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Citation for published version (APA):

Blokhuis, A., Brouwer, A. E., Chowdhury, A., Frankl, P., Mussche, T. J. J., Patkós, B., & Szönyi, T. (2010). A Hilton-Milner theorem for vector spaces. *The Electronic Journal of Combinatorics*, 17(1), R71-1/12.

Document status and date:

Published: 01/01/2010

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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- The final published version features the final layout of the paper including the volume, issue and page numbers.

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A Hilton-Milner Theorem for Vector Spaces

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Submitted: Nov 1, 2009; Accepted: May 4, 2010; Published: May 14, 2010

Mathematics Subject Classification: 05D05, 05A30

Abstract

We show for $k \geq 2$ that if $q \geq 3$ and $n \geq 2k + 1$, or $q = 2$ and $n \geq 2k + 2$, then any intersecting family \mathcal{F} of k -subspaces of an n -dimensional vector space over $GF(q)$ with $\bigcap_{F \in \mathcal{F}} F = 0$ has size at most $\binom{n-1}{k-1} - q^{k(k-1)} \binom{n-k-1}{k-1} + q^k$. This bound is sharp as is shown by Hilton-Milner type families. As an application of this result, we determine the chromatic number of the corresponding q -Kneser graphs.

1 Introduction

1.1 Sets

Let X be an n -element set and, for $0 \leq k \leq n$, let $\binom{X}{k}$ denote the family of all subsets of X of cardinality k . A family $\mathcal{F} \subset \binom{X}{k}$ is called *intersecting* if for all $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 \neq \emptyset$. Erdős, Ko, and Rado [5] determined the maximum size of an intersecting family, and introduced the so-called shifting technique.

Theorem 1.1 (Erdős-Ko-Rado) Suppose $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Excepting the case $n = 2k$, equality holds only if $\mathcal{F} = \{F \in \binom{X}{k} : x \in F\}$ for some $x \in X$.

For any family $\mathcal{F} \subset \binom{X}{k}$, the *covering number* $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. Theorem 1.1 shows that if $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family of maximum size and $n > 2k$, then $\tau(\mathcal{F}) = 1$.

Hilton and Milner [15] determined the maximum size of an intersecting family with $\tau(\mathcal{F}) \geq 2$. Later, Frankl and Füredi [9] gave an elegant proof of Theorem 1.2 using the shifting technique.

Theorem 1.2 (Hilton-Milner) Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $k \geq 2$, $n \geq 2k + 1$, and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Equality holds only if

- (i) $\mathcal{F} = \{F\} \cup \{G \in \binom{X}{k} : x \in G, F \cap G \neq \emptyset\}$ for some k -subset F and $x \in X \setminus F$.
- (ii) $\mathcal{F} = \{F \in \binom{X}{3} : |F \cap S| \geq 2\}$ for some 3-subset S if $k = 3$.

1.2 Vector spaces

Theorem 1.1 and Theorem 1.2 have natural extensions to vector spaces. We let V always denote an n -dimensional vector space over the finite field $GF(q)$. For $k \in \mathbb{Z}^+$, we write $\binom{V}{k}_q$ to denote the family of all k -dimensional subspaces of V . For $a, k \in \mathbb{Z}^+$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} a \\ k \end{bmatrix}_q := \prod_{0 \leq i < k} \frac{q^{a-i} - 1}{q^{k-i} - 1}.$$

A simple counting argument shows that the size of $\binom{V}{k}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From now on, we will omit the subscript q .

If two subspaces of V intersect in the zero subspace, then we say they are *disjoint* or that they *trivially intersect*; otherwise we say the subspaces *non-trivially intersect*. A family $\mathcal{F} \subset \binom{V}{k}$ is called intersecting if any two k -spaces in \mathcal{F} non-trivially intersect. The maximum size of an intersecting family of k -spaces was first determined by Hsieh [16]. For alternate proofs of Theorem 1.3, see [4] and [11]. We remark that there is as yet no analog of the shifting technique for vector spaces.

Theorem 1.3 (Hsieh) Suppose $\mathcal{F} \subset \binom{V}{k}$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Equality holds if and only if $\mathcal{F} = \{F \in \binom{V}{k} : v \subset F\}$ for some one-dimensional subspace $v \subset V$, unless $n = 2k$.

Let the *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F} \subset \binom{V}{k}$ be defined as the minimum dimension of a subspace of V that intersects all elements of \mathcal{F} nontrivially. Theorem 1.3 shows that, as in the set case, if \mathcal{F} is a maximum intersecting family of k -spaces, then $\tau(\mathcal{F}) = 1$. Families satisfying $\tau(\mathcal{F}) = 1$ are known as *point-pencils*.

In this paper, we will extend Theorem 1.2 to vector spaces, and determine the maximum size of an intersecting family $\mathcal{F} \subseteq \binom{V}{k}$ with $\tau(\mathcal{F}) \geq 2$. For two subspaces $S, T \leq V$, we let $S + T \leq V$ denote their linear span. We observe that for a fixed 1-subspace $E \leq V$ and a k -subspace U with $E \not\leq U$, the family

$$\mathcal{F}_{E,U} = \{U\} \cup \{W \in \binom{V}{k} : E \leq W, \dim(W \cap U) \geq 1\}$$

is not maximal as we can add all subspaces in $\binom{E+U}{k}$ that are not in $\mathcal{F}_{E,U}$. We will say that \mathcal{F} is an *HM-type family* if

$$\mathcal{F} = \{W \in \binom{V}{k} : E \leq W, \dim(W \cap U) \geq 1\} \cup \binom{E+U}{k}$$

for some $E \in \binom{V}{1}$ and $U \in \binom{V}{k}$ with $E \not\leq U$. If \mathcal{F} is an HM-type family, then its size is

$$|\mathcal{F}| = f(n, k, q) := \binom{n-1}{k-1} - q^{k(k-1)} \binom{n-k-1}{k-1} + q^k. \quad (1.1)$$

The main result of the paper is the following theorem.

Theorem 1.4 *Suppose $k \geq 3$, and either $q \geq 3$ and $n \geq 2k + 1$, or $q = 2$ and $n \geq 2k + 2$. For any intersecting family $\mathcal{F} \subseteq \binom{V}{k}$ with $\tau(\mathcal{F}) \geq 2$, we have $|\mathcal{F}| \leq f(n, k, q)$ (with $f(n, k, q)$ as in (1.1)). Equality holds only if*

- (i) \mathcal{F} is an HM-type family,
- (ii) $\mathcal{F} = \mathcal{F}_3 = \{F \in \binom{V}{k} : \dim(S \cap F) \geq 2\}$ for some $S \in \binom{V}{3}$ if $k = 3$.

Furthermore, if $k \geq 4$, then there exists an $\epsilon > 0$ (independent of n, k, q) such that if $|\mathcal{F}| \geq (1 - \epsilon)f(n, k, q)$, then \mathcal{F} is a subfamily of an HM-type family.

If $k = 2$, then a maximal intersecting family \mathcal{F} of k -spaces with $\tau(\mathcal{F}) > 1$ is the family of all 2-subspaces of a 3-subspace, and the conclusion of the theorem holds.

After proving Theorem 1.4 in Section 2, we apply this result to determine the chromatic number of q -Kneser graphs. The vertex set of the q -Kneser graph $qK_{n:k}$ is $\binom{V}{k}$. Two vertices of $qK_{n:k}$ are adjacent if and only if the corresponding k -subspaces are disjoint. In [3], the chromatic number of the q -Kneser graph $qK_{n:2}$ is determined, and the minimum colorings are characterized. In [18], the chromatic number of the q -Kneser graph is determined in general for $q > q_k$. In Section 4, we prove the following theorem.

Theorem 1.5 *If $k \geq 3$, and either $q \geq 3$ and $n \geq 2k + 1$, or $q = 2$ and $n \geq 2k + 2$, then the chromatic number of the q -Kneser graph is $\chi(qK_{n:k}) = \binom{n-k+1}{1}$. Moreover, each color class of a minimum coloring is a point-pencil and the points determining a color are the points of an $(n - k + 1)$ -dimensional subspace.*

In Section 5, we prove the non-uniform version of the Erdős-Ko-Rado theorem.

Theorem 1.6 *Let \mathcal{F} be an intersecting family of subspaces of V .*

(i) If n is even, then $|\mathcal{F}| \leq \binom{n-1}{n/2-1} + \sum_{i>n/2} \binom{n}{i}$.

(ii) If n is odd, then $|\mathcal{F}| \leq \sum_{i>n/2} \binom{n}{i}$.

For even n , equality holds only if $\mathcal{F} = \binom{V}{>n/2} \cup \{F \in \binom{V}{n/2} : E \leq F\}$ for some $E \in \binom{V}{1}$, or if $\mathcal{F} = \binom{V}{>n/2} \cup \binom{U}{n/2}$ for some $U \in \binom{V}{n-1}$. For odd n , equality holds only if $\mathcal{F} = \binom{V}{>n/2}$.

Note that Theorem 1.6 follows from the profile polytope of intersecting families which was determined implicitly by Bey [1] and explicitly by Gerbner and Patkós [12], but the proof we present in Section 5 is simple and direct.

2 Proof of Theorem 1.4

This section contains the proof of Theorem 1.4 which we divide into two cases.

2.1 The case $\tau(\mathcal{F}) = 2$

For any $A \leq V$ and $\mathcal{F} \subseteq \binom{V}{k}$, let $\mathcal{F}_A = \{F \in \mathcal{F} : A \leq F\}$. First, let us state some easy technical lemmas.

Lemma 2.1 *Let $a \geq 0$ and $n \geq k \geq a + 1$ and $q \geq 2$. Then*

$$\binom{k}{1} \binom{n-a-1}{k-a-1} < \frac{1}{(q-1)q^{n-2k}} \binom{n-a}{k-a}.$$

Proof. The inequality to be proved simplifies to

$$(q^{k-a} - 1)(q^k - 1)q^{n-2k} < q^{n-a} - 1. \quad \square$$

Lemma 2.2 *Let $E \in \binom{V}{1}$. If $E \not\leq L \leq V$, where L is an l -subspace, then the number of k -subspaces of V containing E and intersecting L is at least $\binom{l}{1} \binom{n-2}{k-2} - q \binom{l}{2} \binom{n-3}{k-3}$ (with equality for $l = 2$), and at most $\binom{l}{1} \binom{n-2}{k-2}$.*

Proof. The k -spaces containing E and intersecting L in a 1-dimensional space are counted exactly once in the first term. Those subspaces that intersect L in a 2-dimensional space are counted $\binom{2}{1} = q + 1$ times in the first term and $-q$ times in the second term, thus once overall. If a subspace intersects L in a subspace of dimension $i \geq 3$, then it is counted $\binom{i}{1}$ times in the first term and $-q \binom{i}{2}$ times in the second term, and hence a negative number of times overall. \square

Our next lemma gives bounds on the size of an HM-type family that are easier to work with than the precise formula mentioned in the introduction.

Lemma 2.3 *Let $n \geq 2k + 1$, $k \geq 3$ and $q \geq 2$. If $\mathcal{F} \subset \binom{V}{k}$ is an HM-type family, then $(1 - \frac{1}{q^3-q}) \binom{k}{1} \binom{n-2}{k-2} < \binom{k}{1} \binom{n-2}{k-2} - q \binom{k}{2} \binom{n-3}{k-3} \leq f(n, k, q) = |\mathcal{F}| \leq \binom{k}{1} \binom{n-2}{k-2}$.*

Proof. Since $q \binom{k}{2} = \binom{k}{1}(\binom{k}{1} - 1)/(q + 1)$ and $n \geq 2k + 1$, the first inequality follows from Lemma 2.1. Let \mathcal{F} be the HM-type family defined by the 1-space E and the k -space U . Then \mathcal{F} contains all k -subspaces of V containing E and intersecting U , so that the second inequality follows from Lemma 2.2. For the last inequality, Lemma 2.2 almost suffices, but we also have to count the k -subspaces of $\binom{E+U}{k}$ that do not contain E . Each $(k - 1)$ -subspace W of U is contained in $q + 1$ such subspaces, one of which is $E + W$. On the other hand, $E + W$ was counted at least $q + 1$ times since $k \geq 3$. This proves the last inequality. \square

Lemma 2.4 *If a subspace S does not intersect each element of $\mathcal{F} \subset \binom{V}{k}$, then there is a subspace $T > S$ with $\dim T = \dim S + 1$ and $|\mathcal{F}_T| \geq |\mathcal{F}_S|/\binom{k}{1}$.*

Proof. There is an $F \in \mathcal{F}$ such that $S \cap F = 0$. Average over all $T = S + E$ where E is a 1-subspace of F . \square

Lemma 2.5 *If an s -dimensional subspace S does not intersect each element of $\mathcal{F} \subset \binom{V}{k}$, then $|\mathcal{F}_S| \leq \binom{k}{1} \binom{n-s-1}{k-s-1}$.*

Proof. There is an $(s + 1)$ -space T with $\binom{n-s-1}{k-s-1} \geq |\mathcal{F}_T| \geq |\mathcal{F}_S|/\binom{k}{1}$. \square

Corollary 2.6 *Let $\mathcal{F} \subseteq \binom{V}{k}$ be an intersecting family with $\tau(\mathcal{F}) \geq s$. Then for any i -space $L \leq V$ with $i \leq s$ we have $|\mathcal{F}_L| \leq \binom{k}{1}^{s-i} \binom{n-s}{k-s}$.* \square

Proof. If $i = s$, then clearly $|\mathcal{F}_L| \leq \binom{n-s}{k-s}$. If $i < s$, then there exists an $F \in \mathcal{F}$ such that $F \cap L = 0$; now apply Lemma 2.4 $s - i$ times. \square

Before proving the q -analogue of the Hilton-Milner theorem, we describe the essential part of maximal intersecting families $\mathcal{F} \subset \binom{V}{k}$ with $\tau(\mathcal{F}) = 2$.

Proposition 2.7 *Let $n \geq 2k$ and let $\mathcal{F} \subset \binom{V}{k}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Define \mathcal{T} to be the family of 2-spaces of V that intersect all subspaces in \mathcal{F} . One of the following three possibilities holds:*

- (i) $|\mathcal{T}| = 1$ and $\binom{n-2}{k-2} < |\mathcal{F}| < \binom{n-2}{k-2} + (q + 1) \left(\binom{k}{1} - 1 \right) \binom{k}{1} \binom{n-3}{k-3}$;
- (ii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, and there is an $(l + 1)$ -space W (with $2 \leq l \leq k$) and a 1-space $E \leq W$ so that $\mathcal{T} = \{M : E \leq M \leq W, \dim M = 2\}$. In this case,

$$\binom{l}{1} \binom{n-2}{k-2} - q \binom{l}{2} \binom{n-3}{k-3} \leq |\mathcal{F}| \leq \binom{l}{1} \binom{n-2}{k-2} + \binom{k}{1} \left(\binom{k}{1} - \binom{l}{1} \right) \binom{n-3}{k-3} + q^l \binom{n-l}{k-l}.$$

For $l = 2$, the upper bound can be strengthened to

$$|\mathcal{F}| \leq (q + 1) \binom{n-2}{k-2} - q \binom{n-3}{k-3} + \binom{k}{1} \left(\binom{k}{1} - \binom{2}{1} \right) \binom{n-3}{k-3} + q^2 \binom{k}{1} \binom{n-3}{k-3};$$

- (iii) $\mathcal{T} = \binom{A}{2}$ for some 3-subspace A and $\mathcal{F} = \{U \in \binom{V}{k} : \dim(U \cap A) \geq 2\}$. In this case, $|\mathcal{F}| = (q^2 + q + 1) \left(\binom{n-2}{k-2} - \binom{n-3}{k-3} \right) + \binom{n-3}{k-3}$.

Proof. Let $\mathcal{F} \subset \binom{V}{k}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. By maximality, \mathcal{F} contains all k -spaces containing a $T \in \mathcal{T}$. Since $n \geq 2k$ and $k \geq 2$, two disjoint elements of \mathcal{T} would be contained in disjoint elements of \mathcal{F} , which is impossible. Hence, \mathcal{T} is intersecting.

Observe that if $A, B \in \mathcal{T}$ and $A \cap B < C < A + B$, then $C \in \mathcal{T}$. As an intersecting family of 2-spaces is either a family of 2-spaces containing some fixed 1-space E or a family of 2-subspaces of a 3-space, we get the following:

(*) : \mathcal{T} is either a family of all 2-subspaces containing some fixed 1-space E that lie in some fixed $(l+1)$ -space with $k \geq l \geq 1$, or \mathcal{T} is the family of all 2-subspaces of a 3-space.

(i) : If $|\mathcal{T}| = 1$, then let S denote the only 2-space in \mathcal{T} and let $E \leq S$ be any 1-space. Since $\tau(\mathcal{F}) > 1$, there exists an $F \in \mathcal{F}$ with $E \not\leq F$, for which we must have $\dim(F \cap S) = 1$. As S is the only element of \mathcal{T} , for any 1-subspace E' of F different from $F \cap S$, we have $\mathcal{F}_{E+E'} \leq \binom{k}{1} \binom{n-3}{k-3}$ by Lemma 2.5. Hence the number of subspaces containing E but not containing S is at most $(\binom{k}{1} - 1) \binom{k}{1} \binom{n-3}{k-3}$. This gives the upper bound.

(ii) : Assume that $\tau(\mathcal{T}) = 1$ and $|\mathcal{T}| > 1$. By (*), \mathcal{T} is the set of 2-spaces in an $(l+1)$ -space W (with $l \geq 2$) containing some fixed 1-space E . Every $F \in \mathcal{F} \setminus \mathcal{F}_E$ intersects W in a hyperplane. Let L be a hyperplane in W not on E . Then \mathcal{F} contains all k -spaces on E that intersect L . Hence the lower bound and the first term in the upper bound come from Lemma 2.2. The second term comes from using Lemma 2.5 to count the k -spaces of \mathcal{F} that contain E and intersect a given $F \in \mathcal{F}$ (not containing E) in a point of $F \setminus W$. If $l \geq 3$, then there are q^l hyperplanes in W not containing E and there are $\binom{n-l}{k-l}$ k -spaces through such a hyperplane; this gives the last term. For $l = 2$, we use the tight lower bound in Lemma 2.2 to count the number of k -spaces on E that intersect L . There are q^2 hyperplanes in W , and they cannot be in \mathcal{T} , so Lemma 2.5 gives the bound.

(iii) : This is immediate. □

Corollary 2.8 *Let $\mathcal{F} \subset \binom{V}{k}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Suppose $q \geq 3$ and $n \geq 2k + 1$, or $q = 2$ and $n \geq 2k + 2$. If \mathcal{F} is at least as large as an HM-type family and $k > 3$, then \mathcal{F} is an HM-type family. If $k = 3$, then \mathcal{F} is an HM-type family or an \mathcal{F}_3 -type family.*

There exists an $\epsilon > 0$ (independent of n, k, q) such that if $k \geq 4$ and $|\mathcal{F}|$ is at least $(1 - \epsilon)$ times the size of an HM-type family, then \mathcal{F} is an HM-type family.

Proof. Apply Proposition 2.7. Note that the HM-type families are precisely those from case (ii) with $l = k$.

Let $n = 2k + r$ where $r \geq 1$. We have $|\mathcal{F}| / \binom{n-2}{k-2} < 1 + \frac{q+1}{(q-1)q^r} \binom{k}{1}$ in case (i) of Proposition 2.7 by Lemma 2.1. We have $|\mathcal{F}| / \binom{n-2}{k-2} < (\frac{1}{q} + \frac{1}{(q-1)q^r}) \binom{k}{1} + \frac{q^2}{(q-1)q^r}$ in case (ii) when $l < k$. In both cases, for $q \geq 3$ and $k \geq 3$, or $q = 2$, $k \geq 4$, and $r \geq 2$, this is less than $(1 - \epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3. Using the stronger estimate in Lemma 2.3, we find the same conclusion for $q = 2$, $k = 3$, and $r \geq 2$.

In case (iii), $|\mathcal{F}_3| = \binom{3}{2} \binom{n-2}{k-2} - \frac{q^3-q}{q-1} \binom{n-3}{k-3}$. For $k \geq 4$, this is much smaller than the size of the HM-type families. For $k = 3$, the two families have the same size. \square

Proposition 2.9 *Suppose that $k \geq 3$ and $n \geq 2k$. Let $\mathcal{F} \subseteq \binom{V}{k}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. Let $3 \leq l \leq k$. If there is an l -space that intersects each $F \in \mathcal{F}$ and*

$$|\mathcal{F}| > \binom{l}{1} \binom{k}{1}^{l-1} \binom{n-l}{k-l}, \quad (2.2)$$

then there is an $(l-1)$ -space that intersects each $F \in \mathcal{F}$.

Proof. By averaging, there is a 1-space P with $|\mathcal{F}_P| \geq |\mathcal{F}| / \binom{l}{1}$. If $\tau(\mathcal{F}) = l$, then by Corollary 2.6, $|\mathcal{F}| \leq \binom{l}{1} \binom{k}{1}^{l-1} \binom{n-l}{k-l}$, contradicting the hypothesis. \square

Corollary 2.10 *Suppose $k \geq 3$ and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$. Let $\mathcal{F} \subseteq \binom{V}{k}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. If $|\mathcal{F}| > \binom{3}{1} \binom{k}{1}^2 \binom{n-3}{k-3}$, then $\tau(\mathcal{F}) = 2$; that is, \mathcal{F} is contained in one of the systems in Proposition 2.7, which satisfy the bound on $|\mathcal{F}|$.*

Proof. By Lemma 2.1 and the conditions on n and q , the right hand side of (2.2) decreases as l increases, where $3 \leq l \leq k$. Hence, by Proposition 2.9, we can find a 2-space that intersects each $F \in \mathcal{F}$. \square

Remark 2.11 For $n \geq 3k$, all systems described in Proposition 2.7 occur.

2.2 The case $\tau(\mathcal{F}) > 2$

Suppose that $\mathcal{F} \subseteq \binom{V}{k}$ is an intersecting family and $\tau(\mathcal{F}) = l > 2$. We shall derive a contradiction from $|\mathcal{F}| \geq f(n, k, q)$, and even from $|\mathcal{F}| \geq (1 - \epsilon)f(n, k, q)$ for some $\epsilon > 0$ (independent of n, k, q).

2.2.1 The case $l = k$

First consider the case $l = k$. Then $|\mathcal{F}| \leq \binom{k}{1}^k$ by Corollary 2.6. On the other hand,

$$|\mathcal{F}| \geq \left(1 - \frac{1}{q^3-q}\right) \binom{k}{1} \binom{n-2}{k-2} > \left(1 - \frac{1}{q^3-q}\right) \binom{k}{1}^{k-1} ((q-1)q^{n-2k})^{k-2}$$

by Lemma 2.3 and Lemma 2.1. If either $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, then either $k = 3$, $(n, k, q) = (9, 4, 3)$, or $(n, k, q) = (10, 4, 2)$. If $(n, k, q) = (9, 4, 3)$ then $f(n, k, q) = 3837721$, and $40^4 = 2560000$, which gives a contradiction. If $(n, k, q) = (10, 4, 2)$, then $f(n, k, q) = 153171$, and $15^4 = 50625$, which again gives a contradiction. Hence $k = 3$. Now $|\mathcal{F}| \geq \left(1 - \frac{1}{q^3-q}\right) \binom{k}{1} \binom{n-2}{k-2}$ gives a contradiction for $n \geq 8$, so $n = 7$. Therefore, if we assume that $n \geq 2k+1$ and either $q \geq 3$, $(n, k) \neq (7, 3)$ or $q = 2$, $n \geq 2k+2$ then we are not in the case $l = k$.

It remains to settle the case $n = 7$, $k = l = 3$, and $q \geq 3$. By Lemma 2.4, we can choose a 1-space E such that $|\mathcal{F}_E| \geq |\mathcal{F}| / \binom{3}{1}$ and a 2-space S on E such that $|\mathcal{F}_S| \geq |\mathcal{F}_E| / \binom{3}{1}$.

Then $|\mathcal{F}_S| > q+1$ since $|\mathcal{F}| > \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}^2$. Pick $F' \in \mathcal{F}$ disjoint from S and define $H := S + F'$. All $F \in \mathcal{F}_S$ are contained in the 5-space H . Since $|\mathcal{F}| > \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, there is an $F_0 \in \mathcal{F}$ not contained in H . If $F_0 \cap S = 0$, then each $F \in \mathcal{F}_S$ is contained in $S + (H \cap F_0)$; this implies $|\mathcal{F}_S| \leq q+1$, which is impossible. Thus, all elements of \mathcal{F} disjoint from S are in H .

Now F_0 must meet F' and S , so F_0 meets H in a 2-space S_0 . Since $|\mathcal{F}_S| > q+1$, we can find two elements F_1, F_2 of \mathcal{F}_S with the property that S_0 is not contained in the 4-space $F_1 + F_2$. Since any $F \in \mathcal{F}$ disjoint from S is contained in H and meets F_0 , it must meet S_0 and also F_1 and F_2 . Hence the number of such F 's is at most q^5 . Altogether $|\mathcal{F}| \leq q^5 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}^2$; the first term comes from counting $F \in \mathcal{F}$ disjoint from S and the second term comes from counting $F \in \mathcal{F}$ on a given one-dimensional subspace $E < S$. This contradicts $|\mathcal{F}| \geq (1 - \frac{1}{q^3-q}) \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

2.2.2 The case $l < k$

Assume, for the moment, that there are two l -subspaces in V that non-trivially intersect all $F \in \mathcal{F}$, and that these two l -spaces meet in an m -space, where $0 \leq m \leq l-1$. By Corollary 2.6, for each 1-subspace P we have $|\mathcal{F}_P| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$, and for each 2-subspace L we have $|\mathcal{F}_L| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$. Consequently,

$$|\mathcal{F}| \leq \begin{bmatrix} m \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix} + (\begin{bmatrix} l \\ 1 \end{bmatrix} - \begin{bmatrix} m \\ 1 \end{bmatrix})^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}. \quad (2.3)$$

The upper bound (2.3) is a quadratic in $x = \begin{bmatrix} m \\ 1 \end{bmatrix}$ and is largest at one of the extreme values $x = 0$ and $x = \begin{bmatrix} l-1 \\ 1 \end{bmatrix}$. The maximum is taken at $x = 0$ only when $\begin{bmatrix} l \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} k \\ 1 \end{bmatrix} > \frac{1}{2} \begin{bmatrix} l-1 \\ 1 \end{bmatrix}$; that is, when $k = l$. Since we assume that $l < k$, the upper bound in (2.3) is largest for $m = l-1$. We find

$$|\mathcal{F}| \leq \begin{bmatrix} l-1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix} + (\begin{bmatrix} l \\ 1 \end{bmatrix} - \begin{bmatrix} l-1 \\ 1 \end{bmatrix})^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}.$$

On the other hand,

$$|\mathcal{F}| \geq (1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} > (1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix} ((q-1)q^{n-2k})^{l-2}.$$

Comparing these, and using $k > l$, $n \geq 2k+1$, and $n \geq 2k+2$ if $q = 2$, we find either $(n, k, l, q) = (9, 4, 3, 3)$ or $q = 2$, $n = 2k+2$, $l = 3$, and $k \leq 5$. If $(n, k, l, q) = (9, 4, 3, 3)$ then $f(n, k, q) = 3837721$, while the upper bound is 3508960, which is a contradiction. If $(n, k, l, q) = (12, 5, 3, 2)$ then $f(n, k, q) = 183628563$, while the upper bound is 146766865, which is a contradiction. If $(n, k, l, q) = (10, 4, 3, 2)$ then $f(n, k, q) = 153171$, while the upper bound is 116205, which is a contradiction. Hence, under our assumption that there are two distinct l -spaces that meet all $F \in \mathcal{F}$, the case $2 < l < k$ cannot occur.

We now assume that there is a unique l -space T that meets all $F \in \mathcal{F}$. We can pick a 1-space $E < T$ such that $|\mathcal{F}_E| \geq |\mathcal{F}| / \begin{bmatrix} l \\ 1 \end{bmatrix}$. Now there is some $F' \in \mathcal{F}$ not on E , so E is in $\begin{bmatrix} k \\ 1 \end{bmatrix}$ lines such that each $F \in \mathcal{F}_E$ contains at least one of these lines. Suppose L is one of these lines and L does not lie in T ; we can enlarge L to an l -space that still does not

meet all elements of \mathcal{F} , so $|\mathcal{F}_L| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l-1 \\ k-l-1 \end{bmatrix}$ by Lemma 2.4 and Lemma 2.5. If L does lie on T , we have $|\mathcal{F}_L| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$ by Corollary 2.6. Hence,

$$|\mathcal{F}| \leq \begin{bmatrix} l \\ 1 \end{bmatrix} |\mathcal{F}_E| \leq \begin{bmatrix} l \\ 1 \end{bmatrix} \left(\begin{bmatrix} l-1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix} \right) + \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l-1 \\ 1 \end{bmatrix} \right) \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l-1 \\ k-l-1 \end{bmatrix} \right) \right).$$

On the other hand, we have $|\mathcal{F}| > \left(1 - \frac{1}{q^3-q}\right) ((q-1)q^{n-2k})^{l-2} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$. Under our standard assumptions $n \geq 2k+1$ and $n \geq 2k+2$ if $q=2$, this implies $q=2$, $n=2k+2$, $l=3$, which gives a contradiction. We showed: If $q \geq 3$ and $n \geq 2k+1$ or if $q=2$ and $n \geq 2k+2$, then an intersecting family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ with $|\mathcal{F}| \geq f(n, k, q)$ must satisfy $\tau(\mathcal{F}) \leq 2$. Together with Corollary 2.8, this proves Theorem 1.4.

3 Critical families

A subspace will be called a *hitting subspace* (and we shall say that the subspace intersects \mathcal{F}), if it intersects each element of \mathcal{F} .

The previous results just used the parameter τ , so only the hitting subspaces of smallest dimension were taken into account. A more precise description is possible if we make the intersecting system of subspaces critical.

Definition 3.1 An intersecting family \mathcal{F} of subspaces of V is *critical* if for any two distinct $F, F' \in \mathcal{F}$ we have $F \not\subseteq F'$, and moreover for any hitting subspace G there is a $F \in \mathcal{F}$ with $F \subset G$.

Lemma 3.2 For every non-extendable intersecting family \mathcal{F} of k -spaces there exists some critical family \mathcal{G} such that

$$\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \exists G \in \mathcal{G}, G \subseteq F\}.$$

Proof. Extend \mathcal{F} to a maximal intersecting family \mathcal{H} of subspaces of V , and take for \mathcal{G} the minimal elements of \mathcal{H} . \square

The following construction and result are an adaptation of the corresponding results from Erdős and Lovász [6]:

Construction 3.3 Let A_1, \dots, A_k be subspaces of V such that $\dim A_i = i$ and $\dim(A_1 + \dots + A_k) = \binom{k+1}{2}$. Define

$$\mathcal{F}_i = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : A_i \subseteq F, \dim A_j \cap F = 1 \text{ for } j > i\}.$$

Then $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ is a critical, non-extendable, intersecting family of k -spaces, and $|\mathcal{F}_i| = \begin{bmatrix} i+1 \\ 1 \end{bmatrix} \begin{bmatrix} i+2 \\ 1 \end{bmatrix} \dots \begin{bmatrix} k \\ 1 \end{bmatrix}$ for $1 \leq i \leq k$.

For subsets Erdős and Lovász proved that a critical, non-extendable, intersecting family of k -sets cannot have more than k^k members. They conjectured that the above construction is best possible but this was disproved by Frankl, Ota and Tokushige [10]. Here we prove the following analogous result.

Theorem 3.4 *Let \mathcal{F} be a critical, intersecting family of subspaces of V of dimension at most k . Then $|\mathcal{F}| \leq \binom{k}{1}^k$.*

Proof. Suppose that $|\mathcal{F}| > \binom{k}{1}^k$. By induction on i , $0 \leq i \leq k$, we find an i -dimensional subspace A_i of V such that $|\mathcal{F}_{A_i}| > \binom{k}{1}^{k-i}$. Indeed, since by induction $|\mathcal{F}_{A_i}| > 1$ and \mathcal{F} is critical, the subspace A_i is not hitting, and there is an $F \in \mathcal{F}$ disjoint from A_i . Now all elements of \mathcal{F}_{A_i} meet F , and we find $A_{i+1} > A_i$ with $|\mathcal{F}_{A_{i+1}}| > |\mathcal{F}_{A_i}| / \binom{k}{1}$. For $i = k$ this is a contradiction. \square

Remark 3.5 For $l \leq k$ this argument shows that there are not more than $\binom{l}{1} \binom{k}{1}^{l-1}$ l -spaces in \mathcal{F} .

If $l = 3$ and $\tau > 2$ then for the size of \mathcal{F} the previous remark essentially gives $\binom{3}{1} \binom{k}{1}^2 \binom{n-3}{k-3}$, which is the bound in Corollary 2.10.

Modifying the Erdős-Lovász construction (see Frankl [7]), one can get intersecting families with many l -spaces in the corresponding critical family.

Construction 3.6 *Let A_1, \dots, A_l be subspaces with $\dim A_1 = 1$, $\dim A_i = k + i - l$ for $i \geq 2$. Define $\mathcal{F}_i = \{F \in \binom{V}{k} : A_i \leq F, \dim(F \cap A_j) \geq 1 \text{ for } j > i\}$. Then $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_l$ is intersecting and the corresponding critical family has at least $\binom{k-l+2}{1} \cdots \binom{k}{1}$ l -spaces.*

For n large enough the Erdős-Ko-Rado theorem for vector spaces follows from the obvious fact that no critical, intersecting family can contain more than one 1-dimensional member. The Hilton-Milner theorem and the stability of the systems follow from (*) which was used to describe the intersecting systems with $\tau = 2$. As remarked above, the fact that the critical family has to contain only spaces of dimension 3 or more limits its size to $O(\binom{n}{k-3})$, if k is fixed and n is large enough. Stronger and more general stability theorems can be found in Frankl [8] for the subset case.

4 Coloring q -Kneser graphs

In this section, we prove Theorem 1.5. We will need the following result of Bose and Burton [2] and its extension by Metsch [17].

Theorem 4.1 (Bose-Burton) *If \mathcal{E} is a family of 1-subspaces of V such that any k -subspace of V contains at least one element of \mathcal{E} , then $|\mathcal{E}| \geq \binom{n-k+1}{1}$. Furthermore, equality holds if and only if $\mathcal{E} = \binom{H}{1}$ for some $(n - k + 1)$ -subspace H of V .*

Proposition 4.2 (Metsch) *If \mathcal{E} is a family of $\binom{n-k+1}{1} - \varepsilon$ 1-subspaces of V , then the number of k -subspaces of V that are disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k)}$.*

Proof of Theorem 1.5. Suppose that we have a coloring with at most $\binom{n-k+1}{1}$ colors. Let G (the good colors) be the set of colors that are point-pencils and let B (the bad colors) be the remaining set of colors. Then $|G| + |B| \leq \binom{n-k+1}{1}$. Suppose $|B| = \varepsilon > 0$. By Proposition 4.2, the number of k -spaces with a color in B is at least $\varepsilon q^{(k-1)(n-k)}$, so that the average size of a bad color class is at least $q^{(k-1)(n-k)}$. This must be smaller than the size of a HM-type family. Thus, by Lemma 2.3,

$$q^{(k-1)(n-k)} \leq \binom{k}{1} \binom{n-2}{k-2}.$$

For $k \geq 3$ and $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, this is a contradiction. (The weaker form of Proposition 4.2, as stated in [17], suffices unless $q = 2$, $n = 2k+2$.) If $|B| = 0$, all color classes are point-pencils, and we are done by Theorem 4.1. \square

5 Proof of Theorem 1.6

Let $a + b = n$, $a < b$ and let $\mathcal{F}_a = \mathcal{F} \cap \binom{V}{a}$ and $\mathcal{F}_b = \mathcal{F} \cap \binom{V}{b}$. We prove

$$|\mathcal{F}_a| + |\mathcal{F}_b| \leq \binom{n}{b} \tag{5.4}$$

with equality only if $\mathcal{F}_a = \emptyset$ and $\mathcal{F}_b = \binom{V}{b}$.

Adding up (5.4) for $n/2 < b \leq n$ gives the bound on $|\mathcal{F}|$ in Theorem 1.6 if n is odd; adding the result of Greene and Kleitman [14] that states $|\mathcal{F}_{n/2}| \leq \binom{n-1}{n/2-1}$ proves it for even n . For the uniqueness part of Theorem 1.6, we only have to note that if n is even then, by results of Godsil and Newman [13], we must have $\mathcal{F}_{n/2} = \{F \in \binom{V}{n/2} : E \leq F\}$ for some $E \in \binom{V}{1}$ or $\mathcal{F}_{n/2} = \binom{U}{n/2}$ for some $U \in \binom{V}{n-1}$.

Now we prove (5.4). Consider the bipartite graph with vertex set $(\binom{V}{a}, \binom{V}{b})$ and join $A \in \binom{V}{a}$ and $B \in \binom{V}{b}$ if $A \cap B = \emptyset$. Observe that $\mathcal{F}_a \cup \mathcal{F}_b$ is an independent set in this graph. Now, this graph is regular with degree q^{ab} . Therefore any independent set in this graph has size at most $\binom{n}{b}$ by König's Theorem. Moreover, independent sets of size $\binom{n}{b}$ can only be $\binom{V}{a}$ or $\binom{V}{b}$, but the former is not an intersecting family. This proves (5.4). \square

Acknowledgements

Ameera Chowdhury thanks the NSF for supporting her and the Rényi Institute for hosting her while she was an NSF-CESRI fellow during the summer of 2008. Balázs Patkós's research was supported by NSF Grant #: CCF-0728928. Tamás Szőnyi gratefully acknowledges the financial support of NWO, including the support of the DIAMANT and Spinoza projects. He also thanks the Department of Mathematics at TU/e for the warm hospitality. He was partly supported by OTKA Grants T 49662 and NK 67867.

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