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Derivatives of Markov kernels and their Jordan decomposition

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Abstract

We study a particular class of transition kernels that stems from differentiating Markov kernels in the weak sense. Sufficient conditions are established for this type of kernels to admit a Jordan–type decomposition. The decomposition is explicitly constructed.

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1 Introduction

Let $P_\theta$ be a family of Markov kernels from a measurable space $(X, \mathcal{X})$ to a locally compact space $Y$ (a precise definition will be given later in the text), with $\theta \in \Theta \subset \mathbb{R}$, and let $\mathcal{C}_c(Y)$ denote the set of continuous real-valued mappings with compact support on $Y$. The Markov kernel $P_\theta$ is called weakly differentiable at $\theta$ if for any $x \in X$ a finite signed measure $P_\theta'(x; \cdot)$ on $(Y, \mathcal{Y})$ exists such that for any $g \in \mathcal{C}_c(Y)$:

$$\frac{d}{d\theta} \int g(y) P_\theta(x; dy) = \int g(y) P_\theta'(x; dy).$$

(1)

This definition of weak differentiability is slightly more general than the original one in [4]: there (1) has to hold for any continuous bounded mapping $g$. Weak differentiability has been successfully applied to the theory of Markov chains. See [1] for an application to a problem in maintenance theory and [2] for an application to option pricing. The concept of weak differentiation is also related to finding optimal statistical tests, see [7]. For Markov chains, the following result is of particular interest: let $\pi_\theta$ denote the (unique) invariant distribution of $P_\theta$ (existence is assumed here), then it can be shown that

$$\pi_\theta' = \pi_\theta \sum_{n=0}^{\infty} P_\theta^n P_\theta^n,$$

(2)

where $P_\theta'$ is defined through (1) and $P_\theta^n$ denotes the $n$ fold product of $P_\theta$, see [4, 3] for a proof and more details on weak differentiability. If $P_\theta'$ exists, then the fact that $P_\theta'(x; \cdot)$ fails to be a probability measure poses the problem of sampling from $P_\theta'$. For $x \in X$ fixed, we can represent $P_\theta'(x; \cdot)$ by its Jordan decomposition as a difference between two probability measures as follows. For a finite signed measure $\mu$ denote its Jordan decomposition by $[\mu]^+ - [\mu]^-$, i.e., $\mu = [\mu]^+ - [\mu]^-$ and $[\mu]^+, [\mu]^-$ are positive measures. Let

$$c_{P_\theta}(x) = [P_\theta]^+(x; X) = [P_\theta]^-(x; X)$$

(3)

and

$$P_\theta^+(x; \cdot) = \frac{[P_\theta]^+(x; \cdot)}{c_{P_\theta}(x)}, \quad P_\theta^-(x; \cdot) = \frac{[P_\theta]^-(x; \cdot)}{c_{P_\theta}(x)},$$

then it holds, for all $g \in \mathcal{C}_c(Y)$, that

$$\int g(y) P_\theta'(x; dy) = c_{P_\theta}(x) \left( \int g(y) P_\theta^+(x; dy) - \int g(y) P_\theta^-(x; dy) \right).$$

(4)

For the above line of argument we fixed $x$. For $P_\theta^+$ and $P_\theta^-$ to be Markov kernels, we have to consider $P_\theta^+$ and $P_\theta^-$ as functions in $x$ and have to establish
measurability of $P^+_\theta(\cdot;A)$ and $P^-_\theta(\cdot;A)$ for any $A \in \mathcal{Y}$. The solution of this problem implies that $cP_\theta(\cdot)$ in (3) is measurable as a mapping from $X$ to $\mathbb{R}$. A representation of $P'_\theta$ through $(cP_\theta(\cdot), P^+_\theta, P^-_\theta)$, with $cP_\theta$ measurable and $P^\pm_\theta$ Markov kernels, is called a weak derivative of $P_\theta$. The existence of a weak derivative is of key importance for the statistical interpretation of (2) and for obtaining efficient unbiased gradient estimators.

In this paper, we give sufficient conditions for $P'_\theta$ to possess a representation as scaled difference of two Markov kernels. Specifically, we show that uniform boundedness of $P'_\theta$ (i.e., the supremum of $|\int g(y)P_\theta(x;dy)|$ over $g \in \mathcal{C}_c(Y)$ with $|g| \leq 1$ and $x \in X$ is finite) is together with a topological condition on $Y$ sufficient for $cP_\theta(\cdot)$ in (3) to be measurable (and for $P^+_\theta$ and $P^-_\theta$ to be Markov kernels again). In conclusion we will show that uniform boundedness is sufficient for $P'_\theta$ to admit a weak derivative.

The paper is organized as follows. Section 1 introduces the basic concepts and definitions. Section 2 shows that, under suitable conditions, the kernel $P'_\theta$ as defined in (1) can be uniquely extended to the bounded Borel–measurable mappings. In Section 3 an explicit construct of a Jordan–type decomposition of $P'_\theta$ is given.

## 2 Conditional Integrals and Kernels

We say that a topological space is second countable if its topology is generated by a countable basis, i.e., if there exists a countable family of open (or closed) sets which generates the topology. Throughout the paper we let $Y$ always denote a locally compact second countable Hausdorff space. We denote by $\mathcal{Y}$ the $\sigma$–field of Baire measurable subsets of $Y$, i.e., the $\sigma$–field generated by the compact subsets of $Y$.

**Remark 1** On a second countable locally compact space the Borel–field (the $\sigma$–field generated by the open or closed sets) and the Baire–field coincide. (This holds true since any open set in a second countable locally compact space is a countable union of compact sets.) Thus, $\mathcal{Y}$ is the $\sigma$–field generated by the family of open sets in $Y$.

For example, the space $\mathbb{R}^n$ and any submanifold of it constitutes a locally compact second countable space.

**Remark 2** Notice that a metrizable space is second countable if and only if it is separable (see [8] Theorem 16.11). Conversely, a locally compact or even a compact space may be separable but not second countable. An example of
a separable compact space that fails to be second countable is provided by the Stone-Cech compactification of the natural numbers.

Let $X$ be an arbitrary set and let $\mathcal{X}$ be an arbitrary $\sigma$–field on $X$. Let $\mathcal{B}_b(Y)$ be the family of real–valued bounded $\mathcal{Y}$–measurable functions on $Y$, let $\mathcal{C}_c$ the family of continuous functions with compact support on $Y$ and let $\mathcal{B}(X)$ denote the family of real–valued $\mathcal{X}$–measurable functions on $X$.

We call a Baire measurable function, say $g$, simple if and only if an integer $n \in \mathbb{N}$ and, for $i \leq n$, sets $B_i \in \mathcal{Y}$ and constants $\gamma_i \in \mathbb{R}$ exist such that

$$g(y) = \sum_{i=1}^{n} \gamma_i 1_{B_i}(y), \quad y \in Y.$$ 

The family of Baire measurable simple functions on $Y$ is denoted by $\mathcal{B}_{\text{simp}}(Y)$.

We note that $\mathcal{C}_c(Y) \subset \mathcal{B}_b(Y)$ and define the supremum norm $\| \cdot \|$ on $\mathcal{B}_b(Y)$ by

$$\|g\| := \sup_{y \in Y} |g(y)|.$$ 

We call a set $\mathcal{G} \subset \mathcal{B}_b(Y)$ uniformly bounded or sup–norm bounded if

$$\sup_{g \in \mathcal{G}} \|g\| < \infty.$$ 

We say that a sequence $(g_n)_{n \in \mathbb{N}}$ of functions $g_n \in \mathcal{B}_b(Y)$ is uniformly bounded if the set $\{g_n \mid n \in \mathbb{N}\}$ is uniformly bounded.

We say that a linear functional $J : \mathcal{C}_c(Y) \to \mathbb{R}$ is an integral if it is bounded on uniformly bounded subsets of $\mathcal{C}_c(Y)$ (such functionals may also be called sup-norm bounded). We say that a linear functional $\tilde{J} : \mathcal{B}_b(Y) \to \mathbb{R}$ is an extended integral if it is bounded on uniformly bounded subsets $\mathcal{G}$ of $\mathcal{B}_b(Y)$.

We say that a sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n$ from some set $S$ to a Hausdorff space $V$ converges point–wise if $\lim_{n \to \infty} f_n(s)$ exists for any $s \in S$.

**Definition 1** A kernel $P(\cdot, \cdot)$ from $X$ to $Y$ is a function $P : X \times Y \to \mathbb{R}$ such that $P(x, \cdot)$ is for any $x \in X$ a finite signed measure on $(Y, \mathcal{Y})$ and $x \mapsto P(x, B)$ is for any $B \in \mathcal{Y}$ a $\mathcal{X}$–measurable function on $X$. We say that the kernel is Markov (or a Markov kernel) if for any $x \in X$ the measure $P(x, \cdot)$ is a probability measure. We denote the space of all kernels from $X$ to $Y$ by $\mathcal{P}(X, Y)$.

**Definition 2** A conditional integral $I(\cdot, \cdot)$ from $X$ to $\mathcal{C}_c(Y)$ is a function $I : X \times \mathcal{C}_c(Y) \to \mathbb{R}$ such that
• \(I(x, \cdot)\) is an integral (i.e. a linear functional on \(C_c(Y)\) which is sup-norm bounded) and

• \(x \mapsto I(x, f)\) is for any \(f \in C_c(Y)\) a \(X\)-measurable function on \(X\).

We denote the space of conditional integrals from \(X\) to \(C_c(Y)\) by \(I(X, Y)\).

**Definition 3** Let \(Z\) denote an arbitrary Hausdorff space. We say that a function \(F : B_b(Y) \to Z\) is point-wise sequentially continuous on uniformly bounded subsets of \(B_b(Y)\) if for any uniformly bounded point-wise convergent sequence \((g_n)_{n \in \mathbb{N}}\) in \(B_b(Y)\) with limit \(g \in B_b(Y)\) we have that \(\lim F(g_n) = F(g)\).

Given a function space \(F \subseteq \mathbb{R}^X\). We say that a set \(S \subseteq F\) is point-wise sequentially closed if \(S\) contains all the limits which are in \(F\) of point-wise convergent sequences \((g_n)_{n \in \mathbb{N}}\) whose elements \(g_n\) are in \(S\). We say that a set \(\overline{S}\) is the point-wise sequential closure of a set \(S\) if \(\overline{S}\) is the smallest point-wise sequentially closed set containing \(S\). A set \(S\) is point-wise sequentially dense in a set \(T\) if \(T\) is a subset of the sequential closure \(\overline{S}\) of \(S\). (For more details on sequential continuity and measurable functions see [5] Section 3.2.)

**Proposition 1** Let \(K \subseteq Y\) be compact and let \(O \subseteq Y\) be open with compact closure such that \(K \subseteq O\). Then there exists a continuous function \(f : Y \to [0, 1]\) such that \(f(K) = 1\) and \(f(Y \setminus O) = 0\).

**Proof.** This follows by an application of the Urysohn Lemma (see [8] 15.6) to \(K\) and \(Y \setminus O \cup \{\infty\}\) in the one-point compactification (see [8] 19.2 and 19A) \(Y \cup \{\infty\}\) of \(Y\), since any compact space is normal (see [8] 17.10).

**Lemma 1** It holds that:

(a) The space \(B(X)\) is point-wise sequentially closed in \(\mathbb{R}^X\).

(b) The function-space \(B_{\text{simp}}(Y)\) is point-wise sequentially dense in \(B_b(Y)\).

(c) The function-space \(C_c(Y)\) is point-wise sequentially dense in \(B_b(Y)\).

**Proof.** (a) Is the well known fact that a limit of a point-wise convergent sequence of measurable functions is again measurable.

(b) Is a re-formulation of the fact that any measurable function is the pointwise limit of a sequence of simple functions. (See for example Corollary 3.2.1 of [5].)

(c) Given an arbitrary compact set \(K\) we can by second countability and local compactness of \(Y\) choose a sequence \((O_n)_{n \in \mathbb{N}}\) of open sets such that
On $n+1 \subset O_n$, $\bigcap_n O_n = K$ and the closures $\overline{O_n}$ are compact. By Proposition 1 we find continuous functions $f_n$ such that $f_n(K) = 1$ and $f_n(Y \setminus O_n) = 0$. Since $\overline{O_n}$ is compact these functions $f_n$ possess compact support. Thus, $1_K = \lim_{n \in \mathbb{N}} f_n(x)$, and $1_K$ lies in the point-wise sequential closure of $C_c(Y)$. Since any open set $O$ is the countable union of compact sets, we see that also any function $1_O$ and thus especially the function $1_Y$ belongs to the sequential closure of $C_c(Y)$. (That $1_Y$ belongs to the sequential closure of $C_c(Y)$ can also be easily seen using a countable partition of unity.) Hence, any finite linear combination of function $1_A$ with $A \in \mathcal{Y}$ belongs to the sequential closure of $C_c(Y)$. So we obtain (c) from (b).

Lemma 2 Any conditional integral $I \in \mathcal{I}(X,Y)$ extends uniquely to a conditional integral $\tilde{I} : X \times B_b(Y) \mapsto \mathbb{R}$ such that for any $x \in X$ the function $\tilde{I}(x,\cdot)$ is point-wise sequentially continuous on uniformly bounded subsets of $B_b(Y)$. Moreover, there exists a one-one correspondence between kernels and conditional integrals $G : \mathcal{P}(X,Y) \mapsto \mathcal{I}(X,Y)$ given by

$$[G(P)](x,f) = \int f(y) \ P(x,dy) \text{ for all } f \in C_c(Y),$$

or, if we prefer to consider the extensions $\tilde{I}$ of the conditional integrals $I$, by

$$[G(P)](x,g) = \int g(y) \ P(x,dy),$$

for all $g \in B_b(Y)$.

We call the above extension $\tilde{I}$ of a conditional integral $I$ the extended conditional integral. By Lemma 1 there is a one–one correspondence between conditional integrals $I$ and their extensions $\tilde{I}$.

Proof of Lemma 2: The proof consists of 3 steps. First we show that for a given conditional integral $I \in \mathcal{I}(X,Y)$ there exists for any $x \in X$ a unique measure $P(x,\cdot)$ on $(Y,\mathcal{Y})$. Then we show that the integrals $I(x,\cdot)$ on $C_c(Y)$ extend for arbitrary $x \in X$ uniquely to extended integrals $\tilde{I}(x,\cdot)$ on $B_b(Y)$.

Step 1: Let $I$ be a given conditional integral. According to the Riesz representation theorem, there exists for any $x \in X$ a unique measure $P(x,\cdot)$ on $(Y,\mathcal{Y})$, such that

$$I(x,f) = \int f(y) \ P(x,dy) \text{ for all } f \in C_c(Y).$$
Thus, there exists for any \( x \in X \) a unique extended integral \( \tilde{I}(x, \cdot) \) such that
\[
\tilde{I}(x, g) = \int g(y) \, P(x, dy) \quad \text{for all } g \in \mathcal{B}_b(Y) .
\] (7)

Note that, by the dominated convergence theorem, \( \tilde{I}(x, \cdot) \) is sequentially point-wise continuous on uniformly bounded sets. \( \tilde{I}(x, \cdot) \) is also the unique extension of \( I(x, \cdot) \) from \( \mathcal{C}_c(Y) \) to \( \mathcal{B}_b(Y) \) which is sequentially point-wise continuous on uniformly bounded sets, since \( \{ f \in \mathcal{C}_c(Y) \mid -1 \leq f \leq 1 \} \) is point-wise sequentially dense in \( \{ g \in \mathcal{B}_b(Y) \mid -1 \leq g \leq 1 \} \) (The fact that \( \{ f \in \mathcal{C}_c(Y) \mid -1 \leq f \leq 1 \} \) is point-wise sequentially dense in \( \{ g \in \mathcal{B}_b(Y) \mid -1 \leq g \leq 1 \} \) is proved completely analogous as we proved (c) in Lemma 1.)

**Step 2:** In the second step we show that the functions \( x \mapsto \tilde{I}(x, g) \) are \( X \)-measurable, for \( g \in \mathcal{B}_b(Y) \) arbitrary, i.e., we show that \( \tilde{I} \) is a conditional integral. Further we show that the unique corresponding function \( P : X \times Y \), defined in the first step, is a kernel.

Let \( \mathbb{R}^X \) be endowed with the topology of point-wise convergence. Define an operator \( T : \mathcal{B}_b(Y) \to \mathbb{R}^X \) by
\[
[T(g)](x) = \tilde{I}(x, g) .
\]
The fact that, for arbitrary \( x \in X \), the integral \( \tilde{I}(x, \cdot) \) is point-wise sequentially continuous on uniformly bounded sets of \( \mathcal{B}_b(Y) \) (where we take \( M = \mathcal{B}_b(Y) \) and \( V = \mathbb{R} \) in Definition 3) implies that \( T \) is also point-wise sequentially continuous (where we take \( M = \mathcal{B}_b(Y) \) and \( V = \mathbb{R}^X \) in Definition 3).

Further, \( f \in \mathcal{C}_c(Y) \) implies by definition of \( T \) and the fact that \( I \in \mathcal{I}(X,Y) \) that
\[
T(f) = [x \mapsto I(x, f)] \in \mathcal{B}(X) ,
\] (8)
i.e., we have that \( T(\mathcal{C}_c(Y)) \subseteq \mathcal{B}(X) \).

By (8) together with Lemma 1 (c) and the point-wise sequential continuity of \( T \), we obtain that \( T(\mathcal{B}_b(Y)) \subseteq \mathcal{B}(X) \). In other words, we obtain that \( g \in \mathcal{B} \) implies that \( x \mapsto \tilde{I}(x, g) \) is \( X \)-measurable. The fact that \( x \mapsto \tilde{I}(x, g) \) is \( X \)-measurable implies in the case that \( g \) is the characteristic function of a set \( B \) that \( x \mapsto P(x, B) \) is \( X \)-measurable. Thus, \( P \) is a kernel and (as already noted in the first step) by the Riesz representation theorem unique.

In the first two steps we have shown that to an integral \( I \in \mathcal{I}(X,Y) \) there corresponds a unique kernel \( P \in \mathcal{P}(X,Y) \) and a unique extended integral \( \tilde{I} \). Further we know by equation (6) and (5) that this correspondence is given by \( G^{-1} \). In the third step we show that to any \( P \in \mathcal{P}(X,Y) \) there corresponds a unique \( I = G(P) \in \mathcal{I}(X,Y) \).
Step 3: We show now that any kernel $P$ corresponds to an unique integral $I$. That any kernel $P$ gives us by formula (7) for any $x$ an extended integral $\tilde{I}(x,.)$ is trivial. To show that $\tilde{I}$ is a conditional extended integral note that for any simple function $g = \sum_{i=1}^{n} \gamma_i 1_{B_i} \in B_{simp}$ we have:

$$\tilde{I}(x, g) = \sum_i \gamma_i P(x, B_i).$$

So for $g \in B_{simp}$ the function $x \mapsto \tilde{I}(x, g)$ is a finite sum of $\mathcal{X}$-measurable functions and thus itself $\mathcal{X}$-measurable. It remains to be shown that $x \mapsto \tilde{I}(x, g)$ is for any $g \in B_b(Y)$ a $\mathcal{X}$-measurable function. We do this by arguments analogous to the arguments provided in step 2 as will be explained in the following.

Let $T$ denote the operator defined in step 2. Recall that $T$ is point-wise sequentially continuous. Furthermore, $f \in B_{simp}(Y)$ implies (by definition of $T$ and the fact that for $g \in B_{simp}(Y)$ the function $x \mapsto \tilde{I}(x, g)$ is $\mathcal{X}$-measurable) that:

$$T(f) = [x \mapsto \tilde{I}(x, f)] \in \mathcal{B}(X),$$

i.e., we have that $T(B_{simp}(Y)) \subseteq \mathcal{B}(X)$.

By (9) together with Lemma 1 (b) and point-wise sequential continuity of $T$, we obtain that $T(B_b(Y)) = \mathcal{B}(X)$. In other words, we obtain that $g \in \mathcal{B}$ implies that $x \mapsto \tilde{I}(x, g)$ is $\mathcal{X}$-measurable. $\square$

Now we define weak differentiability of conditional integrals and kernels.

Definition 4 Let $\Theta$ be an open interval in $\mathbb{R}$ and let $\vartheta \mapsto I_{\vartheta}$ be a path in (mapping from $\Theta$ to) the space $\mathcal{I}(X, Y)$. We say that $\vartheta \mapsto I_{\vartheta}$ is weakly differentiable if

$$\frac{dI_{\vartheta}(x, f)}{d\vartheta}$$

exists for all $(x, f) \in X \times C_c(Y)$

If $\vartheta \mapsto I_{\vartheta}$ is weakly differentiable then we say that it is bounded weakly differentiable if

$$\sup_{f \in C_c(Y) \atop |f| \leq 1} \left| \frac{dI_{\vartheta}(x, f)}{d\vartheta} \right| < \infty,$$

for any $x \in X$.

We say that a path $\theta \mapsto P_\theta$ in the space $\mathcal{P}(X, Y)$ of kernels is bounded differentiable if the corresponding path $\theta \mapsto G(P_\theta)$ in the space $\mathcal{I}(X, Y)$ of conditional integrals is bounded weakly differentiable.
Theorem 1 If the path $\vartheta \mapsto P_\vartheta$ in the space $\mathcal{P}(X, Y)$ is bounded weakly differentiable, then the weak derivative can be represented by a path $\vartheta \mapsto P'_\vartheta$ in the space $\mathcal{P}(X, Y)$. The connection between $\vartheta \mapsto P_\vartheta$ and $\vartheta \mapsto P'_\vartheta$ is given by
\[
\int f(y) P'_\vartheta(x, dy) = \frac{d \int f(y) P_\vartheta(x, dy)}{d\vartheta}.
\]

Proof. Let $I_\vartheta = G(P_\vartheta)$ be the corresponding path in the space of conditional integrals. Define for any $(x, f) \in X \times \mathcal{C}_c(Y)$ the function $I'_\vartheta(x, f)$ by
\[
I'_\vartheta(x, f) = \frac{dI_\vartheta(x, f)}{d\vartheta}.
\]
Let $(h_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of positive reals which goes to 0. Then for $f \in \mathcal{C}_c$ we have:
\[
x \mapsto I'_\vartheta(x, f) = x \mapsto \frac{dI_\vartheta(x, f)}{d\vartheta} = x \mapsto \lim_{n \to \infty} \frac{I_{\vartheta + h_n}(x, f) - I_\vartheta(x, f)}{h_n}.
\]
Thus, $x \mapsto I'_\vartheta(x, f)$ is for $f \in \mathcal{C}_c(Y)$ a limit of a sequence of $\mathcal{X}$-measurable functions and therefore itself $\mathcal{X}$-measurable. Furthermore, $I'(x, \cdot)$ is by the condition of boundedness in the definition of bounded weakly differentiable for any $x \in X$ norm-bounded; i.e., $I'(x, \cdot)$ is bounded on uniformly bounded subsets of $\mathcal{C}_c(Y)$. Thus, $I'(x, \cdot)$ is for any $x \in X$ an integral and $I'('\cdot, \cdot)$ is thus itself a conditional integral. By the correspondence between conditional integrals and kernels we obtain a kernel $P' = G^{-1}(I')$. The formula connecting $P'$ and $P$ is clear from the correspondence between $P'$, $P$ and $I'$, $I$ and the definition of $I'$.

\[\square\]

3 Jordan Decomposition of Weak Derivatives of Markov Kernels

Definition 5 Given a kernel $P \in \mathcal{P}(X, Y)$ we define the absolute value $|P|$ of the kernel as follows:
\[
|P|(x, B) = \sup_{A \subseteq Y : A \subseteq B} 2 \cdot P(x, A) - P(x, B), \quad x \in X, \ B \in \mathcal{Y}.
\]

Lemma 3 The absolute value $|P|$ of a kernel $P \in \mathcal{P}(X, Y)$ is again a kernel.

Proof: That the absolute value $|P|(x, \cdot)$ is a finite measure is a well known fact and it remains to be shown that the function
\[
x \mapsto |P|(x, B)
\]
is \( \mathcal{X} \)-measurable.

Let \( \mathcal{A} \) be the set-field generated by a countable basis of the topology of \( Y \). Then, \( \mathcal{A} \) is countable and generates the \( \sigma \)-field \( \mathcal{Y} \). For any set \( B \in \mathcal{Y} \) and any measure \( \mu \) on \( (Y, \mathcal{Y}) \) there exists a sequence \( (A_n)_{n \in \mathbb{N}} \) of sets \( A_n \in \mathcal{A} \) such that \( \lim \mu(A_n \triangle B) = 0 \) (see [6] Lemma A.24). Thus, the function
\[
x \mapsto |P|(x, B)
\]
is the point-wise supremum over the countable family
\[
\left\{ x \mapsto 2 \cdot P(x, A) - P(x, B) : A \in \mathcal{A} \text{ and } A \subseteq B \right\}
\]
of \( \mathcal{X} \)-measurable functions and thus itself a \( \mathcal{X} \)-measurable function on \( X \). \( \square \)

**Definition 6** We say that a kernel is positive if \( P(x, B) \geq 0 \) for all \( (x, B) \in X \times \mathcal{Y} \). We say that a pair of kernels \((P^+, P^-)\) forms a decomposition of a kernel \( P \) if \( P^+ \) and \( P^- \) are positive kernels and \( P(x, B) = P^+(x, B) - P^-(x, B) \). We say that this decomposition is minimal or Jordan if for any other decomposition \((Q^+, Q^-)\) of \( P \) we have \( P^+(x, B) \leq Q^+(x, B) \) and \( P^-(x, B) \leq Q^-(x, B) \).

**Corollary 1** Any kernel \( P \in \mathcal{P}(X, Y) \) possesses a Jordan decomposition.

**Proof:** For \( (x, B) \in X \times \mathcal{Y} \) define
\[
P^+(x, B) := \frac{|P|(x, B) + P(x, B)}{2}
\]
and
\[
P^-(x, B) := \frac{|P|(x, B) - P(x, B)}{2}.
\]
Then, \( P^+(x, B), P^-(x, B) \geq 0 \) and \( P^+(x, \cdot), P^-(x, \cdot) \) are measures, and \( x \mapsto P^+(x, B) \) as well as \( x \mapsto P^+(x, B) \) are \( \mathcal{X} \)-measurable functions on \( X \). It is also clear that the decomposition is minimal. \( \square \)

**Theorem 2** Suppose that the path \( \vartheta \mapsto P_\vartheta \) in the space \( \mathcal{P}(X, y) \) is bounded weakly differentiable and that for any \( \theta \) the kernel \( P_\theta \) is Markov. Then there exist for any \( \vartheta \) Markov kernels \( Q^+_\vartheta \) and \( Q^-_\vartheta \) from \( X \) to \( Y \) and a \( \mathcal{X} \)-measurable function \( c_\vartheta : X \to \mathbb{R} \) such that the weak derivative \( P'_\vartheta \) of \( P_\vartheta \) decomposes in the form
\[
P_\vartheta(x, B) = c_\vartheta(x) \left( Q^+_\vartheta(x, B) - Q^-_\vartheta(x, B) \right) \quad \forall (x, B) \in X \times \mathcal{Y}.
\]
**Proof:** By Theorem 1, the weak derivative $P'_{\vartheta}$ is for any $\vartheta$ a kernel and by the Corollary 1, $P'_{\vartheta}$ possesses a Jordan decomposition $(P^+_{\vartheta}, P^-_{\vartheta})$, i.e., $P'_{\vartheta} = P^+_{\vartheta} - P^-_{\vartheta}$ and $P^+_{\vartheta}, P^-_{\vartheta}$ are positive kernels. Since the $P_{\vartheta}$ are Markov kernels we have $P^+_{\vartheta}(x, Y) = P^-_{\vartheta}(x, Y)$. Let $c_{\vartheta}: X \rightarrow \mathbb{R}$ be defined by

$$c_{\vartheta}(x) := P^+_{\vartheta}(x, Y) = P^-_{\vartheta}(x, Y).$$

Since $P^+_{\vartheta}$ is a kernel, the function $c(\cdot)$ is $X$-measurable. Let

$$Q^+_{\vartheta}(x, B) := \frac{1}{c(x)} P^+_{\vartheta}(x, B) \quad \text{for all } x \text{ with } c(x) > 0,$$

$$Q^-_{\vartheta}(x, B) := \frac{1}{c(x)} P^-_{\vartheta}(x, B) \quad \text{for all } x \text{ with } c(x) > 0$$

and let for an arbitrary fixed probability measure $\mu$, arbitrary $x$ with $c_{\vartheta}(x) = 0$ and arbitrary $B \in \mathcal{Y}$

$$Q^+_{\vartheta}(x, B) = Q^-_{\vartheta}(x, B) = \mu(B).$$

Then $Q^+_{\vartheta}$ as well as $Q^-_{\vartheta}$ are Markov kernels. □

**Remark 3** This specific decomposition $(c_{\vartheta}(\cdot), Q^+_{\vartheta}, Q^-_{\vartheta})$ is only possible because the kernels $P'_{\vartheta}$ stem from weak differentiation of a Markov kernel valued function $\vartheta \mapsto P_{\vartheta}$.

**References**


