Circuits and trees in oriented linear graphs

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CIRCUITS AND TREES IN ORIENTED LINEAR GRAPHS

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§ 1. $P_n^{(\sigma)}$-cycles.

In this § we state the problem which gave rise to our investigations about graphs. The further contents of the paper are independent of this § 1.

Consider a set of σ figures 1, 2, . . ., σ, and let n be a natural number. A sequence of n figures will be called an n-tuple. Clearly, there are σ^n different n-tuples.

An oriented circular array, consisting of σ^n figures, will be called a P_n-cycle, whenever it has the property that each n-tuple occurs exactly once as a set of n consecutive figures of the cycle. An example, with $\sigma = 3$, $n = 2$ is the cycle 1 1 2 2 3 3 1 3 2. [1])

The existence of $P_n^{(\sigma)}$-cycles, for arbitrary values of σ and n, was proved by M. H. MARTIN [3], I. J. GOOD [2] and D. REES [4]. One of us showed ([1]) that, for $\sigma = 2$, the number of different $P_n^{(2)}$-cycles equals $2^{f(n)}$, $f(n) = 2^{n-1} - n$.

This result was derived as follows. The number of $P_n^{(2)}$-cycles can be interpreted as the number of circuits in a certain graph N_{n+1} (compare also [2]). The graph N_{n+1} can be obtained by a certain operation from N_n, and by a general theorem on circuits in oriented graphs the number of circuits of N_{n+1} could be expressed in the number of circuits of N_n. This theorem on graphs was proved in [1] only for the case that at any vertex 2 edges point outward and 2 inward. In the present paper we shall deal, among other things, with the general case (theorem 4). This result immediately enables us to determine the number of $P_n^{(\sigma)}$-cycles for arbitrary σ. Referring to [1] for details, we only state the result: The number of different $P_n^{(\sigma)}$-cycles is $\sigma^{-n}(\sigma!)^q$, where $q = \sigma^{n-1}$.

For example, there are 24 $P_2^{(3)}$-cycles. Six of them are

1 1 2 3 3 2 2 1 3	1 1 2 2 3 2 1 3 3
1 1 2 3 2 2 1 3 3	1 1 2 2 3 3 1 3 2
1 1 2 2 3 3 2 1 3	1 1 2 2 3 1 3 3 2

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[1]) It has to be understood that these figures have to be placed around an oriented circle. Therefore, 21 is one of the 2 – tuples occurring in the cycle. Naturally, 112233132 and 331321122 are considered as one and the same cycle, but 112313322, which has the reversed order, is a different one.

Another six are obtained from these by interchanging the figures 2 and 3 everywhere. By reversing the orientation, 12 new cycles arise.

§ 2. Preliminaries about permutation groups.

Let $\mathfrak{S}_m$ be the symmetric group of degree m, that is, the group of all $m!$ permutations of a set E_m of m objects. If $\mathfrak{A}$ is a subset of $\mathfrak{S}_m$ then the number of cyclic permutations in $\mathfrak{A}$ will be denoted by $|\mathfrak{A}|$, and the total number of elements in $\mathfrak{A}$ by $n(\mathfrak{A})$.

A subset $\mathfrak{D}$ of $\mathfrak{S}_m$ will be called a *D-set* (in $\mathfrak{S}_m$), whenever it has the property that $|S\mathfrak{D}|$ has the same value for all $S \varepsilon \mathfrak{S}_m$. It is easily seen that, in that case, we have $|S\mathfrak{D}| = m^{-1} . n(\mathfrak{D})$. For, if C is any cyclic permutation, then there are exactly $n(\mathfrak{D})$ possibilities for S such that $S\mathfrak{D}$ contains C, and it follows that $m! \,|S\mathfrak{D}| = (m-1)!\, n(\mathfrak{D})$.

Furthermore, it may be remarked that $|S\mathfrak{D}| = |\mathfrak{D}S|$, since SBS^{-1} is cyclic whenever B is cyclic. Therefore, if $\mathfrak{D}$ is a D-set and if P is an arbitrary element of $\mathfrak{S}_m$, then $\mathfrak{D}P$ is also a D-set.

$\mathfrak{S}_m$ itself clearly is a D-set in $\mathfrak{S}_m$, but theorem 1 will show that non-trivial D-sets exist.

Let E_l be a sub-set of the set of objects E_m, containing l objects. Consider the sub-group $\mathfrak{G} \subset \mathfrak{S}_m$ of all permutations which only permute the elements of E_l, leaving the remaining elements of E_m invariant. If G is any permutation of $\mathfrak{G}$, then $\bar{G}$ denotes the corresponding permutation of the objects of E_l, that is to say, we disregard the objects belonging to $E_m - E_l$ (which are invariant under G). G is defined uniquely by $\bar{G}$, and vice versa. The same notation will be used for sets: if $\mathfrak{A} \subset \mathfrak{G}$, then $\bar{\mathfrak{A}}$ denotes the set of all $\bar{G}$, where $G \varepsilon \mathfrak{A}$.

L e m m a 1. Let $\mathfrak{B}$ be a sub-set of $\mathfrak{G}$ such that $\bar{\mathfrak{B}}$ is a D-set in $\bar{\mathfrak{G}}$, and let $C \varepsilon \mathfrak{S}_m$ be a cyclic permutation. Then we have

$$|\mathfrak{B}C| = l^{-1} . n(\mathfrak{B}).$$

P r o o f. We shall deal with the cyclic representations of the permutations involved. Let $\bar{G}$ be the element of $\bar{\mathfrak{G}}$ whose cyclic representation is obtained by cancelling the objects of $E_m - E_l$ from the cyclic representation of C. Further, let G_1 be an arbitrary permutation of $\mathfrak{G}$. Then it is easily verified that $\bar{G}_1\bar{G}$ (of degree l) shows the same number of cycles as G_1C (of degree m). Hence G_1C is cyclic whenever $\bar{G}_1\bar{G}$ is cyclic. Therefore

$$|\mathfrak{B}C| = |\bar{\mathfrak{B}}\bar{G}| = l^{-1} . n(\bar{\mathfrak{B}}) = l^{-1} . n(\mathfrak{B}).$$

Lemma 2. Let $\mathfrak{B}$ be a subset of $\mathfrak{G}$ such that $\overline{\mathfrak{B}}$ is a D-set in $\overline{\mathfrak{G}}$. Let Q be any arbitrary permutation of $\mathfrak{S}_m$. Then we have

$$\frac{|\mathfrak{B}\,Q|}{n(\mathfrak{B})} = \frac{|\mathfrak{G}\,Q|}{n(\mathfrak{G})}.$$

Proof. If there is no $G \varepsilon \mathfrak{G}$ such that GQ is cyclic, then both sides are equal to zero. Now assume that $G \varepsilon \mathfrak{G}$ is such that GQ is cyclic; put $GQ = C$.

We have $\mathfrak{G}\,Q = \mathfrak{G}\,C$, and $\mathfrak{B}\,Q = (\mathfrak{B}\,G^{-1})\,C$. The set $\overline{\mathfrak{B}}_1 = \overline{\mathfrak{B}}\,\overline{G}^{-1}$ is a D-set in $\overline{\mathfrak{G}}$, since $\overline{\mathfrak{B}}$ was a D-set in $\overline{\mathfrak{G}}$. Now, by lemma 1,

$$|\mathfrak{B}\,Q| = |\mathfrak{B}_1 C| = l^{-1} \,.\, n(\mathfrak{B}_1) = l^{-1} \,.\, n(\mathfrak{B}).$$

Analogously

$$|\mathfrak{G}\,Q| = |\mathfrak{G}\,C| = l^{-1} \,.\, n(\mathfrak{G}),$$

and lemma 2 has been proved.

Let k and n be natural numbers, and take $m = kn$. We consider a set E_m of m objects, divided into k systems, each of them containing n objects. We shall again denote by $\mathfrak{S}_m$ the group of all permutations of E_m. $\mathfrak{H}$ denotes the group consisting of all $k!\,(n!)^k$ permutations H with the property that Ha and Hb belong to the same system whenever a and b belong to the same system. Or, shortly, H transforms systems into systems.

Theorem 1. $\mathfrak{H}$ is a D-set in $\mathfrak{S}_m$.

Proof. If either $k = 1$ or $n = 1$, then we have $\mathfrak{H} = \mathfrak{S}_m$, and the theorem is trivial.

Next we shall deal with the case $k = 2$, $m = 2n$. It has to be shown that $|S\,\mathfrak{H}|$ does not depend on S ($S \varepsilon \mathfrak{S}_{2n}$). Let $\mathfrak{H}_1$ be the set of all permutations mapping the first system onto itself, and let $\mathfrak{H}_2$ be the set of those mapping the first system onto the second. Thus $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2$.

Let p be the number of objects of the first system mapped into the first system by S, then there are $q = n - p$ objects of the first system which are mapped into the second system. Then we have

$$|S\,\mathfrak{H}_1| = q\,\{(n-1)!\}^2.$$

This can be seen, for instance, by interpreting $|S\,\mathfrak{H}_1|$ as the number of circuits (see § 3) in the following graph. Take two vertices, A and B, and $2n$ oriented edges: p of them from A to A, p from B to B, q from A to B and q from B to A. The number of circuits can be shown to be $q\,\{(n-1)!\}^2$. It can be very rapidly determined by theorem 6, for there are exactly q trees with root A.

$\mathfrak{H}_2$ can be written as $S_0\, \mathfrak{H}_1$, where S_0 is an arbitrary element of $\mathfrak{H}_2$. Now $| S\, \mathfrak{H}_2 | = | S_1\, \mathfrak{H}_1 |$, where $S_1 = SS_0$. S_1 has the same nature as S, apart from the fact that p and q changed their roles.

Hence $| S\, \mathfrak{H}_2 | = p\,\{(n-1)!\}^2$, and so

$$| S\, \mathfrak{H} | = | S\, \mathfrak{H}_1 | + | S\, \mathfrak{H}_2 | = (p+q)\,\{\,(n-1)!\}^2 = (n!)^2/n.$$

This does not depend on S, and so our theorem has been proved in the case $k = 2$.

Next we consider the general case $k > 2$. We have to show that $| S_1\, \mathfrak{H} | = | S_2\, \mathfrak{H} |$ for any pair S_1, S_2 ($S_1\, \varepsilon \mathfrak{S}_m$, $S_2\, \varepsilon \mathfrak{S}_m$). Since any $S\, \varepsilon \mathfrak{S}_m$ can be written as a product of transpositions, it is sufficient to prove that $| S\, \mathfrak{H} | = | S\, T\, \mathfrak{H} |$ for all S and for any transposition T. Or, what is the same thing, that

$$| \mathfrak{H}\, Q | = | T\, \mathfrak{H}\, Q | \tag{2.1}$$

for any $Q\, \varepsilon \mathfrak{S}_m$ and any transposition $T\, \varepsilon \mathfrak{S}_m$.

We may assume that T interchanges two symbols belonging to the first and to the second system, respectively (if T interchanges two symbols of the same system, then we have $\mathfrak{H} = T\, \mathfrak{H}$, and (2.1) is trivial). Let $\mathfrak{H}^*$ be the sub-group of $\mathfrak{H}$ consisting of all permutations of $\mathfrak{H}$ which leave all individual elements of the 3^{rd}, 4^{th}, . . ., k^{th} system invariant, and let $\mathfrak{G}$ be the group arising from $\mathfrak{S}_m$ in the same manner. We now apply lemma 2, with $l = 2n$ and $\mathfrak{B} = \mathfrak{H}^*$. Since the theorem has been proved for the case $k = 2$, we know that $\overline{\mathfrak{H}}^*$ is a D-set in $\overline{\mathfrak{G}}$.

Therefore

$$\frac{| \mathfrak{H}^* Q |}{n\,(\mathfrak{H}^*)} = \frac{| \mathfrak{G}\, Q |}{n\,(\mathfrak{G})}, \qquad \frac{| T\, \mathfrak{H}^* Q |}{n\,(\mathfrak{H}^*)} = \frac{| T\, \mathfrak{G}\, Q |}{n\,(\mathfrak{G})}.$$

Evidently $T\, \mathfrak{G} = \mathfrak{G}$, and so $| \mathfrak{H}^* Q | = | T\, \mathfrak{H}^* Q |$ for all $Q\, \varepsilon \mathfrak{S}_m$.

Since $\mathfrak{H}^*$ is a sub-group of $\mathfrak{H}$, we can split $\mathfrak{H}$ into classes, $\mathfrak{H} = \Sigma\, \mathfrak{H}^* Q_i$, and now (2.1) follows immediately. The order of the group $\mathfrak{H}$ is $k!(n!)^k$, and therefore

$$| \mathfrak{H} | = m^{-1}\, n(\mathfrak{H}) = m^{-1}\, k!\, (n!)^k = n^{-1}\, (k-1)!\, (n!)^k. \tag{2.2}$$

Let $\mathfrak{K}$ be the set of all permutations K with the property that the n objects of each system are transformed into objects of n different systems. In other words, K is such that, if a and b belong to the same system, then Ka and Kb belong to different systems. Clearly $\mathfrak{K}$ is empty if $k < n$.

It is not difficult to show that $\mathfrak{K}$ *is a D-set*. For, if H is an arbitrary permutation of $\mathfrak{H}$, we have $\mathfrak{K} = \mathfrak{K}\, H$. It follows that $\mathfrak{K}$ is the sum of a number of left-classes mod $\mathfrak{H}$: $\mathfrak{K} = \Sigma K_i \mathfrak{H}$. Each component $K_i \mathfrak{H}$ is a D-set, by theorem 1. Hence $\mathfrak{K}$ is a D-set.

It is easily seen that in the special case $k = n$ the number of elements in $\mathfrak{K}$ is $(n!)^{2n}$, and so we have

$$|\mathfrak{K}| = (n!)^{2n} n^{-2} \qquad (k = n). \tag{2.3}$$

§ 3. T-Graphs.

In §§ 3, 4, 5, 6 we shall be mainly concerned with a special type of finite oriented linear graphs, called *T-graphs* [1]). These have the property that, at each vertex P, the number σ_i of oriented edges pointing to P_i equals the number of edges pointing away from P_i. For simplicity we assume $\sigma_i > 0$ for all i. If this number happens to be the same for all vertices ($\sigma_i = \sigma$ for all i) then we shall call the graph a $T^{(\sigma)}$ [2]).

We do not exclude the possibility that, in a T-graph, several different edges point from P_i to P_j, and we neither exclude edges pointing from P_i to P_i itself (closed loops).

Therefore, a T-graph can be interpreted as a pair of mappings of a finite set of edges $\{e_1, \ldots, e_m\}$ onto a finite set of vertices $\{P_1, \ldots, P_N\}$ such that each vertex is the image of the same number of edges in both mappings. The first mapping maps every edge onto the point where it starts from, and the second one onto the point where it terminates. We shall call these vertices the *tail* and the *head* of the edge, respectively. If the head of e_i coincides with the tail of e_j, then e_i and e_j will be called *consecutive* (which does not imply that e_j and e_i are consecutive).

By a complete circuit (a *circuit* for short) is meant any cyclic arrangement of the set of edges in such a manner that the head of each edge coincides with the tail of the next one in the circuit. Or, in other words, such that consecutive edges in the circuit are consecutive in the graph.

Naturally, two circuits are considered as identical whenever the first one is a cyclic permutation of the second. It has to be understood that the order of the edges counts, and not only the order of the heads. So, for instance, if $m = 3$, $N = 1$, then P_1 is the head as well as the tail of all edges. There are two different circuits, viz. (e_1, e_2, e_3) and (e_1, e_3, e_2).

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1) Tutte [5] calls them *simple oriented networks*.

2) In the paper [1] the name "T-net" denoted the same thing as $T^{(2)}$ does in our present notation.

The number of circuits of a graph T will be denoted by $|T|$ [1]).

A permutation P of the set of edges $e_1, \ldots, e_m$ will be called *conservative* (with respect to T), whenever $Pe_i = e_j$ always implies that the head of e_i coincides with the tail of e_j. We choose one special conservative permutation A_0, arbitrary, but fixed in the sequel. The set of all conservative permutations of T can be represented as $\mathfrak{G} A_0$, where $\mathfrak{G}$ is the group of all permutations which leave the tails of all edges invariant.

Evidently, any circuit determines a cyclic conservative permutation, and vice versa.

Therefore,

$$|T| = |\mathfrak{G} A_0|.$$

This simple relation between the number of circuits in a graph and the number of cyclic permutations in a set explains why we choose the same notation $|\quad|$ for both.

Consider a vertex P_i where σ_i edges start and σ_i edges terminate. By the *local symmetric group* $\mathfrak{G}_i$ we shall denote the group of all $\sigma_i!$ permutations which permute the σ_i edges whose tail is P_i, but which leave invariant all edges whose tail is not P_i.

Clearly $\mathfrak{G}$ is the direct product of $\mathfrak{G}_1, \ldots, \mathfrak{G}_N$.

§ 4. Traffic regulations.

We shall also consider circuits described under certain restrictive conditions, called traffic regulations.

Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ be sub-sets of $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$, respectively, and construct the set

(4.1) $$\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_N,$$

defined in the same way as the direct product

(4.2) $$\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2 \times \ldots \times \mathfrak{G}_N.$$

Now a circuit described under the traffic regulation $\mathfrak{B}$ is defined as a circuit corresponding to a permutation BA_0, where $B \in \mathfrak{B}$, and A_0 is the fixed permutation chosen in § 3.

Denoting the number of circuits described under the traffic regulation $\mathfrak{B}$ by $|T|_{\mathfrak{B}}$, we have

(4.3) $$|T| = |T|_{\mathfrak{G}}, \quad |T|_{\mathfrak{B}} = |\mathfrak{B} A_0|.$$

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[1]) We have $|T| > 0$ if and only if T is connected (see [2]). For non-connected graphs our theorems are trivial. Nevertheless, all our proofs are valid for that case also.

The traffic regulation (4.1) will be called *regular* if, for each i, $\overline{\mathfrak{B}}_i$ is a D-set in $\overline{\mathfrak{G}}_i$ [1]).

Theorem 2. [2]) If $\mathfrak{B}$ is regular, then we have

$$(4.4) \qquad \frac{1}{n(\mathfrak{B})} \mid T \mid_{\mathfrak{B}} = \frac{1}{n(\mathfrak{G})} \mid T \mid_{\mathfrak{G}},$$

where $n(\mathfrak{B})$ and $n(\mathfrak{G})$ denote the number of elements of $\mathfrak{B}$ and $\mathfrak{G}$, respectively.

Proof. Since $\mathfrak{G}_i$ itself satisfies the condition imposed on $\mathfrak{B}_i$, it is sufficient to show that the value of the left-hand-side of (4.4) does not change if some $\mathfrak{B}_i$ is replaced by the corresponding $\mathfrak{G}_i$. If this has been proved, we can replace all $\mathfrak{B}_i$'s by $\mathfrak{G}_i$'s one after the other, and (4.4) follows.

To this end we consider $\mathfrak{B}$, defined by (4.1) and $\mathfrak{B}^*$, defined by

$$(4.5) \qquad \mathfrak{B}^* = \mathfrak{G}_1 \times \mathfrak{B}_2 \times \mathfrak{B}_3 \times \ldots \times \mathfrak{B}_N,$$

and we have to show that

$$(4.6) \qquad \mid T \mid_{\mathfrak{B}} : \mid T \mid_{\mathfrak{B}^*} = n(\mathfrak{B}_1) : n(\mathfrak{G}_1),$$

for the latter ratio equals $n(\mathfrak{B}) : n(\mathfrak{B}^*)$.

Referring to (4.3) we write

$$(4.7) \quad \mid T \mid_{\mathfrak{B}} = \Sigma \mid \mathfrak{B}_1 B_2 \ldots B_N A_0 \mid,$$
$$\mid T \mid_{\mathfrak{B}^*} = \Sigma \mid \mathfrak{G}_1 B_2 \ldots B_N A_0 \mid,$$

where, in both sums, $B_2, \ldots, B_N$ run independently through the elements of $\mathfrak{B}_2, \ldots, \mathfrak{B}_N$, respectively. If we put $B_2 \ldots B_N A_0 = Q$, then we have, by lemma 2,

$$\mid \mathfrak{B}_1 Q \mid : \mid \mathfrak{G}_1 Q \mid = n(\mathfrak{B}_1) : n(\mathfrak{G}_1).$$

Applying this to each pair of corresponding terms of the sums in (4.7), we obtain (4.6).

§ 5. **Special traffic regulations.**

Let T be a T-graph with N vertices and m edges. Again, the numbers of edges pointing towards $P_1, \ldots, P_N$ are denoted by $\sigma_1, \ldots, \sigma_N$, respectively, and so $m = \sigma_1 + \ldots + \sigma_N$.

Let λ be a positive integer. Then by T^λ [3]) we denote the graph which arises from T if we replace any edge P_iP_j of T by λ edges P_iP_j, with the same orientation. Hence T^λ has N vertices and λm edges. The edges of T^λ arising from one and the same edge of T are said to form a *bundle*.

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[1]) As in § 2, the bar indicates that the permutations are considered as permutations of the σ_i edges whose tail is P_i, whereas the other edges are disregarded.

[2]) This theorem was used implicitly in [1].

[3]) The notations T^σ and $T^{(\sigma)}$ (for the latter see § 3) must not be confused.

In T^λ we shall consider several possible traffic regulations. We first choose a fixed conservative permutation A_0 which transforms bundles into bundles.

A traffic regulation will be obtained by choosing, at each vertex P_i, a set $\mathfrak{B}_i$ of permutations of the $\lambda\sigma_i$ edges starting from that vertex. We shall consider three possibilities, all regular in the sense of § 4.

1°. $\mathfrak{B}_i = \mathfrak{G}_i$, where $\mathfrak{G}_i$ is the local symmetric group (of order $(\lambda\sigma_i)!$).

2°. $\mathfrak{B}_i = \mathfrak{H}_i$. Here $\mathfrak{H}_i$ is the sub-set of $\mathfrak{G}_i$ which transforms bundles into bundles. In other words, as to the edges whose tail is P_i it acts like the group $\mathfrak{H}$ of theorem 1, where the systems are given by the bundles. Thus $n = \lambda$, $k = \sigma_i$.

3°. $\mathfrak{B}_i = \mathfrak{K}_i$. Here $\mathfrak{K}_i$ is the sub-set of $\mathfrak{H}_i$ consisting of the permutations which transform the edges of each outgoing bundle at P_i into sets of edges belonging to λ different bundles (see the end of § 2).

We have, by theorem 2,

$$(5.1) \qquad \frac{|T^\lambda|_{\mathfrak{G}}}{\prod\limits_{i=1}^{N} (\lambda\sigma_i)!} = \frac{|T^\lambda|_{\mathfrak{H}}}{\prod\limits_{i=1}^{N} \sigma_i!\,(\lambda!)^{\sigma_i}} = \frac{|T^\lambda|_{\mathfrak{K}}}{\prod\limits_{i=1}^{N} \varphi(\lambda, \sigma_i)},$$

where $\varphi(\lambda, \sigma_i)$ is the number of elements of $\mathfrak{K}_i$.

As stated in § 4, we have $|T^\lambda|_{\mathfrak{G}} = |T^\lambda|$.

The number $|T^\lambda|_{\mathfrak{H}}$ can be connected with the number of circuits in T itself. To this end we consider a circuit of T^λ described according to the traffic regulation $\mathfrak{H}$. At any stage, the bundle to which an edge belongs only depends on the bundle containing the preceding edge. Therefore, the sequence of bundles described by the circuit is periodic mod m, and any bundle is used exactly λ times. It follows that each circuit under consideration defines a circuit of T. Conversely, it is easily seen that each circuit of T arises from $\lambda^{-1}(\lambda!)^m$ different circuits of T^λ in this manner. Hence

$$(5.2) \qquad |T^\lambda|_{\mathfrak{H}} = \lambda^{-1}(\lambda!)^m \,.\, |T|,$$

and so we obtain from (5.1)

Theorem 3. $\qquad |T^\lambda| = \lambda^{-1} \,.\, |T| \,.\, \prod\limits_{i=1}^{N} \dfrac{(\lambda\sigma_i)!}{\sigma_i!}.$

We shall now make the restriction that T is a $T^{(\sigma)}$, and that $\lambda = \sigma$, that is to say

$$\sigma_1 = \ldots = \sigma_N = \sigma = \lambda, \qquad m = N\sigma.$$

Then we have (see (2.3))

$$\varphi(\lambda, \sigma_i) = (\sigma!)^{2\sigma}$$

and now (5.1) and (5.2) lead to

(5.3) $$|T^\sigma|_{\mathfrak{R}} = |T| \cdot \sigma^{-1} (\sigma!)^m \cdot \left(\frac{(\sigma!)^{2\sigma}}{\sigma!\,(\sigma!)^\sigma}\right)^N = |T| \cdot \sigma^{-1} (\sigma!)^{N(2\sigma-1)}.$$

If T is a $T^{(\sigma)}$, with N vertices and $m = \sigma N$ edges, then by T^* we denote the graph defined as follows. T^* has m vertices $E_1, \ldots, E_m$. Two vertices E_i, E_j are connected in T^* by an edge from E_i to E_j if and only if e_i, e_j are consecutive in T. This process was considered in [1] for the case $\sigma = 2$ only.

Theorem 4. $|T^*| = \sigma^{-1} (\sigma!)^{N(\sigma-1)} \cdot |T|$.

Proof. By "*σ-cycle in T*" is meant a circular array containing each edge of T exactly σ times, such that two edges are consecutive in the array if and only if they are consecutive in T. A σ-cycle will be called *restricted* if it is such that any pair of consecutive edges of T occurs just once as a pair of consecutive elements in the array. It will be clear that any restricted σ-cycle in T defines uniquely a circuit in T^*, and vice versa.

The restricted σ-cycles in T are closely related to the circuits in T^σ described under the traffic regulation $\mathfrak{R}$. Actually, if we identify the edges of each bundle in T^σ a $\mathfrak{R}$-circuit in T^σ becomes a restricted σ-cycle, owing to the definition of $\mathfrak{R}$. Conversely, any restricted σ-cycle gives rise to a large number of $\mathfrak{R}$-circuits. Any bundle occurs σ times in the cycle, and each time an arbitrary edge of the bundle can be chosen.

So we see that $(\sigma!)^m$ different $\mathfrak{R}$-circuits arise from one restricted σ-cycle. Now the theorem follows from (5.3), since $m = N\sigma$.

In a T-graph which is not necessarily a $T^{(\sigma)}$ we can still consider (unrestricted) σ-cycles [1]). The number of different σ-cycles can be determined from theorem 3. A difficulty lies in the fact that a σ-cycle may be periodical with a period md, where d is a proper divisor of σ, which could not happen with a restricted σ-cycle.

If $c(\varrho)$ denotes the number of those ϱ-cycles in T whose period is exactly $m\varrho$, then we have obviously

$$|T^\rho| = \sum_{d|\rho} \frac{d}{\varrho}\, c(d)\, (\varrho!)^m.$$

[1]) And, if T is a $T^{(\sigma)}$, we can consider (unrestricted) ϱ-cycles, for arbitrary values of ϱ.

Hence we obtain from Möbius' inversion formula,

$$c(\varrho) = \sum_{d/\rho} \frac{d}{\varrho} \mu\left(\frac{\varrho}{d}\right) . (d!)^{-m} . \mid T^d \mid ,$$

and so the number of unrestricted ϱ-cycles equals

$$\sum_{d/\rho} c(d) = \frac{1}{\varrho} \sum_{d/\varrho} \varphi\left(\frac{\varrho}{d}\right) (d!)^{-m} d . \mid T^d \mid ,$$

where φ is Euler's indicator. $\mid T^d \mid$ can be evaluated by theorem 3.

Especially, is T is a $T^{(\sigma)}$, then the number of unrestricted ϱ-cycles is

$$\frac{1}{\varrho} \sum_{d/\rho} \varphi\left(\frac{\varrho}{d}\right)\left(\frac{(\sigma d)!}{(d!)^\sigma \sigma!}\right)^N . \mid T \mid .$$

§ 6. Trees in T-graphs.

Let T be a T-graph with N vertices and m edges. The number of edges whose tail is P_i is again denoted by σ_i. Choose an arbitrary vertex; for convenience of notations we take it to be P_1. We shall define the notion: (oriented) tree with root P.

A tree with root P_1 is a sub-set Λ of the set of edges of T, with the following properties.

a. Any vertex $\neq P_1$ is the tail of just one element of Λ.

b. No element of Λ has its tail in P_1.

c. Any vertex can be connected with P_1 by a set of consecutive edges, all belonging to Λ.

It is easily seen that c can be replaced by

c*. Λ contains no closed oriented cycles.

There is a striking relation between trees and circuits. Choose a fixed edge e_1 whose tail is P_1, and consider an arbitrary circuit of T. We traverse it, starting with e_1. Running through the circuit, each vertex P_i will be visited σ_i times. The edge by which we leave P_i after having visited it for the σ_i-th time will be called the *last exit* of P_i.

Theorem 5a. *The set Λ consisting of the last exits of $P_2, \ldots, P_N$ is a tree with root P_1.*

Proof. The properties a and b are trivial. We shall verify c*.

We can number the edges of T according to the order in the circuit, with indices $1, \ldots, m$; e_1 gets the index 1.

If e_i and e_j both belong to Λ, and if e_i and e_j are consecutive in T, then we have $i < j$. For, e_{i+1} has the same tail as e_j, and j is the maximal value of the indices of all the edges with this tail.

Consequently, A does not contain any closed cycle; the indices in such a cycle would increase indefinitely.

Theorem 5b. If a tree A with root P_1 is given, and if e_1 is given, then there are exactly

$$(6.1) \qquad \prod_{i=1}^{N} (\sigma_i - 1)!$$

circuits of T whose set of last exits coincides with A.

Proof. At any vertex we number the outgoing edges [1], with the following restrictions: At P_1 the edge e_1 gets the number 1; at P_i $(i > 1)$ the edge belonging to A gets the highest possible number, that is σ_i. The number of ways in which this can be arranged is expressed by (6.1). It remains to be shown that, for each numbering of this type, there exists a circuit (and not more than one circuit) corresponding with this numbering.

First thing it will be clear that we have no choice at all if we try to traverse a circuit according to this numbering. Starting with e_1, we arrive at a vertex P_2, say. It is prescribed which outgoing edge we have to take first, etc. If we meet a vertex for a second time, we are forced to leave it by the edge bearing the number 2, and so on. The process has to stop somewhere, the graph being finite. The only reason why it should stop is, that we arrive at a vertex where all outgoing edges have already been taken before. This must be P_1, for all other vertices have been entered at least as often as they have been left.

We can show that at this moment all edges of T have been used, each exactly once of course, which means that a circuit has been described. Assume that a certain edge is *vacant*, that means that it has not yet been used. Considering its head, there is a vacant entry and hence there is a vacant exit. Especially, the exit belonging to A has to be vacant, since it has the highest number. This vacant edge of A leads into another vacant edge of A, and so on. By *c*, we eventually arrive at P_1, and we find that there is a vacant outgoing edge. This contradicts the fact that the process stopped.

From Theorem 5a and 5b we immediately obtain.

Theorem 6. *The number of trees in T with a given root is*

$$|T| \cdot \{ \prod_{i=1}^{N} (\sigma_i - 1)! \}^{-1},$$

[1] This way of numbering is different from the one considered in the proof of theorem *5a*.

which does not depend on the vertex chosen as the root. As before, $|T|$ *denotes the number of circuits of* T.

Theorem 6 furnishes a new proof of theorem 3. For, there is a simple relation between the number of trees in T and in T^λ. Any tree in T^λ gives rise to a tree in T, by the mapping $T^\lambda \to T$ which maps entire bundles of T^λ into the corresponding edges of T. Conversely, in any bundle of T^λ an edge can be chosen in λ ways. Any tree in T contains $N—1$ edges, and so we have

(6.2) $$t(T^\lambda) = t(T) \,.\, \lambda^{N-1},$$

where $t(T)$ and $t(T^\lambda)$ denote the number of trees with a given root, in T and T^λ, respectively. By theorem 6 we have

(6.3) $$|T^\lambda| = t(T^\lambda) \,.\, \prod_{i=1}^{N} (\lambda\sigma_i - 1)\,!,$$

(6.4) $$|T| = t(T) \,.\, \prod_{i=1}^{N} (\sigma_i - 1)\,!\,.$$

Theorem 3 follows from (6.2), (6.3) and (6.4).

§ 7. **Trees in arbitrary oriented graphs.**

We consider an oriented graph G, with N vertices $P_1, \ldots, P_N$. We no longer require that it is a T-graph, that is to say, the number of edges starting from P_i need not be the same as the number of edges pointing towards P_i.

Again, we can consider (oriented) trees, with a given root. Tutte [5] showed, that the number of trees in T with a given root can be interpreted as the value of a certain determinant. Since his result is in several ways connected with the results of the present paper, we give a full account of his theorem, with a new proof.

Let (a_{ij}) $(i, j = 1, \ldots, N)$ be the following matrix. If $i \neq j$, then $a_{ij} = - b_{ij}$, where b_{ij} denotes the number of oriented edges from P_i to P_j (P_i is the tail and P_j is the head of these edges) Further a_{ii} is such that $\sum_{j=1}^{N} a_{ij} = 0$.

T h e o r e m 7 (Tutte). The number of trees with the given root P_i equals the minor of a_{ii} in the matrix (a_{ij}).

P r o o f. For simplicity of notation we take $i = 1$.

We first consider a special graph, where each vertex $\neq P_1$ is the tail of just one edge, and where no edge leaves P_1. This graph is either a tree or it is not; the possibility of constructing more than one tree in this graph does not exist. We shall show that the minor of a_{11} is 1 or 0 according to whether the graph is or is not a tree.

First assume that the graph is a tree. We shall apply induction with respect to N; for $N = 2$ the result is trivial. Take $N > 2$. There is at least one vertex which is not the head of an edge. This is the case with P_2, say. Then the second column of the matrix reads 0, 1, 0, . . ., 0. Hence the value of the minor of a_{11} is not altered if the second row and the second column are both cancelled. The new matrix corresponds to the graph which results by cancelling P_2 and the edge starting from P_2. This new graph is still a tree, and the induction is completed.

Next assume that the graph is not a tree. Then it shows somewhere a cycle of edges not containing P_1. For example, let the cycle consist of the edges P_2P_3, P_3P_4, P_4P_3. Then, in the matrix, the 2^{nd}, 3^{rd}, and 4^{th} row are linearly dependent, for their sum vanishes. It follows that the minor of a_{11} equals zero. This completes the proof of the theorem for our special graph.

The general case is easily reduced to this one by repeated application of the following operation. Divide the set of edges starting from a certain edge, P_2, say, into two groups. Now construct two graphs; the first one arises from the original graph by cancelling the edges of the first group, the second one by cancelling the edges of the second group. The matrices of the graphs are such that the second row of the original matrix equals the sum of the corresponding rows in the new matrices; all other rows are identical in the three matrices. Therefore, the minor of a_{11} in the original matrix is the sum of the minors of a_{11} in the new matrices. On the other hand, the number of trees in the original graph is the sum of the numbers of trees in both graphs. This proves the theorem.

Theorem 6 shows that in a T-graph the number of trees does not depend on the choice of the root. Tutte deduced the same fact from theorem 7. We repeat his argument. Assume that the graph considered in theorem 7 is a T-graph. Then we have $a_{ii} = \sigma_i - \varrho_i$ where ϱ_i is the number of edges from P_i to P_i. Therefore, we also find that the sum of the elements in each column of the matrix is equal to zero. It is a well-known fact that if in a square matrix the sum of the elements in each row and in each column vanishes, then the cofactors of all elements have the same value. Especially, the minor of a_{ii} does not depend on i.

We again consider an arbitrary graph G, which need not be a T-graph. Let $P_1, \ldots, P_n$ be its vertices, and let σ_i be the number of edges starting from P_i, and τ_i the number of edges pointing towards P_i. Furthermore, b_{ij} denotes the number of oriented edges from P_i to P_j. Hence $\sigma_i = \sum_j b_{ij}$, $\tau_j = \sum_i b_{ij}$.

Next we consider a permutation S of the N objects 1, 2, ..., N. Let G_S be the graph arising from G in the following manner: replace each edge $P_i P_j$ of G_S by an edge $P_i P_{Sj}$, where Sj is the result of S applied to the object j. Therefore, if the analogues of σ_i, τ_j, b_{ij} for the graph G_S are denoted by $\sigma_i^{(S)}$, $\tau_i^{(S)}$, $b_{ij}^{(S)}$, respectively, then we have

$$\sigma_i^{(S)} = \sigma_i \ , \ \tau_{Sj}^{(S)} = \tau_j \ , \ b_{i,Sj}^{(S)} = b_{ij}. \tag{7.1}$$

Let $t_i\,(G_S)$ denote the number of oriented trees in G_S whose root is P_i, and let $\mathfrak{S}_N$ denote the group of all $N!$ permutations of the objects $1, \ldots, N$. Then we have

Theorem 8. $\sum_{S \in \mathfrak{S}_N} t_i\,(G_S) = (N-1)\,! \prod_{k \neq i} \sigma_k.$

Proof. We may and do assume $i = 1$. We shall apply theorem 7. To this end, we consider the matrix

$$M_S = (\lambda_i \delta_{ij} - b_{ij}^{(S)}) \qquad (i, j = 1, \ldots, N)$$

where δ_{ij} is Kronecker's symbol. Its determinant $\det M_S$ is a multilinear polynomial in the variables $\lambda_1, \ldots, \lambda_N$:

$$\det M_S = f_S\,(\lambda_1, \ldots, \lambda_N),$$

and it will be clear from theorem 7 that

$$t_1\,(G_S) = \frac{\partial}{\partial \lambda_1} f_S\,(\lambda_1, \sigma_2, \sigma_3, \ldots, \sigma_N). \tag{7.2}$$

We put

$$P\,(\lambda_1, \ldots, \lambda_N) = \sum_{S \in \mathfrak{S}_N} f_S\,(\lambda_1, \ldots, \lambda_N). \tag{7.3}$$

In the first place we can show that $P\,(\lambda_1, \ldots, \lambda_N)$ does not contain terms of degree $< N-1$. For instance, consider the term with $\lambda_3 \lambda_4 \ldots \lambda_N$, which does not contain either λ_1 or λ_2. Let T be the transposition of the objects 1 and 2. Then the coefficient of $\lambda_3 \ldots \lambda_N$ in $\det M_{TS}$ is easily seen to be the opposite of the coefficient of $\lambda_3 \ldots \lambda_N$ in $\det M_S$. If S runs through $\mathfrak{S}_N$, then TS does the same, and so the coefficient of $\lambda_3 \ldots \lambda_N$ in $P\,(\lambda_1, \ldots, \lambda_N)$ turns out to be zero.

We next deal with the terms of degree $N-1$, and therefore we consider $\lambda_1 \lambda_2 \dots \lambda_{i-1} \lambda_{i+1} \dots \lambda_N$. Its coefficient in f_S equals $-b_{ii}{}^{(S)}$. Consequently, its coefficient in $P(\lambda_1, \dots, \lambda_N)$ is

$$-\sum_{S\in\mathfrak{S}_N} b_{ii}{}^{(S)} = -\sum_{S\in\mathfrak{S}_N} b_{i,S^{-1}i} = -(N-1)!\sum_j b_{ij} = -(N-1)!\,\sigma_i.$$

Finally, the coefficient of $\lambda_1 \dots \lambda_N$ in f_S equals 1, and in $P(\lambda_1, \dots, \lambda_N)$ it is $N!$ Thus we have proved that

$$P(\lambda_1, \dots, \lambda_N) = \lambda_1 \dots \lambda_N \,.\, (N-1)!\,\{N - \sum_i \sigma_i/\lambda_i\}.$$

From (7.2) and (7.3) we now deduce

$$\sum_{S\in\mathfrak{S}_N} t_1(G_S) = \frac{\partial}{\partial\lambda} P(\lambda_1, \sigma_2, \dots, \sigma_N) = \sigma_2 \dots \sigma_N \,.\, (N-1)!$$

Theorem 8 is, in some sense, a generalization of theorem 1. For, if we apply theorem 8 to a graph which is a $T^{(\sigma)}$, then we have, by theorem 6,

$$(7.4) \qquad \sum_{S\in\mathfrak{S}_N} |G_S| = (N-1)!\,\sigma^{N-1}\{(\sigma-1)!\}^N = (N-1)!\,\frac{1}{\sigma}\,.\,(\sigma!)^N.$$

It is not difficult to see that (7.4) is equivalent with theorem 1 (take $n = \sigma$, $k = N$).

Note added in proof. By theorems 6 and 7 the number of circuits in a T-graph can be expressed as a determinant. For the special case that T is a $T^{(2)}$, this result was announced by W. T. Tutte and C. A. B. Smith (On unicursal paths in a network of degree 4, Amer. Math. Monthy **48**, 233—237 (1941)).

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