Three-body dispersion contributions to the thermodynamic properties and effective pair interactions in liquid argon

Citation for published version (APA):

DOI:
10.1063/1.479390

Document status and date:
Published: 01/01/1999

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.
Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
Three-body dispersion contributions to the thermodynamic properties and effective pair interactions in liquid argon

Martin A. van der Hoef
Department of Chemical Engineering, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

Paul A. Madden
Physical and Theoretical Chemistry Laboratory, University of Oxford, South Parks Road, Oxford OX1 3QZ, United Kingdom

(Received 29 March 1999; accepted 20 April 1999)

The contributions of three-body triple dipole and dipole-dipole-quadrupole dispersion interactions to the thermodynamic properties of liquid argon are examined, using a recently introduced simulation scheme which contains an explicit, quantum mechanical representation of the underlying electronic structure [Mol. Phys. 94, 417 (1998)]. The experimental pressure and energy at a series of liquid densities are shown to be quite accurately reproduced by a combination of the best available pair potential (Aziz) plus these three-body terms. The extent to which these many-body effects can be encompassed by an effective pair potential is then discussed. The nonuniqueness of such an effective potential is reiterated. It is shown that in the dense liquid, the three-body contribution to the effective pair potential \( \phi(r) \) varies approximately linearly with density and is almost temperature independent. It is shown how the addition of \( \phi(r) \) to the Aziz pair potential moves the latter toward the widely used Lennard-Jones (12-6) potential.

I. INTRODUCTION

The role of many-body contributions to the interatomic potential in determining the properties of atomic fluids is a subject which has attracted much attention over a long period of time, from theory, simulation and experiment. Foremost among the many-body terms are the three-body contributions to the dispersion interactions, for which the Axilrod–Teller (AT) expression gives the lowest-order (tripole, or DDD) contributions:

\[
u_{123}^{\text{DDD}} = C_g(1 + 3 \cos \gamma_1 \cos \gamma_2 \cos \gamma_3) r_{12}^{-3} r_{13}^{-3} r_{23}^{-3},
\]

where \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are the interior angles of the triangle formed by atoms 1, 2, and 3 and \( r_{ij} \) is an interatomic separation. Other many-body effects include higher-order dispersion terms (dipole-dipole-quadrupole, or DDQ, etc.) as well as many-body contributions which enter on the range of the overlap of the electron densities of the constituent atoms. Explicit expressions for the higher-order dispersion terms do exist, but become increasingly complicated and their effect on liquid-state properties has not been characterized. Very little is known about the short-range terms; in part these must “damp” the asymptotic expressions for the dispersion interactions at short-range, but other effects must occur in the repulsive interaction between a collection of three or more atoms.

Recently, we introduced a novel simulation method which allows the dispersion effects to be represented. In it the dispersion interaction is not modeled via the classical expressions for the pair- and three-body potential (e.g., the London and Axilrod–Teller formulas), but from an explicit description of the electronic structure in the spirit of the Car–Parrinello method. The advantage of treating dispersion interactions in this way, rather than by use of the analytical expression, is that the description at the quantum mechanical level allows for proper treatment of both dispersion damping and periodic boundary conditions at the three-body level. The latter makes this method suitable for performing simulations of bulk system with full three-body interactions, whereas the use of an explicit three-body potential, like the AT, always introduces problems of truncating the interactions.

In the present paper, we apply this method to the classical problem of the many-body dispersion contributions to the properties of argon. In it we will examine these contributions to the energy, equation-of-state (EoS) and radial distribution function (RDF). We will then also consider how the many-body potential terms contribute to a (temperature and density dependent) effective pair potential and, ultimately, discuss how the combination of this with the “exact” pair potential compares with the much studied Lennard-Jones pair potential.

A partial motivation for this first application of the code is to validate it on a well-studied problem. However, compared to previous studies our work contains a number of additional features. First, we characterize the significance of the DDQ terms, in addition to the DDD ones and thereby hint at the convergence of the three-body dispersion terms at the level of properties. Second, we are using a liquid structure which is fully consistent with the implicit many-body potential we are using, rather than evaluating the three-body effects using structures deduced with a pair potential, or doing a perturbation calculation. Third, we evaluate the effect...
tive pair potentials at a wider range of temperature and density than has been done previously. We note that, although the problem has been much studied, there remains a great deal of ignorance about what has been learned from these studies. For example, although it is well-recognized that an effective potential must depend on the thermodynamic state, it is little appreciated that there is no such thing as the effective pair potential at a given state point—two different pair potentials are required to reproduce energy and EoS. In the face of this, perhaps some repetition of long-standing results can be tolerated.

II. SIMULATIONS OF THE TWO- AND THREE-BODY DISPERSION EFFECTS

A. The simulation model

In this section, we will briefly describe the simulation model that is being used: for full details we refer to Ref. 19.

The method rests on a simplified representation of the internal electronic structure of the atoms by a single quantum particle in an effective three-level system: an s-type ground state, three p-type excited states, and five d-type excited states. This system is then handled in a fully quantum mechanical way, using an N-atom wave function which is expanded (configuration interaction) about the noninteracting ground state over all possible pairs of excited states, where the N-atom functions are constructed from the single atom states by simple product. If combined with an electronic Hamiltonian containing all dipole-dipole, dipole-quadrupole states by simple product. If combined with an electronic Hamiltonian containing all dipole-dipole, dipole-quadrupole states by simple product. If combined with an electronic Hamiltonian containing all dipole-dipole, dipole-quadrupole states by simple product.

$U_{\text{disp}} = \langle e_{\text{CPA}} \rangle_{p,T}$, $3P_{\text{disp}}V = \langle w_{\text{CPA}} \rangle_{p,T}$.

B. Simulation details

We have performed simulations of argon at densities ranging from $\rho^* = 0.1$ to $\rho^* = 0.9$ and at temperatures $T^* = 1.1$, $T^* = 1.6$ and $T^* = 2.0$, where we have used reduced LJ units ($\epsilon/k = 119.8$ K, $\sigma = 0.3412$ nm) to indicate the state point. We stress that this is purely for convenience, as no corresponding state relations hold in the presence of three-body forces.

The purpose of the calculations is to characterize the effects of the three-body interactions, with respect to the structure and properties predicted for the true pair potential, as represented by the Aziz HFD-B potential. In practice, we have used a simpler form:

$$u(r_{12}) = a \exp(-b r_{12}) + C_{6} r_{12}^{-6} + (C_{8} + c) r_{12}^{-8}$$

with $a = 400.1$, $b = 1.927$, $c = 1454$, $C_{6} = 64.3$, $C_{8} = 1169.9$ (all values in atomic units), where the $C_{6}$ and $C_{8}$ coefficients were set equal to the best literature values, and the other three parameters ($a$, $b$, and $c$), where chosen such that $u(r)$ fits the Aziz potential for all relevant distances.

The dispersive parts of the two- and three-body interactions were taken care of by the two-level quantum mechanical model described in Sec. II A. The remaining part of the potential (a $\alpha \text{exp}(-br) + c r^{-8}$) was added as a classical potential in the molecular dynamics simulation. Four parameters, $a_{1}, a_{2}, u_{1}, u_{2}$, determine the magnitude of the various dispersion terms (see Ref. 19). They were determined from values of the classical dispersion parameters $C_{6}$, $C_{8}$, $C_{11}$ [Eqs. (1) and (3)] and $C_{11}$ (the coefficient of the classical DDQ term) by means of relations (23) and (30) in Ref. 19. Values for $C_{6}$ and $C_{8}$ are given above, and a value for $C_{11}$ of 518.3 a.u. was found by Kumar and Meath. The coefficient $C_{11}$ was chosen equal to 668.16 au, which follows from the best estimates for the $C_{6}$ and $C_{8}$ coefficients and the dipole and quadrupole polarizabilities (see Ref. 19). This value was found to be within a few percent of the best literature value from Ref. 25. The resulting values of $a_{1}, a_{2}, u_{1}$, and $u_{2}$ were 10.748, 48.2, 0.742 and 0.5067 a.u., respectively, yielding the proper numerical values for $C_{6}$, $C_{8}$, $C_{9}$ and $C_{11}$ by means of relations (23) and (30) in Ref. 19. Test runs showed that the DDQ and QQK contributions are negligible compared to the DDD and DDQ, and therefore we did not consider any interaction beyond DDQ.

The systems studied contained 125 particles, and we used a timestep of 2.5 fs, which is about 4 times smaller than in conventional Lennard-Jones simulations. After equilibration the total production runs lasted 30000 timesteps, during which pressure, energy and RDFs were measured, as well as the effective pair interactions discussed in Sec. III.

C. Results for the pressure, energy and $g(r)$

We begin by showing in Fig. 1 results for the RDF at three different reduced densities (0.1, 0.3, 0.8) and at a reduced temperature of 1.1. RDFs have been calculated for the full simulation model, including the three-body DDD and DDQ terms, and for a simulation with the Aziz pair potential.
where these three-body terms are omitted; the differences, \( \Delta g(r) \), between the RDFs of these calculations are shown in the bottom part of the figure. The results show that at the relatively low density of 0.3 (similar to the expected critical density), the three-body terms have a noticeable effect on the RDF. At the higher density of 0.8, close to the triple point density, and at the lowest density of 0.1, the difference between the RDFs obtained with and without the three-body effects is very small. At the high density, the liquid structure is dominated by packing effects, and the dispersion terms are of little importance. At low density, the three-body contribution becomes small because of the relatively small number of three-atom encounters. The largest effects are found for densities around 0.3, which is in the vicinity of the critical density.

The reduction of the RDF by the three-body forces is still much larger than reported by Levesque and Weis for krypton (which has stronger three-body effects than argon). They found that the RDF was reduced by about 0.02 in a small region about \( r = 0.3 \). A similar reduction was found by Barker, for a slightly different potential (K2 by Barker, Klein, Bobetic) plus AT. In a more recent study by Meroni et al., the three-body AT terms were evaluated from a pair HNC theory for krypton. Although their results showed the same trend as in our simulations, the three-body contributions were not large enough to get the Aziz + AT results into good agreement with experiment. The conclusion from our simulations fully underscores Barker’s statement in Ref. 10: ‘‘the net contribution of other many-body interactions to the properties of solid and dense fluid argon, krypton, xenon must be very small.’’ A similar conclusion was drawn more recently by Anta et al. with regard to the phase-equilibrium properties of argon.

**III. THE REPRESENTATION OF THREE-BODY EFFECTS IN AN EFFECTIVE PAIR POTENTIAL**

A question of general interest is the way in which the three-body effects contribute to an effective pair potential which might be used in a simulation or theoretical calculation to reproduce some property of the simulated fluid exactly. It is not generally recognized that this requires a different potential for each property calculated. In order to emphasize this point, we begin by obtaining in a unified way the relationship between the effective potentials which repro-
duce the energy and EoS of the liquid with the full many-body potential, as has previously been done by Barker, Rowlinson and co-workers, among others.

### A. Nonuniqueness of the effective pair potential

We start from the exact expressions for the potential energy $U$ and the excess pressure $P_{ex}$ in a fluid at a particular state point $(\rho, T)$, where the particles interact via a two-body potential $u_{12} = u(r_{12})$ and a three-body potential $u_{123} = u(r_{12}, r_{13}, r_{23})$:

$$U = \frac{1}{2} \rho^2 \int \int dr_1 dr_2 g_{12}(\rho_{12})$$

$$+ \frac{1}{6} \rho^3 \int \int \int dr_1 dr_2 dr_3 g_{123}(\rho_{123})$$

$$3P_{ex} V = \frac{1}{2} \rho^2 \int \int dr_1 dr_2 g_{12}(\rho_{12})$$

$$+ \frac{1}{6} \rho^3 \int \int \int dr_1 dr_2 dr_3 g_{123}(\rho_{123})$$

(4)

with

$$w_{12} = -r_{12} \frac{\partial h_{12}}{\partial r_{12}}$$

$$w_{123} = -\left( r_{12} \frac{\partial}{\partial r_{12}} + r_{13} \frac{\partial}{\partial r_{13}} + r_{23} \frac{\partial}{\partial r_{23}} \right) u_{123}$$

and where $g_{12} = g(r_{12})$ and $g_{123} = g(r_{12}, r_{13}, r_{23})$ are the pair and three-body radial distribution functions, respectively. By writing $g_{123}$ in the following form (which defines $g'_{123}$)

$$g_{123} = g_{12} g_{13} g_{23}$$

we can evaluate the three-body part of Eq. (5) as

$$3 \times \frac{1}{2} \rho^3 \int \int dr_1 dr_2 g_{12}(\rho_{12}) \frac{\partial}{\partial r_{12}} \int dr_3 g_{13} g_{23}(\rho_{13}) u_{123}$$

(7)

where the factor of 3 follows from the three terms which constitute $w_{123}$. As a final result, we may write the excess pressure in the following form:

$$3P_{ex} V = \frac{1}{2} \rho^2 \int \int dr_1 dr_2 g_{12}(\rho_{12}) \frac{\partial}{\partial r_{12}} [u_{12} + \varphi_{12}]$$

(8)

where

$$\varphi_{12} = \varphi(r_{12})$$

$$\varphi_{12} = \varphi(r_{12}: \rho, T) = \rho \int dr_3 g_{13} g_{23}(\rho_{13}) u_{123}$$

(9)

In the subsequent we can rewrite the potential energy Eq. (4) as

$$U = \frac{1}{2} \rho^2 \int \int dr_1 dr_2 g_{12}(\rho_{12}) [u_{12} + \varphi_{12}]$$

(10)

with

$$\varphi_{12} = \varphi(r_{12}: \rho, T) = \frac{1}{3} \rho \int dr_3 g_{13} g_{23}(\rho_{13}) u_{123}$$

(11)

From Eqs. (9) and (11) follows that we may define two effective pair potentials $u_{12} + \varphi(r_{12}: \rho, T)$ and $u_{12} + \varphi(r_{12}: \rho, T)$, where the corrections to the true pair potential differ by a factor of 3:

$$\varphi(r_{12}: \rho, T) = \frac{1}{3} \rho \int dr_3 g_{13} g_{23}(\rho_{13}) u_{123}$$

(12)

and yield the correct three-body pressure and energy, respectively. In other words, the effective pair potential deduced from three-body interactions is not a unique quantity. In the following we will consider the three-body contribution to the effective pair potential, relevant for the energy. Note that such terms are defined as an average over the radial distribution functions and are therefore state dependent. In the limit of $r \to \infty$, the AT form may be used to show that $\varphi_{12}$ can be written as an explicit function of the distance and density:

$$\varphi(r_{12}: \rho, T) = f(r_{12}) = \frac{8 \pi}{9} C_0 \rho r^{-6}$$

(13)

### B. Effective pair potentials from the simulation

In Ref. 19 it is shown that the dispersion energy $e_{CPA}$ in our simulation model can be written in the following way:

$$e_{CPA}(\{c\}, \{r\}) = \sum_{ij} e_{ij}^{(2)} + \sum_{ij} e_{ij}^{(3)}$$

(14)

where $\Sigma_{ij} e_{ij}^{(2)}$ is a two-body type energy, and $\Sigma_{ij} e_{ij}^{(3)}$ is a three-body type energy. From Eq. (15) we can thus evaluate the three-body contributions to an effective pair potential in the simulation:

$$\phi'(r_{ij}: \rho, T) = \frac{1}{2} \sum_{ij} (\delta(r_{ij}) e_{ij}^{(3)} - 1) |_{\rho, T, hB}$$

(15)
where the subscript $3B$ indicates that this average is obtained over the course of an MD simulation in the presence of full three-body forces. We have to add a small correction $\Delta \phi$ to this term because of the fact that a strict separation into pure two- and three-body terms as indicated in Eq. (15) cannot be made in our variational equations, due to the CPA normalization factor which appears both in two- and three-body type terms; a detailed comparison with perturbation theory shows that also $e^{(3)}_{ij}$ contains some (small) three-body contributions. The corresponding correction $\Delta \phi$ can be estimated from a comparison with a true two-body run, i.e., by evaluating

$$
\Delta \phi(r; \rho, T) = \left( \frac{1}{2} \sum_{ij} \delta(r - r_{ij})e^{(2)}_{ij} \right)_{\rho, T, 3B} - \left( \frac{1}{2} \sum_{ij} \delta(r - r_{ij})e^{(2)}_{ij} \right)_{\rho, T, 2B}.
$$

The final three-body contribution to the effective pair potential measured is then given by

$$
\phi(r; \rho, T) = \phi'(r; \rho, T) + \Delta \phi(r; \rho, T).
$$

We note that the RDF is measured in the simulation via

$$
g(r) = \langle \sum_{ij} \delta(r - r_{ij}) \rangle / 2\pi r^2 N, \tag{16}
$$

from the same data as Eq. (16); therefore, by construction $\phi$ yields the exact three-body energy, when integrated over the radial distribution function, and is therefore directly comparable with the effective pair potential defined by Eq. (11). As shown in Ref. 19, DDD, DDQ appear as separate terms in $e^{(3)}_{ij}$ so the corresponding effective pair potentials measured via Eq. (16) might also be specified as

$$
\phi = \phi^{DDD} + \phi^{DDQ}.
$$

**C. Results for the effective pair potential**

In Fig. 3 we show results for the three-body correction to the effective pair potential at several densities and fixed temperature of 1.1. The results show $\phi$ normalized by the asymptotic form $f(r; \rho) = (8\pi\rho/3)C_6 r^{-6}$ and indicates a nontrivial density dependence due to the changing local liquid structure. The results may be compared with theoretical estimates of $\phi^{DDD}$ obtained by use of

$$
\beta \phi_{12} = -\frac{1}{3} \int d_{13} [\exp(-\beta u_{123}) - 1] g_{13} g_{23},
$$

which (without the factor $1/3$) yields the correct RDF to linear order in the density.\(^1\)\(^2\) Note that to lowest order in the expansion of the exponential, Eq. (20) is equivalent to Eq. (11), if the Kirkwood superposition approximation ($g_{123} = 1$) is used. Applications of (20) can be found in the literature, using approximate expressions for the pair distribution function. In Refs. 2, 4, $g(r)$ is estimated from $g(r) = \exp(-\beta u_{12}(r))$, i.e., the lowest density approximation. The resulting $\phi^{DDD}$ (dashed line) is compared with the simulation results in the figure, and seem to agree well with them at sufficiently low density ($\rho = 0.1$). In Ref. 6, $g(r)$ is calculated from a pair-hypernetted chain approximation. The result of this approximation at $\rho = 0.7$ (open squares) compares very well with the simulation results at the corresponding density.

Figure 4 illustrates the temperature dependence of $\phi^{DDD}$, which is seen to be very low at liquid densities and only to become appreciable at low densities where the temperature dependence of the liquid structure becomes noticeable.

**D. A thermodynamically consistent EPP**

For simulation or theory it might be of interest to construct an effective pair potential $\tilde{\phi}^{DDD}$ which is thermodynamically consistent, i.e., a single effective potential which reproduces both the correct three-body energy and three-body forces. We have to add a small correction $\Delta \phi$ to this term because of the fact that a strict separation into pure two- and three-body terms as indicated in Eq. (15) cannot be made in our variational equations, due to the CPA normalization factor which appears both in two- and three-body type terms; a detailed comparison with perturbation theory shows that also $e^{(3)}_{ij}$ contains some (small) three-body contributions. The corresponding correction $\Delta \phi$ can be estimated from a comparison with a true two-body run, i.e., by evaluating

$$
\Delta \phi(r; \rho, T) = \left( \frac{1}{2} \sum_{ij} \delta(r - r_{ij})e^{(2)}_{ij} \right)_{\rho, T, 3B} - \left( \frac{1}{2} \sum_{ij} \delta(r - r_{ij})e^{(2)}_{ij} \right)_{\rho, T, 2B}.
$$

The final three-body contribution to the effective pair potential measured is then given by

$$
\phi(r; \rho, T) = \phi'(r; \rho, T) + \Delta \phi(r; \rho, T).
$$

We note that the RDF is measured in the simulation via

$$
g(r) = \langle \sum_{ij} \delta(r - r_{ij}) \rangle / 2\pi r^2 N, \tag{16}
$$

from the same data as Eq. (16); therefore, by construction $\phi$ yields the exact three-body energy, when integrated over the radial distribution function, and is therefore directly comparable with the effective pair potential defined by Eq. (11). As shown in Ref. 19, DDD, DDQ appear as separate terms in $e^{(3)}_{ij}$ so the corresponding effective pair potentials measured via Eq. (16) might also be specified as

$$
\phi = \phi^{DDD} + \phi^{DDQ}.
$$

FIG. 3. Effective DDD pair potential $\phi^{DDD}(r)$ normalized by its functional form at infinity $f(r) = \frac{2\pi}{3} C_6 r^{-6}$, at a reduced temperature of 1.1. Solid lines: simulation results for densities ranging from 0.1 (top curve) to 0.9 (bottom curve), with an increment of 0.1. Dashed line: theoretical result obtained from Eq. (20) with the low-density estimate $g(r) = \exp(-\beta u_{12}(r))$ for the RDF (Ref. 4); open squares: theoretical result where the RDF is estimated from a pair HNC approximation at density 0.7 (Ref. 6).

FIG. 4. Simulation results for the effective DDD pair potential $\phi^{DDD}(r)$ normalized by its functional form at infinity $f(r) = \frac{2\pi}{3} C_6 r^{-6}$, at reduced temperatures of 1.1 (solid lines), 1.6 (dashed lines) and 2.0 (gray lines), at reduced densities 0.3 (left), 0.6 (middle) and 0.9 (right).
TABLE II. Ratio of the value of the $C_{9}^{\text{eff}}$ coefficient obtained from the pressure equation (22), to the value obtained from the energy equation (23), as a function of the power $N$ for three different densities, at a reduced temperature of $T^* = 1.1$. For all densities studied, the two values coincide at $N=9$, indicating the an effective pair potential of the form $r^{-9}$ would give a consistent three-body energy and pressure.

<table>
<thead>
<tr>
<th>Power $N$</th>
<th>$\rho^* = 0.3$</th>
<th>$C_{9}^{\text{eff},P} / C_{9}^{\text{eff},U}$</th>
<th>$\rho^* = 0.6$</th>
<th>$C_{9}^{\text{eff},P} / C_{9}^{\text{eff},U}$</th>
<th>$\rho^* = 0.9$</th>
<th>$C_{9}^{\text{eff},P} / C_{9}^{\text{eff},U}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1.495</td>
<td>1.487</td>
<td>1.484</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.288</td>
<td>1.282</td>
<td>1.278</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.128</td>
<td>1.123</td>
<td>1.120</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.003</td>
<td>0.999</td>
<td>0.996</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.903</td>
<td>0.899</td>
<td>0.896</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.821</td>
<td>0.817</td>
<td>0.815</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.752</td>
<td>0.749</td>
<td>0.747</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

body pressure, at the DDD level. As was suggested by Barker, such a potential should be a homogeneous function of the same degree ($-9$) in the coordinates as the Axilrod–Teller energy, i.e., we should try a potential

$$\tilde{\phi}_{\text{DDD}}(r; \rho, T) = C_{9}^{\text{eff}}(\rho, T)r^{-9}$$

(21)

and determine the $C_{9}^{\text{eff}}$ coefficient such that

$$U_{\text{DDD}} = \frac{1}{2} \rho^2 \int \int d\mathbf{r}_1 \, d\mathbf{r}_2 g_{12}^{\text{eff}} r^{-9}.$$  

(22)

$$3 \rho_{\text{ex}}^3 V = \frac{1}{2} \rho^2 \int \int d\mathbf{r}_1 \, d\mathbf{r}_2 g_{12}^{\text{eff}} C_{9}^{\text{eff}} r^{-9}.$$  

(23)

If the potential is thermodynamically consistent, the $C_{9}^{\text{eff}}$ coefficient determined from Eqs. (22) should be equivalent to the one determined from Eq. (23). This was found to be true for all densities and temperatures studied, as shown in Table II, in which we show the ratio of the coefficients of an effective potential of the form $r^{-N}$ determined both from the total DDD pressure ($C_{N}^{\text{eff},P}$) and the total DDD energy ($C_{N}^{\text{eff},U}$), as a function of the power $N$, for various reduced densities and temperature 1.1. It is shown that the two coefficients coincide at $N=9$, which was also found for all other densities and temperatures studied.

In Fig. 5 we show the resulting $C_{9}^{\text{eff}}$ determined from Eq. (22) [or Eq. (23)] as a function of the reduced density, for temperatures $T^* = 1.1$, $T^* = 1.6$ and $T^* = 2.0$. An extensive range of simulations would be required to determine a fit of the functional form of $C_{9}^{\text{eff}}(\rho, T)$; still, from the figure a simple estimate of the coefficient could be obtained in the density range $[0.0, 1.0]$ and the temperature range $[1.0, 2.0]$.

It should be stressed, however, that such a potential would only reproduce the exact three-body energy and pressure if averaged over the RDF of the full three-body system. In order to see how well this effective pair potential would reproduce the structure of the 3B-fluid, we have performed a simulation with a potential obtained by combining the Aziz and the effective $r^{-9}$ potential discussed above, at density $\rho^* = 0.3$ and temperature $T^* = 1.1$, and compared (Fig. 6) the result with those obtained from the Aziz potential alone and from the full three-body simulation (both as in Fig. 1). It can be seen that the Aziz + effective $r^{-9}$ potential reproduces the full three-body RDF very well, a much larger discrepancy is found with the pure Aziz potential. Nevertheless, the small difference in structure causes the three-body energy and pressure from this simulation to be slightly different from those of the full three-body calculation. In principle, the $C_{9}^{\text{eff}}$ coefficient could be re-evaluated in the two-body simulation via Eq. (22), the value of which could then be used in a new two-body simulation, etc., until convergence is reached.
progressively reduces the depth of the effective pair potential with increasing density, without shifting the position of the minimum, so that in the dense fluid the depth of the effective pair potential corresponds quite well with that of the Lennard-Jones potential. The relative importance of the DDD and DDQ effects is illustrated by the figure, which indicates a rapid convergence of the three-body effects on the energy [cf. Eq. (13)]. The Lennard-Jones potential, which gives a reasonable compromise potential for the pressure and energy, while agreeing well with the effective potential in the region of the repulsive wall and the minimum, is significantly deeper than the effective potentials at larger $r$ and closer to the pair potential there.

IV. DISCUSSION—RELATIONSHIP OF THE TRUE AND LENNARD-JONES PAIR POTENTIALS

We conclude by showing in Fig. 7 a comparison of the true pair potential [$u_{12}$, Eq. (3)], the much studied Lennard-Jones potential ($e/k=119.8$ K, $\sigma=0.3412$ nm) and the effective pair potentials $u_{12}+\varphi_{\text{DDD}}(r;\rho,T)$ and $u_{12}+\varphi_{\text{DDD}}(r;\rho,T)+\varphi_{\text{DDQ}}(r;\rho,T)$ at $\rho=0.9$ and $T=1.1$ (i.e., close to the triple point). Including the three-body effects reached; the difference with the present estimate for the $C_9^{\text{eff}}$ coefficient is expected to be small, however.

The $C_9^{\text{eff}} r^{-9}$ potential is not the true three-body correction to the effective pair potential; the latter is shown in Fig. 3, and will not fit to a simple functional form; the $r^{-9}$ potential should be viewed as a practical alternative for modeling three-body DDD interactions, reproducing the energy, pressure and RDF with reasonable accuracy. It should be emphasized that the $r^{-9}$ potential has been fitted to bulk thermodynamic properties, and will fail at surfaces and interfaces, as has been shown by Barker. In his paper, Barker demonstrated that the $r^{-9}$ potential results in a surface free energy which disagrees by as much as 19% (argon) to 35% (xenon) with the experimental data. Inclusion of the full three-body interaction brought the disagreement down to 2%.

FIG. 7. A comparison of the true pair potential (Aziz), the Lennard-Jones potential and the effective pair potentials at the DDD and DDD+DDQ levels for argon at $\rho^*=0.9$ and $T^*=1.1$. 