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LYAPUNOV METHODS FOR TIME-INVARIANT DELAY DIFFERENCE INCLUSIONS[*]

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Abstract. Motivated by the fact that delay *difference* inclusions (DDIs) form a rich modeling class that includes, for example, uncertain time-delay systems and certain types of networked control systems, this paper provides a comprehensive collection of Lyapunov methods for DDIs. First, the Lyapunov–Krasovskii approach, which is an extension of the classical Lyapunov theory to time-delay systems, is considered. It is shown that a DDI is $\mathcal{KL}$-stable if and only if it admits a Lyapunov–Krasovskii function (LKF). Second, the Lyapunov–Razumikhin method, which is a type of small-gain approach for time-delay systems, is studied. It is proved that a DDI is $\mathcal{KL}$-stable if it admits a Lyapunov–Razumikhin function (LRF). Moreover, an example of a linear delay difference equation which is globally exponentially stable but does not admit an LRF is provided. Thus, it is established that the existence of an LRF is not a necessary condition for $\mathcal{KL}$-stability of a DDI. Then, it is shown that the existence of an LRF is a sufficient condition for the existence of an LKF and that only under certain additional assumptions is the converse true. Furthermore, it is shown that an LRF induces a family of sets with certain contraction properties that are particular to time-delay systems. On the other hand, an LKF is shown to induce a type of contractive set similar to those induced by a classical Lyapunov function. The class of quadratic candidate functions is used to illustrate the results derived in this paper in terms of both LKFs and LRFs, respectively. Both stability analysis and stabilizing controller synthesis methods for linear DDIs are proposed.

Key words. stability theory, Lyapunov functions and stability, time-delay systems, invariant sets

AMS subject classifications. 39A30, 37B25, 37L25

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1. Introduction. Systems affected by time delay can be found within many applications in the control field; see, e.g., [25] for an extensive list of examples. Delay difference inclusions (DDIs) form a rich modeling class that includes, for example, uncertain systems, time-delay systems, and certain types of networked control systems [14, 45]. However, while stability analysis of delay-free systems is often based on the existence of a Lyapunov function (LF) (see, e.g., [1]), for systems affected by delays the classical Lyapunov theory does not apply straightforwardly. This is due to the fact that the influence of the delayed states can cause a violation of the monotonic decrease condition that a standard LF obeys. To solve this issue, two types of functions were proposed: the Lyapunov–Krasovskii function (LKF) [27], which is an extension of the classical LF to time-delay systems, and the Lyapunov–Razumikhin function (LRF) (see, e.g., [16]), which is a function that is constructed based [40] on a type of small-gain condition for time-delay systems. The main focus of this paper is on discrete-time systems. Therefore, for continuous-time systems, we give only a brief account of some Lyapunov theorems and refer the reader to [15, 16, 25, 27, 36] and the references

therein for further reading. Theorem 4.1.3 in [25] establishes that a time-delay system is globally exponentially stable (GES) if and only if it admits an LKF. Furthermore, Theorem 5.19 in [15] establishes that any linear delay differential equation that is GES admits a quadratic LKF. This result was partially extended to linear delay differential inclusions in [24]. However, for LRFs such converse results are missing. Moreover, LRFs can be considered [25] as particular cases of LKFs. Also, it is known [23] that any quadratic LRF yields a particular quadratic LKF.

It is not immediately clear how the Lyapunov–Razumikhin method and Lyapunov–Krasovskii approach are to be used for stability analysis of discrete-time systems. One of the most commonly used approaches [2] to stability analysis of DDIs is to augment the state vector with all delayed states/inputs that affect the current state, which yields a standard difference inclusion of higher dimension. Thus, stability analysis methods for difference inclusions based on Lyapunov theory (see, e.g., [1, 21]), become applicable. Recently, in [17] it was pointed out that such an LF for the augmented state system provides an LKF for the original system affected by delay. Moreover, in [17] it was also shown that all existing methods based on the Lyapunov–Krasovskii approach provide a particular type of LF for the augmented state system. As such, an equivalent notion of LKFs for discrete-time systems was obtained. Examples of controller synthesis methods based on this approach can be found in, among many others, [7, 8, 12, 26, 43]. However, converse results for the Lyapunov–Krasovskii approach, such as the ones mentioned above for continuous-time systems, are missing. For LRFs the situation is more complicated. The exact translation of this approach to discrete-time systems yields a noncausal constraint [11, 44]. An alternative, Razumikhin-like condition for discrete-time systems was proposed in [33], where the LRF was required to be less than the maximum over its past values for the delayed states. Stability analysis and controller synthesis methods based on the existence of an LRF can be found in, e.g., [13, 32, 34]. For discrete-time systems, a result on the connection between LKFs and LRFs is missing. Moreover, for both continuous- and discrete-time systems, it remains an open question whether there exist systems that are $\mathcal{KL}$-stable or even GES but do not admit an LRF.

Given that DDIs form a rich and relevant modeling class (that was recently shown to include networked control systems) while an overview of the corresponding counterpart of the Lyapunov methods for delay differential inclusions is missing, the purpose of this paper is to provide a comprehensive collection of Lyapunov methods for DDIs. To this end, first, using the augmented state system, a converse Lyapunov theorem for the Lyapunov–Krasovskii approach is established. Second, for the Lyapunov–Razumikhin method, the results of [11] and [33] are extended to DDIs. Third, via an example of a linear delay difference equation that is GES but does not admit an LRF, it is shown that the existence of an LRF is a sufficient condition but not a necessary condition for $\mathcal{KL}$-stability of DDIs. Then, it is established that the existence of an LRF is a sufficient condition for the existence of an LKF and that only under certain additional assumptions is the converse true. Furthermore, it is shown that an LRF induces a family of sets with certain contraction properties that are particular to time-delay systems. On the other hand, an LKF is shown to induce a standard contractive set for the augmented state system, similar to the contractive set induced by a classical LF. The class of quadratic candidate functions is used to illustrate the application of the results derived in this paper to both stability analysis and stabilizing controller synthesis for linear polytopic DDIs in terms of LKFs as well as LRFs.

The remainder of the paper is organized as follows. Section 2 contains some useful

preliminaries. Sections 3.1 and 3.2 present stability conditions in terms of LKFs and LRFs, respectively. In section 4 relations between LKFs and LRFs are discussed. Next, section 5 deals with contractive sets for DDIs. In section 6 synthesis techniques are provided for quadratic LKFs and LRFs, respectively. Moreover, in section 6.1 these techniques are illustrated via an example. Conclusions are drawn in section 7, and the two appendices contain the proof of a technical lemma and some numerical data, respectively.

2. Preliminaries.

2.1. Notation and basic definitions. Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{Z}_+$ denote the field of real numbers, the set of nonnegative reals, the set of integers, and the set of nonnegative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$, define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$. Furthermore, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a vector $x \in \mathbb{R}^n$, let $[x]_i$, $i \in \mathbb{Z}_{[1,n]}$ denote the ith component of x and let $\|x\|_p := (\sum_{i=1}^n |[x]_i|^p)^{\frac{1}{p}}$, $p \in \mathbb{Z}_{>0}$, denote an arbitrary p-norm. Moreover, let $\|x\|_\infty := \max_{i \in \mathbb{Z}_{[1,n]}} |[x]_i|$ denote the infinity norm. Let $\mathbf{x} := \{x(l)\}_{l \in \mathbb{Z}_+}$ with $x(l) \in \mathbb{R}^n$ for all $l \in \mathbb{Z}_+$ denote an arbitrary sequence and define $\|\mathbf{x}\| := \sup\{\|x(l)\| \mid l \in \mathbb{Z}_+\}$. Furthermore, $\mathbf{x}_{[c_1,c_2]} := \{x(l)\}_{l \in \mathbb{Z}_{[c_1,c_2]}}$, with $c_1, c_2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c_1,c_2]}$. Similarly, $\mathrm{col}(\{x(l)\}_{l \in \mathbb{Z}_{[c_1,c_2]}}) := \begin{bmatrix} x(c_2)^\top & \dots & x(c_1)^\top \end{bmatrix}^\top$ is also ordered monotonically (albeit in a decreasing fashion from top to bottom) with respect to the index l. For a symmetric matrix $Z \in \mathbb{R}^{n \times n}$, let $Z \succ 0$ ($Z \prec 0$) denote that Z is positive (negative) definite and let $\lambda_{\max}(Z)$ ($\lambda_{\min}(Z)$) denote the largest (smallest) eigenvalue of Z. Moreover, $*$ is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b^\top \\ b & c \end{bmatrix} = \begin{bmatrix} a & * \\ b & c \end{bmatrix}$. Let $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix and let $0_{n \times m} \in \mathbb{R}^{n \times m}$ denote a matrix with all elements equal to zero. Let $\mathbb{S}^h := \mathbb{S} \times \cdots \times \mathbb{S}$ for any $h \in \mathbb{Z}_{\geq 1}$ denote the h-times cross-product of an arbitrary set $\mathbb{S} \subseteq \mathbb{R}^n$. Moreover, let $\mathrm{int}(\mathbb{S})$ denote the interior of $\mathbb{S}$, let $\partial\mathbb{S}$ denote the boundary of $\mathbb{S}$, and let $\mathrm{cl}(\mathbb{S})$ denote the closure of $\mathbb{S}$. For a $\lambda \in \mathbb{R}$ define $\lambda\mathbb{S} := \{\lambda x \mid x \in \mathbb{S}\}$. Let $\mathrm{co}(\cdot)$ denote the convex hull. A continuous function $\varphi : \mathbb{R}_{[0,a)} \to \mathbb{R}_+$, for some $a \in \mathbb{R}_{>0}$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\varphi(0) = 0$. Moreover, $\varphi \in \mathcal{K}_\infty$ if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi \in \mathcal{K}$, and $\lim_{r \to \infty} \varphi(r) = \infty$. A continuous function $\beta : \mathbb{R}_{[0,a)} \times \mathbb{R}_+ \to \mathbb{R}_+$, for some $a \in \mathbb{R}_{>0}$, is said to belong to class $\mathcal{KL}$ if for each fixed $s \in \mathbb{R}_+$, $\beta(r,s) \in \mathcal{K}$ with respect to r and for each fixed $r \in \mathbb{R}_{[0,a)}$, $\beta(r,s)$ is decreasing with respect to s and $\lim_{s \to \infty} \beta(r,s) = 0$.

2.2. DDIs. Consider the DDI

$$(2.1) \qquad\qquad x(k+1) \in F(\mathbf{x}_{[k-h,k]}), \quad k \in \mathbb{Z}_+,$$

where $\mathbf{x}_{[k-h,k]} \in (\mathbb{R}^n)^{h+1}$, $h \in \mathbb{Z}_{\geq 1}$ is the maximal delay, and $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ is a *set-valued* map with the origin as equilibrium point, i.e., $F(\mathbf{0}_{[k-h,k]}) = \{0\}$. Next, consider the following standing assumption, which is a common assumption for difference inclusions without delay as well; see, e.g., [21].

Assumption 1. The set $F(\mathbf{x}_{[-h,0]}) \subset \mathbb{R}^n$ is compact and nonempty for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$.

Note that while the DDI (2.1) is time invariant, uncertain time-varying delays can be incorporated, similarly as in, e.g., [17]. It is worth pointing out that the aforementioned technique does not introduce any conservatism since the map F is not required to be convex.

Let $\mathcal{S}(\mathbf{x}_{[-h,0]})$ denote the set of all trajectories of (2.1) that correspond to initial condition $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. Furthermore, let $\Phi(\mathbf{x}_{[-h,0]}) := \{\phi(k, \mathbf{x}_{[-h,0]})\}_{k \in \mathbb{Z}_{\geq -h}} \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ denote a trajectory of (2.1) such that $\phi(k, \mathbf{x}_{[-h,0]}) = x(k)$ for all $k \in \mathbb{Z}_{[-h,0]}$ and $\phi(k+1, \mathbf{x}_{[-h,0]}) \in F(\phi_{[k-h,k]}(\mathbf{x}_{[-h,0]}))$ for all $k \in \mathbb{Z}_+$. Above, the notation $\phi_{[k-h,k]}(\mathbf{x}_{[-h,0]}) := \{\phi(l, \mathbf{x}_{[-h,0]})\}_{l \in \mathbb{Z}_{[k-h,k]}}$ was used.

DEFINITION 2.1.   (i) *System* (2.1) *is called a linear delay difference equation (DDE) if* $F(\mathbf{x}_{[k-h,k]}) := \{\sum_{\theta=-h}^{0} A_\theta x(k+\theta)\}$ *where* $A_\theta \in \mathbb{R}^{n \times n}$ *for all* $\theta \in \mathbb{Z}_{[-h,0]}$.

(ii) *System* (2.1) *is called a linear DDI if* $F(\mathbf{x}_{[k-h,k]}) := \{\sum_{\theta=-h}^{0} A_\theta x(k+\theta) \mid A_\theta \in \mathrm{cl}(\mathcal{M}_\theta)\}$ *with* $\mathcal{M}_\theta \subset \mathbb{R}^{n \times n}$ *and* $\mathcal{M}_\theta$ *bounded for all* $\theta \in \mathbb{Z}_{[-h,0]}$.

DEFINITION 2.2. *System* (2.1) *is called* $\mathcal{D}$*-homogeneous of order* t, $t \in \mathbb{Z}_+$, *if for any* $s \in \mathbb{R}$ *it holds that* $F(s\mathbf{x}_{[-h,0]}) = s^t F(\mathbf{x}_{[-h,0]})$ *for all* $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$.

DEFINITION 2.3. *Let* $\lambda \in \mathbb{R}_{[0,1)}$. *A convex and compact set* $\mathbb{X} \subset \mathbb{R}^n$ *with* $0 \in \mathrm{int}(\mathbb{X})$ *is called* λ*-$\mathcal{D}$-contractive for the DDI* (2.1) *if* $F(\mathbf{x}_{[-h,0]}) \subseteq \lambda \mathbb{X}$ *for all* $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$.

Moreover, consider the following notions of stability.

DEFINITION 2.4.   (i) *The origin of the DDI* (2.1) *is called* globally attractive *if* $\lim_{k \to \infty} \|\phi(k, \mathbf{x}_{[-h,0]})\| = 0$ *for all* $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ *and all* $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$.

(ii) *The origin of* (2.1) *is called* Lyapunov stable (LS) *if for every* $\varepsilon \in \mathbb{R}_{>0}$ *there exists a* $\delta(\varepsilon) \in \mathbb{R}_{>0}$ *such that if* $\|\mathbf{x}_{[-h,0]}\| \leq \delta$, *then* $\|\phi(k, \mathbf{x}_{[-h,0]})\| \leq \varepsilon$ *for all* $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ *and all* $k \in \mathbb{Z}_+$.

(iii) *System* (2.1) *is called* globally asymptotically stable (GAS) *if its origin is both globally attractive and LS.*

DEFINITION 2.5.   (i) *System* (2.1) *is called* $\mathcal{KL}$*-stable if there exists a function* $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $\beta \in \mathcal{KL}$, *such that* $\|\phi(k, \mathbf{x}_{[-h,0]})\| \leq \beta(\|\mathbf{x}_{[-h,0]}\|, k)$ *for all* $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, *all* $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, *and all* $k \in \mathbb{Z}_+$.

(ii) *System* (2.1) *is called GES if it is* $\mathcal{KL}$*-stable with* $\beta(r, s) := cr\mu^s$ *for some* $c \in \mathbb{R}_{\geq 1}$ *and* $\mu \in \mathbb{R}_{[0,1)}$.

Note that the above definitions define global and strong properties, i.e., properties that hold for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. The following lemma relates DDIs that are GAS to DDIs that are $\mathcal{KL}$-stable.

LEMMA 2.6. *The following two statements are equivalent:*
  (i) *The DDI* (2.1) *is GAS, and* $\delta(\varepsilon)$ *in Definition* 2.4 *can be chosen to satisfy* $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$.
 (ii) *The DDI* (2.1) *is* $\mathcal{KL}$*-stable.*

The proof of Lemma 2.6 can be obtained mutatis mutandis from the proof of Lemma 4.5 in [22], a result for continuous-time systems without delay. The relevance of the result of Lemma 2.6 comes from the fact that $\mathcal{KL}$-stability, as opposed to mere global asymptotic stability, is a standard assumption in converse Lyapunov theorems; see, e.g., [1, 21, 38]. Note that if the DDI (2.1) is upper semicontinuous [20], then it can be shown, similarly to Proposition 6 in [20], that global asymptotic stability is equivalent to $\mathcal{KL}$-stability.

With the above equivalence established, in the next section various conditions under which a DDI is $\mathcal{KL}$-stable are established.

3. Stability of DDIs.

3.1. The Lyapunov–Krasovskii approach. As pointed out in the introduction, a standard approach for studying stability of delay discrete-time systems is to augment the state vector and then to obtain an LF for the resulting augmented state

system. Hence, let $\xi(k) := \mathrm{col}(\{x(l)\}_{l \in \mathbb{Z}_{[k-h,k]}})$ and consider the difference inclusion

$$(3.1) \qquad\qquad \xi(k+1) \in \bar{F}(\xi(k)), \quad k \in \mathbb{Z}_+,$$

where the map $\bar{F} : \mathbb{R}^{(h+1)n} \rightrightarrows \mathbb{R}^{(h+1)n}$ is obtained from the map F in (2.1), i.e., $\bar{F}(\xi) = \mathrm{col}(\{x(l)\}_{l \in \mathbb{Z}_{[-h+1,0]}}, F(x(-h), \dots, x(0)))$, with $\xi = \mathrm{col}(\{x(l)\}_{l \in \mathbb{Z}_{[-h,0]}})$. Therefore, $\bar{F}(\xi)$ is compact and nonempty for all $\xi \in \mathbb{R}^{(h+1)n}$ and $\bar{F}(0) = \{0\}$. We use $\bar{S}(\xi)$ to denote the set of all trajectories of (3.1) from initial condition $\xi \in \mathbb{R}^{(h+1)n}$. Let $\bar{\Phi}(\xi) := \{\bar{\phi}(k,\xi)\}_{k \in \mathbb{Z}_+} \in \bar{S}(\xi)$ denote a trajectory of (3.1) such that $\bar{\phi}(0,\xi) = \xi$ and $\bar{\phi}(k+1,\xi) \in \bar{F}(\bar{\phi}(k,\xi))$ for all $k \in \mathbb{Z}_+$.

DEFINITION 3.1. *A function $g : \mathbb{R}^l \rightrightarrows \mathbb{R}^p$, possibly set valued, is called homogeneous (positively homogeneous) of order t, $t \in \mathbb{Z}_+$, if $g(sx) = s^t g(x)$ $(g(sx) = |s|^t g(x))$ for all $x \in \mathbb{R}^l$ and all $s \in \mathbb{R}$.*

DEFINITION 3.2. *Let $\lambda \in \mathbb{R}_{[0,1)}$. A convex and compact set $\bar{\mathbb{X}} \subset \mathbb{R}^{(h+1)n}$ with $0 \in \mathrm{int}(\bar{\mathbb{X}})$ is called λ-contractive for system (3.1) if $\bar{F}(\xi) \subseteq \lambda\bar{\mathbb{X}}$ for all $\xi \in \bar{\mathbb{X}}$.*

REMARK 1. *Throughout this paper, uniformly strict Lyapunov conditions are sought, as opposed to classical Lyapunov conditions. Such conditions yield uniformly strict LFs, which in turn induce contractive sets, as opposed to merely invariant sets. The reader interested in more details on uniformly strict LFs is referred to [31].*

The following lemma relates stability of the DDI (2.1) to stability of the difference inclusion (3.1). Thus, stability of the set-valued map $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ is related to stability of the set-valued map $\bar{F} : \mathbb{R}^{(h+1)n} \rightrightarrows \mathbb{R}^{(h+1)n}$.

LEMMA 3.3. *The following claims are true:*
  (i) *The DDI (2.1) is GAS if and only if the difference inclusion (3.1) is GAS.*
  (ii) *The DDI (2.1) is $\mathcal{KL}$-stable if and only if the difference inclusion (3.1) is $\mathcal{KL}$-stable.*
  (iii) *The DDI (2.1) is GES if and only if the difference inclusion (3.1) is GES.*

The proof of Lemma 3.3 can be found in Appendix A. In the standard approach, as in, e.g., [7, 8, 12, 14, 17, 26, 43], an LF for the difference inclusion (3.1) is obtained. This LF is then used to conclude that the DDI (2.1) is $\mathcal{KL}$-stable. Lemma 3.3 enables a formal characterization of this conjecture. Moreover, the converse is also obtained.

THEOREM 3.4. *Let $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$. The following statements are equivalent:*
  (i) *There exist a function $\bar{V} : \mathbb{R}^{(h+1)n} \to \mathbb{R}_+$ and a constant $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that*

$$(3.2a) \qquad\qquad \bar{\alpha}_1(\|\xi\|) \le \bar{V}(\xi) \le \bar{\alpha}_2(\|\xi\|),$$
$$(3.2b) \qquad\qquad \bar{V}(\xi^+) \le \bar{\rho}\bar{V}(\xi)$$

  for all $\xi \in \mathbb{R}^{(h+1)n}$ and all $\xi^+ \in \bar{F}(\xi)$.
  (ii) *The difference inclusion (3.1) is $\mathcal{KL}$-stable.*
  (iii) *The DDI (2.1) is $\mathcal{KL}$-stable.*

Proof. The equivalence of (i) and (ii) was proved in [21, Theorem 2.7], under the additional assumptions that the map $\bar{F}$ is upper semicontinuous and the function V is smooth. However, these assumptions were used only to prove certain robustness properties and can therefore be omitted. Alternatively, this equivalence can be shown following mutatis mutandis the reasoning used in the proof of Lemma 4 in [38], which is a result for difference equations. Furthermore, the equivalence of (ii) and (iii) follows from Lemma 3.3. ☐

A function $\bar{V}$ that satisfies the hypothesis of Theorem 3.4 is called an LKF for the DDI (2.1). From Theorem 3.4 the following two corollaries are obtained.

COROLLARY 3.5. *Let $c_1 \in \mathbb{R}_{>0}$ and let $c_2 \in \mathbb{R}_{\geq c_1}$. Suppose that the DDI (2.1) is a linear DDE and hence also that the corresponding system (3.1) is a linear difference equation. Then the following statements are equivalent:*

(i) *There exist a quadratic function $\bar{V}(\xi) := \xi^\top \bar{P} \xi$, for some symmetric matrix $\bar{P} \in \mathbb{R}^{(h+1)n \times (h+1)n}$, and a constant $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that*

$$c_1 \|\xi\|_2^2 \leq \bar{V}(\xi) \leq c_2 \|\xi\|_2^2, \tag{3.3a}$$

$$\bar{V}(\xi^+) \leq \bar{\rho} \bar{V}(\xi) \tag{3.3b}$$

for all $\xi \in \mathbb{R}^{(h+1)n}$ and all $\xi^+ \in \bar{F}(\xi)$.

(ii) *The linear difference equation (3.1) is GES.*

(iii) *The linear DDE (2.1) is GES.*

COROLLARY 3.6. *Let $c_1 \in \mathbb{R}_{>0}$, $c_2 \in \mathbb{R}_{\geq c_1}$ and let $p \in \mathbb{Z}_{\geq (h+1)n}$. Suppose that the DDI (2.1) is a linear DDI and hence also that the corresponding system (3.1) is a linear difference inclusion. Then the following statements are equivalent:*

(i) *There exist a polyhedral function $\bar{V}(\xi) := \|\bar{P}\xi\|_\infty$, for some $\bar{P} \in \mathbb{R}^{p \times (h+1)n}$, and a constant $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that*

$$c_1 \|\xi\|_\infty \leq \bar{V}(\xi) \leq c_2 \|\xi\|_\infty, \tag{3.4a}$$

$$\bar{V}(\xi^+) \leq \bar{\rho} \bar{V}(\xi) \tag{3.4b}$$

for all $\xi \in \mathbb{R}^{(h+1)n}$ and all $\xi^+ \in \bar{F}(\xi)$.

(ii) *The linear difference inclusion (3.1) is GES.*

(iii) *The linear DDI (2.1) is GES.*

The proof of Corollary 3.5 follows from Corollary 3.1* in [19] and Lemma 3.3. Furthermore, the proof of Corollary 3.6 follows from the corollary in [3] and Lemma 3.3. Note that the set $\mathrm{cl}(\mathcal{M}_\theta)$ is closed and bounded by assumption but not necessarily convex, which is exactly what is required for the corollary in [3]. A function $\bar{V}(\xi) = \xi^\top \bar{P} \xi$ that satisfies the hypothesis of Corollary 3.5 is called a quadratic Lyapunov–Krasovskii function (qLKF). Moreover, a function $\bar{V}(\xi) = \|\bar{P}\xi\|_\infty$ that satisfies the hypothesis of Corollary 3.6 is called a polyhedral Lyapunov–Krasovskii function (pLKF). The following example illustrates the results derived above.

EXAMPLE 1. *Consider the linear DDE*

$$x(k+1) = ax(k) + bx(k-1), \quad k \in \mathbb{Z}_+, \tag{3.5}$$

where $\mathbf{x}_{[k-1,k]} \in \mathbb{R} \times \mathbb{R}$ and $a, b \in \mathbb{R}$. Let $\xi(k) := [x(k), x(k-1)]^\top$, which yields

$$\xi(k+1) = \bar{A}\xi(k), \quad k \in \mathbb{Z}_+, \tag{3.6}$$

where $\bar{A} = \left[\begin{smallmatrix} a & b \\ 1 & 0 \end{smallmatrix}\right]$. Note that for all $b \in \mathbb{R}$ with $|b| < 1$ and all $a \in \mathbb{R}$ with $|a| < 1 - b$, the spectral radius of $\bar{A}$ is strictly less than one and hence (3.6) is GES; see, e.g., [19]. Therefore, it follows from Corollary 3.5 that if $a, b \in \mathbb{R}$ with $|b| < 1$ and $|a| < 1 - b$, then there exist a $\bar{\rho} \in \mathbb{R}_{[0,1)}$ and a symmetric $\bar{P} \in \mathbb{R}^{2 \times 2}$ such that

$$\bar{A}^\top \bar{P} \bar{A} - \bar{\rho} \bar{P} \prec 0, \quad \bar{P} \succ 0. \tag{3.7}$$

Moreover, it also follows from Corollary 3.5 that if $a, b \in \mathbb{R}$ with $|b| < 1$ and $|a| < 1-b$, then (3.5) is GES and admits a qLKF. For example, let $a = 1$ and $b = -0.5$. As $\bar{\rho} = 0.95$ and $\bar{P} = \left[\begin{smallmatrix} 1.3 & -0.5 \\ -0.5 & 0.7 \end{smallmatrix}\right]$ is a solution to (3.7), system (3.6), with $a = 1$ and

$b = -0.5$, is GES. Hence, the linear DDE (3.5), with $a = 1$ and $b = -0.5$, is GES. Moreover, the function $\bar{V}(\xi) = \xi^\top \bar{P}\xi$ is a quadratic LF for (3.6) and the function $\bar{V}(\xi) = \bar{V}(\mathbf{x}_{[-1,0]}) = 1.3x(0)^2 - x(0)x(-1) + 0.7x(-1)^2$ is a qLKF for (3.5).

Unfortunately, the sublevel sets of an LKF do not provide a contractive set in the original state space, i.e., $\mathbb{R}^n$, but rather a contractive set in the higher dimensional state space corresponding to the augmented state system, i.e., $\mathbb{R}^{(h+1)n}$ or equivalently $(\mathbb{R}^n)^{h+1}$. Moreover, as the LKF is a function of the current state and all delayed states, it becomes increasingly complex when the size of the delay, i.e., $h \in \mathbb{Z}_{\geq 1}$, increases. Therefore, it would be desirable to construct a function satisfying particular Lyapunov conditions that involve the nonaugmented system, rather than the augmented one.

3.2. The Lyapunov–Razumikhin approach. The Razumikhin approach is a Lyapunov technique for time-delay systems that satisfies Lyapunov conditions that directly involve the DDI (2.1), as opposed to the augmented state system (3.1).

THEOREM 3.7. *Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\pi(s) > s$ for all $s \in \mathbb{R}_{>0}$ and $\pi(0) = 0$. Suppose that there exist a function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a constant $\rho \in \mathbb{R}_{[0,1)}$ such that*

$$(3.8a) \qquad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathbb{R}^n,$$

and, for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, if $\pi(V(x^+)) \geq \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))$, then

$$(3.8b) \qquad V(x^+) \leq \rho V(x(0)) \quad \forall x^+ \in F(\mathbf{x}_{[-h,0]}).$$

Then, the DDI (2.1) is $\mathcal{KL}$-stable.

The proof of the above theorem, which is omitted here for brevity, is similar in nature to the proof of Theorem 6 in [11] by replacing mutatis mutandis the difference *equation* with the difference *inclusion* as in (2.1). It is obvious that the LRF defined in Theorem 3.7 is noncausal; i.e., (3.8b) imposes a condition on $V(x^+)$ if $V(x^+)$ satisfies some other condition. Note that the corresponding Lyapunov–Razumikhin theorem for continuous-time systems, e.g., Theorem 4.1 in [16], is causal, because it imposes a condition on *the derivative of $V(x)$* if $V(x)$ satisfies a certain condition. Next, an extension of Theorem 3.2 in [33], which provides a causal sufficient condition for stability of the DDI (2.1), will be presented.

THEOREM 3.8. *Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. If there exist a function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a constant $\rho \in \mathbb{R}_{[0,1)}$ such that*

$$(3.9a) \qquad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathbb{R}^n,$$

$$(3.9b) \qquad V(x^+) \leq \rho \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x^+ \in F(\mathbf{x}_{[-h,0]})$, then the DDI (2.1) is $\mathcal{KL}$-stable.

Proof. Suppose that $\rho \neq 0$. Let $\hat{\rho} := \rho^{\frac{1}{h+1}} \in \mathbb{R}_{(0,1)}$ and let

$$\theta_{\mathrm{opt}}(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})) := \arg \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+\theta)} V(\phi(k+\theta, \mathbf{x}_{[-h,0]})),$$

$$(3.10) \qquad U(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})) := \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+\theta)} V(\phi(k+\theta, \mathbf{x}_{[-h,0]})),$$

where $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and $k \in \mathbb{Z}_+$. Next, it will be proved that

$$(3.11) \qquad U(k+1, \boldsymbol{\phi}_{[k-h+1,k+1]}(\mathbf{x}_{[-h,0]})) \leq U(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]}))$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and all $k \in \mathbb{Z}_+$. There-fore, suppose that $\theta_{\mathrm{opt}}(k+1, \boldsymbol{\phi}_{[k-h+1,k+1]}(\mathbf{x}_{[-h,0]})) = 0$ for some $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and $k \in \mathbb{Z}_+$. Then, (3.9b) yields

$$U(k+1, \boldsymbol{\phi}_{[k-h+1,k+1]}(\mathbf{x}_{[-h,0]})) = \hat{\rho}^{-(k+1)} V(\phi(k+1, \mathbf{x}_{[-h,0]}))$$
$$\le \hat{\rho}^{-(k+1)} \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{(h+1)} V(\phi(k+\theta, \mathbf{x}_{[-h,0]}))$$

(3.12) $$\le \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+\theta)} V(\phi(k+\theta, \mathbf{x}_{[-h,0]})) = U(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})).$$

Furthermore, if $\theta_{\mathrm{opt}}(k+1, \boldsymbol{\phi}_{[k-h+1,k+1]}(\mathbf{x}_{[-h,0]})) \in \mathbb{Z}_{[-h,-1]}$, it holds that

$$U(k+1, \boldsymbol{\phi}_{[k-h+1,k+1]}(\mathbf{x}_{[-h,0]})) = \max_{\theta \in \mathbb{Z}_{[-h,-1]}} \hat{\rho}^{-(k+\theta+1)} V(\phi(k+\theta+1, \mathbf{x}_{[-h,0]}))$$
$$= \max_{\theta \in \mathbb{Z}_{[-h+1,0]}} \hat{\rho}^{-(k+\theta)} V(\phi(k+\theta, \mathbf{x}_{[-h,0]}))$$

(3.13) $$\le U(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})).$$

Therefore, from (3.12) and (3.13) it follows that (3.11) holds. Applying (3.11) recursively yields

(3.14) $$U(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})) \le U(0, \boldsymbol{\phi}_{[-h,0]}(\mathbf{x}_{[-h,0]})) \le \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta)).$$

Next, combining (3.10) and (3.14) yields

$$V(\phi(k, \mathbf{x}_{[-h,0]})) \le \hat{\rho}^k U(k, \boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})) \le \hat{\rho}^k \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and all $k \in \mathbb{Z}_+$. Observing that $\max_{\theta \in \mathbb{Z}_{[-h,0]}} \alpha_2(\|x(\theta)\|) = \alpha_2(\|\mathbf{x}_{[-h,0]}\|)$ and applying (3.9a) yields

(3.15) $$\|\phi(k, \mathbf{x}_{[-h,0]})\| \le \alpha_1^{-1}(\hat{\rho}^k \alpha_2(\|\mathbf{x}_{[-h,0]}\|))$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and all $k \in \mathbb{Z}_+$. As $\beta(r, s) := \alpha_1^{-1}(\alpha_2(r)\hat{\rho}^s) \in \mathcal{KL}$, it follows that (2.1) is $\mathcal{KL}$-stable.

Suppose that $\rho = 0$. Then, it follows from (3.9b) and (3.9a) that $\|x^+\| = 0$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Hence, $\|\phi(k, \mathbf{x}_{[-h,0]})\| \le \|\mathbf{x}_{[-h,0]}\|\frac{1}{2}^k$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and all $k \in \mathbb{Z}_+$. Observing that $\beta(r, s) := r\frac{1}{2}^s \in \mathcal{KL}$ completes the proof. $\square$

A function that satisfies the hypothesis of Theorem 3.7 is called a *noncausal* LRF and one that satisfies the hypothesis of Theorem 3.8 is called an LRF. The following corollary follows directly from (3.15).

COROLLARY 3.9. *Let $c_1 \in \mathbb{R}_{>0}$, $c_2 \in \mathbb{R}_{\ge c_1}$, and $\lambda \in \mathbb{Z}_{>0}$. If there exist a function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a constant $\rho \in \mathbb{R}_{[0,1)}$ that satisfy the hypothesis of Theorem 3.8 with $\alpha_1(s) = c_1 s^\lambda$ and $\alpha_2(s) = c_2 s^\lambda$, then the DDI (2.1) is GES.*

Next, Example 1 is used to show that the converse of Theorems 3.7 and 3.8 is not true in general.

PROPOSITION 3.10. *Consider the linear DDE (3.5) and suppose that $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$. Then, the following statements are true:*
   (i) *The linear DDE (3.5) is GES.*

(ii) *The linear DDE* (3.5) *does not admit a noncausal LRF.*

(iii) *The linear DDE* (3.5) *does not admit an LRF.*

Proof. It was shown in Example 1 that the DDE (3.5) with $a, b \in \mathbb{R}$ and such that $|b| < 1$ and $|a| < 1 - b$ admits a qLKF. Hence, the DDE (3.5) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ admits a qLKF. Therefore, it follows from Corollary 3.5 that the DDE (3.5) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ is GES. The proofs of claims (ii) and (iii) proceed by contradiction.

To prove claim (ii), suppose that there exists a noncausal LRF $V : \mathbb{R} \to \mathbb{R}_+$ for the DDE (3.5) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$. Let $x(0) = 1$, $x(-1) = 0$ and let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be any function such that $\pi(s) > s$ for all $s \in \mathbb{R}_{>0}$ and $\pi(0) = 0$. Hence, (3.5) yields that $x(1) = 1$. As

$$\pi(V(x(1))) = \pi(V(1)) \geq \max_{\theta \in \mathbb{Z}_{[-1,0]}} V(x(\theta)) = V(1),$$

it follows from (3.8b) that

$$V(x(1)) = V(1) \leq \rho V(x(0)) = \rho V(1).$$

Obviously, as $\rho \in \mathbb{R}_{[0,1)}$ a contradiction has been reached, and hence V is not a noncausal LRF for the DDE (3.5). As the functions V and π and the constant ρ were chosen, with the restriction that $\pi(s) > s$ for all $s \in \mathbb{R}_{>0}$ and that $\rho \in \mathbb{R}_{[0,1)}$, arbitrarily, it follows that the second claim has been established.

The same initial conditions as those used in the proof of claim (ii) can be used to establish, by contradiction, claim (iii). ∎

While it can be verified using the conditions in Theorem 3.7 whether a function is a noncausal LRF, these conditions cannot be reformulated into an optimization problem which can be used to obtain a noncausal LRF. The conditions in Theorem 3.8, on the other hand, can be reformulated as a semidefinite programming problem whose solution yields an LRF, as will be shown in section 6. Therefore, in what follows, we will focus on LRFs and disregard noncausal LRFs. The interested reader is referred to [33] for a detailed discussion on LRFs and noncausal LRFs and their differences. Therein, it is indicated why LRFs form a less conservative test for stability when compared to noncausal LRFs, which provides another reason for disregarding noncausal LRFs. In the next section, it will be shown that the existence of an LRF implies the existence of an LKF and that only under certain additional assumptions is the converse true.

4. Relations between LKFs and LRFs. For delay differential equations, i.e., delay continuous-time systems, it was shown in [25, section 4.8] that LRFs form a particular case of LKFs, when only Lyapunov stability (see Definition 2.4) rather than $\mathcal{KL}$-stability is of concern. A similar reasoning as that used in [25] can be applied to DDIs as well. Suppose that the function V satisfies the hypothesis of Theorem 3.8 *with* $\rho = 1$. Then, it can be easily verified that

$$\bar{V}(\mathbf{x}_{[-h,0]}) = \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))$$

satisfies the hypothesis of Theorem 3.4 *with* $\bar{\rho} = 1$. Thus, it follows from (3.2b) that

$$\bar{V}(\boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})) \leq \bar{V}(\mathbf{x}_{[-h,0]}) \quad \forall \mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}, \quad \forall \Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]}),$$

and for all $k \in \mathbb{Z}_+$. From this observation one can show, using (3.2a), that (2.1) is LS. However, the same candidate LKF does not satisfy the assumptions of Theorem 3.4

for $\bar{\rho} \in \mathbb{R}_{[0,1)}$, i.e., when $\mathcal{KL}$-stability is imposed. Furthermore, in [23] an example was provided where the above result was generalized to $\bar{\rho} \in \mathbb{R}_{[0,1)}$ for quadratic candidate functions and continuous-time systems. Next, it will be shown how the continuous-time result of [25] can be extended for DDIs to allow for $\bar{\rho} \in \mathbb{R}_{[0,1)}$, via a more complex candidate LKF.

THEOREM 4.1. *Suppose that $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfies the hypothesis of Theorem 3.8. Then,*

$$\bar{V}(\mathbf{x}_{[-h,0]}) := \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho_{h+1+\theta} V(x(\theta)), \tag{4.1}$$

where $\rho_i := \frac{\rho+i}{i+1}$, $i \in \mathbb{Z}_{[1,h]}$, and $\rho_{h+1} := 1$ satisfy the hypothesis of Theorem 3.4.

Proof. First, it is established that

$$\rho < \rho_1 < \cdots < \rho_h < \rho_{h+1} = 1. \tag{4.2}$$

As $\rho < 1$ it holds that $\rho < 1 = (i+1)^2 - (i+2)i$, which is equivalent to

$$(i+2)(\rho+i) < (i+1)(\rho+(i+1)).$$

Therefore, it follows that $\rho_i < \rho_{i+1}$ for all $i \in \mathbb{Z}_{[1,h]}$. Obviously, $\rho_i < \frac{1+i}{i+1} = 1$, which establishes that (4.2) holds. Next, let $\pi_i := \frac{\rho_{i-1}}{\rho_i}$, $i \in \mathbb{Z}_{[1,h+1]}$, and let $\rho_0 := \rho$. Then, as $\rho_{i-1} < \rho_i$ it follows that $\pi_i < \frac{\rho_i}{\rho_i} = 1$. Letting $\pi := \max_{i \in \mathbb{Z}_{[1,h+1]}} \pi_i$ yields $\pi < 1$.

Next, consider any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. Then,

$$\bar{V}(\{\mathbf{x}_{[-h+1,0]}, x^+\}) = \max\{\rho_{h+1} V(x^+), \max_{\theta \in \mathbb{Z}_{[-h+1,0]}} \rho_{h+\theta} V(x(\theta))\}$$

$$\leq \max\{\max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(x(\theta)), \max_{\theta \in \mathbb{Z}_{[-h+1,0]}} \rho_{h+\theta} V(x(\theta))\}$$

$$= \max\{\rho V(x(-h)), \max_{\theta \in \mathbb{Z}_{[-h+1,0]}} \rho_{h+\theta} V(x(\theta))\}$$

$$= \max_{\theta \in \mathbb{Z}_{[-h,0]}} \pi_{h+\theta+1} \rho_{h+\theta+1} V(x(\theta)) \leq \pi \bar{V}(\mathbf{x}_{[-h,0]})$$

for all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Let $\bar{\rho} := \pi$, $\bar{\alpha}_1(s) := \rho_1 \alpha_1(s)$, and $\bar{\alpha}_2(s) := \alpha_2(s)$. As $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ and $\bar{\rho} < 1$, it follows that $\bar{V}$ satisfies the hypothesis of Theorem 3.4. $\square$

Next, it is established under what conditions the existence of an LKF implies the existence of an LRF.

PROPOSITION 4.2. *Suppose that $\bar{V} : \mathbb{R}^{(h+1)n} \to \mathbb{R}_+$ satisfies the hypothesis of Theorem 3.4. Moreover, let $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ be such that $\alpha_3(s) \leq \alpha_4(s)$ and $\alpha_3(\rho s) \geq \bar{\rho}\alpha_4(s)$ for some $\rho \in \mathbb{R}_{[0,1)}$ and all $s \in \mathbb{R}_{>0}$. If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying (3.9a) and*

$$\sum_{\theta=-h}^{0} \alpha_3(V(x(\theta))) \leq \bar{V}(\mathbf{x}_{[-h,0]}) \leq \sum_{\theta=-h}^{0} \alpha_4(V(x(\theta))), \tag{4.3}$$

then V satisfies the hypothesis of Theorem 3.8.

Proof. Applying (4.3) in (3.2b) yields

$$\alpha_3(V(x^+)) - \bar{\rho}\alpha_4(V(x(-h))) + \sum_{\theta=-h+1}^{0} \alpha_3(V(x(\theta))) - \bar{\rho}\alpha_4(V(x(\theta))) \leq 0 \tag{4.4}$$

for all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Note that $\alpha_3(s) > \bar{\rho}\alpha_4(s)$ for all $s \in \mathbb{R}_+$ and hence

$$(4.5) \qquad \sum_{\theta=-h+1}^{0} \alpha_3(V(x(\theta))) - \bar{\rho}\alpha_4(V(x(\theta))) > 0.$$

The inequality (4.5) in combination with $V(x(-h)) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))$ yields that (4.4) is a sufficient condition for

$$(4.6) \qquad \alpha_3(V(x^+)) - \bar{\rho}\alpha_4\left(\max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))\right) \leq 0$$

for all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Then, using that there exists a $\rho \in \mathbb{R}_{[0,1)}$ such that $\rho s \geq \alpha_3^{-1}(\bar{\rho}\alpha_4(s))$ yields

$$V(x^+) - \rho \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta)) \leq 0$$

for all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Hence, the hypothesis of Theorem 3.8 is satisfied and the proof is complete. $\qquad\square$

The following corollary is a slight modification of Proposition 4.2.

COROLLARY 4.3. *Suppose that the hypothesis of Proposition 4.2 holds with (4.3) replaced by*

$$(4.7) \qquad \max_{\theta \in \mathbb{Z}_{[-h,0]}} \alpha_3(V(x(\theta))) \leq \bar{V}(\mathbf{x}_{[-h,0]}) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \alpha_4(V(x(\theta))).$$

Then V satisfies the hypothesis of Theorem 3.8.

Proof. Using the bounds (4.7) in (3.2b) yields

$$(4.8) \qquad \max\left\{\max_{\theta \in \mathbb{Z}_{[-h+1,0]}} \alpha_3(V(x(\theta))), \alpha_3(V(x^+))\right\} - \bar{\rho} \max_{\theta \in \mathbb{Z}_{[-h,0]}} \alpha_4(V(x(\theta))) \leq 0$$

for all $x^+ \in F(\mathbf{x}_{[-h,0]})$. As $\max\{s_1, s_2\} \geq s_2$ for any $s_1, s_2 \in \mathbb{R}_+$, (4.8) is sufficient for

$$\alpha_3(V(x^+)) - \bar{\rho}\alpha_4\left(\max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta))\right) \leq 0$$

for all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Hence, (4.6) is recovered, which completes the proof. $\qquad\square$

The hypotheses and conclusions of Theorem 4.1, Proposition 4.2, and Corollary 4.3 might not seem very intuitive. However, when quadratic or polyhedral candidate functions are considered, these results do provide valuable insights. For example, suppose that $V(x) = \|Px\|_\infty$ is a polyhedral Lyapunov–Razumikhin function (pLRF). Then, it follows from Theorem 4.1 that

$$(4.9) \qquad \bar{V}(\mathbf{x}_{[-h,0]}) = \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho_{h+1+\theta}\|Px(\theta)\|_\infty = \left\|\begin{bmatrix} \rho_{h+1}P & & 0 \\ & \ddots & \\ 0 & & \rho_1 P \end{bmatrix}\xi\right\|_\infty$$

is a pLKF. Conversely, suppose that the function (4.9) is a pLKF for some $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that $\bar{\rho} < \rho_1$. Then, it follows from Corollary 4.3, i.e., by taking $\alpha_3(s) = \rho_1 s$ and $\alpha_4(s) = s$, that $V(x) = \|Px\|_\infty$ is a pLRF.

In contrast, given a quadratic Lyapunov–Razumikhin function (qLRF), Theorem 4.1 does not yield a qLKF but rather a more complex LKF, i.e., the maximum
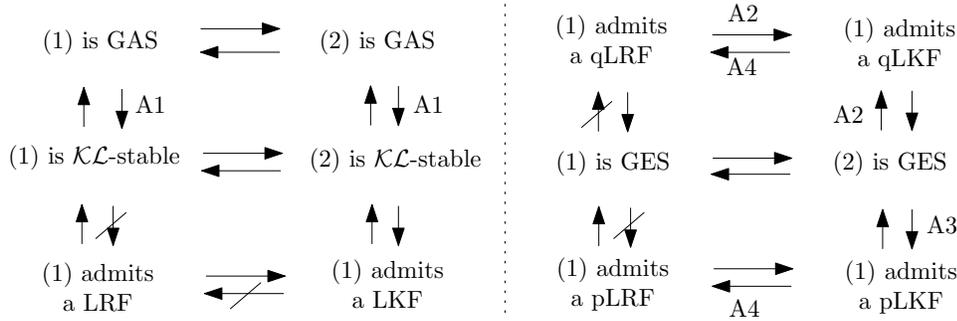
FIG. 4.1. *A schematic overview of all relations established in this paper. $B \to C$ means that B implies C, $B \nrightarrow C$ means that B does not necessarily imply C, and $B \xrightarrow{A} C$ means that B implies C under the additional assumption A. The employed assumptions are as follows: (A1)—$\delta(\varepsilon)$ in Definition 2.4 can be chosen to satisfy $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$; (A2)—the DDI (2.1) is a linear DDE; (A3)—the DDI (2.1) is a linear DDI; (A4)—the LKF has certain structural properties.*

over a set of quadratic functions. On the other hand, Proposition 4.2 can provide a qLRF constructed from a qLKF. Indeed, consider the qLKF

$$\bar{V}(\mathbf{x}_{[-h,0]}) = \sum_{\theta=-h}^{0} x(\theta)^\top P x(\theta) = \xi^\top \begin{bmatrix} P & & 0 \\ & \ddots & \\ 0 & & P \end{bmatrix} \xi;$$

then it follows from Proposition 4.2 that $V(x) = x^\top P x$ is a qLRF.

Figure 4.1 presents a schematic overview of all results derived in sections 2, 3.1, 3.2, and 4.

Interestingly, the existence of a qLRF implies the existence of a qLKF under the additional assumption that the system under study is a linear DDE only. The existence of an LRF and the existence of a pLRF, on the other hand, do imply the existence of an LKF and a pLKF, respectively, for general DDIs (as opposed to for linear DDEs only).

In the next section results on contractive sets for DDIs will be established.

5. Contractive sets for DDIs. Contractive sets are at the basis of many control techniques (see, e.g., [5]), and it is well known that the sublevel sets of an LF are λ-contractive sets. Next, it is established that the existence of a λ-contractive set and a λ-$\mathcal{D}$-contractive set is equivalent to the existence of an LKF and an LRF, respectively. Both results are established via the sublevel sets of an LKF and an LRF, respectively. Recall that a contractive set is by assumption a convex and compact set with the origin in its interior; see Definitions 2.3 and 3.2.

PROPOSITION 5.1. *Suppose that system (3.1) is homogeneous[1] of order 1. The following two statements are equivalent:*
  (i) *The difference inclusion (3.1) admits a continuous and convex LF that is positively homogeneous of order t for some $t \in \mathbb{Z}_{\geq 1}$.*
  (ii) *The difference inclusion (3.1) admits a λ-contractive set for some $\lambda \in \mathbb{R}_{[0,1)}$.*

The proof of Proposition 5.1 can be obtained from the results derived in [4, 5, 37]. Note that the most common LF candidates, such as quadratic and norm-based functions, are inherently continuous and convex. Moreover, continuity is a desirable

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[1]For example, linear difference inclusions are homogeneous of order 1.

property, as continuous LFs guarantee that the corresponding type of stability does not have zero robustness; see, e.g., [31].

Unfortunately, it remains unclear what a contractive set $\bar{\mathbb{V}} \subset \mathbb{R}^{(h+1)n}$ implies for the DDI (2.1) and for the trajectories $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ in the original state space $\mathbb{R}^n$, in particular. The above observation indicates an important drawback of LKFs. While the DDI (2.1) admits an LKF if and only if the system is $\mathcal{KL}$-stable, an LKF does not provide a contractive set in the original, nonaugmented state space. An LRF is based on particular Lyapunov conditions that involve the nonaugmented system, rather than the augmented one. As such, in contrast to an LKF, an LRF, if it exists, provides a type of contractive set for the nonaugmented system. The above discussion indicates, apart from a lower complexity, another advantage of the Lyapunov–Razumikhin method over the Lyapunov–Krasovskii approach.

PROPOSITION 5.2. *Suppose that the DDI (2.1) is $\mathcal{D}$-homogeneous[2] of order 1. The following two statements are equivalent:*

(i) *The DDI (2.1) admits a continuous and convex LRF that is positively homogeneous of order t for some $t \in \mathbb{Z}_{\geq 1}$.*

(ii) *The DDI (2.1) admits a λ-$\mathcal{D}$-contractive set for some $\lambda \in \mathbb{R}_{[0,1)}$.*

Proof. First, the relation (i)$\Rightarrow$(ii) is proved. Consider a sublevel set of V, i.e., $\mathbb{V} := \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$. As $V : \mathbb{R}^n \to \mathbb{R}_+$ is continuous and convex the set $\mathbb{V}$ is [6] closed and convex, respectively. Moreover, boundedness follows from the $\mathcal{K}_\infty$ upper bound on the function V. Furthermore, if $\max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta)) \leq 1$, then it follows from (3.9b) that $V(x^+) \leq \rho$. Hence, as V is positively homogeneous, $V(\rho^{-\frac{1}{t}}x^+) \leq 1$, which yields $x^+ \in \rho^{\frac{1}{t}}\mathbb{V}$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{V}^{h+1}$ and all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Hence, $\mathbb{V}$ is a λ-$\mathcal{D}$-contractive set with $\lambda := \rho^{\frac{1}{t}}$ for the DDI (2.1).

Next, the relation (ii)$\Rightarrow$(i) is proved. Let $\mathbb{V}$ denote a λ-$\mathcal{D}$-contractive set for the DDI (2.1) and consider the *Minkowski function* (see, e.g., [35]) of $\mathbb{V}$, i.e.,

(5.1) $$V(x) := \inf\{\mu \in \mathbb{R}_{>0} \mid x \in \mu\mathbb{V}\}.$$

Then, it follows from claim 4 and claims 2 and 3 of Lemma 5.12.1 in [35] that the function V is continuous and convex, respectively. Furthermore, letting $a_1 := \max_{x \in \mathbb{V}} \|x\| > 0$ and $a_2 := \min_{x \in \partial\mathbb{V}} \|x\| > 0$ yields

$$a_1^{-1}\|x\| \leq V(x) \leq a_2^{-1}\|x\|.$$

Next, consider any $\nu \in \mathbb{R}_{>0}$ and let $\mathbf{x}_{[-h,0]} \in (\nu\mathbb{V})^{h+1}$. Then, $\nu^{-1}\mathbf{x}_{[-h,0]} \in \mathbb{V}^{h+1}$ and therefore $F(\nu^{-1}\mathbf{x}_{[-h,0]}) \subseteq \lambda\mathbb{V}$. As the DDI (2.1) is assumed to be $\mathcal{D}$-homogeneous of order 1, it follows that $F(\mathbf{x}_{[-h,0]}) = \nu F(\nu^{-1}\mathbf{x}_{[-h,0]}) \subseteq \lambda(\nu\mathbb{V})$. Thus, it was shown that if $\mathbb{V}$ is a λ-$\mathcal{D}$-contractive set, then $\nu\mathbb{V}$ is a λ-$\mathcal{D}$-contractive set as well. As the set $\nu\mathbb{V}$ is λ-$\mathcal{D}$-contractive for all $\nu \in \mathbb{R}_{>0}$, it follows that if $x^+ \in \partial(\mu\mathbb{V})$, for some $\mu \in \mathbb{R}_{>0}$ and some $x^+ \in F(\mathbf{x}_{[-h,0]})$, then there exists a $\theta \in \mathbb{Z}_{[-h,0]}$ such that $x(\theta) \in \partial(\mu(\lambda^{-1}\mathbb{V}))$. The above implies that

$$V(x^+) = \inf\{\mu \in \mathbb{R}_{>0} \mid x^+ \in \mu\mathbb{V}\}$$
$$\leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \inf\{\mu \in \mathbb{R}_{>0} \mid x(\theta) \in \mu(\lambda^{-1}\mathbb{V})\} = \max_{\theta \in \mathbb{Z}_{[-h,0]}} \lambda V(x(\theta))$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x^+ \in F(\mathbf{x}_{[-h,0]})$. Therefore, the candidate function (5.1) satisfies the hypothesis of Theorem 3.8 with $\alpha_1(s) := a_1^{-1}s \in \mathcal{K}_\infty$, $\alpha_2(s) :=$

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[2]For example, linear DDIs are $\mathcal{D}$-homogeneous of order 1.

$a_2^{-1}s \in \mathcal{K}_\infty$, and $\rho := \lambda \in \mathbb{R}_{[0,1)}$. As (5.1) satisfies $V(sx) = sV(x)$ for all $s \in \mathbb{R}_+$, the proof is complete. □

Note that the assumptions under which the statements of Propositions 5.1 and 5.2 were proved, i.e., regarding the properties of the contractive sets and the homogeneity of the systems, are standard assumptions for the type of results derived in this section; see, e.g., [4, 5, 37]. Furthermore, Proposition 5.2 recovers Proposition 5.1 and similar results in [4, 5, 37] as a particular case, i.e., for $h = 0$.

Suppose that the DDI (2.1) and system (3.1) are $\mathcal{D}$-homogeneous and homogeneous of order 1, respectively. Moreover, suppose that the DDI (2.1) admits a set $\mathbb{V} \subset \mathbb{R}^n$ which is λ-$\mathcal{D}$-contractive. Then, it follows from Proposition 5.2 that the DDI (2.1) admits an LRF. Moreover, it follows from Theorem 4.1 that the DDI (2.1) admits an LKF which in turn, via Proposition 5.1, guarantees the existence of a λ-contractive set for the augmented state system (3.1).

Suppose again that the DDI (2.1) is $\mathcal{D}$-homogeneous of order 1 and it admits an LKF that satisfies the hypothesis of Proposition 4.2 or Corollary 4.3. Then, from Proposition 4.2 or Corollary 4.3 it follows that there exists an LRF and hence a $\mathbb{V} \subset \mathbb{R}^n$ which is λ-$\mathcal{D}$-contractive.

In the next section we proceed to the illustration of the applicability of the developed Lyapunov methods to stability analysis and stabilizing controller synthesis for linear polytopic DDIs.

6. Synthesis of quadratic Lyapunov functions. The synthesis problem for a quadratic LF can be solved efficiently via semidefinite programming. Therefore, in what follows we restrict ourselves to this class of candidate functions. However, the results derived in the preceding sections are not restricted to a particular type of LF candidate. In fact, since the augmented state system (3.1) is a standard difference inclusion, synthesis techniques for LF candidates such as polyhedral LFs [5, 30], composite LFs [18], and polynomial LFs [39] can be applied directly to obtain an LKF of a corresponding type. In what follows we consider the linear DDI

$$(6.1) \qquad x(k+1) \in \left\{ \sum_{i=-h}^{0} (A_i x(k+i) + B_i u(k+i)) \middle| (A_i, B_i) \in \mathcal{M}_i, \, i \in \mathbb{Z}_{[-h,0]} \right\},$$

with $k \in \mathbb{Z}_+$ and where $\mathcal{M}_i := \mathrm{co}(\{(\hat{A}_{i,l_i}, \hat{B}_{i,l_i})\}_{l_i \in \mathbb{Z}_{[0,L_i]}}) \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, $L_i \in \mathbb{Z}_{\geq 1}$, and $i \in \mathbb{Z}_{[-h,0]}$.

REMARK 2. *Linear DDIs, such as (6.1), can be found within many fields. Apart from the obvious class of uncertain linear systems, networked control systems can be modeled [14, 45] by linear DDIs as well.*

Next, several hypotheses which include a linear matrix inequality that yields, if feasible, an LKF for system (6.1) will be presented. First, stability analysis of system (6.1) with zero input, i.e., $u(k) = 0$ for all $k \in \mathbb{Z}_{\geq -h}$, is discussed. Therefore, let

$$\bar{A}_{l_0,\ldots,l_{-h}} := \begin{bmatrix} \hat{A}_{0,l_0} & \cdots & \hat{A}_{-h+1,l_{-h+1}} & \hat{A}_{-h,l_{-h}} \\ & I_{hn} & & 0_{hn \times n} \end{bmatrix}$$

and let $\bar{\rho} \in \mathbb{R}_{[0,1)}$. Recall that $I_n \in \mathbb{R}^{n \times n}$ and $0_{n \times m} \in \mathbb{R}^{n \times m}$ denote the nth dimensional identity matrix and a rectangular matrix with all elements equal to zero, respectively.

PROPOSITION 6.1. *If there exists a symmetric matrix $\bar{P} \in \mathbb{R}^{(h+1)n \times (h+1)n}$ such that*

$$(6.2) \qquad \begin{bmatrix} \bar{\rho}\bar{P} & * \\ \bar{P}\bar{A}_{l_0,\dots,l_{-h}} & \bar{P} \end{bmatrix} \succ 0 \quad \forall l_i \in \mathbb{Z}_{[0,L_i]}, \quad \forall i \in \mathbb{Z}_{[-h,0]},$$

then system (6.1) *with zero input is GES.*

   Proof. Letting $\xi(k) = \text{col}(\{x(l)\}_{l \in \mathbb{Z}_{[k-h,k]}})$ yields

$$(6.3) \qquad \xi(k+1) \in \left\{ \bar{A}\xi(k) \mid \bar{A} \in \bar{\mathcal{M}} \right\}, \quad k \in \mathbb{Z}_+,$$

where $\bar{\mathcal{M}} := \text{co}(\{\bar{A}_{l_0,\dots,l_{-h}}\}_{(l_0,\dots,l_{-h}) \in \mathbb{Z}_{[0,L_0]} \times \cdots \times \mathbb{Z}_{[0,L_{-h}]}})$. Applying the Schur complement to (6.2) yields $\bar{P} \succ 0$ and

$$\bar{A}_{l_0,\dots,l_{-h}}^{\top} \bar{P} \bar{A}_{l_0,\dots,l_{-h}} - \bar{\rho}\bar{P} \prec 0 \quad \forall l_i \in \mathbb{Z}_{[0,L_i]}, \quad \forall i \in \mathbb{Z}_{[-h,0]}.$$

As all $\bar{A} \in \bar{\mathcal{M}}$ are a convex combination of $\bar{A}_{l_0,\dots,l_{-h}}$, $l_i \in \mathbb{Z}_{[0,L_i]}$, $i \in \mathbb{Z}_{[-h,0]}$, it follows that the candidate LKF $\bar{V}(\mathbf{x}_{[-h,0]}) = \bar{V}(\xi) = \xi^{\top}\bar{P}\xi$ satisfies (3.2b) for system (6.3). Moreover, this candidate LKF also satisfies (3.2a) with $\alpha_1(s) := \lambda_{\min}(\bar{P})s^2$ and $\alpha_2(s) := \lambda_{\max}(\bar{P})s^2$. From Corollary 3.5 it then follows that system (6.1) with zero input, i.e., $u(k) = 0$, $k \in \mathbb{Z}_{\geq -h}$, is GES. □

   When stabilizing controller synthesis is of concern, different augmentations of the state vector lead to different controller synthesis problems. First, let $\bar{\rho} \in \mathbb{R}_{[0,1)}$, $\xi(k) = \text{col}(\{x(l)\}_{l \in \mathbb{Z}_{[k-h,k]}})$ and let $\tilde{A}_{i,l_i} := \hat{A}_{i,l_i}G + \hat{B}_{i,l_i}Y$.

PROPOSITION 6.2. *Suppose there exist a symmetric matrix $\bar{P} \in \mathbb{R}^{(h+1)n \times (h+1)n}$, a matrix $G \in \mathbb{R}^{n \times n}$, and a matrix $Y \in \mathbb{R}^{m \times n}$ such that*

$$\begin{bmatrix} \bar{\rho}\bar{P} & * \\ \begin{bmatrix} \tilde{A}_{0,l_0} & \cdots & \tilde{A}_{-h+1,l_{-h+1}} & \tilde{A}_{-h,l_{-h}} \\ G & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & G & 0 \end{bmatrix}^{\top} \begin{bmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{bmatrix} + \begin{bmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{bmatrix}^{\top} - \bar{P} \end{bmatrix} \succ 0$$

for all $l_i \in \mathbb{Z}_{[0,L_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Then, system (6.1) *in closed loop with the controller $u(k) = Kx(k)$, $k \in \mathbb{Z}_+$, where $K = YG^{-1}$, is GES.*

   Proof. Substituting $Y = KG$, transposing, and using Theorem 1 in [10] or Theorem 3 in [9] yields $\bar{P} \succ 0$ and

$$\begin{bmatrix} (\hat{A}_{0,l_0} + \hat{B}_{0,l_0}K)^{\top} & \\ \vdots & I_{hn} \\ \vdots & \\ (\hat{A}_{-h,l_{-h}} + \hat{B}_{-h,l_{-h}}K)^{\top} & 0_{n \times hn} \end{bmatrix} \bar{P} \begin{bmatrix} (\hat{A}_{0,l_0} + \hat{B}_{0,l_0}K)^{\top} & \\ \vdots & I_{hn} \\ \vdots & \\ (\hat{A}_{-h,l_{-h}} + \hat{B}_{-h,l_{-h}}K)^{\top} & 0_{n \times hn} \end{bmatrix}^{\top} - \bar{\rho}\bar{P} \prec 0$$

for all $l_i \in \mathbb{Z}_{[0,L_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. The remainder of the proof can then be obtained from the proof of Proposition 6.1. □

   Augmenting the state vector with the delayed states and the delayed inputs, i.e., $\xi(k) = \text{col}(\{u(l)\}_{l \in \mathbb{Z}_{[k-h,k-1]}}, \{x(l)\}_{l \in \mathbb{Z}_{[k-h,k]}})$, yields

$$(6.4) \qquad \xi(k+1) \in \left\{ \bar{A}\xi(k) + \bar{B}u(k) \mid (\bar{A}, \bar{B}) \in \tilde{\mathcal{M}} \right\}, \quad k \in \mathbb{Z}_+,$$

where $\tilde{\mathcal{M}} := \mathrm{co}(\{(\tilde{A}_{l_0,\ldots,l_{-h}}, \tilde{B}_{l_0})\}_{(l_0,\ldots,l_{-h})\in\mathbb{Z}_{[0,L_0]}\times\cdots\times\mathbb{Z}_{[0,L_{-h}]}})$ and

$$\tilde{A}_{l_0,\ldots,l_{-h}} := \begin{bmatrix} \hat{A}_{0,l_0} & \ldots & \hat{A}_{-h,l_{-h}} & \hat{B}_{-1,l_{-1}} & \ldots & \hat{B}_{-h,l_{-h}} \\ I_{hn} & 0_{hn\times n} & & 0_{hn\times hm} & \\ 0_{hm\times(h+1)n} & & 0_{m\times m} & 0_{(h-1)m\times m} \\ & & I_{(h-1)m} & 0_{(h-1)m\times m} \end{bmatrix}, \quad \tilde{B}_{l_0} := \begin{bmatrix} \hat{B}_{0,l_0} \\ 0_{hn\times m} \\ I_m \\ 0_{(h-1)m\times m} \end{bmatrix}.$$

PROPOSITION 6.3. *Let $\bar{\rho} \in \mathbb{R}_{[0,1)}$. Suppose there exist a matrix $Y \in \mathbb{R}^{m\times((h+1)n+hm)}$ and a symmetric matrix $Z \in \mathbb{R}^{((h+1)n+hm)\times((h+1)n+hm)}$ such that*

$$\begin{bmatrix} \bar{\rho}Z & * \\ \tilde{A}_{l_0,\ldots,l_{-h}}Z + \tilde{B}_{l_0}Y & Z \end{bmatrix} \succ 0$$

for all $l_i \in \mathbb{Z}_{[0,L_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Then, system (6.4) in closed loop with the controller $u(k) = K\xi(k)$, $k \in \mathbb{Z}_+$, where $K = YZ^{-1}$, is GES.

Proof. Substituting $Y = KZ$, applying a congruence transformation with a matrix that has Z^{-1} on its diagonal and zero elsewhere, and applying the Schur complement yields $Z^{-1} \succ 0$ and

$$(\tilde{A}_{l_0,\ldots,l_{-h}} + \tilde{B}_{l_0}K)^\top Z^{-1}(\tilde{A}_{l_0,\ldots,l_{-h}} + \tilde{B}_{l_0}K) - \bar{\rho}Z^{-1} \prec 0$$

for all $l_i \in \mathbb{Z}_{[0,L_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Let $\bar{V}(\xi) = \xi^\top Z^{-1}\xi$. Pre- and postmultiplying the above inequality with $\xi^\top$ and ξ, respectively, yields that $\bar{V}(\xi^+) - \bar{\rho}\bar{V}(\xi) \le 0$ for all $\xi^+ \in \{(\bar{A} + \bar{B}K)\xi \mid (\bar{A}, \bar{B}) \in \tilde{\mathcal{M}}\}$. Standard Lyapunov arguments can then be used to show that system (6.4) is GES. $\square$

As $\xi(k)$ is also a function of $u(k)$, Lemma 3.3 cannot be used and hence it is established only that the augmented state system (6.4) is GES. To establish stability of the DDI (6.1) requires a modified version of Lemma 3.3 which is omitted here for brevity.

Next, a controller synthesis algorithm based on the existence of an LRF will be presented. Note that therein both the stability analysis and the controller synthesis problem are equivalent, as opposed to the LKF setup presented above. Let $\rho \in \mathbb{R}_{[0,1)}$.

PROPOSITION 6.4. *Suppose there exist $\delta_i \in \mathbb{R}_+$, $i \in \mathbb{Z}_{[-h,0]}$, a symmetric matrix $Z \in \mathbb{R}^{n\times n}$, and a matrix $Y \in \mathbb{R}^{n\times m}$ such that $\sum_{i=-h}^0 \delta_i \le 1$ and*

$$(6.5) \quad \begin{bmatrix} \rho\delta_0 Z & & 0 & * \\ & \ddots & & \\ 0 & & \rho\delta_{-h}Z & * \\ \hat{A}_{0,l_0}Z + \hat{B}_{0,l_0}Y & \ldots & \hat{A}_{-h,l_{-h}}Z + \hat{B}_{-h,l_{-h}}Y & Z \end{bmatrix} \succ 0$$

for all $l_i \in \mathbb{Z}_{[0,L_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Then system (6.1) in closed loop with controller $u(k) = Kx(k)$, $k \in \mathbb{Z}_+$, where $K = YZ^{-1}$, is GES.

Proof. Applying a congruence transformation with a matrix that has Z^{-1} on its diagonal and zero elsewhere, substituting $Z^{-1} = P$, $YZ^{-1} = K$, and applying the Schur complement to (6.5) yields $P \succ 0$ and

$$\begin{bmatrix} \rho\delta_0 P & & 0 \\ & \ddots & \\ 0 & & \rho\delta_{-h}P \end{bmatrix} - \begin{bmatrix} \left(\hat{A}_{0,l_0} + \hat{B}_{0,l_0}K\right)^\top \\ \vdots \\ \left(\hat{A}_{-h,l_{-h}} + \hat{B}_{-h,l_{-h}}K\right)^\top \end{bmatrix} P \begin{bmatrix} \left(\hat{A}_{0,l_0} + \hat{B}_{0,l_0}K\right)^\top \\ \vdots \\ \left(\hat{A}_{-h,l_{-h}} + \hat{B}_{-h,l_{-h}}K\right)^\top \end{bmatrix}^\top \succ 0$$

for all $l_i \in \mathbb{Z}_{[0,L_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Next, consider the candidate LRF $V(x) := x^\top P x$. As for all $i \in \mathbb{Z}_{[-h,0]}$ all $(A_i, B_i) \in \mathcal{M}_i$ are a convex combination of $(\hat{A}_{i,l_i}, \hat{B}_{i,l_i})$, $l_i \in \mathbb{Z}_{[0,L_i]}$, it follows that system (6.1) satisfies

$$(6.6) \qquad V(x^+) - \rho \sum_{i=-h}^{0} \delta_i V(x(i)) \leq 0$$

for all $(A_i, B_i) \in \mathcal{M}_i$ and all $i \in \mathbb{Z}_{[-h,0]}$. As

$$\sum_{i=-h}^{0} \delta_i V(x(i)) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta)),$$

it is obtained that the candidate LRF $V(x) = x^\top P x$ satisfies (3.9b) for system (6.1). Moreover, the above candidate LRF also satisfies (3.9a) with $\alpha_1(s) := \lambda_{\min}(P)s^2$ and $\alpha_2(s) := \lambda_{\max}(P)s^2$. Thus, it follows from Corollary 3.9 that system (6.1) is GES. $\square$

The matrix inequality (6.5) is bilinear in the scalars δ_i and the matrix Z. The set $\mathbb{R}_{[0,1]}^{h+1}$, where the sequence of scalar variables $\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ is allowed to take values, can be discretized using a gridding technique. Then, solving (6.5) for each point in the resulting grid amounts to solving a linear matrix inequality. Thus, a feasible solution to (6.5) can be obtained by solving a sequence of linear matrix inequalities. Observe that if Z, Y, and $\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ satisfy (6.5) with $\sum_{i=-h}^{0} \delta_i < 1$, then there exist $\{\hat{\delta}_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ such that $\sum_{i=-h}^{0} \hat{\delta}_i = 1$ and such that Z, Y, and $\{\hat{\delta}_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ also satisfy (6.5). Therefore, it suffices to consider only those points in the grid such that $\sum_{i=-h}^{0} \delta_i = 1$. Alternatively, also after defining a grid for the scalars δ_i, $i \in \mathbb{Z}_{[0,h]}$, branch and bound optimization algorithms [29] can be used to obtain a computationally more efficient solution to the matrix inequality (6.5).

6.1. Illustrative example. Next, the various synthesis techniques presented in this section are applied to control a DC motor over a communication network. This is a benchmark example for networked control systems, taken from [42]. We examine a network which introduces uncertain time-varying input delays, which yields

$$(6.7) \qquad \begin{aligned} \begin{bmatrix} \dot{i}_a(t) \\ \dot{\omega}(t) \end{bmatrix} &= \begin{bmatrix} -27.47 & -0.09 \\ 345.07 & -1.11 \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} 5.88 \\ 0 \end{bmatrix} e_a(t), \\ e_a(t) &= u(k) \quad \forall t \in [t_k + \tau(k), t_{k+1} + \tau(k+1)), \end{aligned}$$

where i_a is the armature current, ω is the angular velocity of the motor, and the input signal is the armature voltage e_a. Furthermore, $t_k = kT_s$, $k \in \mathbb{Z}_+$, is the sample time, $T_s \in \mathbb{R}_+$ denotes the sampling period, and $u(k) \in \mathbb{R}^m$ is the control action generated at time $t = t_k$. $\tau(k) \in \mathbb{R}_{[0,\bar{\tau}]}$ denotes the input delay at time $k \in \mathbb{Z}_+$ and $\bar{\tau} \in \mathbb{R}_{[0,T_s]}$ is the maximal delay induced by the network. It is assumed that $\bar{\tau} \leq T_s$; i.e., the delay is always smaller than or equal to the sampling period. Furthermore, it is assumed that $u(t) = u_{\text{initial}}$ for all $t \in \mathbb{R}_{[0,\tau(0))}$, where $u_{\text{initial}} \in \mathbb{R}^m$ is a predetermined constant vector. Note that, as all controllers in this section are time invariant, both delays on the measurement link and delays on the link from the controller to the plant can be lumped [45] into a single delay on the latter link and hence output delays are implicitly taken into account. For a sampling time $T_s = 0.01$s the matrices $A_d \in \mathbb{R}^{n \times n}$, $B_d \in \mathbb{R}^{n \times m}$, which define the corresponding discrete-time model of the system, were

obtained, and their numerical values can be found in Appendix B. Moreover, the time-varying delay can be overapproximated by a polytope [14] which yields

$$(6.8) \qquad x(k + 1) \in \{A_d x(k) + B_0 u(k) + B_{-1} u(k - 1) \mid B_0 \in B_d - \Delta, \ B_{-1} \in \Delta\},$$

with $k \in \mathbb{Z}_+$, where $\Delta := \mathrm{co}(\{\hat{\Delta}_l\}_{l \in \mathbb{Z}_{[0,L]}})$. The matrices $\hat{\Delta}_l$ were obtained using the Cayley–Hamilton technique presented in [14], and numerical results for various values of $\bar{\tau}$ can be found in Appendix B.

Using Proposition 6.1, it can be established[3] that system (6.8) is open-loop stable and an LKF can be obtained for $\bar{\rho} \geq 0.956$. For $\bar{\rho} < 0.956$, no LKF could be obtained using Proposition 6.1. Therefore, $\rho = 0.8$ is chosen to impose a faster convergence via controller synthesis. First, taking $\tau(k) \in \mathbb{R}_{[0,0.48T_s]}$ and using Proposition 6.2 yields the qLKF matrix and corresponding controller matrix

$$P_{\mathrm{LKF}} = \begin{bmatrix} 25.1650 & 3.3515 & 17.2159 & 1.8791 \\ 3.3515 & 0.5249 & 2.3463 & 0.3022 \\ 17.2159 & 2.3463 & 51.7357 & 6.9770 \\ 1.8791 & 0.3022 & 6.9770 & 1.1497 \end{bmatrix},$$

$$K_{\mathrm{LKF}} = \begin{bmatrix} -14.9462 & -1.7247 \end{bmatrix}.$$

However, for $\bar{\tau} > 0.48T_s$ Proposition 6.2 no longer provides a feasible solution. Second, taking $\tau(k) \in \mathbb{R}_{[0,0.424T_s]}$, Proposition 6.4 yields the LRF matrix and corresponding controller matrix

$$P_{\mathrm{LRF}} = \begin{bmatrix} 7.0084 & 0.5100 \\ 0.5100 & 0.0380 \end{bmatrix},$$

$$K_{\mathrm{LRF}} = \begin{bmatrix} -10.9567 & -0.8047 \end{bmatrix},$$

along with $\delta = 0.75$. However, for $\bar{\tau} > 0.424T_s$ Proposition 6.4 no longer provides a feasible solution. Thirdly, using Proposition 6.3 for $\tau(k) \in \mathbb{R}_{[0,T_s]}$ we find the LKF matrix and corresponding controller matrix

$$P_{\mathrm{LKF},[x;u]} = \begin{bmatrix} 78.6145 & 8.4224 & 5.9068 \\ 8.4224 & 1.5622 & 0.7916 \\ 5.9068 & 0.7916 & 1.0523 \end{bmatrix},$$

$$K_{\mathrm{LKF},[x;u]} = \begin{bmatrix} -5.7555 & -0.6024 & -0.0886 \end{bmatrix}.$$

Hence, for the Lyapunov–Krasovskii approach, it can be concluded that the stabilizing controller synthesis conditions presented in Proposition 6.2 are more conservative than those presented in Proposition 6.3. For the Lyapunov–Razumikhin method, it is worth pointing out that for $\delta = 0.5$ and $\bar{\tau} > 0.35T_s$ no stabilizing controller is obtained via Proposition 6.4. This indicates the additional freedom provided by the introduction of δ as a free variable. Furthermore, it can also be observed from the example that the Lyapunov–Krasovskii approach can be used to find a stabilizing controller, i.e., via Proposition 6.2, for a larger range of time-varying delays when compared to the Lyapunov–Razumikhin approach, i.e., via Proposition 6.4. This observation confirms the results that were derived in this paper. Obviously, this does not discard the Lyapunov–Razumikhin method as a valuable technique, as the Lyapunov–Razumikhin method has a smaller computational complexity when compared to Lyapunov–Krasovskii approach and provides a contractive set that is particular to time-delay systems.

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[3]Numerical results were obtained using the Multi-Parametric Toolbox v.2.6.2 and SeDuMi v.1.1.

7. Conclusions. A comprehensive collection of Lyapunov techniques that can be used for stability analysis of DDIs was presented. Both the Lyapunov–Krasovskii approach and the Lyapunov–Razumikhin method were discussed. It was shown that a DDI is $\mathcal{KL}$-stable if and only if it admits an LKF. Moreover, it was shown that the existence of an LRF is a sufficient condition but not a necessary condition for $\mathcal{KL}$-stability. Furthermore, it was shown that the existence of an LRF is a sufficient condition for the existence of an LKF and that only under certain additional assumptions is the converse true. Then, it was shown that an LRF induces a family of sets with certain contraction properties that are particular to time-delay systems, while the LKF was shown to induce a type of contractive set similar to those induced by a classical LF. For linear DDIs, the class of quadratic Lyapunov functions was used to illustrate the application of the results derived in this paper in terms of stability analysis and controller synthesis for both LKFs and LRFs, respectively.

While time-invariant DDIs were considered in this paper, uncertain time-varying delays can be incorporated, without any conservatism, similarly as in, e.g., [17]. A further extension of the results in this paper to time-varying systems can be attained, under some additional assumptions, using the techniques explained in, e.g., [41].

Appendix A. Proof of Lemma 3.3. First, some preliminary results are derived. Let $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and let $\xi := \mathrm{col}(\{x(l)\}_{l \in \mathbb{Z}_{[-h,0]}})$. On a finite dimensional vector space $\mathbb{R}^n$ all norms are equivalent [28]; i.e., for any two norms $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$, there exist constants $\underline{c}, \overline{c} \in \mathbb{R}_{>0}$ such that $\underline{c}\|x\|_{p_1} \le \|x\|_{p_2} \le \overline{c}\|x\|_{p_1}$ for all $x \in \mathbb{R}^n$. Hence, for any norm $\|\cdot\|_{p_1}$ there exist constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\text{(A.1)} \quad \|\mathbf{x}_{[-h,0]}\|_{p_1} = \left\| \begin{bmatrix} \|x(0)\|_{p_1} \\ \vdots \\ \|x(-h)\|_{p_1} \end{bmatrix} \right\|_\infty \ge \left\| c_1 \begin{bmatrix} \|x(0)\|_\infty \\ \vdots \\ \|x(-h)\|_\infty \end{bmatrix} \right\|_\infty = c_1\|\xi\|_\infty \ge c_1 c_2 \|\xi\|_{p_1}.$$

Similarly, there exist constants $c_3, c_4 \in \mathbb{R}_{>0}$ such that

$$\text{(A.2)} \quad \|\mathbf{x}_{[-h,0]}\|_{p_1} = \left\| \begin{bmatrix} \|x(0)\|_{p_1} \\ \vdots \\ \|x(-h)\|_{p_1} \end{bmatrix} \right\|_\infty \le \left\| c_3 \begin{bmatrix} \|x(0)\|_\infty \\ \vdots \\ \|x(-h)\|_\infty \end{bmatrix} \right\|_\infty = c_3\|\xi\|_\infty \le c_3 c_4 \|\xi\|_{p_1}.$$

Furthermore, the definition of the p-norm yields

$$\text{(A.3)} \qquad \|\xi\|_p^p = \sum_{i=1}^{(h+1)n} |[\xi]_i|^p = \sum_{i=-h}^{0} \sum_{j=1}^{n} |[x(i)]_j|^p \ge \sum_{j=1}^{n} |[x(0)]_j|^p = \|x(0)\|_p^p.$$

From (A.3) and the fact that $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(s) := s^{\frac{1}{p}}$ and $p \in \mathbb{Z}_{>0}$ is strictly increasing, it follows that $\|x(0)\|_p \le \|\xi\|_p$ for all $p \in \mathbb{Z}_{>0}$. It is straightforward to see from the definition of the infinity norm that $\|x(0)\|_\infty \le \|\xi\|_\infty$ holds as well. In what follows, let $\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi)$ correspond to $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$.

Proof of claim (i). Suppose that the DDI (2.1) is GAS. As the DDI (2.1) is globally attractive, it follows from (A.1) that there exists a $c_5 \in \mathbb{R}_{>0}$ such that

$$\lim_{k \to \infty} \|\bar{\phi}(k,\xi)\| \le \lim_{k \to \infty} c_5 \|\phi_{[k-h,k]}(\mathbf{x}_{[-h,0]})\| = 0$$

for all $\xi \in \mathbb{R}^{(h+1)n}$ and all $\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi)$. Thus, we obtain that the origin of (3.1) is globally attractive. Furthermore, as the DDI (2.1) is LS, it follows from (A.1) that for all $\varepsilon \in \mathbb{R}_{>0}$ there exist $\delta, c_5 \in \mathbb{R}_{>0}$, with $\delta \le \varepsilon$, such that if $\|\mathbf{x}_{[-h,0]}\| \le \delta$, then

$$\|\bar{\phi}(k,\xi)\| \le c_5 \|\phi_{[k-h,k]}(\mathbf{x}_{[-h,0]})\| \le c_5 \varepsilon$$

for all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ and all $k \in \mathbb{Z}_+$. Moreover, it follows from (A.2) that there exists a $c_6 \in \mathbb{R}_{>0}$ such that $\|\mathbf{x}_{[-h,0]}\| \le c_6\|\xi\|$. Therefore, we conclude that for every $\bar{\varepsilon} := c_5\varepsilon \in \mathbb{R}_{>0}$ there exists a $\bar{\delta} := \frac{1}{c_6}\delta \in \mathbb{R}_{>0}$ such that if $\|\xi\| \le \bar{\delta}$, and hence $\|\mathbf{x}_{[-h,0]}\| \le \delta$, then

$$\|\bar{\phi}(k,\xi)\| \le c_5\|\boldsymbol{\phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]})\| \le c_5\varepsilon = \bar{\varepsilon}$$

for all $\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi)$ and all $k \in \mathbb{Z}_+$. Thus, it was shown that (3.1) is LS and hence that (3.1) is GAS.

Next, suppose that system (3.1) is GAS. As the difference inclusion (3.1) is globally attractive, it follows from (A.3) that

$$\lim_{k\to\infty} \|\phi(k,\mathbf{x}_{[-h,0]})\| \le \lim_{k\to\infty} \|\bar{\phi}(k,\xi)\| = 0$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Thus, we obtain that the origin of (2.1) is globally attractive. Furthermore, using (A.3) and as (3.1) is LS, it follows that for all $\bar{\varepsilon} \in \mathbb{R}_{>0}$ there exists a $\bar{\delta} \in \mathbb{R}_{>0}$ such that if $\|\xi\| \le \bar{\delta}$, then

$$\|\phi(k,\mathbf{x}_{[-h,0]})\| \le \|\bar{\phi}(k,\xi)\| \le \bar{\varepsilon}$$

for all $\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi)$ and all $k \in \mathbb{Z}_+$. Moreover, it follows from (A.1) that there exists a $c_7 \in \mathbb{R}_{>0}$ such that $\|\xi\| \le c_7\|\mathbf{x}_{[-h,0]}\|$. Therefore, we conclude that for every $\varepsilon := \bar{\varepsilon} \in \mathbb{R}_{>0}$ there exists a $\delta := \frac{1}{c_7}\bar{\delta} \in \mathbb{R}_{>0}$ such that if $\|\mathbf{x}_{[-h,0]}\| \le \delta$, and hence $\|\xi\| \le \bar{\delta}$, then

$$\|\phi(k,\mathbf{x}_{[-h,0]})\| \le \|\bar{\phi}(k,\xi)\| \le \bar{\varepsilon} = \varepsilon$$

for all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ and all $k \in \mathbb{Z}_+$. Thus, it was shown that (2.1) is LS and hence that (2.1) is GAS, which proves claim (i).

Proof of claim (ii). Suppose that the DDI (2.1) is $\mathcal{KL}$-stable. Then, it follows from Lemma 2.6 that the DDI (2.1) is GAS and that for $\delta(\varepsilon)$, $\lim_{\varepsilon\to\infty}\delta(\varepsilon) = \infty$ is an admissible choice. Hence, as $\bar{\delta} = \frac{1}{c_6}\delta$ and $\bar{\varepsilon} := c_5\varepsilon$ with $c_5, c_6 \in \mathbb{R}_{>0}$, it follows that $\lim_{\bar{\varepsilon}\to\infty}\bar{\delta}(\bar{\varepsilon}) = \infty$ is an admissible choice as well. Thus, using Lemma 2.6 again, it follows that system (3.1) is $\mathcal{KL}$-stable.

Conversely, suppose that system (3.1) is $\mathcal{KL}$-stable. Then, Lemma 2.6 yields that system (3.1) is GAS and that $\lim_{\bar{\varepsilon}\to\infty}\bar{\delta}(\bar{\varepsilon}) = \infty$ is an admissible choice. Hence, as $\delta = \frac{1}{c_7}\bar{\delta}$ with $c_7 \in \mathbb{R}_{>0}$ it follows that $\lim_{\varepsilon\to\infty}\delta(\varepsilon) = \infty$ is an admissible choice as well. Thus, using Lemma 2.6 again, it follows that the DDI (2.1) is $\mathcal{KL}$-stable.

Proof of claim (iii). Suppose that (2.1) is GES. Then

$$\|\phi(k,\mathbf{x}_{[-h,0]})\| \le c\|\mathbf{x}_{[-h,0]}\|\mu^k \quad \forall\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}, \quad \forall\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]}),$$

for all $k \in \mathbb{Z}_+$, and for some $c \in \mathbb{R}_{\ge 1}$ and $\mu \in \mathbb{R}_{[0,1)}$. It then follows from (A.1) and (A.2) that there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\|\bar{\phi}(k,\xi)\| \le c_1\|\boldsymbol{\phi}_{[k,k-h]}(\mathbf{x}_{[-h,0]})\| \le c_1 c_2 c\|\xi\|\mu^{k-h} \quad \forall\xi \in \mathbb{R}^{(h+1)n},$$

for all $\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi)$, and for all $k \in \mathbb{Z}_+$. As $\bar{c} := cc_1c_2\mu^{-h} \in \mathbb{R}_{\ge 1}$ and $\bar{\mu} := \mu \in \mathbb{R}_{[0,1)}$, it follows that (3.1) is GES.

Conversely, suppose that (3.1) is GES. Then

$$\|\bar{\phi}(k,\xi)\| \le \bar{c}\|\xi\|\bar{\mu}^k \quad \forall\xi \in \mathbb{R}^{(h+1)n}, \quad \forall\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi), \quad \forall k \in \mathbb{Z}_+,$$

and for some $\bar{c} \in \mathbb{R}_{\geq 1}$ and $\bar{\mu} \in \mathbb{R}_{[0,1)}$. It then follows from (A.3) and (A.1) that there exists a $c_3 \in \mathbb{R}_{>0}$ such that

$$\|\phi(k, \mathbf{x}_{[-h,0]})\| \leq \|\bar{\phi}(k, \xi)\| \leq c_3 c \|\mathbf{x}_{[-h,0]}\| \bar{\mu}^k \quad \forall \mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1},$$

for all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, and for all $k \in \mathbb{Z}_+$. As $c := c_3 \bar{c} \in \mathbb{R}_{\geq 1}$ and $\mu := \bar{\mu} \in \mathbb{R}_{[0,1)}$, it follows that (2.1) is GES, which completes the proof. $\quad\square$

Appendix B. Numerical values of $\boldsymbol{\Delta}$. The matrices A_d and B_d can be computed via MATLAB, which yields

$$A_d = e^{A_c T_s} = \begin{bmatrix} 0.7586 & -0.0008 \\ 2.9984 & 0.9876 \end{bmatrix}, \quad B_d = \int_0^{T_s} e^{A_c(T_s - \theta)} d\theta B_c = \begin{bmatrix} 0.0514 \\ 0.0924 \end{bmatrix}.$$

For $\bar{\tau} = 0.48 T_s$ the Cayley–Hamilton method presented in [14] yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0229 \\ 0.0663 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0053 \\ 0.0663 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0282 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that these are the generators of the polytope denoted by equation (9) in [14]. Letting $\bar{\tau} = 0.424 T_s$ yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0201 \\ 0.0605 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0048 \\ 0.0605 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0249 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and letting $\bar{\tau} = T_s$ yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0514 \\ 0.0924 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0074 \\ 0.0924 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0588 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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