 0 \),

\[
\bar{p}(x, y)_{\sigma < \omega} \sim \bar{p}_{\text{inner}}(\sigma < \omega)
\]

\[
\simeq -\frac{1}{2\pi} i (\lambda + 1) \rho_0 U_0 \frac{e^{-k_0 y}}{K_-(k_0)} \frac{1}{-k_0} \left[ e^{\frac{\omega + \sigma}{a} \int_{-\infty}^{0} e^{-i|k|x} \frac{|k|}{|k|^{\frac{1}{2} + i\delta}} \text{dk} + \frac{\omega - \sigma}{a - b} \int_{0}^{\infty} e^{-i|k|x} \frac{|k|^{\frac{1}{2} + i\delta}}{|k|^{\frac{1}{2} + i\delta}} \text{dk} \right]
\]

\[
= \frac{\zeta c_1}{2\pi} (\lambda + 1) \rho_0 U_0 \frac{e^{-k_0 y}}{-k_0 K_-(k_0)} \left[ (-1)^{\frac{1}{2} + i\delta} \int_{0}^{\infty} k^{-\frac{1}{2} - i\delta} e^{i k z} \text{dk} - \int_{0}^{\infty} k^{-\frac{1}{2} - i\delta} e^{-i k z} \text{dk} \right]
\]

where \( z = x + i y \). The integrals converge, and can be evaluated like

\[
\int_{0}^{\infty} e^{i k z} \frac{k^{\frac{1}{2} + i\delta}}{k^{\frac{1}{2} + i\delta}} \text{dk} = \frac{\Gamma\left(\frac{1}{2} - i\delta\right)}{(-i z)^{\frac{1}{2} - i\delta}}.
\]

The net innerfield pressure is then given by

\[
\bar{p}_{\text{inner}}(\sigma < \omega) \simeq i^{-\frac{1}{2} - i\delta} \Gamma\left(\frac{1}{2} - i\delta\right) \frac{\zeta c_1}{2\pi} (\lambda + 1) \rho_0 U_0 \frac{e^{-k_0 y}}{-k_0 K_-(k_0)} \left( z^{-\frac{1}{2} + i\delta} - z^{-\frac{1}{2} + i\delta} \right)
\]

with \( z = r e^{i\theta} \). The pressure decays as \( r^{-\frac{1}{2}} \), which thus limits its effective acoustic source strength.

(b) **High shear, } \sigma > \omega \):

The singularity in this case is stronger than the one in the previous case, which enables us to assess that the radiated pressure \( \bar{p}_{\text{inner}}(\sigma > \omega) \) field must be stronger. The asymptotic behaviour of the integral (28) at \( k \to 0 \) is essentially the singularity taken out from the integral in (30). Hence the outer limit \( r \to \infty \) of the inner pressure field \( \bar{p} \) (with \( z = r e^{i\theta} \)) is given by (31) as:

\[
\bar{p}_{\text{inner}}(\sigma > \omega) \simeq i^{(i\delta)} \Gamma(-i\delta) \frac{\zeta c_1}{2\pi} (\lambda + 1) \rho_0 U_0 \frac{e^{-k_0 y}}{-k_0 K_-(k_0)} \left( z^{i\delta} - z^{-i\delta} \right).
\]

An important difference is that the modulus of the pressure field varies with \( r \) like \( |r^{-i\delta}| = 1 \), i.e. remains constant rather than decaying, and is therefore much stronger than in the previous case. Physically, it means that there is a strong interaction between the edge and the interphase. In case of low shear, this interaction is weak so that the reflected waves have small order of magnitude compared to the transmitted waves. But in the case of high shear, this interaction is strong hence the acoustic energy is contained in the boundary layer. Hence we conclude that an infinite linear shear profile model is an inconsistent modelling assumption for high shear case.

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Figure 5. The solution fields $u$, $v$ and $p$ for low shear $\sigma = 4 < \omega = 5$, $y_0 = 1.25$ (left) and high shear $\sigma = 5 > \omega = 4$, $y_0 = 1$ (right), while $\zeta = \frac{1}{2}(1+i), U_0 = 5.$

The above far field limit is taken inside the uniform shear flow, which means that we have a diverging mean flow velocity $U = \sigma y \to \infty$ as $y \to \infty$ which is not very physical. Although both (34) and (35) do satisfy the prevailing equations, we have to make sure that no unphysical artefacts are created. So we curtail the shear at height $h$ and define the mean flow being a constant $U_\infty$ beyond $y > h$. This is explained in the next section.

V.D. Far field of inner solution – outside shear layer

In order to approximate the solution outside the shear layer we assume a piecewise smooth transition of the shear layer at $y = h$ where it becomes straight as shown in figure 6, i.e.

$$U = \sigma y, \quad y < h,$$
$$U = U_\infty, \quad y \geq h.$$

Let us assume that $h \gg y_0$, so that the source does not interfere with the transition layer. The assumption is based on the physical understanding that the vortical field decays exponentially off the line $y = y_0$. Under this assumption, the incident field $p_{\text{in}}$ is negligible near the interface, while the inner pressure field $p_{\text{in}}$ is reflected back as $p_{\text{ref}}$ without further interaction with the wall, and transmitted as $p_{\text{tra}}$ into the far field. Hence, we may match the outer acoustic field to $p_{\text{tra}}$ in order to obtain a more realistic value of the far field sound. In order to obtain $p_{\text{tra}}$, we apply the continuity of pressure and velocity at the boundary $y = h$. In the Fourier domain, we
have for $y < h$ representation (28), which is for the Fourier transforms
\[
\tilde{p}(k, y) = \rho_0 D (\Omega_\infty - \text{sign}(\text{Re} \, k) \sigma) e^{-|k|(y-h)}, \quad \tilde{v}(k, y) = -i D|k| e^{-|k|(y-h)},
\]
\[
D = -i(\lambda + 1) U_0 e^{-k_0 y_0} \frac{K_-(k)}{K_-(k_0)} |k|(k - k_0) e^{-|k|h}, \quad \Omega_\infty = \omega - kU_\infty.
\]

The reflected and transmitted variables are given as
\[
\tilde{p}_{\text{ref}}(k, y) = \rho_0 R(\Omega_\infty + \text{sign}(\text{Re} \, k) \sigma) e^{|k|(y-h)}, \quad \tilde{p}_{\text{tra}}(k, y) = \rho_0 T \Omega_\infty e^{-|k|(y-h)}
\]
\[
\tilde{v}_{\text{ref}}(k, y) = i R|k| e^{|k|(y-h)}, \quad \tilde{v}_{\text{tra}}(k, y) = -i T|k| e^{-|k|(y-h)}
\]
where reflection and transmission coefficients $R$ and $T$ are obtained from the conditions of continuity of pressure and velocity at $y = h$
\[
\tilde{p}(k, h) + \tilde{p}_{\text{ref}}(k, h) = \tilde{p}_{\text{tra}}(k, h)
\]
\[
\tilde{v}(k, h) + \tilde{v}_{\text{ref}}(k, h) = \tilde{v}_{\text{tra}}(k, h).
\]

The two linear equations in variables $T$ and $R$
\[
\rho_0 D (\Omega_\infty - \text{sign}(\text{Re} \, k) \sigma) + \rho_0 R(\Omega_\infty + \text{sign}(\text{Re} \, k) \sigma) = \rho_0 T \Omega_\infty,
\]
\[
-i D|k| + i R|k| = -i T|k|
\]
can be solved to yield
\[
T = \frac{\Omega_\infty}{\Omega_\infty + \frac{1}{2} \text{sign}(\text{Re} \, k) \sigma}, \quad R = \frac{\frac{1}{2} \text{sign}(\text{Re} \, k) \sigma}{\Omega_\infty + \frac{1}{2} \text{sign}(\text{Re} \, k) \sigma}.
\]

The inner pressure transmitted outside the shear is then
\[
\tilde{p}_{\text{tra}}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -i(\lambda + 1) \rho_0 U_0 e^{-k_0 y_0} \left[ \frac{\Omega_\infty^2}{\Omega_\infty + \frac{1}{2} \text{sign}(\text{Re} \, k) \sigma} \right] \frac{K_-(k) e^{-|k|x - |k|y}}{|k|(k - k_0)} \, dk.
\]
\[
(36)
\]
If we write $\Omega_\infty = \omega - k\sigma h$, the outer limit of the inner pressure can be obtained by the asymptotic evaluation of the integral (36) in the limit $k \to 0$,
\[
\tilde{p}_{\text{tra}}(x, y) = -i(\lambda + 1) \rho_0 U_0 \frac{e^{-k_0 y_0}}{2\pi} \frac{e^{-k_0 y_0}}{-k_0 K_-(k_0)} \int_{-\infty}^{0} \left[ \frac{\omega^2}{\omega - \frac{1}{2} \sigma} \right] \frac{K_-(k)}{|k|} e^{-|k|x - |k|y} \, dk
\]
\[
- i(\lambda + 1) \rho_0 U_0 \frac{e^{-k_0 y_0}}{2\pi} \frac{e^{-k_0 y_0}}{-k_0 K_-(k_0)} \int_{0}^{\infty} \left[ \frac{\omega^2}{\omega + \frac{1}{2} \sigma} \right] \frac{K_-(k)}{|k|} e^{-|k|x - |k|y} \, dk \quad (37)
\]

Figure 6. Inner pressure reflected and transmitted at interface $y = h$. 

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In the case of $\sigma < \omega$, using (29) and (33), we obtain
\[
\bar{p}_{\text{tra}}(\sigma<\omega) = i^{-\frac{1}{2}+i\delta} \Gamma\left(\frac{3}{2} - i\delta\right) \frac{\zeta c_1}{2\pi} \rho_0 U_0 \frac{e^{-k_0 y_0}}{-k_0 K_-(k_0)} \left[ \frac{\omega}{\sigma + \omega - \frac{1}{2}\sigma} z^{-\frac{1}{2}+i\delta} + \frac{\omega}{\sigma - \omega + \frac{1}{2}\sigma} \right].
\]

In the other case, i.e. $\sigma > \omega$, using (29) and appendix [E], we have
\[
\bar{p}_{\text{tra}}(\sigma>\omega) = i^{1+\delta} \Gamma(-i\delta) \frac{\zeta c_1}{2\pi} \rho_0 U_0 \frac{e^{-k_0 y_0}}{-k_0 K_-(k_0)} \left[ \frac{\omega}{\sigma + \omega - \frac{1}{2}\sigma} z^{-i\delta} + \frac{\omega}{\sigma - \omega + \frac{1}{2}\sigma} \right].
\]

where $z = re^{i\delta}$. We conclude from (34), (35), (38) and (39) that the inclusion of the transition layer does not change the functional relationship of the sound radiated to farfield and differ by only a constant. We will match the outerfield acoustic solution to both inner fields in the next section.

V.E. Far field of pressure release wall solution $\bar{p}_{\text{inner}}$ – inside shear layer

The solution set (28) is valid for the pressure release wall case as well and the integrals in (28) can be analytically evaluated like in [2]. Once the analytic evaluation is done, we can take the farfield limit $z \to \infty$ and obtain the outer limit.

V.E.1. Low shear case

Substituting (72) into (28) coupled with the contour integral like in [2] and farfield limit, we obtain the low shear case solution continuous across $x = 0$
\[
\bar{\tau} \sim \frac{(\lambda + 1) U_0}{2\pi} e^{-k_0 y_0 - \frac{3}{2}\pi i} \Gamma\left(\frac{3}{2} - i\delta\right) \left(\frac{\omega - \sigma}{\omega + \sigma}\right) \left[ (k_0 z)^{-\frac{1}{2}+i\delta} + (k_0 \bar{\tau})^{-\frac{1}{2}+i\delta} \right],
\]
\[
\bar{\nu} \sim \frac{i(\lambda + 1) U_0}{2\pi} e^{-k_0 y_0 - \frac{3}{2}\pi i} \Gamma\left(\frac{3}{2} - i\delta\right) \left(\frac{\omega - \sigma}{\omega + \sigma}\right) \left[ (k_0 z)^{-\frac{1}{2}+i\delta} - (k_0 \bar{\nu})^{-\frac{1}{2}+i\delta} \right],
\]
\[
\bar{p} \sim i(\lambda + 1) \frac{\rho_0 U_0^2}{\omega} e^{-k_0 y_0 - \frac{3}{2}\pi i} \Gamma\left(\frac{1}{2} - i\delta\right) \left(\frac{\omega - \sigma}{\omega + \sigma}\right) \left[ (\omega + \sigma)(k_0 z)^{-\frac{1}{2}+i\delta} + (\omega - \sigma)(k_0 \bar{p})^{-\frac{1}{2}+i\delta} \right].
\]

Also,
\[
\bar{p}(x < 0, 0) = 0 \Rightarrow p_m(x < 0, 0) + \bar{p}(x < 0, 0) = 0
\]
\[
\bar{p}(x > 0, 0) = i(\lambda + 1) U_0 e^{-k_0 y_0 - ik_0 x} \Rightarrow \nu_m(x > 0, 0) + \bar{\nu}(x > 0, 0) = 0.
\]

Hence, the boundary condition are satisfied by the solution. The velocities ($\bar{\tau}, \bar{\nu}$) and pressure $\bar{p}$ decays as $\sim r^{-\frac{1}{2}-i\delta}$ and $\sim r^{-\frac{1}{2}-i\delta}$ respectively similar to the finite impedance case. Since the incompressible field is necessarily derived for a linear mean flow $\propto y$, which evidently does not connect smoothly with a uniform mean flow, it can not be matched to compressible acoustic outer field in a strict asymptotic sense. However, this can be repaired by adding a uniform flow on top of the shear flow, with an interface in between. As long as the backreaction from the interface is weak, the difference is small. The far field asymptotic expressions of $\bar{\tau}$ and $\bar{\nu}$ in (40) are harmonic functions and can be written as the gradient of a potential $\phi$. For a bounded mean flow, the pressure and radial velocity $\bar{\nu}$ would then for large $r$ be given by
\[
\bar{p} = -\rho_0 (i\omega + U \frac{\partial \phi}{\partial x}) \phi \simeq -i \rho_0 \omega \phi, \quad \bar{\nu} \simeq \frac{\partial \phi}{\partial x} \phi.
\]
Consider the potential function with \((\Omega, \overline{\Omega}) = \nabla \phi\) given by
\[
\phi = -(\lambda + 1) \frac{U_0^2}{2\pi \omega} e^{-k_0 y_0} e^{-\frac{i}{2} \pi i} \Gamma\left(\frac{1}{2} - i\delta\right) \left(\frac{\omega - \sigma}{\omega + \sigma}\right)^{\frac{1}{4}} \left[(k_0 z)^{-\frac{1}{2} + i\delta} + (k_0 \overline{z})^{-\frac{1}{2} + i\delta}\right]. \tag{43}
\]
From (42) and (43), we have
\[
\overline{\Omega} = i(\lambda + 1) \frac{\rho_0 U_0^2}{2\pi} e^{-k_0 y_0} e^{-\frac{i}{2} \pi i} \Gamma\left(\frac{1}{2} - i\delta\right) \left(\frac{\omega - \sigma}{\omega + \sigma}\right)^{\frac{1}{4}} \left[(k_0 z)^{-\frac{1}{2} + i\delta} + (k_0 \overline{z})^{-\frac{1}{2} + i\delta}\right],
\tag{44}
\]
which is not exactly same as the \(\overline{\Omega}\) in (40) because in the farfield with uniform flow, the linear velocity profile term \(\sigma y\) in (40) will disappear and an expression similar to (44) is retrieved. The pressure release wall solution satisfies the boundary conditions and is analytically exact. It resembles with the finite impedance solution, (34) as well. Hence we conclude that the asymptotic limit \(k \to 0\) of (28) is most reasonable to obtain the outer limit of the pressure solution and the regularizations so far, are correct.

**V.E.2. High shear case**

Similar to the previous case, substituting (74) into (28) coupled with the integral contour like [2] and farfield limit, we obtain the high shear case solution continuous across \(x = 0\)
\[
\Omega \sim (\lambda + 1) \frac{U_0}{2\pi} e^{-k_0 y_0} \Gamma(1 - i\delta) \left(\frac{\sigma - \omega}{\sigma + \omega}\right)^{\frac{1}{4}} \left[(k_0 z)^{-1 + i\delta} + (k_0 \overline{z})^{-1 + i\delta}\right],
\]
\[
\overline{\Omega} \sim (\lambda + 1) \frac{U_0}{2\pi} e^{-k_0 y_0} \Gamma(1 - i\delta) \left(\frac{\sigma - \omega}{\sigma + \omega}\right)^{\frac{1}{4}} \left[(k_0 z)^{-1 + i\delta} - (k_0 \overline{z})^{-1 + i\delta}\right],
\]
\[
\overline{\Omega} \sim i(\lambda + 1) \frac{\rho_0 U_0^2}{2\pi \omega} e^{-k_0 y_0} \Gamma(-i\delta) \left(\frac{\sigma - \omega}{\sigma + \omega}\right)^{\frac{1}{4}} \left[(\omega + \sigma)(k_0 z)^{i\delta} + (\omega - \sigma)(k_0 \overline{z})^{i\delta} + k_0 \sigma y\delta \left[(k_0 z)^{-1 + i\delta} + (k_0 \overline{z})^{-1 + i\delta}\right]\right]. \tag{45}
\]
Also,
\[
\overline{\Omega}(x < 0, 0) = 0 \Rightarrow p_w(x < 0, 0) + \overline{\Omega}(x < 0, 0) = 0
\]
\[
\overline{\Omega}(x > 0, 0) = i(\lambda + 1) U_0 e^{-k_0 y_0 - ik_0 x} \Rightarrow v_{\text{in}}(x > 0, 0) + \overline{\Omega}(x > 0, 0) = 0. \tag{46}
\]
Hence, the boundary conditions are satisfied by the solution. The velocities \((\Omega, \overline{\Omega})\) and pressure \(\overline{\Omega}\) decays as \(r^{-1+i\delta}\) and \(r^{i\delta}\) respectively, similar to the finite impedance wall. Finally, like the low shear case, we define a potential function and obtain the pressure in the uniform flow region as
\[
\phi = -(\lambda + 1) \frac{U_0^2}{2\pi \omega} e^{-k_0 y_0} \Gamma(-i\delta) \left(\frac{\sigma - \omega}{\sigma + \omega}\right)^{\frac{1}{4}} \left[(k_0 z)^{i\delta} + (k_0 \overline{z})^{i\delta}\right] \tag{47}
\]
\[
\overline{\Omega} \sim i(\lambda + 1) \frac{\rho_0 U_0^2}{2\pi} e^{-k_0 y_0} \Gamma(-i\delta) \left(\frac{\sigma - \omega}{\sigma + \omega}\right)^{\frac{1}{4}} \left[(k_0 z)^{i\delta} + (k_0 \overline{z})^{i\delta}\right] \tag{48}
\]
The expression in (48) is similar to the finite impedance solution (35) and thus, we conclude that the limit \(k \to 0\) in (28), in order to know the farfield behaviour works precisely and the regularizations of poles and zeros so far, are correct.
VI. Outer solution and asymptotic matching

Since the mean flow Mach number is small, the inner problem is incompressible. We assume the outer acoustic field, where the mean flow velocity profile changed from linear $U(y) = \sigma y$ to a constant, compressible but with negligible mean flow. Then we have the equation

$$\nabla^2 p + \kappa^2 p = 0, \quad \kappa = \frac{\omega}{c_0}.$$ 

With a point source in $x = y = 0$, assuming a certain symmetry in $r$ and $\theta$ (where $x = r \cos \theta$ and $y = r \sin \theta$), we search for solutions of the form

$$p(r, \theta) = \gamma(r) \beta(\theta).$$

If we substitute this in the equations we find

$$\gamma'' + \frac{1}{r} \gamma' + \kappa^2 \gamma = \nu^2 \frac{\gamma}{r^2}, \quad \beta'' + \nu^2 \beta = 0,$$

such that

$$\beta(\theta) = B_1 e^{i \nu \theta} + B_2 e^{-i \nu \theta}$$

and

$$\gamma(r) = m H_{\nu}^{(2)}(kr) + n H_{\nu}^{(2)}(kr) = m H_{\nu}^{(2)}(kr) + n e^{-i \nu \pi} H_{\nu}^{(2)}(kr) = MH_{\nu}^{(2)}(kr)$$

with the relationship $H_{\nu}^{(2)}(kr) = e^{-i \nu \pi} H_{\nu}^{(2)}(kr)$ [14]. Clearly, $M$ is superfluous but is kept for convenience. The constants $B_1$, $B_2$ and $\nu$ are to be determined from the matching condition at $r \to 0$ where the Hankel function has the following asymptotic behaviour [14]

$$H_{\nu}^{(2)}(kr) \simeq \frac{1}{\pi} \Gamma(\nu) \left( \frac{1}{2} kr \right)^{-\nu} = \alpha r^{-\nu}$$

with $\text{Re}(\nu) > 0$ and constant $\alpha = i \pi^{-1} \Gamma(\nu) \left( \frac{1}{2} \pi \right)^{-\nu}$. If $\nu$ is purely imaginary, the $r^{-\nu}$-term does not dominate any more for small $r$ and we find [14]

$$H_{\nu}^{(2)}(kr) \simeq \frac{1}{\pi} \left( \Gamma(\nu) \frac{1}{2} kr \right)^{-\nu} + e^{i \nu \pi} \Gamma(-\nu) \left( \frac{1}{2} \pi \right)^{\nu} = \alpha r^{-\nu} + \tilde{\alpha} r^{\nu},$$

with $\tilde{\alpha} = i \pi^{-1} e^{i \nu \pi} \Gamma(-\nu) \left( \frac{1}{2} \pi \right)^{\nu}$. From (49), (50) and (51) or (52), we have for $r \to 0$

$$p(r, \theta) \simeq \alpha r^{-\nu}(B_1 e^{i \nu \theta} + B_2 e^{-i \nu \theta}), \quad \text{resp.} \quad (\alpha r^{-\nu} + \tilde{\alpha} r^{\nu})(B_1 e^{i \nu \theta} + B_2 e^{-i \nu \theta}),$$

(53)

to be matched with the outer limit of the inner solutions (34), (38), (35) and (39).

VI.A. Farfield sound, low shear case

For low shear, $\sigma < \omega$, the asymptotic matching of (53) with (34) or (38) leads to the following expression of $\nu$ and $M$, given by

$$\nu = \frac{1}{2} - i \delta, \quad M = i^{-\frac{1}{2} + i \delta} (\lambda + 1) \frac{c_1}{2} \rho_0 U_0 \frac{e^{-k_0 \rho_0}}{-k_0 K_-(k_0)} \left( \frac{1}{2} \pi \right)^{\frac{1}{2} - i \delta},$$

(54)

while $B_1$ and $B_2$ represent the matching with the inner pressure $P_{\text{inner}}(\sigma < \omega)$ inside, or $\tilde{P}_{\text{tra}}(\sigma < \omega)$ outside the shear layer respectively.

$$B_1 = \frac{\omega}{\sigma - \omega - \frac{1}{2} \sigma} \quad \text{and} \quad B_2 = \frac{\omega}{\sigma + \omega - \frac{1}{2} \sigma} \quad \text{matched with} \quad \tilde{P}_{\text{inner}}(\sigma < \omega)$$

$$B_1 = \frac{\omega}{\sigma - \omega - \frac{1}{2} \sigma} \quad \text{and} \quad B_2 = \frac{\omega}{\sigma + \omega - \frac{1}{2} \sigma} \quad \text{matched with} \quad \tilde{P}_{\text{tra}}(\sigma < \omega).$$

(55)
Eventually, the farfield sound is given by

\[ p(r, \theta) = i^{-\frac{3}{2} + i\delta} (\lambda + 1) \frac{\zeta c_1}{2} \rho_0 U_0 e^{-2k_0 y_0} \left( \frac{1}{2} + \frac{1}{2} \right) \times \]

\[ H^2_{\nu}(\kappa r) \left( - e^{i\left(\frac{3}{2} - i\delta\right)\theta} + e^{-i\left(\frac{3}{2} - i\delta\right)\theta} \right) \]

when matched with the inner pressure \( \overline{p}_{\text{inner}} (\sigma < \omega) \) inside the shear layer, or

\[ p(r, \theta) = i^{-\frac{3}{2} + i\delta} (\lambda + 1) \frac{\zeta c_1}{2} \rho_0 U_0 e^{-2k_0 y_0} \left( \frac{1}{2} + \frac{1}{2} \right) \times \]

\[ H^2_{\nu}(\kappa r) \left( \frac{\omega}{\sigma - \omega + \frac{1}{2}\sigma} e^{i\left(\frac{3}{2} - i\delta\right)\theta} + \frac{\omega}{\sigma + \omega - \frac{1}{2}\sigma} e^{-i\left(\frac{3}{2} - i\delta\right)\theta} \right) \]

when matched with the inner pressure \( \overline{p}_{\text{tra}} (\sigma < \omega) \) transmitted outside the layer. Shown in figure 7 is the farfield sound obtained by above two different matchings. The difference is indeed very small. The soundfield behaves like \( r^{-\frac{1}{2} + i\delta} \) and is absolutely similar to the hard-soft case except that the \( \delta \) becomes \(-\delta\) now which is the effect of the boundary condition reversal.

**VI.A.1. Pressure release wall matching**

Similar to previous, the pressure release wall solution \( (40) \) can also be matched. From \( (51) \) and \( (44) \), we have \( \nu = \frac{1}{2} - i\delta \), and

\[ B_0 = (\lambda + 1) \frac{\rho_0 U_0^2}{2} e^{-2k_0 y_0 - \frac{1}{2} \pi i} \left( \frac{\omega - \sigma}{\omega + \sigma} \right)^\nu \]

\[ B_1 = 1 \quad \text{and} \quad B_2 = 1, \]

and hence

\[ \overline{p} = B_0 H^2_{\nu}(\kappa r) (B_1 e^{i\nu\theta} + B_2 e^{-i\nu\theta}), \]

\[ \overline{w} = \frac{1}{\rho_0 c_0} B_0 H^2_{\nu}(\kappa r) (B_1 e^{i\nu\theta} + B_2 e^{-i\nu\theta}), \]

\[ H^2_{\nu}(\kappa r) \sim \left( \frac{2}{\pi \kappa r} \right)^{\frac{1}{2}} e^{-i\kappa r + \frac{1}{2}i\nu\pi + \frac{1}{2}i\pi}, \quad H^2_{\nu}(\kappa r) \sim -i \left( \frac{2}{\pi \kappa r} \right)^{\frac{1}{2}} e^{-i\kappa r + \frac{1}{2}i\nu\pi + \frac{1}{2}i\pi}, \]

we can obtain the time averaged radial acoustic intensity in the farfield as

\[ \overline{\text{Re}(\overline{p} \overline{w}^*)} \simeq \frac{(\lambda + 1)^2 \rho_0 c_0^3 y_0^3}{8\pi c_0^3 \kappa r} e^{-2k_0 y_0} \left( e^{2\delta \theta} + e^{-2\delta \theta} - 2 \cos \theta \right). \]

Integrated over \( 0 < \theta < \pi \) we obtain the interesting expression of the radiated acoustic power

\[ \frac{1}{2} \int_0^\pi \text{Re}(\overline{p} \overline{w}^*) r d\theta = \rho_0 c_0^3 y_0 \left( \frac{U_0}{c_0} \right)^4 \frac{e^{-2\omega/\sigma}}{2\pi \delta} \left( \frac{\omega^2 \sigma^2}{(\omega^2 - \sigma^2)(\omega + \sigma)^2} \right), \]

(62)

to be multiplied by the square of the small dimensionless amplitude of the incident vorticity \((9)\). The radiating acoustic power behaves as \( \sim U_0^4 \).

**VI.B. Farfield sound, high shear case**

The successful matching of the low shear case can not be continued to the high shear case \( \sigma > \omega \). As announced in \( (53) \), the inner field that behaves like \( r^{i\delta} \) has to match with an acoustic field that behaves like \( \sigma r^{-i\delta} + \bar{\alpha} r^{i\delta} \), which is apparently not possible here. Similar behaviour
was found in the hard soft problem as well. For the record and completeness, we neglect the term \( r^{-i\delta} \) and find that

\[
\nu = -i\delta, \quad M = i^{-1+i\delta} (\lambda + 1) \frac{\zeta c_1}{2} \rho_0 U_0 e^{-k_0 y_0} \left( \frac{1}{2\kappa} \right)^{-i\delta} \]

with \( B_1 \) and \( B_2 \) remaining the same as in the previous case (65). Hence, our expression of the farfield sound is given by

\[
p(r, \theta) = i^{-1+i\delta} (\lambda + 1) \frac{\zeta c_1}{2} \rho_0 U_0 e^{-k_0 y_0} \left( \frac{1}{2\kappa} \right)^{-i\delta} H^{(2)}_{\nu}(\kappa r) (-e^{-i(-i\delta)\theta} + e^{-i(-i\delta)\theta})
\tag{64}
\]

when matched with the inner pressure \( p_{\text{inner}}(\sigma>\omega) \), inside the shear layer. Or we have

\[
p(r, \theta) = -i^{-1+i\delta} (\lambda + 1) \frac{\zeta c_1}{2} \rho_0 U_0 e^{-k_0 y_0} \left( \frac{1}{2\kappa} \right)^{-i\delta} \times H^{(2)}_{\nu}(\kappa r) \left( \frac{\sigma - \omega}{\sigma + \omega} \frac{\omega}{\sigma + \omega} e^{i(-i\delta)\theta} + \frac{\omega}{\sigma - \omega} e^{-i(-i\delta)\theta} \right)
\tag{65}
\]

when matched with the inner pressure \( \bar{p}_{\text{tra}}(\sigma>\omega) \), transmitted outside the layer. Shown in figure 8 is the farfield sound obtained by above two different matching. It can be seen that the soundfield behaves like \( r^{i\delta} \) and is similar to the hard soft transition except that the \( \delta \) is negative of the previous delta, which is the effect of boundary condition reversal.

**VII. Conclusions**

A systematic and analytically exact solution is obtained by means of the Wiener-Hopf technique of the problem of vorticity, convected by a linearly sheared mean flow, scattered by the soft-hard transition of the wall and the resulting soundfield associated to the scattering process.
It is illustrated by numerical examples. A particular feature is the fact that the Wiener-Hopf kernel can be split exactly. This enables us to find in rather detail the functional relationship of the hydrodynamic far field and hence the associated acoustic source strength. The farfield and resulting sound field are based upon the asymptotic behaviour of the solution in terms of Fourier integrals and the reliability is confirmed by the limiting case \( Z = 0 \) where the solution is explicit and exact.

Like the hard-soft transition, The problem appears to be distinguished into two different classes, based upon the relative size of problem parameters \( \sigma \) (the mean flow shear \( \bar{U}' \)) and \( \omega \) (the perturbation frequency), and not (for example) of the impedance of the wall. If the mean shear is relatively weak, i.e. if \( \sigma < \omega \), the hydrodynamic far field varies as the inverse square root of the distance from the hard-soft singularity including \( U_0^4 \) relation for radiated acoustic power, consistent to the hard-soft transition [1]. If the mean shear is relatively strong, i.e. if \( \sigma > \omega \), the hydrodynamic far field tends (in modulus) to a constant which confirms a strong interaction between the edge and the interphase that leads to the linear infinite shear as an inconsistent modelling assumption.

The functional relationship of the solution of soft - hard transition is similar to the hard - soft transition. The only difference is that the new \( \delta \) is negative of the previous one, which results from the boundary condition reversal. Consistent to the hard - soft transition, the high shear case is found to be inconclusive. A more realistic flow boundary layer profile could be useful to model this case.

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**Appendix**

**A. Wiener-Hopf formulation of pressure release wall**

Introduce the half-range Fourier transforms

\[
F_-(k) = \int_{-\infty}^{0} \varphi(x,0) e^{ikx} \, dx, \quad G_+(k) = \int_{0}^{\infty} \varphi(x,0) e^{ikx} \, dx
\] (66)

that are analytic in \( \text{Im}(k) < \varepsilon \) and \( \text{Im}(k) > -\varepsilon \) and assumed to be analytic in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \) respectively. We have

\[
F_-(k) = -i|k|A(k) + (\lambda + 1)U_0 \frac{e^{-k_0y}}{(k - k_0)},
\]

Furthermore, we have

\[
G_+(k) = \int_{0}^{\infty} \varphi(x,0) e^{ikx} \, dx = \int_{-\infty}^{\infty} \varphi(x,0) e^{ikx} \, dx = \rho_0A(k)\mu K(k)
\] (67)

with Wiener-Hopf kernel

\[
K(k) = \frac{\omega\mu - \sigma k}{k^2 + \varepsilon^2}, \quad (68)
\]
where $\mu = |k|$. With $\varepsilon = 0$ and $\omega \neq \sigma$, $K(k)$ is free from zeros however, for $\varepsilon > 0$ there are zero’s as shown in Fig. 9, namely

$$k^\pm_l = + 0 \pm i \varepsilon \frac{\omega}{\sqrt{\omega^2 - \sigma^2}} \quad \text{if} \quad \sigma < \omega,$$

$$k_h = \varepsilon \frac{\omega}{\sqrt{\sigma^2 - \omega^2}} \quad \text{if} \quad \sigma > \omega.$$ 

In the low-shear case $\sigma < \omega$, $|k^\pm_l| > \varepsilon$, so the zeros are imaginary, outside $S$ and located on the right side of the branchcuts of $\mu$. In the high-shear case $\sigma > \omega$, however, there is only one zero, which is real and therefore always inside the strip and is needed to be cancelled out as shown in Appendix B. From (67), we arrive at the Wiener-Hopf equation

$$F_-(k) = -i \frac{G_+(k)}{\rho_0} \frac{1}{K(k)} + (\lambda + 1)U_0 \frac{e^{-k_0y_0}}{(k - k_0)}$$

which is to be solved in the standard way [6] by writing

$$K(k) = \frac{K_+(k)}{K_-(k)},$$

where splitfunction $K_+$ is analytic and nonzero in $\mathbb{C}^+$ and $K_-$ is analytic and nonzero in $\mathbb{C}^-$. In the high-shear case, we have to remove the zero $k_h$ first. In the low-shear case, these split-functions are constructed in the usual way [11].

![Figure 9. Contour $C$ and zeros of $K(k)$.](image)

### B. Evaluation of the split functions for pressure release wall

We introduce two complex power functions, one analytic in the upper and the other analytic in the lower half plane. They are defined with branch cuts along the negative, respectively positive, imaginary axis, and equal to its principal branch at the right complex half plane. In order to be as explicit as possible we define the functions via principal value logarithms $\log(\cdot)$ as follows.

$$(z)^a_+ \overset{\text{def}}{=} e^{a \log(-iz) + \frac{1}{2} \pi ia}$$

$$(z)^a_- \overset{\text{def}}{=} e^{a \log(iz) - \frac{1}{2} \pi ia}$$

This amounts to

$$(z)^a_+ = \begin{cases} 
  z^a & \text{if} \quad \text{Re } z > 0, \\
  (-z)^a e^{\pi ia} & \text{if} \quad \text{Re } z < 0, 
\end{cases}$$

$$(z)^a_- = \begin{cases} 
  z^a & \text{if} \quad \text{Re } z > 0, \\
  (-z)^a e^{-\pi ia} & \text{if} \quad \text{Re } z < 0. 
\end{cases}$$
Note that we can create a function, discontinuous across the imaginary axis, by the quotient

\[
\frac{(z)^a}{(z)^a} = \begin{cases} 
1 & \text{if } \Re z > 0, \\
e^{2\pi i a} & \text{if } \Re z < 0.
\end{cases}
\]

and

\[
\frac{(z)^{\frac{1}{2}+a}}{(z)^{\frac{1}{2}+a}} = \begin{cases} 
1 & \text{if } \Re z > 0, \\
-e^{2\pi i a} & \text{if } \Re z < 0.
\end{cases}
\]

**B.A. Low-shear case** \(\sigma < \omega\)

Now we rewrite

\[
L(k) = \frac{\omega \mu - \sigma k}{k^2 + \varepsilon^2} = \mathcal{L}(k, \varepsilon)(\omega - \sigma)(k + i\varepsilon)_+^{\frac{1}{2} - i\delta}(k - i\varepsilon)_-^{\frac{1}{2} + i\delta},
\]

where

\[
\delta = \frac{1}{2\pi} \log \left| \frac{\omega + \sigma}{\omega - \sigma} \right|.
\]

Hence we have

\[
\mathcal{L}(k, \varepsilon) = \frac{\omega \mu - \sigma k}{\omega - \sigma}(k + i\varepsilon)_+^{\frac{1}{2} - i\delta}(k - i\varepsilon)_-^{\frac{1}{2} + i\delta}.
\]

It is convenient later to write this in the scaled variable \(t = k/\varepsilon\) by \(G(t) = \mathcal{L}(\varepsilon t, \varepsilon)\), such that parameter \(\varepsilon\) divides out into

\[
G(t) = \frac{\omega \sqrt{t^2 + 1} - \sigma t}{\omega - \sigma} \left( (t + i\varepsilon)_+^{\frac{1}{2} - i\delta}(t - i\varepsilon)_-^{\frac{1}{2} + i\delta} \right),
\]

and \(G\) is independent of \(\varepsilon\). Note that \(G(t) \to 1\) for \(t \to \pm\infty\). More in particular

\[
\log G(t) = \log \left(1 - \frac{2\delta}{t} + \ldots\right) = -\frac{2\delta}{t} + \ldots, \quad t \to \pm\infty.
\]

As a result we have (assume \(\Im z > 0\)) for a \(\mathcal{L}_+\) function

\[
2\pi i \log \mathcal{L}_+(z, \varepsilon) = \int_{-\infty}^{\infty} \frac{\log \mathcal{L}(x, \varepsilon)}{x - z} \, dx = \int_{-\infty}^{\infty} \frac{\log G(t)}{t - z/\varepsilon} \, dt
\]

For the limit of \(\varepsilon \to 0\) we split the integral into

\[
\int_{-\infty}^{-1/\sqrt{z}} + \int_{1/\sqrt{z}}^{1/\sqrt{z}} + \int_{1/\sqrt{z}}^{\infty} \frac{\log G(t)}{t - z/\varepsilon} \, dt \simeq \\
\int_{-\infty}^{-1/\sqrt{z}} - \frac{2\delta}{t} \, dt + \int_{1/\sqrt{z}}^{\infty} - \frac{2\delta}{t} \, dt - \frac{\varepsilon}{z} \int_{1/\sqrt{z}}^{1/\sqrt{z}} \log G(t) \, dt \simeq \\
- 4\delta \varepsilon \int_0^{\infty} \frac{1}{x^2 - z^2} \, dx - \frac{\varepsilon}{z} \int_{-1/\sqrt{z}}^{1/\sqrt{z}} \log G(t) \, dt \to 0
\]

Since the integral tends to zero we have to conclude that

\[\mathcal{L}_+(z, 0) = 1 \quad \text{and} \quad \mathcal{L}_-(z, 0) = 1.\]

As a result we have thus for \(K = K_+/K_-\) (the limit \(\varepsilon \to 0\) of \(L\))

\[
K_+(k) = (\omega - \sigma)(k)_+^{\frac{1}{2} - i\delta}, \quad K_-(k) = (k)_-^{\frac{1}{2} - i\delta}.
\]

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B.B. High-shear case $\sigma > \omega$

Now we rewrite

$$L(k) = -\frac{\sigma k - \omega \mu}{k^2 + \varepsilon^2} = -(\sigma - \omega) L(k, \varepsilon)(k - k_h)(k + i\varepsilon)^{-1-i\delta}(k - i\varepsilon)^{-1+i\delta}, \quad (73)$$

where $(k - k_h)$ divides out the real zero of $L$ in the strip:

$$k_h = \varepsilon t_0, \quad t_0 = \frac{\omega}{\sqrt{\sigma^2 - \omega^2}}$$

Hence we have

$$L(k, \varepsilon) = \frac{\sigma k - \omega \mu}{(\sigma - \omega)(k - k_h)}(k + i\varepsilon)^{i\delta}(k - i\varepsilon)^{-i\delta}.$$ 

It is convenient later to write this in the scaled variable $t = k/\varepsilon$ by $G(t) = L(\varepsilon t, \varepsilon)$, such that parameter $\varepsilon$ divides out into

$$G(t) = \frac{\sigma t - \omega \sqrt{t^2 + 1}}{(\sigma - \omega)(t - t_0)}(t + i\delta)(t - i\delta),$$

such that $G$ is independent of $\varepsilon$. Note that

$$G(t) \rightarrow 1 \quad \text{for} \quad t \rightarrow \pm \infty$$

More in particular we have

$$\log G(t) = \log \left(1 + \frac{t_0 - 2\delta}{t} + \ldots\right) = \frac{t_0 - 2\delta}{t} + \ldots, \quad t \rightarrow \pm \infty$$

An interesting property is

$$G(-t) = \frac{1}{G(t)}.$$ 

As a result we have (assume $\Im z > 0$) for a $F_+$ function

$$2\pi i \log L_+(z, \varepsilon) = \int_{-\infty}^{\infty} \frac{\log L(x, \varepsilon)}{x - z} dx = \int_{-\infty}^{0} -\log L(-x, \varepsilon) dx + \int_{0}^{\infty} \log L(x, \varepsilon) dx = \int_{0}^{\infty} \log G(t) \frac{2t}{t^2 - (z/\varepsilon)^2} dt.$$ 

For the limit of $\varepsilon \rightarrow 0$ we split the integral into

$$\int_{0}^{1/\sqrt{\pi}} \log G(t) \frac{2t}{t^2 - (z/\varepsilon)^2} dt + \int_{1/\sqrt{\pi}}^{\infty} \log G(t) \frac{2t}{t^2 - (z/\varepsilon)^2} dt \approx -\frac{2\varepsilon^2}{z^2} \int_{0}^{1/\sqrt{\pi}} t \log G(t) dt + 2(t_0 - 2\delta) \int_{1/\sqrt{\pi}}^{\infty} \frac{1}{t^2 - (z/\varepsilon)^2} dt.$$ 

From noting that $t \log G(t)$ tends to a constant so the first integral is $O(\varepsilon \sqrt{\varepsilon})$, which is smaller than the second of $O(\varepsilon)$, we have

$$\approx 2(t_0 - 2\delta) \int_{1/\sqrt{\pi}}^{\infty} \frac{1}{t^2 - (z/\varepsilon)^2} dt \approx 2(t_0 - 2\delta) \varepsilon \int_{0}^{\infty} \frac{1}{x^2 - z^2} dx = (t_0 - 2\delta) \varepsilon \pi i |z| \rightarrow 0.$$ 

21 of [25]
We have to conclude that
\[ \mathcal{L}_+(z, 0) = 1. \]
This leads to also
\[ \mathcal{L}_-(z, 0) = 1. \]
As a result we have thus for \( K = K_+ / K_- \) (the limit \( \varepsilon \to 0 \) of \( L \))
\[ K_+(k) = (\omega - \sigma)(k)^{-1-i\delta}, \quad K_-(k) = (k)^{-i\delta}, \quad (74) \]
Please note that the pole \((k - k_h)\) in (73) must be multiplied out and can not stay in the denominator to have the strip \( S \) free from zeros. Hence, we have to multiply it with \( K_- \) instead of \( K_+ \) when we take the limit \( \varepsilon \to 0 \). This process is essentially different from the Hard - Soft (pressure release) wall transition.

C. Analytic split functions and their asymptotic behaviour

The analytic evaluation of the integral \( I \) is performed in [1] and is borrowed directly. For the record, it is given as
\[
2\pi i \log K_+(k) = I = -\text{dilog}\left(\frac{k}{k + a - b}\right) + \text{dilog}\left(\frac{k}{k + a + b}\right) \\
+ \frac{1}{2} \log^2\left(\frac{k + a - b}{k}\right) - \frac{1}{2} \log^2\left(\frac{k + a + b}{k}\right) \\
+ \log\left(\frac{k + a - b}{k}\right) \log\left(\frac{b - a}{k + a - b}\right) - \log\left(\frac{k + a + b}{k}\right) \log\left(\frac{-b - a}{k + a + b}\right) \\
- 2\pi i C_1 \log\left(\frac{k + a - b}{k}\right) + 2\pi i C_2 \log\left(\frac{k + a + b}{k}\right), \quad (75)
\]
where \( C_1 = C(k, a - b) \) and \( C_2 = C(-k, -a - b) \) is the contribution of the poles and can be 0 or -1, please check [1] for the details.

C.A. Asymptotic behaviour of the split functions for \( k \) near 0

The behaviour of the integral \( I(k) \) and \( K_+(k) \) in the limit \( k \to 0 \) is different for high shear \((\sigma > \omega)\) and low shear \((\sigma < \omega)\) cases. hence, they are presented in separate sections.

C.B. High shear case

The asymptotic analysis of (75) for \( k \to 0 \) and \( \text{Im } k = +0 \) relates to the high shear \((\sigma > \omega)\) cases 1 and 2 in table 2. The following asymptotic behaviour is confirmed by [1],
\[
K_+(k) \sim c_1 k^{-i\delta} \quad \text{and} \quad K_-(k) \sim \frac{c_1}{(a - \text{sign}(\text{Re } k)b)k^{1-i\delta}}, \quad (76)
\]
where \( \delta \) is real positive constant and and \( c_1 \) is a complex constant given by
\[
\delta = \frac{1}{2\pi} \log\left|\frac{\sigma + \omega}{\sigma - \omega}\right|, \quad c_1 = e^{\frac{1}{2\pi}\left[\frac{1}{2} \log^2(a-b) - \frac{1}{2} \log^2(a+b) + \pi i \log(\frac{a+b}{a-b})\right]} \quad (77)
\]
We immediately notice that the expression in (76) is not analogous to (74). This situation arises because the zero \( k_h \) at \( k = 0 \) in (73) is contained by \( K_+ \) rather than \( K_- \) because \( k = 0 \) was a zero in the H-S kernel rather than a pole and hence, was regularized to stay within the
numerator. However, since our kernel is inverse of the previous kernel, the zero at \( k = 0 \) becomes the pole and should be canceled by multiplying it with \( K_- \) to have the strip \( S \) free from poles in order to have the analytic continuation of our split functions. We see from (73) that this pole must be associated with \( K_- \). This is precisely the reason that the pressure release limit is useful and is done here. Hence, we conclude that the pole in \( k = 0 \) stays with \( K_- \) and finally we obtain
\[
K_+(k) \sim c_1 k^{-1-i\delta} \quad \text{and} \quad K_-(k) \sim \frac{c_1}{(a - \text{sign}(Re\ k)b)} k^{-i\delta}.
\] (78)

**C.C. Low shear case**

The asymptotic analysis of (75) for \( k \to 0 \) and \( \text{Im}(k) = +0 \) considers the low shear (\( \sigma < \omega \)) cases 3 and 4 of table 2. The following asymptotic behavior is confirmed by [1]
\[
K_+(k) \sim c_1 k^{-\frac{1}{2}-i\delta} \quad \text{and} \quad K_-(k) \sim \frac{c_1}{(a - \text{sign}(Re\ k)b)} k^{\frac{1}{2}-i\delta}.
\] (79)

where (same as before) \( \delta \) is real positive and \( c_1 \) is a complex constant given by
\[
\delta = \frac{1}{2\pi} \log\left| \frac{\sigma + \omega}{\sigma - \omega} \right|, \quad c_1 = e^{\frac{\pi}{2\pi} \left( \frac{1}{2} (\log^2(a-b) - \log^2(a+b)) + \pi i \log\left( \frac{a+b}{a-b} \right) \right)}.
\] (80)

In this case, all the zeros and poles are out of the strip \( S \) and that justifies that the expression in (79) is analogous to (72).

**C.D. Asymptotic analysis for \( k \) large**

The analysis for \( k \to \infty \) is useful to derive the edge condition in the next section. Again, we consider \( \text{Im}(k) = +0 \). Noting that for \( z \to 0 \) we have \( \text{dilog}(1 - z) \simeq z + O(z^2) \) and \( \log(1 + z) = z + O(z^2) \), we may obtain for \( k \to \infty \)
\[
I \simeq \frac{2b}{k} \log k + \frac{a - b}{k} (\log(b - a) - 2\pi i C_1) - \frac{a + b}{k} (\log(-b - a) - 2\pi i C_2).
\]

Overall, the dominating term is \( \frac{2b}{k} \log k \).

**D. Evaluation of entire function \( E \)**

\( E \) can be determined from the condition at infinity. In order to obtain \( E(k) \) for \( k \to \infty \), we need the asymptotic behaviour of \( K_+, k \to \infty \). From C.D we have
\[
\lim_{k \to \infty} \log K_+(k) = \lim_{k \to \infty} \frac{2b}{2\pi i k} \log k = 0
\] (81)

so \( K_+(k) \to 1 \).

The asymptotic behaviour of \( G_+(k) \) in the limit \( k \to \infty \) is found from the so-called edge condition for \( r \to 0 \) where \( r \) is the distance from the edge. Consider a pressure distribution \( p \) at a small distance \( r \) from the discontinuity at \( r = 0 \), such that \( p \) is dominated by some power of \( r \), say \( p = O(r^\alpha) \). From the momentum equation it follows that the (radial) velocity, say \( w \), should be \( w = O(r^{\alpha-1}) \). The outward energy flux \( \Phi(r) \) across a small circular arc, centred at the edge at radius \( r \) (see figure 10) is then given by
\[
\Phi(r) \sim \int_0^\pi \pi r \, dr \, d\theta \sim \pi r^\alpha r^{\alpha-1} r \sim r^{2\alpha}.
\] (82)
Figure 10. Energy flux across a small semi-circle of radius \( r \) around the singularity.

In the absence of a physical source at \( r = 0 \), the energy flux should vanish for \( r \downarrow 0 \). Hence we must have \( \alpha > 0 \).

The function \( G_+(k) \) from (66) is therefore

\[
G_+(k \to \infty) \sim \int_0^\infty (x^\alpha + Z x^{\alpha-1}) e^{ikx} \, dx = k^{-1+\alpha} \Gamma(1+\alpha) e^{\frac{i\pi(\alpha+1)}{2}} + k^{-\alpha} \Gamma(\alpha) e^{\frac{i\pi\alpha}{2}} \tag{83}
\]

From (26), (81) and (83), we have

\[
E(k) = \frac{G_+(k)}{\rho_0 \zeta K(k)} + O(1/k) \sim k^{-\alpha} \to 0 \quad (k \to \infty). \tag{84}
\]

Thus the function \( E(k) \) vanishes at \( k \to \infty \) and since it is an entire function, it should vanish everywhere, i.e. \( E(k) = 0 \).

E. Regularisation of the diverging integral

We want to assign a meaning to

\[
\psi(x, y) = \int_0^\infty \frac{1}{k^{1-i\delta}} e^{ikz} \, dk
\]

where \( z = x + iy \) with \( y > 0 \) and \( \delta \) is real and nonzero. The integral converges for \( k \to \infty \) but not for \( k = 0 \). Following Lighthill-Jones [12,13], we define the function \( H(k) k^{-1+i\delta} \) as the generalised derivative

\[
\frac{H(k)}{k^{1-i\delta}} \overset{df}{=} \frac{d}{dk} \left( \frac{H(k)}{i\delta k^{-i\delta}} \right)
\]

and the integral

\[
\psi(x, y) = \int_{-\infty}^{\infty} \frac{d}{dk} \left( \frac{H(k)}{i\delta k^{-i\delta}} \right) e^{ikz} \, dk = -\int_{-\infty}^{\infty} \frac{zH(k)}{\delta k^{-i\delta}} e^{ikz} \, dk = -z\delta^{-1} \int_0^{\infty} k^{i\delta} e^{ikz} \, dk = -i\delta^{-1} \Gamma(1+i\delta) (-iz)^{-i\delta} = \Gamma(i\delta) (-iz)^{-i\delta}.
\]

This result is unique and independent of scaling.

References


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