

A unified approach to the restoration of lost samples in discrete-time signals.

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A UNIFIED APPROACH TO THE RESTORATION OF LOST
SAMPLES IN DISCRETE-TIME SIGNALS

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Abstract

We consider the problem of estimating lost sample values in discrete-time signals. The problem is treated as a linear minimum variance estimation problem, which, in principle, requires knowledge of the signal's autocorrelation coefficients. We show that, starting from this general principle, several sample restoration methods studied separately in the literature, can be derived. As particular examples we consider sample restoration methods for band-limited signals, multiple sinusoids, autoregressive processes and (quasi periodic) speech signals. The restoration method for multiple sinusoids is new, and has not, to our knowledge, been published before.

1 Introduction

In this paper we discuss some methods for the restoration of a number of lost or unknown samples occurring in a discrete-time signal. It is assumed that the positions of the unknown samples are known. Also, it is assumed that the unknown samples are embedded in a sufficiently large neighbourhood of known ones. There are no restrictions on the patterns of the unknown samples; they may occur in bursts as well as in more random patterns.

We start by deriving linear minimum variance estimates for the lost samples. The lost samples are estimated as linear combinations of known

samples. Therefore, for the computation of every lost sample a set of weighting coefficients is required. The optimal coefficients, giving the estimates with the minimum error variance, are obtained as the solutions of sets of equations similar to the well-known Wiener-Hopf equations. They can be arranged conveniently in a matrix form, involving the signal's autocorrelation matrix. In principle, we have two possibilities. The first is that the signal's autocorrelation matrix is regular. Sample restoration methods for autoregressive processes [1,2] and for speech signals [3] are examples of restoration methods that can be applied in this case. The second possibility is that the signal's autocorrelation matrix is singular. Sample restoration methods for band-limited signals [4] and for multiple sinusoids are examples of methods that can be applied in that case. For both possibilities we give an analysis of the restoration error. The restoration method for multiple sinusoids is new, and has not, to our knowledge, been published before.

In the case of an autoregressive process where both the parameters and some samples are unknown we present an iterative procedure to estimate both parameters and unknown samples. This method can be applied successfully for the restoration of unknown samples in audio signals. We also present a similar procedure for the case of multiple sinusoids with unknown frequencies.

We conclude the paper by presenting some simulation results.

2 A linear minimum variance estimate

Let $\underline{s} = [s_1, \dots, s_N]^T$ be a finite vector of samples taken from a stationary discrete-time stochastic signal that has zero mean. The superscript T denotes transposition. Assume that $s_{t(1)}, \dots, s_{t(m)}$ are unknown. Here $1 \leq t(1) \leq \dots \leq t(m) \leq N$ are known. The unknown samples are estimated by

$$(1) \quad s_{t(i)} = \sum_{\substack{j=1 \\ j \neq t(1), \dots, t(m)}}^N H_{ij} s_j, \quad i=1, \dots, m.$$

The coefficients H_{ij} from (1) are chosen such that they minimize the total expected quadratic restoration error

$$(2) \quad W(t(1), \dots, t(m)) = E \left[\sum_{i=1}^m (s_{t(i)} - \hat{s}_{t(i)})^2 \right].$$

The H_{ij} that minimize $W(t(1), \dots, t(m))$ are solutions of

$$(3) \quad \sum_{\substack{j=1 \\ j \neq t(1), \dots, t(m)}}^N H_{ij} r(k-j) - r(t(i)-k) = 0,$$

where $i=1, \dots, m$, $k=1, \dots, N$, $k \neq t(1), \dots, t(m)$ and $r(j) = E[s_k s_{k+j}]$ is the j^{th} autocorrelation coefficient of \underline{s} .

Expression (3) can be brought into a more convenient form as follows. Set

$H = (H_{ij})_{i=1, \dots, m, j=1, \dots, N}$ where $(H_{it(j)})_{i,j=1, \dots, m} = -I$. Furthermore, let $R = (R_{ij})_{i,j=1, \dots, N} = (r(i-j))_{i,j=1, \dots, N}$ and let R' be the $(N-m) \times N$ -submatrix of R obtained by deleting the rows of R with indices $t(1), \dots, t(m)$. Then for (3) we can write

$$(4) \quad (a) R'H^T = [0], \quad (b) (H_{it(j)})_{i,j=1, \dots, m} = -I,$$

$[0]$ being the $(N-m) \times N$ all-zero matrix.

To find a matrix H , satisfying (4), we have to distinguish between the case where the autocorrelation matrix R is regular and the case where R is (almost) singular. We first consider the case where R is regular.

The $m \times N$ matrix G that satisfies

$$(5) \quad RG^T = [e_{t(1)}, \dots, e_{t(m)}],$$

where $e_{t(i)}$ is the $t(i)^{\text{th}}$ unit vector of length N , can be used to find the solution H of (4). Indeed, let

$$(6) \quad G' = (G_{it(j)})_{i,j=1, \dots, m}$$

Then it can be shown that H , defined by

$$(7) \quad H = -G'^{-1}G$$

solves (4) provided that G' is non-singular. Now we have that

$$(8) \quad G' = ((R^{-1})_{t(i)t(j)})_{i,j=1, \dots, m}$$

and G' is positive definite and non-singular since R is.

If R is singular, a matrix G satisfying (5) cannot always be found, but solutions for (4) may still exist. We assume that R has rank $N-m$ or less. This assumption is not necessary, but it simplifies the following results. In that case, the null-space of R has at least dimension m and a matrix G can be found satisfying

$$(9) \quad RG^T = [0].$$

To exclude $G = [0]$ as a solution, we require in addition that the norms of the rows of G must all be unequal to 0. If G'^{-1} exists, H as defined in (7) is a solution of (4). In general G'^{-1} does not always exist, but it does in the special cases discussed in this paper.

The N vector \underline{v} is obtained from \underline{s} by substituting zeros for the unknown samples. If

$$\underline{\hat{x}} = [s_{t(1)}, \dots, s_{t(m)}]^T, \quad \text{then}$$

$$(10) \quad \underline{\hat{x}} = H\underline{v} = -G'^{-1}G\underline{v},$$

where G is the solution of either (5) or (9). In the adaptive cases, that are discussed later, or if the error pattern is not known in advance, G is not known in advance. Then it is more convenient to solve $\underline{\hat{x}}$ from

$$(11) \quad G'\underline{\hat{x}} = -G\underline{v} =: \underline{z}$$

than to calculate G'^{-1} . The vector \underline{z} is referred to as the syndrome.

We can derive an expression for the error covariance matrix $D := E[(\hat{x} - x)(\hat{x} - x)^T]$. It follows from (10) and (4) and the definition of \underline{y} that

$$(12) \quad \underline{\hat{x}} = H\underline{s} + \underline{x},$$

with $\underline{x} = [s_{t(1)}, \dots, s_{t(m)}]^T$. Therefore,

$$(13) \quad D = E[H\underline{s}\underline{s}^T H^T] = HRH^T.$$

If R is regular, it follows on substituting (7) into (13) and using (5) that

$$(14) \quad D = G'^{-T} = G'^{-1},$$

If R has a rank $N-m$ or less, it follows on substituting (7) into (13) and using (9) that

$$(15) \quad D = [0].$$

This shows that in principle in this case an errorless restoration is possible.

So far the number of samples used to estimate the unknown samples was finite. If R, H and G are allowed to be infinite matrices the results previously obtained also apply for the estimation of unknown samples in an infinite sequence. In that case there are some additional results. For G we have

$$(16) \quad G_{ij} = g_{j-t(i)}, \quad i=1, \dots, m, j=-\infty, \dots, \infty,$$

with the rows of G being shifted versions of an infinite vector \underline{g} , which, in the case of a regular R , is defined by

$$(17) \quad g_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\theta)} \exp(j\theta k) d\theta, \quad k=-\infty, \dots, \infty,$$

where $S(\theta) = \sum_{k=-\infty}^{\infty} r(k) \exp(-j\theta k)$ is the spectrum of the signal $(s_k)_{k=-\infty, \dots, \infty}$. In the case of a singular R , \underline{g} is defined, but not uniquely, by

$$(18) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\theta) S(\theta) d\theta = 0,$$

where $G(\theta) = \sum_{k=-\infty}^{\infty} g_k \exp(-j\theta k)$ is the Fourier transform of $(g_k)_{k=-\infty, \dots, \infty}$.

For G' we have

$$(19) \quad G'_{ij} = g_{t(i)-t(j)}, \quad i, j=1, \dots, m,$$

and for the \underline{z} of (11)

$$(20) \quad z_j = \sum_{k=-\infty}^{\infty} g_{t(i)-k} v_k, \quad i, j=1, \dots, m.$$

In general \underline{g} as defined in (17), has infinite length. Therefore, in practical applications a finite approximation must be used to calculate \underline{z} in (20). However, if $(s_k)_{k=-\infty, \dots, \infty}$ is an autoregressive process of finite order p or the sum of a finite number of sinusoids, then \underline{g} has finite length.

In the following, we discuss four particular examples of the general minimum variance sample restoration method. The restoration of unknown samples in autoregressive processes and in speech signals, in which cases R is a regular matrix, is discussed in Sections 3 and 4 respectively. The restoration of unknown samples in multiple sinusoids and in band-limited signals, in which cases R is a singular matrix, is discussed in Sections 5 and 6 respectively.

In all the examples to be discussed here, a vector \underline{g} is derived either from (17) or (18), by making use of the signal's spectral properties. The system (11) is constructed by using (19) and (20) and is solved. The quality of a restoration depends on the error covariance matrix, given by (13) and either (14) or (15), but also on the robustness of the restoration method. This robustness is characterized for instance by the sensitivity to noise and the condition of the system (11). For some of the examples an analysis of these items is given in [2,3,4].

Part of the results of this section can be found in [1,5,6].

3 Sample restoration in autoregressive processes, [1,2]

For an autoregressive process $(s_k)_{k=-\infty, \dots, \infty}$ of order p and with prediction coefficients a_0, a_1, \dots, a_p , $a_0=1$, we have

$$(21) \quad \sum_{j=0}^p a_j s_{k-j} = e_k, \quad k=-\infty, \dots, \infty,$$

where $(e_k)_{k=-\infty, \dots, \infty}$ is a white noise process with zero mean and variance σ_e^2 . The signal spectrum $S(\theta)$ is given by

$$(22) \quad S(\theta) = \sigma_e^2 \left(\sum_{k=-p}^p b_k \exp(-j\theta k) \right)^{-1},$$

where

$$(23) \quad b_k = \sum_{j=0}^{p-|k|} a_j a_{j+|k|}, \quad k=-p, \dots, p.$$

On substituting (22) into (17) we have for \underline{g}

$$(24) \quad g_k = \begin{cases} \sigma_e^{-2} b_k, & |k| \leq p, \\ 0, & \text{otherwise.} \end{cases}$$

Also, $G' = \sigma_e^{-2} B$, with

$$(25) \quad B_{ij} = \begin{cases} b_{t(i)-t(j)}, & |t(i)-t(j)| \leq p, \\ 0, & \text{otherwise.} \end{cases}$$

If, in this case, one defines the syndrome \underline{z} by

$$(26) \quad z_i = - \sum_{k=-p}^p b_k v_{t(i)-k},$$

then $\underline{\hat{x}}$ can be obtained as the solution of

$$(27) \quad B \underline{\hat{x}} = \underline{z}.$$

In [2] the solution of this system is discussed in detail. There it is shown that this system is generally very well-conditioned.

In practical situations the order of prediction and the prediction coefficients are often unknown and have to be estimated from the data. In that case the following approach [2] can be applied.

It is assumed that $t(1) \geq p+1$ and that $t(m) \leq N-p$. Although several algorithms exist for the estimation of p [7], the rather arbitrary choice $p \approx 3m$ is used instead. It produces satisfactory restoration results for the experiments done with digitized music and speech. The estimates $\underline{\hat{a}}$ and $\underline{\hat{x}}$ are the \underline{a} and \underline{x} that minimize

$$(28) \quad Q(\underline{a}, \underline{x}) = \sum_{k=p+1}^N \left| \sum_{j=0}^p a_j s_{k-j} \right|^2 = \sum_{k=p+1}^N |e_k|^2,$$

with $s_{t(i)} = x_i$, for $i=1, \dots, m$. This choice can be

motivated by the facts that a) minimizing $Q(\underline{a}, \underline{x})$ as a function of \underline{x} under the assumption that \underline{a} is known leads to an estimate $\underline{\hat{x}}$ solving (27), and b) minimizing $Q(\underline{a}, \underline{x})$ as a function of \underline{a} under the assumption that \underline{x} is known and that

$(s_k)_{k=-\infty, \dots, \infty}$ is a Gaussian process is similar to a maximum likelihood estimation procedure for \underline{a} [2].

Since $Q(\underline{a}, \underline{x})$ contains 4th order terms, this minimization is a non-trivial problem. The following iterative procedure can be used. Starting with a zeroth estimate $\underline{\hat{x}}^{(0)}$, ($\underline{\hat{x}}^{(0)} = 0$, for instance), one produces a first estimate $\underline{\hat{a}}^{(1)}$ for \underline{a} by minimizing $Q(\underline{a}, \underline{\hat{x}}^{(0)})$ as a function of \underline{a} . This, in fact, comes down to the well-known autocovariance method for estimating prediction coefficients. Then, by minimizing $Q(\underline{\hat{a}}^{(1)}, \underline{x})$ as a function of \underline{x} , one produces a first estimate $\underline{\hat{x}}^{(1)}$ for \underline{x} , which comes down to solving the system (27). This can be repeated to obtain second estimates $\underline{\hat{a}}^{(2)}$ and $\underline{\hat{x}}^{(2)}$ and so on. It is clear that $Q(\underline{a}, \underline{x})$ decreases to some non-negative number but it seems hard to determine whether this number is a global minimum or not.

For digitized music, if $m \leq 16$, and for digitized speech, if $m \leq 100$, the iterative procedure just described already produces good results without audible distortion after only one iteration, although more iterations can improve the results.

4 Sample restoration in speech signals, [3]

The restoration method discussed here is essentially a restoration method for quasi-periodic signals, but it has been developed for the restoration of lost samples in speech. It can be successfully applied in Mobile Automatic Telephony (MAT) systems, where, due to fading, burst errors of up to 12.5 ms occur in the received speech signal. At a sample rate of 8 kHz, this amounts to a burst of 100 unknown samples. Because this method is greatly simplified if we deal with burst errors only, and these errors are realistic, we restrict ourselves to this type of error.

The sample restoration method presented here is based on the LPC model, [9]. In this model it is

assumed that a speech signal can be described as the output signal of an all-pole filter, having a transfer function $1/A(z)$. This filter is excited either by a white noise signal, or by a (nearly) periodic signal, consisting of (nearly) equidistant pulses. In the first case one speaks of unvoiced speech, in the second case of voiced speech. In the voiced case, the distance between the pulses is called the pitch period. The pitch period is usually between 2 and 20 ms. At a sample rate of 8 kHz, this implies that the pitch period contains a number of samples, further denoted by q , that is between $q_{\min}=16$ and $q_{\max}=160$.

The samples in \underline{s} originate either from voiced or from unvoiced speech. In principle, voiced and unvoiced speech require different sample restoration methods. Here only a restoration method for voiced speech is presented. It is found in practice that this method can also be used satisfactorily on unvoiced speech. In this way a complicated and usually unreliable voiced/unvoiced decision is avoided.

A basic characteristic of voiced speech is its periodicity, the period being the pitch period. For the samples $(s_k)_{k=-\infty, \dots, \infty}$ we may assume that

$$(29) \quad s_k = c s_{k-q} + e_k, \quad k = -\infty, \dots, \infty,$$

where c is a positive constant $0 < c < 1$, close to 1, known as the pitch coefficient. The signal e_k is a white noise process. Note that this expression is similar to (21). Therefore the same approach is applied here. Assume that c and q are known. Then, as in Section 3, we can derive a sequence

$$(30) \quad b_k = \begin{cases} 1, & k = 0, \\ a = -c/(1 + c^2), & |k| = q, \\ 0, & \text{otherwise.} \end{cases}$$

For the $m \times m$ matrix B , we then have

$$(31) \quad B_{ij} = \begin{cases} 1, & i = j, \\ a, & |i - j| = q, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix is sparse, containing only three non-zero diagonals. The m vector \underline{z} is also simple:

$$(32) \quad z_i = -a (v_{t(i)-q} + v_{t(i)+q}), \quad i=1, \dots, m,$$

with the N vector \underline{v} defined in Section 2. Finally, \underline{x} can be solved from (27). The factor a used in (31) and (32) is defined in (30). The simplicity of the matrix B brings down the complexity of solving the system (27) from $O(m^2)$ to $O(m)$, as is discussed in [3]. This makes the restoration of large bursts feasible in real time. Note that if $m \leq q$, G is the identity matrix and the restoration problem is solved by putting $\underline{x} = \underline{z}$. The result of this procedure is that the restored signal segment conforms to the assumed periodicity as well as possible in a quadratic sense.

In the previous part of this section it was assumed that c and q are known. Of course, c and q vary in time and have to be estimated before the system (27) can be solved. The procedure of estimating q is generally known as pitch estimation. Several suitable pitch estimation methods are discussed in [9]. A well-known method for pitch estimation that also provides an estimate for c uses the estimate

$$(33) \quad \hat{r}_j = 1/N \sum_{k=1}^{N-j} s_k s_{k+j}, \quad j = q_{\min}, \dots, q_{\max}$$

for the autocorrelation function of the speech signal. It attains its global maximum at $j=0$. If $(s_k)_{k=1, \dots, N}$ are samples from voiced speech, with a pitch period of q samples, the next maximum is at $j=q$. Since $q_{\min} < q < q_{\max}$, q is the index of the global maximum in the interval $[q_{\min}, q_{\max}]$. The constant c in (29) can be estimated as $c = \hat{r}_q / \hat{r}_0$, but experiments have shown that without loss of restoration quality, it may be fixed to a value less than but close to 1. Before calculating \hat{r}_j , the unknown samples have to be set equal to zero. In [3] a method is given to simplify the calculation of the autocorrelation coefficients.

5 Sample restoration in multiple sinusoids

For a signal $(s_k)_{k=-\infty, \dots, \infty}$ consisting of t sinusoids, with random amplitudes $(A_j)_{j=1, \dots, t}$, random phases $(\phi_j)_{j=1, \dots, t}$, uniformly distributed over the interval $[-\pi, \pi)$, and frequencies $(\theta_j)_{j=1, \dots, t}$, the spectrum is given by

$$(34) \quad s(\theta) = \sum_{j=1}^t \frac{\pi}{2} A_j^2 (\delta(\theta - \theta_j) + \delta(\theta + \theta_j)).$$

The autocorrelation function of the signal is given by

$$(35) \quad r_k = \sum_{j=1}^t \frac{1}{2} A_j^2 \cos(k\theta_j).$$

It is well known that the rank of the autocorrelation matrix R of this signal is at most $2t$, independent of the dimensions [10]. This implies that if $N > 2t + m$, a matrix G has to be found that satisfies (9). In the following we assume that $N > 4t + m$, and that $t(1) > 2t + 1$ and $t(m) > N - 2t$.

Let the $2t + 1$ vector $\underline{u} = (u_k)_{k=0, \dots, 2t}$ be a vector in the null-space of the $(2t + 1) \times (2t + 1)$ autocorrelation matrix R . If we define the sequence

$$(36) \quad (g_k)_{k=-2t, \dots, 2t} \text{ by } g_k = \sum_{j=1}^{2t+1-|k|} u_j u_{j+|k|}, \quad k=-2t, \dots, 2t,$$

then it can be shown that $\underline{g} = (g_k)_{k=-2t, \dots, 2t}$ is in the null-space of the $(4t + 1) \times (4t + 1)$ autocorrelation matrix R . Moreover, since the Fourier transform $U(\theta)$ of $(u_k)_{k=0, \dots, 2t}$ has zeros at frequencies $(+\theta_j)_{j=1, \dots, t}$, $G(\theta) = |U(\theta)|^2$ satisfies (18) and G' and G'' can be obtained by using (18) and (19). The unknown samples can be calculated by constructing the syndrome \underline{z} according to (20) and by solving the system (11).

In the previous derivation, \underline{u} could have been used instead of \underline{g} , since replacing $G(\theta)$ in (18) by $U(\theta)$ gives the same result. However, by using \underline{g} as defined in (36) it is guaranteed that the matrix G' in (11) is positive definite and therefore the system (11) can be solved.

In this case the restoration method can also be made adaptive. This is useful if the number of sinusoids in the signal and their frequencies are unknown and both \underline{u} and \underline{x} have to be estimated from the available data. Let in this case t be an upper bound for the maximum number of sinusoids in the signal. The estimates $\hat{\underline{u}}$ and $\hat{\underline{x}}$ can be found by minimizing

$$(37) \quad P(\lambda, \underline{u}, \underline{x}) = \sum_{k=2t+1}^N \left| \sum_{j=0}^{2t} u_j s_{k-j} \right|^2 + \lambda (\underline{u}^T \underline{u} - 1)$$

as a function of λ , \underline{u} and \underline{x} , with $s_{t(i)} = x_i$, for $i=1, \dots, m$. This choice can be motivated by the facts that a) minimizing $P(\lambda, \underline{u}, \underline{x})$ as a function of \underline{x} under the assumption that λ and \underline{u} are known leads to an estimate $\hat{\underline{x}}$ solving (11), and b) minimizing $P(\lambda, \underline{u}, \underline{x})$ as a function of λ and \underline{u} under the assumption that \underline{x} is known leads to estimates $\hat{\lambda}$ and $\hat{\underline{u}}$ for the minimum eigenvalue and the corresponding eigenvector of the $(2t + 1) \times (2t + 1)$ autocovariance matrix $C = (c_{ij})_{i,j=0, \dots, 2t}$ with

$$(38) \quad c_{ij} = \sum_{k=2t+1}^N s_{k-i} s_{k-j}, \quad i, j=0, \dots, 2t.$$

As in the case of the sample restoration in Section 3 minimizing $P(\lambda, \underline{u}, \underline{x})$ can be done iteratively. Starting with a zeroth estimate $\hat{\underline{x}}^{(0)}$, ($\hat{\lambda}^{(0)} = 0$, for instance), one produces first estimates $\hat{\lambda}^{(1)}$ and $\hat{\underline{u}}^{(1)}$ for \underline{u} by minimizing $P(\lambda, \underline{u}, \hat{\underline{x}}^{(0)})$ as a function of λ and \underline{u} . Then, by minimizing $P(\hat{\lambda}^{(1)}, \hat{\underline{u}}^{(1)}, \underline{x})$ as a function of \underline{x} , one produces a first estimate $\hat{\underline{x}}^{(1)}$ for \underline{x} . This can be repeated to obtain second estimates $\hat{\lambda}^{(2)}$, $\hat{\underline{u}}^{(2)}$ and $\hat{\underline{x}}^{(2)}$ and so on. Again it is clear that, since C is positive definite, $P(\lambda, \underline{u}, \underline{x})$ decreases to some non-negative number but it seems hard to determine whether this number is a global minimum or not.

If the iteration process converges to the correct minimum, $\hat{\lambda}^{(i)}$ will converge to zero, because the covariance matrix becomes singular. The value of $\hat{\lambda}^{(i)}$ can then be used as a criterion to stop the iteration process.

6 Sample restoration in band-limited signals, [4]

In the case of a band-limited signal, the signal spectrum $S(\theta)$ is zero on one or more subintervals of the interval $[-\pi, \pi)$. Any function $G(\theta)$ that is zero on the subintervals where $S(\theta)$ is non-zero satisfies (18). To ensure that G' in (11) is positive definite we demand that $G(\theta)$ is positive

on the subintervals where $S(\theta)$ is zero. Unfortunately, a sequence that has a Fourier transform that is zero over some subinterval of $[-\pi, \pi)$ has in principle infinite length. The finite sequence $(g_k)_{k=-p, \dots, p}$, for some p that is used in practical cases, is always an approximation of the ideal infinite sequence.

As an example we consider a low-pass signal that is band-limited to the subinterval $(-\alpha\pi, \alpha\pi)$, $0 < \alpha < 1$. The ideal $G(\theta)$ is a high-pass filter, being zero on the subinterval $(-\alpha\pi, \alpha\pi)$, $0 < \alpha < 1$, and one on the subintervals $[-\pi, -\alpha\pi)$ and $(\alpha\pi, \pi)$. For g_k we then have

$$(39) \quad g_k = \delta_k - \frac{\sin(\alpha k \pi)}{k \pi}, \quad k = -\infty, \dots, \infty.$$

In [4] the robustness of this method has been investigated for bursts of unknown samples in low-pass signals. In particular the condition of the system (11) is considered. The main conclusion is that for larger amounts of unknown samples this method is very sensitive to out-of-band components in the signal. It can be shown that in the case of a band-limited signal, corrupted by white noise with a variance σ^2 , the restoration error $W(t(1), \dots, t(m))$ as defined in (2) is given by

$$(40) \quad W(t(1), \dots, t(m)) = \sigma^2 \text{trace}(G^t)^{-1},$$

where $t(i) = t(i-1) + 1$, $i = 2, \dots, m$. In [4] it has been shown that $\text{trace}(G^t)^{-1}$ increases roughly as $\exp(\alpha m \pi / 2)$. This shows that even for small bursts and small α the restoration error can become large.

7 Results

In [1,2,3,4] an extensive account is given of the performance of the sample restoration methods for autoregressive processes, speech signals and band-limited signals. The main conclusions are summarized here. The use of the restoration method for speech signals is restricted to speech signals only. For these signals the method performs very well. The restoration method for autoregressive processes is more general, because many signals can

be modeled in that way. It has turned out that especially for signals with a peaky spectrum, such as music, speech and multiple sinusoids the method performs very well. An advantage is that it is relatively insensitive to the presence of noise. The restoration method for band-limited signals only works if the input signal has no out-of-band components and if the product of bandwidth and number of unknown samples is small. In other cases large errors are made.

Here we present a comparison of the restoration method for autoregressive processes, being the best of the known methods discussed here, and the new restoration method for multiple sinusoids. The test signals are the same sinusoids as in [2]. The noiseless signal is given by

$$(41) \quad s_k = 100 \sin(0.23\pi k + 0.3\pi) + 60 \sin(0.4\pi k + 0.3\pi), \quad k = 1, \dots, N,$$

two other test sequences were generated by adding Gaussian white noise. The signal-to-noise ratios are respectively 40dB and 20dB. The pattern of unknown samples is a burst of length $m=16$. The methods were both tried on a short segment of length $N=64$ and on a longer one of length $N=512$. For both methods results are given after one and after three iterations. In the case of the restoration method for autoregressive processes p denotes the assumed order of the process, in the case of the restoration method for multiple sinusoids $p=2t$, where t is the assumed upper bound for the number of sinusoids. For p the values $p=4$, corresponding to the true number of sinusoids, and $p=10$ were tried. The restoration errors e , defined by

$$(42) \quad e = \frac{|\underline{x} - \underline{x}|^2 / m}{|\underline{s}|^2 / N},$$

are presented in Table 1. The restoration errors obtained by the restoration method for autoregressive processes are denoted by a_i , the restoration errors obtained by the restoration method for multiple sinusoids are denoted by s_i . Here the subscript i denotes the number of iterations.

SNR	m	p	N	a_1	a_3	s_1	s_3
inf.	16	4	64	0.56E+00	0.59E-20	0.19E-02	0.16E-27
inf.	16	10	64	0.90E-02	0.77E-26	0.13E-03	0.15E-30
40dB	16	4	64	0.57E+00	0.48E-02	0.58E-02	0.48E-02
40dB	16	10	64	0.73E-02	0.13E-03	0.40E-01	0.12E-01
20dB	16	4	64	0.76E+00	0.27E+00	0.76E+00	0.35E+00
20dB	16	10	64	0.18E-01	0.13E-01	0.19E+01	0.66E+00
inf.	16	4	512	0.64E-02	0.28E-28	0.65E-08	0.61E-28
inf.	16	10	512	0.15E-06	XXXXXXXX	0.15E-07	XXXXXXXX
40dB	16	4	512	0.24E-01	0.90E-02	0.94E-02	0.92E-02
40dB	16	10	512	0.21E-03	0.11E-03	0.23E-02	0.12E-02
20dB	16	4	512	0.58E+00	0.54E+00	0.96E+00	0.90E+00
20dB	16	10	512	0.91E-02	0.11E-01	0.46E+00	0.18E+00

Table 1 Restoration errors in multiple sinusoids, given by (40), for various signal to noise ratios. In the cases denoted by XXXXXXXX the restoration error could not be calculated, because the routines calculating the prediction coefficients and the eigenvalues failed. This was caused by singularity of the matrix C in (38).

From Table 1 it can be seen that the restoration method for sinusoids performs better, especially with respect to the convergence rate, on the noise-free test signals. If the signal is noisy the assumed order must be correct, or at least not too high, otherwise the performance of the restoration method for multiple sinusoids decreases significantly compared to the other method. Inspection of $G(\theta)$ in this case shows that in the case of the restoration method for multiple sinusoids, many spurious dips appear. This leads to other frequency components in the restored part. The restoration method for autoregressive processes, on the contrary, shows an increased performance if the assumed order is increased.

If we consider the complexity of both methods, then it is clear that the restoration method for multiple sinusoids is the most complex one, because it requires the computation of an eigenvalue and an eigenvector. The order of complexity of this is $O(p^4)$, whereas the order of complexity of the calculation of the prediction coefficients in the restoration method for autoregressive processes is $O(p^2)$. For the eigenvector problem iterative methods, which take fewer operations, have been proposed in the literature, e.g. [11].

Owing to the higher complexity and the fact that the performance decreases if the number of sinus-

oids is assumed too high, the restoration method for multiple sinusoids is most suitable in cases where the number of sinusoids is known and low. In those cases, the convergence rate is high.

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