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Abstract

We discuss a moving boundary problem modeling tumor growth in in vitro tissue cultures. It is shown that the unique flat steady state solution is exponentially stable with respect to general (sufficiently smooth) perturbations. Furthermore, it is also shown that any solution to the problem becomes instantaneously real analytic in space and in time.

Key Words and Phrases: Moving boundary problem, elliptic-parabolic, asymptotic stability, tumor growth.

2000 Mathematics Subject Classification: 35R35, 35B32

1 Introduction

In this paper we consider the following moving boundary problem:

$$\left.\begin{array}{rcll}
\frac{1}{\beta}\sigma_t - \Delta\sigma & = & -\sigma & \text{in } \Omega_{\rho(t)}, \quad t > 0, \\
-\Delta p & = & \mu(\sigma - \widetilde{\sigma}) & \text{in } \Omega_{\rho(t)}, \quad t \geq 0, \\
\sigma & = & \overline{\sigma} & \text{on } \Gamma_{\rho(t)}, \quad t \geq 0, \\
V_n & = & -\partial_n p & \text{on } \Gamma_{\rho(t)}, \quad t > 0, \\
p & = & \gamma\kappa & \text{on } \Gamma_{\rho(t)}, \quad t \geq 0, \\
\sigma_y & = & 0 & \text{on } \Gamma_0, \\
p_y & = & 0 & \text{on } \Gamma_0,
\end{array}\right\} \tag{1.1}$$

where $\sigma = \sigma(t,x,y)$ and $p = p(t,x,y)$ are defined on the time-space manifold $\cup_{t\geq 0}(\{t\} \times \Omega_{\rho(t)})$ with a spatially periodic cell of the form

$$\Omega(t) := \Omega_{\rho(t)} := \{(x,y) \,|\, x \in \mathbb{T}^1, 0 < y < \rho(t,x)\}, \qquad t \geq 0.$$

Here $\mathbb{T}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ stands for the 1-dimensional torus and the function $\rho \in C(\mathbb{R}_+ \times \mathbb{T}^1)$ with $\rho(t,x) > 0$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^1$ describes the moving boundary

$$\Gamma(t) := \Gamma_{\rho(t)} := \{(x,y) \,|\, x \in \mathbb{T}^1, y = \rho(t,x)\}, \qquad t \geq 0$$

of the system (1.1). The other component of the boundary of $\Omega_{\rho(t)}$ is denoted by

$$\Gamma_0 := \{(x,y)\,|\,x\in\mathbb{T}^1, y=0\}.$$

It is obviously independent of the time variable t. In the above system, $\beta,\gamma,\overline{\sigma},\widetilde{\sigma},\mu > 0$ are positive constants, ∂_n denotes the outer normal derivative, and κ is the curvature of Γ_ρ, taken negative where ρ is convex. Moreover, V_n denotes the normal velocity of the moving boundary $t\mapsto\Gamma_{\rho(t)}$. The model has to be completed by describing the initial domain via a given positive function $\rho_0\in C(\mathbb{T}^1)$ and a corresponding initial value σ_0 defined on Ω_{ρ_0}. i.e., we further impose

$$\rho(0,\cdot)=\rho_0 \quad \text{on} \quad \mathbb{T}^1, \qquad \sigma(0,\cdot,\cdot)=\sigma_0 \quad \text{in} \quad \Omega_{\rho_0}. \tag{1.2}$$

The above system is the mathematical formulation of in vitro tissue culture models describing tumor growth, cf. [15, 16, 18]. In this model $\Omega(t)$ stands for the domain occupied by the tumor at time t. Moreover, σ and p represent the concentration of a nutrient (e.g. oxygen or glucose) diffusing inside the tumor body and the internal pressure distribution. The number β represents the ratio of the cell-tumor doubling timescale (typically ≈ 1 day) and nutrient diffusion timescale (≈ 1 minute). The consumption rate is normalized to the value 1. In the second equation of (1.1) μ stands for the proliferation rate of the system, while the constant $\widetilde{\sigma}$ plays the role of a threshold value for the tumor cell proliferation: In regions where $\sigma>\widetilde{\sigma}$ there is sufficient nutrient available so that the tumor grows there, while in regions with $\sigma<\widetilde{\sigma}$ there is not enough nutrient to sustain tumor cells alive and the tumor volume is decreasing. It is assumed that there is a constant supply of nutrient through the moving upper part $\Gamma(t)$ of the boundary. Writing $\overline{\sigma}$ for the rate of this supply, we get the first boundary condition on $\Gamma(t)$. The motion of the free interface is governed by Darcy's law which leads to the second boundary condition on $\Gamma(t)$. Furthermore, surface tension effects counteract the internal pressure, which is modelled by the third boundary condition on $\Gamma(t)$, where γ stands for the surface tension coefficient. Finally, the last two conditions reflect the fact that neither nutrient nor tumor cells can pass through the lower boundary.

In the limit case $\beta\to\infty$ we get the following quasi-stationary approximation of (1.1):

$$\left.\begin{array}{rcll}
\Delta\sigma &=& \sigma & \text{in } \Omega_{\rho(t)}, \quad t>0,\\
-\Delta p &=& \mu(\sigma-\widetilde{\sigma}) & \text{in } \Omega_{\rho(t)}, \quad t\geq 0,\\
\sigma &=& \overline{\sigma} & \text{on } \Gamma_{\rho(t)}, \quad t\geq 0,\\
V_n &=& -\partial_n p & \text{on } \Gamma_{\rho(t)}, \quad t>0,\\
p &=& \gamma\kappa & \text{on } \Gamma_{\rho(t)}, \quad t\geq 0,\\
\sigma_y &=& 0 & \text{on } \Gamma_0,\\
p_y &=& 0 & \text{on } \Gamma_0,\\
\rho(0,\cdot) &=& \rho_0 & \text{on } \mathbb{T}^1,
\end{array}\right\} \tag{1.3}$$

which was studied in [5] and [7]. In particular, dynamical properties of equilibria to (1.3) have been investigated. In this connection, a triple $(\rho^\star,\sigma^\star,p^\star)$ is called an equilibrium to (1.3) if it satisfies the free boundary problem obtained from

(1.3) by replacing the boundary condition $V_n = -\partial_n p$ on the interface $\Gamma(t)$ by the condition $\partial_n p = 0$. It was shown in [5] that (1.3) possesses equilibria only if $\overline{\sigma} > \widetilde{\sigma}$ and that there exists a unique flat one (i.e. an equilbrium where $\rho^\star$ is constant and consequently $\sigma^\star$ and $p^\star$ are independent of x). Moreover this flat equlibrium is asymptotically stable with respect to (small) perturbations ρ_0 of class $h^{4+\alpha}$, provided the surface tension coefficient γ is large enough, cf. Theorem 1.1 in [5]. Here, $h^{m+\alpha}$ denotes the scale of little Hölder spaces, see Section 2 for a definition.

In the present paper we study the situation when β is finite. We show in our main result that if γ/μ is sufficiently large then the unique flat equilibriuim of (1.1), (1.2) is asymptotically stable with respect to perturbations (ρ_0, σ_0) belonging to the space $h^{4+\alpha} \times W^{2,q}$. The main technical tool here is the principle of linearized stability whose applicability is based on maximal regularity results for the linearized problem.

Clearly, this is in accordance with the stability result obtained in [5] in the limit case $\beta \to \infty$ for large values of γ. Asymptotic stability results of spherically symmetric stationary solutions for a similar but different tumor model have been obtained for a quasi-stationary approximation with large γ in [6] and for small proliferation rates μ in [13].

Due to the parabolic character of the evolution we get a smoothing of the boundary up to analyticity. To prove this result, we apply a technique originally due to Angenent ([1, 2], see also [9, 12] for related results) which allows to conclude this result from the Implicit Function theorem in Banach spaces and the invariance of the nonlinear problem under spatial translations. The functional analytic framework for this is again given by maximal regularity results for the linearization.

2 Transformations and abstract setting

Fix $\alpha \in (0,1)$ and $q > 2/(2-\alpha)$. Throughout this paper we use the symbol $h^{m+\alpha}$ with $m \in \mathbb{N}$ to indicate a little Hölder space of class $m+\alpha$, i.e. the closure of the space of smooth functions with respect to the usual $C^{m+\alpha}$-Hölder norm. Standard Sobolev spaces are denoted by $W^{m,q}$.

In order to transform our moving boundary problem (1.1) to a nonlinear problem on a fixed domain we introduce the following notation:

$$\Omega := \mathbb{T}^1 \times (0,1), \quad \Gamma_i := \mathbb{T}^1 \times \{i\},\ i = 0, 1,$$
$$h_+^{l+\alpha}(\mathbb{T}^1) := \{\rho \in h^{l+\alpha}(\mathbb{T}^1) \,|\, \rho > 0\},\ l = 3, 4.$$

Moreover, we will denote the trace operator from (function spaces on) Ω to (function spaces on) Γ_i by tr_i. For $\rho \in h_+^{4+\alpha}(\mathbb{T}^1)$ we define $\Theta_\rho \in \mathrm{Diff}^{4+\alpha}(\Omega, \Omega_\rho)$ by (cf. [5])

$$\Theta_\rho(x, y) = (x, \rho(x)y)$$

and introduce the transformed differential operators

$$\mathcal{A}(\rho)v \quad := \quad \Theta_\rho^* \Delta \Theta_{\rho_*} v,$$

$$\mathcal{B}(\rho)v \quad := \quad \Theta_\rho^*(\mathrm{tr}\nabla\Theta_{\rho_*}v\cdot\nu_\rho),$$

where tr denotes the trace operator from Ω_ρ to Γ_ρ and $\nu_\rho := (-\rho', 1)$ is an exterior normal to Γ_ρ.

It is easy to check that these operators have explicit representations given by (cf. [5])

$$\mathcal{A}(\rho)v = v_{xx} - 2y\rho'\rho^{-1}v_{xy} + (1+(y\rho')^2)\rho^{-2}v_{yy} + y(2(\rho')^2 - \rho\rho'')\rho^{-2}v_y \tag{2.1}$$

$$\mathcal{B}(\rho)v = -(\mathrm{tr}_1 v_x - \rho'\rho^{-1}\mathrm{tr}_1 v_y)\rho' + \rho^{-1}\mathrm{tr}_1 v_y \tag{2.2}$$

and that (with obvious identifications)

$$(\mathcal{A},\mathcal{B}) \in C^\omega(h_+^{l+\alpha}(\mathbb{T}^1), \mathcal{L}(h^{k+\alpha}(\Omega), h^{k-2+\alpha}(\Omega)\times h^{k-1+\alpha}(\Gamma_1))), \tag{2.3}$$

$l = 3, 4$, $k = 2, \ldots, l$. Recalling that the curvature κ of Γ_ρ is given by

$$\kappa = -\rho''(1+(\rho')^2)^{-3/2}$$

we are able to reformulate (1.1) as

$$\left.\begin{array}{rcll} \widehat{\sigma}_t - \beta\mathcal{A}(\rho)\widehat{\sigma} &=& -\beta(\widehat{\sigma} - \overline{\sigma} + \mathcal{R}(\rho)[\widehat{\sigma},\widehat{p}]) & \text{in } \Omega, \\ -\mathcal{A}(\rho)\widehat{p} &=& \mu(\widehat{\sigma} + \overline{\sigma} - \widetilde{\sigma}) & \text{in } \Omega, \\ \widehat{\sigma} &=& 0 & \text{on } \Gamma_1, \\ \rho_t &=& -\mathcal{B}(\rho)\widehat{p} & \text{on } \Gamma_1, \\ \widehat{p} &=& -\gamma\rho''(1+(\rho')^2)^{-3/2} & \text{on } \Gamma_1, \\ \widehat{\sigma}_y &=& 0 & \text{on } \Gamma_0, \\ \widehat{p}_y &=& 0 & \text{on } \Gamma_0 \end{array}\right\} \tag{2.4}$$

with $\widehat{\sigma} := \theta_\rho^*\sigma - \overline{\sigma}$, $\widehat{p} := \theta_\rho^* p$, and

$$\mathcal{R}(\rho)[\widehat{\sigma},\widehat{p}](x,y) = \widehat{\sigma}_y(x,y)y\rho^{-1}(x)(\mathcal{B}(\rho)\widehat{p})(x).$$

For the sake of simplicity, we will abuse notation and write σ, p instead of $\widehat{\sigma}, \widehat{p}$.

For $\rho \in h_+^{3+\alpha}(\mathbb{T}^1)$, the triple $(-\mathcal{A}(\rho), \mathrm{tr}_1, \mathrm{tr}_0\partial_y)$ constitutes a regular elliptic boundary system which provides an isomorphism between appropriate function spaces. In particular, we can define solution operators $\mathcal{S}$ and $\mathcal{T}$ by (cf. [5])

$$\begin{aligned} \mathcal{S}(\rho) &:= (-\mathcal{A}(\rho), \mathrm{tr}_1, \mathrm{tr}_0\partial_y)^{-1}(\cdot, 0, 0) \\ \mathcal{T}(\rho) &:= (-\mathcal{A}(\rho), \mathrm{tr}_1, \mathrm{tr}_0\partial_y)^{-1}(0, \cdot, 0) \end{aligned}$$

for which we have

$$\mathcal{S} \in C^\omega\big(h_+^{l+\alpha}(\mathbb{T}^1), \mathcal{L}(W^{k,q}(\Omega), W^{k+2,q}(\Omega))\big),\ l = 3,4,\ k = 0,\ldots,l-2, \tag{2.5}$$

$$\mathcal{T} \in C^\omega\big(h_+^{l+\alpha}(\mathbb{T}^1), \mathcal{L}(h^{k+\alpha}(\Gamma_1), h^{k+\alpha}(\Omega))\big),\quad l = 3,4,\ k = 2,\ldots,l. \tag{2.6}$$

Using these operators, we can write

$$p = p(\rho,\sigma) = \mu\mathcal{S}(\rho)(\sigma + \overline{\sigma} - \widetilde{\sigma}) - \gamma\mathcal{T}(\rho)\left[\rho''\left(1+\rho'^2\right)^{-3/2}\right] \tag{2.7}$$

and reformulate our moving boundary problem as an evolution equation for the pair (ρ, σ):

$$\partial_t \begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \mathcal{F}(\rho, \sigma) := \begin{pmatrix} -\mathcal{B}(\rho)p \\ \beta\left(\mathcal{A}(\rho)\sigma - \sigma - \overline{\sigma} + \mathcal{R}(\rho)[\sigma, p]\right) \end{pmatrix}. \tag{2.8}$$

We introduce the following setting to formulate and prove our main result: Define the function spaces

$$\begin{aligned} W_B^{2,q}(\Omega) &:= \{v \in W^{2,q}(\Omega) \,|\, \mathrm{tr}_0 v_y = 0,\ \mathrm{tr}_1 v = 0\}, \\ E_0 &:= h^{1+\alpha}(\mathbb{T}^1) \times L^q(\Omega), \\ E_1 &:= h^{4+\alpha}(\mathbb{T}^1) \times W_B^{2,q}(\Omega), \end{aligned}$$

and the set

$$E_1^+ := \{(\rho, \sigma) \in E_1 \,|\, \rho > 0\}$$

which is open in E_1.

Our first result concerns the smoothness of $\mathcal{F}$ with respect to the chosen spaces.

Lemma 2.1 *(Smoothness of $\mathcal{F}$)*
We have

$$\mathcal{F} \in C^\omega\left(E_1^+, E_0\right).$$

Proof: Interpreting p via (2.7) as an operator acting on (ρ, σ), we find from (2.5), (2.6) and the continuity of the embedding $W^{4,q}(\Omega) \hookrightarrow h^{2+\alpha}(\Omega)$

$$[(\rho, \sigma) \mapsto p] \in C^\omega\left(E_1^+, h^{2+\alpha}(\Omega)\right),$$

and, consequently, from (2.3)

$$[(\rho, \sigma) \mapsto \mathcal{B}(\rho)p] \in C^\omega\left(E_1^+, h^{1+\alpha}(\mathbb{T}^1)\right).$$

This proves the smoothness of the first component of $\mathcal{F}$ and

$$[(\rho, \sigma) \mapsto \mathcal{R}(\rho)[\sigma, p]] \in C^\omega\left(E_1^+, L^q(\Omega)\right).$$

Together with the fact that

$$\mathcal{A} \in C^\omega\left(h_+^{4+\alpha}(\mathbb{T}^1), \mathcal{L}(W^{2,q}(\Omega), L^q(\Omega))\right),$$

this implies the smoothness of the second component of $\mathcal{F}$. ∎

3 Linearization

Given $(\rho, \sigma) \in E_1^+$, we write

$$A := \mathcal{F}'(\rho, \sigma)$$

for the Fréchet derivative of $\mathcal{F}$ at (ρ, σ). If X and Y are Banach spaces, we will denote the class of compact linear operators from X to Y by $\mathcal{K}(X, Y)$.

Lemma 3.1 *(The linearization)*
Given $(h,k) \in E_1$, *we have*

$$A\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

with

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^0 & 0 \\ A_{21} & A_{22}^0 \end{pmatrix} + K, \qquad K \in \mathcal{K}(E_1, E_0), \tag{3.1}$$

$$\begin{aligned} A_{11}^0 h &:= \gamma \mathcal{B}(\rho)\mathcal{T}(\rho)[(1+{\rho'}^2)^{-3/2}h''], \\ A_{22}^0 k &:= \beta\left(\mathcal{A}(\rho)k - k\right). \end{aligned}$$

Proof: Note at first that from (2.1), (2.2) we get

$$\begin{aligned} \mathcal{A}'(\rho)[h] &= -2y(\rho^{-1}h' - \rho'\rho^{-2}h)v_{xy} \\ &\quad +(2y^2\rho'h'\rho^{-2} - 2(1+(y\rho')^2)\rho^{-3})v_{yy} \\ &\quad +y((-\rho h'' + 4\rho'h' - \rho''h)\rho^{-2} - 2(2(\rho')^2 - \rho\rho'')\rho^{-3}h)v_y, \quad (3.2) \\ \mathcal{B}'(\rho)[h] &= -(\mathrm{tr}_1 v_x - \rho'\rho^{-1}\mathrm{tr}_1 v_y)h' + (\rho^{-1}h' - \rho'\rho^{-2}h)\rho'\mathrm{tr}_1 v_y \\ &\quad -\rho^{-2}h\,\mathrm{tr}_1 v_y. \quad (3.3) \end{aligned}$$

To differentiate the solution operator $\mathcal{S}$, note that $u(\rho) = \mathcal{S}(\rho)f$ means

$$\left.\begin{aligned} -\mathcal{A}(\rho)u(\rho) &= f && \text{in } \Omega, \\ u(\rho) &= 0 && \text{on } \Gamma_1, \\ \partial_y u(\rho) &= 0 && \text{on } \Gamma_0. \end{aligned}\right\}$$

Differentiation with respect to ρ yields

$$\left.\begin{aligned} -\mathcal{A}(\rho)u'(\rho)[h] &= \mathcal{A}'(\rho)[h]u(\rho) && \text{in } \Omega, \\ u'(\rho)[h] &= 0 && \text{on } \Gamma_1, \\ \partial_y u'(\rho)[h] &= 0 && \text{on } \Gamma_0. \end{aligned}\right\}$$

Hence

$$\mathcal{S}'(\rho)[h]f = \mathcal{S}(\rho)(\mathcal{A}'(\rho)[h]\mathcal{S}(\rho)f). \tag{3.4}$$

Analogously,

$$\mathcal{T}'(\rho)[h]g = \mathcal{S}(\rho)(\mathcal{A}'(\rho)[h]\mathcal{T}(\rho)g). \tag{3.5}$$

Straightforward calculations yield

$$A_{11}h = -\partial_\rho(\mathcal{B}(\rho)p(\rho,\sigma))[h] = A_{11}^0 h + K_{11}h,$$

where

$$\begin{aligned} K_{11}h &= \gamma\mathcal{B}(\rho)\mathcal{T}'(\rho)[h](\rho''(1+{\rho'}^2)^{-3/2}) - \mathcal{B}'(\rho)[h]p(\rho,\sigma) \\ &\quad -3\gamma\mathcal{B}(\rho)\mathcal{T}(\rho)(\rho''(1+{\rho'}^2)^{-5/2}\rho'h') \\ &\quad -\mu\mathcal{B}(\rho)\mathcal{S}'(\rho)[h](\sigma+\overline{\sigma}-\widetilde{\sigma}). \end{aligned} \tag{3.6}$$

As $\rho > 0$ we have $\rho''(1+\rho'^2)^{-3/2}$, $\rho''(1+\rho'^2)^{-5/2}\rho' \in h^{2+\alpha}(\mathbb{T}^1)$, and therefore, by (3.5), the mapping properties of $\mathcal{T}$ and $\mathcal{S}$, the Banach algebra property of $h^{2+\alpha}(\mathbb{T}^1)$, and the compactness of the embedding $h^{3+\alpha}(\mathbb{T}^1) \hookrightarrow\!\!\hookrightarrow h^{2+\alpha}(\mathbb{T}^1)$ we get

$$\begin{aligned} h \mapsto \mathcal{T}'(\rho)[h](\rho''(1+\rho'^2)^{-3/2}) &\in \mathcal{K}(h^{4+\alpha}(\mathbb{T}^1), h^{2+\alpha}(\Omega)), \\ h \mapsto \mathcal{T}(\rho)(\rho''(1+\rho'^2)^{-5/2}\rho' h') &\in \mathcal{K}(h^{4+\alpha}(\mathbb{T}^1), h^{2+\alpha}(\Omega)). \end{aligned}$$

Applying now (2.3) we find that the first an the third term on the right in (3.6) are in $\mathcal{K}(h^{4+\alpha}(\mathbb{T}^1), h^{1+\alpha}(\mathbb{T}^1))$. By similar arguments for the others we get $K_{11} \in \mathcal{K}(h^{4+\alpha}(\mathbb{T}^1), h^{1+\alpha}(\mathbb{T}^1))$.

Furthermore,

$$A_{12}k = K_{12}k = -\mu\mathcal{B}(\rho)\mathcal{S}(\rho)k,$$

and from (2.5), the compact embedding $W^{4,q}(\Omega) \hookrightarrow\!\!\hookrightarrow h^{2+\alpha}(\Omega)$ and (2.3) we get

$$A_{12} = K_{12} \in \mathcal{K}(W^{2,q}_B, h^{1+\alpha}(\mathbb{T}^1)). \tag{3.7}$$

Finally,

$$A_{22} = A^0_{22} + K_{22},$$

where

$$K_{22}k = \mathcal{R}(\rho)[k, p(\rho,\sigma)] + \mu\mathcal{R}(\rho)[\sigma, \mathcal{S}(\rho)k],$$

and by parallel arguments $K_{22} \in \mathcal{K}(W^{2,q}_B, L^q(\Omega))$. Summarizing,

$$K = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix} \in \mathcal{K}(E_1, E_0)$$

as stated. ■

For a pair (X_0, X_1) of Banach spaces we will write $X_1 \overset{d}{\hookrightarrow} X_0$ iff X_1 is continuously and densely embedded in X_0. Note that $h^{4+\alpha}(\mathbb{T}^1) \overset{d}{\hookrightarrow} h^{1+\alpha}(\mathbb{T}^1)$ and $W^{2,q}_B(\Omega) \overset{d}{\hookrightarrow} L^q(\Omega)$, therefore $E_1 \overset{d}{\hookrightarrow} E_0$. If $X_1 \overset{d}{\hookrightarrow} X_0$, we denote by $\mathcal{H}(X_1, X_0)$ the class of operators $A \in \mathcal{L}(X_1, X_0)$ such that $-A$, considered as a densely defined operator in X_0 with domain X_1, is the generator of a strongly continuous analytic semigroup on X_0.

Lemma 3.2 *(Generator property of A)*
We have

$$-A \in \mathcal{H}(E_1, E_0).$$

Proof: It has been shown in [11], Theorem 4.1,and in [3], Theorem 4.1, respectively, that

$$-A^0_{11} \in \mathcal{H}(h^{4+\alpha}(\mathbb{T}^1), h^{1+\alpha}(\mathbb{T}^1)), \quad -A_{22} \in \mathcal{H}(W^{2,q}_B(\Omega), L^q(\Omega)).$$

Therefore, by [4], Theorem I.1.6.1,

$$-\begin{pmatrix} A^0_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{H}(E_1, E_0). \tag{3.8}$$

Together with (3.1) this implies the assertion as $\mathcal{H}(E_1, E_0)$ is stable under perturbations from $\mathcal{K}(E_1, E_0)$. ■

4 Stability of flat equilibria

The stationary problem corresponding to the evolution equation (2.8) has been investigated in [5]. It has been proved there that equilibria exist only if $\overline{\sigma} > \widetilde{\sigma}$. In this case, there is precisely one flat equilibrium (i.e. an equilibrium where ρ is constant and consequently σ and p are independent of x) which is given by

$$\rho = \rho^\star, \quad \sigma^\star(y) = \overline{\sigma}\left(\frac{\cosh(\rho^\star y)}{\cosh \rho^\star} - 1\right), \quad p^\star(y) = \mu\left(\frac{\widetilde{\sigma}{\rho^\star}^2}{2}(y^2-1) - \sigma^\star(y)\right), \tag{4.1}$$

where $\rho^\star$ is the (unique) positive constant satisfying

$$\frac{\tanh \rho^\star}{\rho^\star} = \frac{\widetilde{\sigma}}{\overline{\sigma}}.$$

Note that $\rho^\star$ and $\sigma^\star$ are both independent of μ and γ.

In this section we assume $\overline{\sigma} > \widetilde{\sigma}$.

Theorem 4.1 *(Exponential stability of equilibria)*

Let $\widetilde{\sigma}$, $\overline{\sigma} \in (0,\infty)$ with $\widetilde{\sigma} < \overline{\sigma}$ be fixed. Let further β, γ, $\mu \in (0,\infty)$ be given.

There exists a positive constant q_0 depending only on $\widetilde{\sigma}, \overline{\sigma}$ such that if

$$\gamma/\mu > q_0$$

then there is a $\delta > 0$ such that for any $(\rho_0, \sigma_0) \in E_1$ satisfying

$$\|\rho_0 - \rho^\star\|_{h^{4+\alpha}(\mathbb{T}^1)} + \|\sigma_0 - \sigma^\star\|_{W^{2,q}(\Omega)} < \delta,$$

the evolution problem

$$\partial_t(\rho(t),\sigma(t))^T = \mathcal{F}(\rho(t),\sigma(t)), \quad (\rho(0),\sigma(0)) = (\rho_0,\sigma_0) \tag{4.2}$$

has precisely one solution

$$(\rho(\cdot),\sigma(\cdot)) \in C^1([0,\infty),E_0) \cap C([0,\infty),E_1).$$

It satisfies an exponential decay estimate

$$\begin{aligned} &\|\rho(t) - \rho^\star\|_{h^{4+\alpha}(\mathbb{T}^1)} + \|\sigma(t) - \sigma^\star\|_{W^{2,q}(\Omega)} \\ \leq\ & Ce^{-\zeta t}\left(\|\rho_0 - \rho^\star\|_{h^{4+\alpha}(\mathbb{T}^1)} + \|\sigma_0 - \sigma^\star\|_{W^{2,q}(\Omega)}\right). \end{aligned}$$

with some $\zeta > 0$, $C > 0$.

This theorem will be proved by applying the principle of linearized stability. For this purpose, we have to identify the linearization of $\mathcal{F}$ at the equilibrium $(\rho^\star, \sigma^\star)$ and to investigate its spectral properties.

Let

$$A^\star := \mathcal{F}'(\rho^\star, \sigma^\star).$$

As $\rho^\star$, $\sigma^\star$, and $p^\star$ are all independent of x, the structure of the linearization at $(\rho^\star, \sigma^\star)$ is simpler than in the general case. Using the differential equations satisfied by $\sigma^\star$ and $p^\star$, we get

$$\mathcal{A}'(\rho^\star)[h]\sigma^\star = -\frac{2h}{\rho^\star}(\sigma^\star + \overline{\sigma}) - \frac{y}{\rho^\star}\sigma_y^* h'', \tag{4.3}$$

$$\mathcal{A}'(\rho^\star)[h]p^\star = \frac{2\mu h}{\rho^\star}(\sigma^\star + \overline{\sigma} - \widetilde{\sigma}) - \frac{y}{\rho^\star}p_y^\star h''. \tag{4.4}$$

The boundary condition $\mathcal{B}(\rho^\star)p^\star = 0$ implies

$$\mathcal{B}'(\rho)[h]p^\star = 0, \tag{4.5}$$

$$\mathcal{R}(\rho^\star)[\sigma, p^\star] = 0, \tag{4.6}$$

$$\mathcal{R}'(\rho^\star)[h][\sigma, p^\star] = 0. \tag{4.7}$$

Eqns. (3.4) and (4.4) imply

$$\partial_1 p(\rho^\star, \sigma^\star)[h] = \mathcal{S}(\rho^\star)\left(\frac{2\mu h}{\rho^\star}(\sigma^\star + \overline{\sigma} - \widetilde{\sigma}) - \frac{y}{\rho^\star}p_y^\star h''\right) - \gamma\mathcal{T}(\rho^\star)h''.$$

Now using(4.3) and (4.5)–(4.7), we straightforwardly get

$$A^\star = \begin{pmatrix} A_{11}^\star & A_{12}^\star \\ A_{21}^\star & A_{22}^\star \end{pmatrix}$$

with

$$\begin{aligned}
A_{11}^\star h &= -\mathcal{B}(\rho^\star)\left(\mathcal{S}(\rho^\star)\left(\frac{2\mu}{\rho^\star}(\sigma^\star + \overline{\sigma} - \widetilde{\sigma})h - \frac{y}{\rho^\star}p_y^\star h''\right) - \gamma\mathcal{T}(\rho^\star)h''\right), \\
A_{12}^\star k &= -\mu\mathcal{B}(\rho^\star)\mathcal{S}(\rho^\star)k, \\
\frac{1}{\beta}A_{21}^\star h &= -\frac{2h}{\rho^\star}(\sigma^\star + \overline{\sigma}) - \frac{y}{\rho^\star}\sigma_y^* h'' + \frac{y\sigma_y^*}{\rho^\star}A_{11}^\star h, \\
\frac{1}{\beta}A_{22}^\star k &= k_{xx} + \frac{1}{\rho^{\star 2}}k_{yy} - k.
\end{aligned}$$

Lemma 4.2 *(The spectrum of $A_{11}^\star$)*

(i) *The spectrum of $A_{11}^\star$, considered as an (unbounded) operator in $h^{1+\alpha}(\mathbb{T}^1)$ with $D(A_{11}^\star) = h^{4+\alpha}(\mathbb{T}^1)$ consists of real eigenvalues.*

(ii) *$0 \in \sigma(A_{11}^\star)$, and the corresponding eigenspace is given precisely by the constants.*

(iii) *The nonzero eigenvalues can be arranged in a sequence $\lambda_1, \lambda_2, \ldots$ such that*

$$\lambda_l \leq -\gamma C_1 l^3 + \mu C_2 l^2, \qquad l \geq 1 \tag{4.8}$$

with constants $C_{1,2}$ independent of γ, μ, and l.

Proof: It follows from the proof of Lemma 3.2 that

$$-A^\star_{11} \in \mathcal{H}(h^{4+\alpha}(\mathbb{T}^1), h^{1+\alpha}(\mathbb{T}^1)).$$

As $h^{4+\alpha}(\mathbb{T}^1) \hookrightarrow\hookrightarrow h^{1+\alpha}(\mathbb{T}^1))$, $A^\star_{11}$ has compact resolvent, so its spectrum consists only of eigenvalues.

As in [5] we are going to work with Fourier representations. We represent $g \in L^2(\mathbb{T}^1)$ as

$$g(x) = g_0 + \sum_{l\geq 1} g_l(x), \quad g_l(x) = g^0_l \cos(lx) + g^1_l \sin(lx). \tag{4.9}$$

Then it is straightforward to calculate that

$$\begin{aligned} g'' &= -\sum_{l\geq 1} l^2 g_l, \\ (\mathcal{T}(\rho^\star)g)(x,y) &= g_0 + \sum_{l\geq 1} g_l(x)\frac{\cosh(l\rho^\star y)}{\cosh(l\rho^\star)}, \end{aligned}$$

and, with an arbitrary integrable function f on $[0,1]$,

$$\begin{aligned} (\mathcal{S}(\rho^\star)[fg])(x,y) &= g_0{\rho^\star}^2 \left(-\int_0^y f(\eta)(y-\eta)\,d\eta + \int_0^1 f(\eta)(1-\eta)\,d\eta\right) \\ &+\sum_{l\geq 1} g_l(x)\frac{\rho^\star}{l}\left(-\int_0^y f(\eta)\sinh(l\rho^\star(y-\eta))\,d\eta \right.\\ &\qquad \left. +\frac{\cosh(l\rho^\star y)}{\cosh(l\rho^\star)}\int_0^1 f(\eta)\sinh(l\rho^\star(1-\eta))\,d\eta\right). \end{aligned}$$

Moreover, in the notation (4.9),

$$A^\star_{11} g_0 = -\frac{2g_0}{{\rho^\star}^2}(\partial_y \mathcal{S}(\rho^\star)[\mu(\sigma^\star + \overline{\sigma} - \widetilde{\sigma})])|_{y=1} = -\frac{2g_0}{\rho^\star}{p^\star}'(1) = 0.$$

Consequently,

$$A^\star_{11} g = \sum_{l\geq 1} \lambda_l g_l, \tag{4.10}$$

where

$$\begin{aligned} \lambda_l &= -\gamma l^3 \tanh(\rho^\star l) + \mu\widetilde{\lambda}_l = -\gamma(l^3\tanh(\rho^\star l) - (\gamma/\mu)^{-1}\widetilde{\lambda}_l), \\ \widetilde{\lambda}_l &= \int_0^1 T_l(\eta)\left(2(\sigma^\star(\eta) + \overline{\sigma} - \widetilde{\sigma}) + l^2(\widetilde{\sigma}{\rho^\star}^2\eta^2 - \eta\sigma^\star_\eta(\eta))\right)\,d\eta, \\ T_l(\eta) &= \cosh(l\rho^\star(1-\eta)) - \tanh(l\rho^\star)\sinh(l\rho^\star(1-\eta)). \end{aligned}$$

As the T_l are uniformly bounded independently of l, we get $\widetilde{\lambda}_l = O(l^2)$ for l large. This proves the estimate (4.8).

From (4.10) we find that the eigenvalues of $A^\star_{11}$ (as an operator in $L^2(\mathbb{T}^1)$) are given by $\{0, \lambda_1, \lambda_2, \ldots\}$. Any $h^{1+\alpha}$-eigenfunction of $A^\star_{11}$ is also an L^2-eigenfunction, hence our assertions follow. ■

Remark: Actually, we even have $\widetilde{\lambda}_l = O(1)$, but we refrain from performing the necessary calculations as we do not need this here. ■

In analogy to (4.9) we write for $k \in L^2(\Omega)$

$$k(x,y) = k_0(y) + \sum_{l\geq 1} k_l(x,y), \quad k_l(x,y) = k_l^0(y)\cos(lx) + k_l^1(y)\sin(lx).$$

As a further preparation for the estimation of the eigenvalues of $A^\star$ we introduce on E_1 the inner product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_M$ given (with obvious notation) by

$$\begin{aligned}
\left\langle \begin{pmatrix} h \\ k \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_M &:= M\langle h,\phi\rangle^\Gamma + \langle k,\psi\rangle^\Omega, \\
\langle h,\phi\rangle^\Gamma &:= h_0\phi_0 + \sum l^2 h_l^i \phi_l^i, \\
\langle k,\psi\rangle^\Omega &:= \int_0^1 \left(b k_0 \psi_0 + \sum k_l^i \psi_l^i \right) dy,
\end{aligned}$$

where

$$b = b(y) = \frac{M\mu {\rho^\star}^2}{2\beta(\sigma^\star(y)+\overline{\sigma})} > \delta > 0$$

and M is a positive constant to be determined later. If there is no other indication, sums have to be taken over $l \geq 1$ and $i = 0, 1$ here and in the sequel.

Lemma 4.3 *(Degenerate positivity of $-A^\star$)*
There is a $q_0 > 0$ such that if $\gamma/\mu > q_0$ then there are positive constants M and c such that

$$\left\langle A^\star \begin{pmatrix} h \\ k \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \right\rangle_M \leq -c\|k\|^2_{H^1(\Omega)} \qquad (h,k) \in E_1.$$

Proof: 1. We have

$$\begin{aligned}
\left\langle A^\star \begin{pmatrix} h \\ k \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \right\rangle_M &= M\langle A^\star_{11}h,h\rangle^\Gamma + M\langle A^\star_{12}k,h\rangle^\Gamma \\
&\quad + \langle A^\star_{21}h,k\rangle^\Omega + \langle A^\star_{22}k,k\rangle^\Omega,
\end{aligned}$$

with

$$\begin{aligned}
\langle A^\star_{11}h,h\rangle^\Gamma &= \sum l^2 \lambda_l {h_l^i}^2, \\
\langle A^\star_{12}k,h\rangle^\Gamma &= \mu\rho^\star \left(h_0 \int_0^1 k_0\, dy + \sum l^2 h_l \int_0^1 T_l k_l^i\, dy \right), \\
\langle A^\star_{21}h,k\rangle^\Omega &= -\frac{2\beta h_0}{\rho^\star} \int_0^1 b(\sigma^\star + \overline{\sigma}) k_0\, dy
\end{aligned}$$

$$
\begin{aligned}
& +\beta \sum h_l^i \int_0^1 \left(-\frac{2}{\rho^\star}(\sigma^\star + \overline{\sigma}) + \frac{y\sigma_y^\star}{\rho^\star}(l^2+\lambda_l) \right) k_l^i \, dy, \\
\langle A_{22}^\star k, k\rangle^\Omega \quad = \quad & \beta \int_0^1 b \left(\frac{1}{{\rho^\star}^2} k_0'' - k_0 \right) k_0 \, dy \\
& -\beta \sum \int_0^1 \left((l^2+1){k_l^i}^2 + \frac{1}{{\rho^\star}^2}({k_l^i}')^2 \right) dy,
\end{aligned}
$$

where integration by parts and the boundary conditions satisfied by k have been applied to obtain the last term.

2. Consider first the terms corresponding to $l = 0$. Due to our choice of b,

$$
M\langle A_{12}^\star k_0, h_0\rangle^\Gamma + \langle A_{21}^\star h_0, k_0\rangle^\Omega = 0,
$$

and after integrating by parts twice and using that $b'(0) = 0$ we get

$$
\langle A_{22}^\star k_0, k_0\rangle^\Omega = -\beta \int_0^1 \left(\left(b - \frac{b''}{2{\rho^\star}^2} \right) k_0^2 + \frac{b}{{\rho^\star}^2}(k_0')^2 \right) dy.
$$

As

$$
b(y) - \frac{b''(y)}{2{\rho^\star}^2} = \frac{M\mu {\rho^\star}^2 \cosh \rho^\star}{4\beta\overline{\sigma}} \frac{2+\cosh(\rho^\star y)^2}{\cosh(\rho^\star y)^3} > \delta > 0,
$$

we get

$$
\left\langle A^\star \begin{pmatrix} h_0 \\ k_0 \end{pmatrix}, \begin{pmatrix} h_0 \\ k_0 \end{pmatrix} \right\rangle_M \le -c\|k_0\|_{H^1(\Omega)}^2.
$$

3. Assume now $l \ge 1$. We straightforwardly estimate

$$
\begin{aligned}
|\langle A_{21}^\star h_l, k_l\rangle^\Omega| \quad & \le \quad \beta C \sum_i l^3 |h_l^i| \|k_l^i\|_{L^2(0,1)} \le \sum_i \left(\beta C_3 l^4 |h_l^i|^2 + \frac{\beta l^2}{2} \|k_l^i\|_{L^2(0,1)}^2 \right), \\
M|\langle A_{12}^\star k_l, h_l\rangle^\Gamma| \quad & \le \quad \mu M C_4 l^2 \sum_i |h_l^i| \|k_l^i\|_{L^2(0,1)} \\
& \le \quad \mu M C_4 l^2 \sum_i \left(|h_l^i|^2 + \|k_l^i\|_{L^2(0,1)}^2 \right)
\end{aligned}
$$

with constants $C_{3,4}$ independent of l, β, γ, and μ. Invoking the first and the last expression in (i), we find a constant c such that

$$
\begin{aligned}
& \left\langle A^\star \begin{pmatrix} h_l \\ k_l \end{pmatrix}, \begin{pmatrix} h_l \\ k_l \end{pmatrix} \right\rangle_M \\
\le \quad & l^2 \left(M\lambda_l + \mu M C_4 + \beta C_3 l^2 \right) \sum_i |h_l^i|^2 \\
& + l^2 \left(\mu M C_4 - \frac{\beta}{2} \right) \sum_i \|k_l^i\|_{L^2(0,1)}^2 - \; c\|k_l\|_{H^1(\Omega)}^2. \qquad (4.11)
\end{aligned}
$$

4. With $C_{1,2}$ from (4.8), set

$$q_0 := \max\left\{\frac{4C_3C_4 + C_2}{C_1}, \max_{l\geq 1}\frac{2C_4 + C_2 l^2}{C_1 l^3}\right\}$$

and assume $\gamma/\mu \geq q_0$. Then

$$\lambda_l \leq -\gamma C_1 l^3 + \mu C_2 l^2 \leq -2\mu C_4 \qquad (l \geq 1)$$

and consequently

$$\lambda_l + \mu C_4 \leq \frac{\lambda_l}{2} \qquad (l \geq 1). \tag{4.12}$$

Furthermore,

$$\frac{l^2}{-\lambda_l} \leq \frac{1}{\gamma C_1 l - \mu C_2} \leq \frac{1}{\gamma C_1 - \mu C_2} \leq \frac{1}{4\mu C_3 C_4} \qquad (l \geq 1).$$

Due to this estimate, it is possible to choose M such that

$$2\beta C_3 \max_{l\geq 1}\frac{l^2}{-\lambda_l} \leq M \leq \frac{\beta}{2\mu C_4}.$$

From this and (4.12) we get

$$\mu M C_4 \leq \frac{\beta}{2}$$

and

$$M\lambda_l + \mu M C_4 + \beta C_3 l^2 \leq M\frac{\lambda_l}{2} + \beta C_3 l^2 \leq 0.$$

The statement of the lemma follows from this and (4.11) by summing over l. ■

Remark: We clearly cannot expect a stronger estimate with any norm of h on the right hand side as

$$\left\langle A^\star \begin{pmatrix} h \\ 0 \end{pmatrix}, \begin{pmatrix} h \\ 0 \end{pmatrix} \right\rangle_M = 0$$

for any constant h. However, the estimate above is sufficient to show the spectral properties of $A^\star$ we need. ■

Lemma 4.4 *(The spectrum of $A^\star$)*
Under the assumptions of Lemma 4.3 *we have*

$$\sup_{\lambda\in\sigma(A^\star)} \operatorname{Re}\lambda < 0.$$

Proof: Note first that due to Lemma 3.2 and the compactness of the embedding $E_1 \hookrightarrow\hookrightarrow E_0$, the spectrum of $A^\star$ consists of eigenvalues only.

1. Let λ be an eigenvalue of $A^\star$ and let $z := (h\ k)^T \neq 0$ be a corresponding eigenvector. (As usual, we work with the complexification of our originally real

Banach spaces. Note that both λ and z may be complex.) We start by showing $\operatorname{Re}\lambda < 0$.

Assume $k = 0$ first. Then λ is an eigenvalue of $A^\star_{11}$. By Lemma 4.2, either $\operatorname{Re}\lambda < 0$ or $\lambda = 0$. In the second case, h is a nonzero constant and, consequently, $A^\star_{21}h \neq 0$, which contradicts $k = 0$.

Assume $k = k_1 + ik_2 \neq 0$, $k_j \in E_1$. On the complexification of E_1 we consider the complexified inner product $\langle\cdot,\cdot\rangle^{\mathbb{C}}$ defined by

$$\langle u, v\rangle^{\mathbb{C}} = \langle u_1 + iu_2, v_1 + iv_2\rangle^{\mathbb{C}} := \langle u_1, v_1\rangle + \langle u_2, v_2\rangle + i(\langle u_2, v_1\rangle - \langle u_1, v_2\rangle),$$

$u_j, v_j \in E_1$. Writing $z = (h_1 + ih_2\ k_1 + ik_2)^T$ with $h_j, k_j \in E_1$ we get by Lemma 4.3 (with M chosen appropriately)

$$\begin{aligned} &\operatorname{Re}\lambda\langle z, z\rangle^{\mathbb{C}} = \operatorname{Re}\langle A^\star z, z\rangle^{\mathbb{C}} \\ = \ & \left\langle A^\star \begin{pmatrix} h_1 \\ k_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ k_1 \end{pmatrix}\right\rangle_M + \left\langle A^\star \begin{pmatrix} h_2 \\ k_2 \end{pmatrix}, \begin{pmatrix} h_2 \\ k_2 \end{pmatrix}\right\rangle_M \\ \leq \ & -c(\|k_1\|^2_{H^1(\Omega)} + \|k_2\|^2_{H^1(\Omega)}) < 0. \end{aligned}$$

Therefore $\operatorname{Re}\lambda < 0$.

2. As $-A^\star \in \mathcal{H}(E_1, E_0)$ is sectorial, $\sigma(A^\star)$ is contained in the sector

$$S_{a,\theta} := \{a + re^{i\phi} \mid r \geq 0, \phi \in (\theta, 2\pi - \theta)\}$$

for some $a \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$. In particular, $K := \sigma(A^\star) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq -1\}$ is a compact subset of $\mathbb{C}$. Either $K = \varnothing$, or the continuous function $\operatorname{Re}: K \longrightarrow \mathbb{R}$ takes its maximum at some element of K. In both cases, the lemma is proved. ∎

Proof of Theorem 4.1: The theorem follows now from applying [17] Theorem 9.1.2. to (4.2). The assumptions of that theorem are satisfied due to Lemmas 2.1, 3.2, and 4.4. ∎

5 Analyticity of the moving boundary

We return to the general situation with $\overline{\sigma}, \widetilde{\sigma} > 0$ and $(\rho, \sigma) \in E_1^+$. Recall that $A = \mathcal{F}'(\rho, \sigma) \in \mathcal{L}(E_1, E_0)$.

In addition to the function spaces used before, we introduce the little Nikolskii spaces $n^{s,q}(\Omega)$ as the closure of $C^\infty(\Omega)$ with respect to the norm of the Besov space $B^s_{q,\infty}$ (see e.g. [3, 14, 19]).

Furthermore, for $s > 1 + 1/q$ we define

$$n^{s,q}_B(\Omega) := \{v \in n^{s,q}(\Omega) \mid \mathrm{tr}_0 v_y = 0, \mathrm{tr}_1 v = 0\}.$$

We fix

$$\theta \in \left(0, \min\left\{\frac{1}{2q}, \frac{1}{3}\left(2 - \frac{2}{q} - \alpha\right), \frac{1}{3}(1-\alpha)\right\}\right)$$

and define

$$\begin{aligned} F_0 &:= h^{1+\alpha+3\theta}(\mathbb{T}^1) \times n^{2\theta,q}(\Omega), \\ F_1 &:= h^{4+\alpha+3\theta}(\mathbb{T}^1) \times n_B^{2+2\theta,q}(\Omega), \\ F_1^+ &:= \{(\rho,\sigma) \in F_1 \,|\, \rho > 0\}. \end{aligned}$$

Our application of the little Nikolskii spaces is based on the fact that they appear as continuous interpolation spaces between Sobolev spaces. Let $(\cdot,\cdot)^0_{\eta,\infty}$ denote the continuous interpolation functor, see [8, 17]. Then, by [3] Theorem 5.2 and Remark 5.3(c),

$$(L^q(\Omega), W^{k,q}(\Omega))^0_{\eta,\infty} = n^{k\eta,q}(\Omega), \qquad k \in \mathbb{N}_+,\ \eta \in (0,1),\ \eta k \notin \mathbb{N}, \tag{5.1}$$

$$(L^q(\Omega), W_B^{2,q}(\Omega))^0_{\eta,\infty} = n^{2\eta,q}(\Omega), \qquad \eta \in (0, \frac{1}{2q}). \tag{5.2}$$

(Here and in the sequel, equality between Banach spaces is understood to include equivalence of the respective norms.) From (5.1) it easily follows that in analogy to (2.3)

$$\mathcal{A} \in C^\omega\left(h_+^{4+\alpha+3\theta}(\mathbb{T}^1), \mathcal{L}(n^{2+2\theta}(\Omega), n^{2\theta}(\Omega))\right). \tag{5.3}$$

The mapping property (2.5) (with α replaced by $\alpha+3\theta$), the trivial embedding $n^{2\theta}(\Omega) \hookrightarrow L^q(\Omega)$, and the sharpened embedding

$$W^{4,q}(\Omega) \hookrightarrow h^{2+\alpha+3\theta}(\Omega) \tag{5.4}$$

imply

$$\mathcal{S} \in C^\omega\left(h_+^{4+\alpha+3\theta}(\mathbb{T}^1), \mathcal{L}(W^{2,q}(\Omega), h^{2+\alpha+3\theta}(\Omega))\right). \tag{5.5}$$

From (5.3) and (5.5) we find by repeating the aguments in the proof of Lemma 2.1

$$\mathcal{F} \in C^\omega(F_1^+, F_0) \tag{5.6}$$

Let A_θ be the $(E_0,E_1)^0_{\theta,\infty}$-realization A_θ of A, and let $D(A_\theta)$ be its domain of definition. (See (A.3) in the appendix, where the abstract aspects of the construction presented here are discussed in some more detail.)

Define the spaces $\mathbb{F}_0$ and $\mathbb{F}_1$ as in (A.1). We refer also to the appendix for the definition of (continuous) maximal regularity.

Lemma 5.1 *(Interpolation and properties of A_θ)*

(i) $(E_0,E_1)^0_{\theta,\infty} = F_0$,

(ii) $D(A_\theta) = F_1$,

(iii) $-A_\theta \in \mathcal{H}(F_0,F_1)$,

(iv) $(\mathbb{F}_0,\mathbb{F}_1)$ *is a pair of maximal regularity for* $-A_\theta$.

Proof: We recall that the little Hölder spaces are stable under continuous interpolation, in particular,

$$(h^{1+\alpha}(\mathbb{T}^1), h^{4+\alpha}(\mathbb{T}^1))^0_{\theta,\infty} = h^{1+\alpha+3\theta}(\mathbb{T}^1).$$

(i) follows from this and (5.2). To show (ii), observe that from (5.6) and (i) we get $A|_{F_1} \in \mathcal{L}(F_1, (E_0, E_1)^0_{\theta,\infty})$, hence $F_1 \subset D(A_\theta)$. To show the opposite inclusion, let $(h, k) \in E_1$ be such that $A(h, k) =: (f, g) \in F_0 = (E_0, E_1)^0_{\theta,\infty}$, i.e,

$$f \in h^{1+\alpha+3\theta}(\mathbb{T}^1), \quad g \in n^{2\theta,q}(\Omega).$$

Using (5.4) and arguing as in the proof of (3.7) we get $A_{12}k \in h^{1+\alpha+3\theta}(\mathbb{T}^1)$ and hence

$$A_{11}h = f - A_{12}k \in h^{1+\alpha+3\theta}(\mathbb{T}^1). \tag{5.7}$$

Replacing α by $\alpha + 3\theta$ and repeating the corresponding arguments in the proofs of Lemmas 3.1 and 3.2 we get (with restriction suppressed in the notation)

$$\begin{aligned} A_{11} &= A^0_{11} + K_{11}, \\ K_{11} &\in \mathcal{K}(h^{4+\alpha+3\theta}(\mathbb{T}^1), h^{1+\alpha+3\theta}(\mathbb{T}^1)), \\ A^0_{11} &\in \mathcal{H}(h^{4+\alpha+3\theta}(\mathbb{T}^1), h^{1+\alpha+3\theta}(\mathbb{T}^1)), \end{aligned}$$

and hence

$$A_{11} \in \mathcal{H}(h^{4+\alpha+3\theta}(\mathbb{T}^1), h^{1+\alpha+3\theta}(\mathbb{T}^1)).$$

This and (5.7) implies

$$h \in h^{4+\alpha+3\theta}(\mathbb{T}^1). \tag{5.8}$$

Furthermore, by this and (5.6)

$$A_{21}h \in n^{2\theta,q}(\Omega)$$

and, in the notation of Lemma 3.1,

$$K_{22}k \in W^{1,q}(\Omega) \hookrightarrow n^{2\theta,q}(\Omega).$$

Therefore

$$\mathcal{A}(\rho)k = g + k - K_{22}k - A_{21}h =: \tilde{g} \in n^{2\theta,q}(\Omega)$$

or

$$k = -S(\rho)\tilde{g}$$

As $S(\rho) \in)\mathcal{L}(L^q(\Omega), W^{2,q}(\Omega))$ and $S(\rho)|_{W^{1,q}(\Omega)} \in \mathcal{L}(W^{1,q}(\Omega), W^{2,q}(\Omega))$ we get by interpolation $S(\rho)|_{n^{2\theta,q}(\Omega)} \in \mathcal{L}(n^{2\theta,q}(\Omega), n^{2+2\theta,q}(\Omega))$ and therefore

$$k \in n^{2+2\theta,q}(\Omega) \cap W^{2,q}_B(\Omega) = n^{2+2\theta,q}_B(\Omega).$$

Together with (5.8) this proves $(h, k) \in F_1$, and (ii) is proved.

Statement (iii) follows from (ii), Lemma 3.2 and Lemma A.1 by abstract interpolation arguments, see [3], Sect 6. Finally, statement (iv) follows now from Corollary A.2. ∎

We are in position now to apply an existence and uniqueness result by Da Prato and Grisvard [8] to our problem. To shorten notation, we will write $z := (\rho, \sigma)$ in the sequel.

Proposition 5.2 *(Short-time existence and uniqueness)*

For any $z_0 \in F_1^+$ there exists a $t^+ = t^+(u) > 0$ and a unique maximal solution

$$z = z(\cdot, z_0) \in C([0,t^+), F_1^+) \cap C^1([0,t^+), F_0)$$

to the evolution problem

$$\partial_t z = \mathcal{F}(z), \qquad z(0) = z_0. \tag{5.9}$$

Proof: By Lemma 5.1 (i),(ii), the F_0-realization of $\mathcal{F}'(z_0)$ (considered as an operator in $\mathcal{L}(E_1, E_0)$) coincides with its restriction to F_1. From this fact, together with (5.6) and Lemma 5.1 (iv), it follows that [8] Théorème 4.1 is applicable to (5.9). This yields the result. ■

On F_0 we introduce the family of translation operators $\{T_\nu \mid \nu \in \mathbb{R}\}$ by

$$\begin{aligned} T_\nu(\rho, \sigma) &:= (T_\nu^{(1)}\rho, T_\nu^{(2)}\sigma), \\ T_\nu^{(1)}\rho(x) &:= \rho(x+\nu), \\ T_\nu^{(2)}\sigma(x,y) &:= \sigma(x+\nu, y). \end{aligned}$$

It is easily verified that $\{T_\nu\}$ acts as a group of norm preserving isomorphisms on F_0 and on F_1 and that F_1^+ is invariant under the action of T_ν. For $\mu \in \mathbb{R}$ we define a corresponding differential operator D_μ by

$$D_\mu z := \partial_t (T_{\mu t} z)|_{t=0}$$

and note that

$$[\mu \mapsto D_\mu] \in \mathcal{L}(\mathbb{R}, \mathcal{L}(F_1, F_0)). \tag{5.10}$$

Due to the invariance of our moving boundary problem with respect to translations in x we have

$$T_\nu \circ \mathcal{F} = \mathcal{F} \circ T_\nu, \qquad \nu \in \mathbb{R}. \tag{5.11}$$

This observation is the basis for the proof of parabolic smoothing of the moving boundary.

Fix $z_0 \in F_1^+$ and denote by t^+ the maximal existence time corresponding to these initial data. For the corresponding solution $z(\cdot) = (\rho(\cdot), \sigma(\cdot))$ we define

$$\hat\rho(t,x) := \rho(t)(x), \qquad (t,x) \in (0,t^+) \times \mathbb{T}^1.$$

Theorem 5.3 *(Analyticity of the moving boundary)*

We have

$$\hat\rho \in C^\omega\big((0,t^+) \times \mathbb{T}^1\big),$$

i.e. the moving boundary $t \mapsto \Gamma_{\rho(t)}$ is analytic jointly in space and time for positive times.

Proof: Fix $(t_0, x_0) \in (0,t^+) \times \mathbb{T}^1$ and $t^* \in (t_0, t^+)$. Let Λ be a small open neighborhood of $(1,0)$ in $\mathbb{R}^2$ and define for $(\lambda, \mu) \in \Lambda$

$$z_{\lambda,\mu} \in C([0,t^*], F_1^+) \cap C^1([0,t^*], F_0)$$

by

$$z_{\lambda,\mu}(t) := T_{t\mu} z(\lambda t), \qquad t \in [0, t^*].$$

Then

$$\partial_t z_{\lambda,\mu}(t) = D_\mu z_{\lambda,\mu}(t) + \lambda T_{t\mu} \mathcal{F}(z(\lambda t)), \qquad t \in [0, t^*],$$

and therefore by (5.11)

$$\partial_t z_{\lambda,\mu}(t) = D_\mu z_{\lambda,\mu}(t) + \lambda \mathcal{F}(z_{\lambda,\mu}(t)) = \mathcal{F}_{\lambda,\mu}(z_{\lambda,\mu}(t)), \tag{5.12}$$

where

$$\mathcal{F}_{\lambda,\mu}(\zeta) := \lambda \mathcal{F}(\zeta) + D_\mu \zeta, \qquad \zeta \in F_1^+.$$

Observe that due to (5.6) and (5.10) we have

$$[((\lambda, \mu), \zeta) \mapsto \mathcal{F}_{\lambda,\mu}(\zeta)] \in C^\omega(\Lambda \times F_1^+, F_0). \tag{5.13}$$

Recall the definitions of $\mathbb{F}_0$, $\mathbb{F}_1$, and tr_t from (A.1),(A.2) and set $T = t^*$ there. Define additionally

$$\mathbb{F}_1^+ := C([0, t^*], F_1^+) \cap C^1([0, t^*], F_0).$$

Note that due to (5.12), $w = z_{\lambda,\mu}$ is the solution of the operator equation

$$\mathcal{G}((\lambda, \mu), w) := (\partial_t - \mathcal{F}_{\lambda,\mu}, \mathrm{tr}_t)(w) = (0, z_0). \tag{5.14}$$

Statement (5.13) and a compactness argument yield

$$\mathcal{G} \in C^\omega(\Lambda \times \mathbb{F}_1^+, \mathbb{F}_0),$$

cf. [10] Lemma 3.5. Furthermore, for the Fréchet derivative of $\mathcal{G}$ with respect to the second argument at the original solution we get

$$D_2\mathcal{G}((1, 0), z) = (\partial_t - A_\theta, \mathrm{tr}_t) \in \mathcal{L}_{is}(\mathbb{F}_1, \mathbb{F}_0 \times F_1)$$

because of Lemma 5.1 (iv). Applying now the Implicit Function theorem to (5.14) yields that there is a neighborhood $\Lambda_0 \subset \Lambda$ of $(1, 0)$ in $\mathbb{R}^2$ such that

$$[(\lambda, \mu) \mapsto z_{\lambda,\mu}] \in C^\omega(\Lambda_0, \mathbb{F}_1). \tag{5.15}$$

Let $E \in \mathcal{L}(\mathbb{F}_1, \mathbb{R})$ be the evaluation operator defined by

$$Ew := w_1(t_0)(x_0), \qquad w \in \mathbb{F}_1,$$

where w_1 denotes the first component of w.

For (t, x) in a suitable neighborhood of (t_0, x_0) in $(0, t^*)$ we have

$$(t/t_0, x - x_0) \in \Lambda_0, \quad \hat{\rho}(t, x) = E z_{t/t_0, x - x_0}.$$

The result follows now from the continuity of E and (5.15). ∎

A Appendix: Continuous maximal regularity by extrapolation

Let $F_1 \overset{d}{\hookrightarrow} F_0$ be a pair of Banach spaces with dense and continuous enbedding. For $T > 0$ define

$$\mathbb{F}_0 := C([0,T], F_0), \qquad \mathbb{F}_1 := C([0,T], F_1) \cap C^1([0,T], F_0) \tag{A.1}$$

and the evaluation operator $\mathrm{tr}_t \in \mathcal{L}(\mathbb{F}_1, F_1)$ by

$$\mathrm{tr}_t(u) := u(0). \tag{A.2}$$

We say that $(\mathbb{F}_0, \mathbb{F}_1)$ is a pair of (continuous) maximal regularity for the operator $\hat{A} \in \mathcal{H}(F_1, F_0)$ iff

$$\left(\partial_t + \hat{A}, \mathrm{tr}_t\right) \in \mathcal{L}_{is}(\mathbb{F}_1, \mathbb{F}_0 \times F_1),$$

i.e. iff the initial value problem

$$\begin{aligned} \frac{du}{dt} + \hat{A}u &= f, \\ u(0) = u_0 & \end{aligned}$$

has a unique solution $u \in \mathbb{F}_1$ for any given $f \in \mathbb{F}_0$, $u_0 \in F_1$.

Starting from any pair $E_1 \overset{d}{\hookrightarrow} E_0$ of densely and continuously embedded Banach spaces and an operator $A \in \mathcal{H}(E_1, E_0)$ it is possible to obtain a related pair of spaces for which (the corresponding restriction of) A has the property of maximal regularity. This is done as follows: Let

$$E_2 := D(A^2) = \{x \in E_1 \mid Ax \in E_1\}$$

(with the graph norm) and fix $\theta \in (0,1)$. Define

$$F_0 := E_\theta := (E_0, E_1)^0_{\theta,\infty}, \qquad F_1 := E_{1+\theta} := (E_1, E_2)^0_{\theta,\infty}.$$

Let $\hat{A} := A|_{E_{1+\theta}}$. Then, by interpolation arguments, $\hat{A} \in \mathcal{H}(E_{1+\theta}, E_\theta)$. Moreover, by Théorème 3.1 in [8], $(\mathbb{F}_0, \mathbb{F}_1)$ is a pair of (continuous) maximal regularity for $\hat{A}$.

The practical applicability of this result is sometimes restricted by the difficulty to characterize $D(A^2)$ and, consequently, $E_{1+\theta}$ from the above definition. The following lemma provides an alternative characterization of $E_{1+\theta}$ as domain of the E_θ-realization A_θ of A. More precisely, let

$$D(A_\theta) := \{x \in E_1 \mid Ax \in E_\theta\}, \quad A_\theta x = Ax. \tag{A.3}$$

Observe that A_θ is closed as an (unbounded) operator on E_θ and consider $D(A_\theta)$ as Banach space with the corresponding graph norm.

Lemma A.1 *(Characterization of $E_{1+\theta}$)*
We have $D(A_\theta) = E_{1+\theta}$ with equivalence of the respective norms.

Proof: Without loss of generality, we assume $A \in \mathcal{L}_{is}(E_1, E_0)$. Then $A|_{E_2} \in \mathcal{L}_{is}(E_2, E_1)$ and by interpolation $A|_{E_{1+\theta}} \in \mathcal{L}_{is}(E_{1+\theta}, E_\theta)$. This immediately gives $E_{1+\theta} \subset D(A_\theta)$. To see the opposite inclusion, pick $x \in D(A_\theta)$. Then $Ax \in E_\theta$ and hence $z := (A|_{E_{\theta+1}})^{-1}Ax \in E_{\theta+1}$. Applying $A|_{E_{\theta+1}}$ on both sides yields $Az = Ax$ and hence $x = z \in E_{\theta+1}$. The equivalence of the corresponding norms follows from the Closed Graph theorem. ∎

Remark: Lemma A.1 holds independently of the interpolation method. The result is implicitly contained in the statements given in [3], Sect. 6.

For the sake of clarity, we summarize the result:

Corollary A.2 *Under the assumptions given above, $(\mathbb{F}_0, \mathbb{F}_1)$ is a pair of maximal regularity for A_θ with*

$$\mathbb{F}_0 = C([0,T], (E_0, E_1)^0_{\theta,\infty}), \qquad \mathbb{F}_1 = C([0,T], D(A_\theta)) \cap C^1([0,T], (E_0, E_1)^0_{\theta,\infty}).$$

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