

Mobius transformation between reflection coefficients at upstream and downstream sides of flame in thermoacoustics systems

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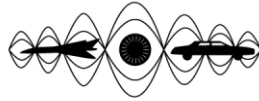
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Annual Congress of the International Institute of Acoustics and Vibration (IIAV)

MOBIUS TRANSFORMATION BETWEEN REFLECTION COEFFICIENTS AT UPSTREAM AND DOWNSTREAM SIDES OF FLAME IN THERMOACOUSTICS SYSTEMS

Mohammad Kojourimanesh, Viktor Kornilov and Philip de Goey

Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands.

email: m.kojourimanesh@tue.nl

Ines Lopez Arteaga

Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands.

KTH Royal Institute of Technology, Department of Aeronautical and Vehicle Engineering, Sweden.

An alternative approach to assess the conditions for the onset of thermoacoustics combustion instability is proposed. The method is based on the analysis of the relation between the reflection coefficient of combustor upstream section (subsystems) with respect to the burner/flame, R_{up} , and the input reflection coefficient, R_{in} , as observed from the inlet of the burner with flame and all downstream subsystems of a combustion appliance. The instability of the combustor can be determined by evaluating the system dispersion relation expressed in the terms of these two reflection coefficients. The properties of the relation between R_{in} , the reflection coefficient of the burner downstream subsystems, R_{dn} , and elements of the flame transfer matrix, TM , of the burner are investigated. This relation has the form of the well-known Mobius transformation. Using the well-developed theory of the transformation, we derive the necessary conditions of R_{dn} to ensure that the magnitude of R_{in} becomes less than 1 in a frequency range. This condition results in a passive thermoacoustics stability of the system's operation. Furthermore, an optimum value of R_{dn} is derived which provides a minimum value of R_{in} at given entries of the burner TM . The practical application of the developed theory provides suitable criteria and guidelines for designing passive acoustic properties at the downstream side of the combustion appliance.

Keywords: Combustion, Flame transfer function, Instability, Passive termination.

1. Introduction

During the development of a combustion appliance, thermo-acoustic stability of the system is a crucial requirement for designers. Acoustic network modelling is a powerful tool for analysing this type of instability [1–4]. Inputs of this analysis are the burner with flame transfer matrix with its entries defined in the complex domain, the physical properties of the unburnt and burned gases, and the acoustic characteristics of the upstream and downstream sides of the flame. Usually, the burner transfer matrix can be related to the burner transfer function which is defined as the ratio of the burner heat release rate oscillation (response) to the acoustic perturbation of the flow (stimulus). To obtain the flame transfer function of a burner with a flame, the common approach is determining the frequency response of the subsystem either experimentally or numerically and fitting the data with a properly selected function to extend it to the complex domain [3]. Besides, acoustic properties at both sides of the burner can be represented by

their reflection coefficients and could also be obtained by measuring them in the frequency domain supplemented by the procedure to extend them to the complex domain. Outputs of the acoustic network modelling tool are roots ($s_0 = \sigma_0 + i \omega_0$) of the determinant of the system's matrix. The real part of the derived root, σ_0 , indicates the instability ($\sigma_0 > 0$) or stability ($\sigma_0 < 0$) rate and the imaginary part defines eigenfrequencies of the system. This procedure gives a reasonably accurate prediction of (in)-stability, and it has been successfully applied in numerous studies before [4–6].

However, the conventional modelling approach does not provide a good overview of conditions and guidelines for designing the upstream and downstream sides of the flame such that the system would be stable. Also, the direct application of the conventional approach requires technically challenging measurements of acoustic variables (pressure p' and/or velocity v') at the hot side of the burner [4,7–10].

The main idea of the present contribution is to propose an alternative approach which needs measurement of two reflection coefficients, which will be signified as R_{up} and R_{in} , at the cold side of the flame instead of measuring the burner transfer function and reflection coefficient at the downstream (hot) side of the flame, which will be named R_{dn} . Then, the stability of the system is analysed by using the argument principle and the Nyquist plot of the products of these two reflection coefficients R_{up} and R_{in} without extending them into the complex domain. It will be derived that the reflection coefficient of the upstream of the flame, R_{up} , is an independent parameter while the input reflection coefficient, R_{in} , depends on the downstream reflection coefficient, R_{dn} , and the entries of the flame transfer matrix, TM [11]. The relation has the form of the well-known Mobius transformation. The principal results of this paper are based on the analysis of aforementioned dependency and introduction of simple geometric criteria that can be used to design an optimal termination at the downstream side of the flame.

In this paper, after providing the theoretical background of network modelling of thermo-acoustic systems, useful properties of the Mobius transformation are summarized. Then, a topological technique is applied to derive all possible cases between the R_{dn} and R_{in} planes. Next, the practical application of the developed theory is shown in an experimental example. Finally, some conclusions are drawn.

2. Formulation of dispersion relation in the terms of reflection coefficients

In this study, network modelling is used to describe the relationship between the upstream and downstream sides of the flame in a thermo-acoustic system. By considering linear acoustical plane waves with zero mean flow Mach number; the relations between upstream (f) and downstream (g) propagation waves, known as Riemann invariants, at both sides of the burner are [2,12]:

$$\begin{cases} f_1 = R_{up}(s) \cdot g_1 \\ f_2 = T(s)_{1,1} \cdot f_1 + T(s)_{1,2} \cdot g_1 \\ g_2 = T(s)_{2,1} \cdot f_1 + T(s)_{2,2} \cdot g_1 \\ g_2 = R_{dn}(s) \cdot f_2 \end{cases} \quad (1)$$

where R_{up} and R_{dn} are upstream and downstream reflection coefficients from the burner location; $T_{i,j}$ are the elements of a flame transfer matrix shown in Fig. 1. All coefficients of this system are functions of the complex frequency $s = \sigma + i\omega$.

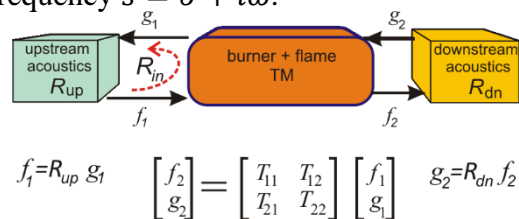


Figure 1: Generic representation of the thermo-acoustic model of a combustion system.

The nontrivial solution of the system of Eq. (1) and consequently the eigenvalues of this system could be obtained if the determinant of the corresponding system matrix becomes zero, so [3]

$$\Delta(s) \triangleq T(s)_{2,2} - R_{dn}(s)T(s)_{1,2} + R_{up}(s)T(s)_{2,1} - R_{up}(s)R_{dn}(s)T(s)_{1,1} = 0. \quad (2)$$

If $T(s)_{2,2} - R_{dn}(s)T(s)_{1,2} \neq 0$ then it can be written in the form of

$$\Delta(s) \triangleq 1 - R_{up}(s) R_{in}(s) = 0, \quad (3)$$

where

$$R_{in} = \frac{T(s)_{1,1} R_{dn}(s) - T(s)_{2,1}}{-R_{dn}(s) T(s)_{1,2} + T(s)_{2,2}}. \quad (4)$$

The expression for R_{in} , which is essentially the reflection coefficient of the burner terminated by R_{dn} , can be also derived from its definition $R_{in} \triangleq \frac{g_1}{f_1}$ by substituting in Eq. (1). In other words, R_{in} defines the ratio of the out-coming (reflecting) wave from the flame, g_1 , and the corresponding incoming (stimulus) wave to the flame, f_1 depicted in Fig. 1.

The analogy of Eq. (3) is widely used in the microwave active circuit design theory to seek for unconditional stability conditions of a transistor formed as two-port stability factors [13–15] and it has already been considered in the context of thermo-acoustic problems (see for instance [3,16–18]). The crucial advantage of this expression is that the instability of the combustion appliance can be determined by measuring two reflection coefficients at the cold side of the burner.

If Eq. (3) has *zeros* in the right half-plane (RHP) of the complex domain, then the system would be unstable and *vice versa*. If one is only interested in whether there is a *zero* of Eq. (3) located in the RHP or not, frequency responses of R_{up} and R_{in} are enough for analysing the system. Based on the argument principle [6], if the polar plot of $1 - R_{up}(i\omega) R_{in}(i\omega)$ turns around the origin, it means that a *zero* is located in the RHP so the system will be unstable. Due to the passive acoustical termination at the upstream side of the flame, i.e. $|R_{up}(i\omega)| \leq 1$, the magnitude of $R_{in}(i\omega)$ less than 1 results in a high chance of thermo-acoustic stability of the system's operation [11]. Hence, the magnitude of $R_{in}(i\omega)$ is a crucial factor in determining the stability of the system. Because $R_{in}(i\omega)$ depends on $R_{dn}(i\omega)$ and $T(i\omega)$ the analysis of this relation opens new options for the design of the system (in)-stability. This is the subject of next part of the current contribution.

2.1 Analysis of relation between R_{in} and R_{dn} .

For a fixed value of the frequency, the entries of the transfer matrix TM are fixed complex numbers and then Eq. (4) can be treated as a mapping of R_{dn} to R_{in} . The mapping has a form of the so-called Mobius (also known as bi-linear) transformation between R_{in} and R_{dn} , with the elements of the flame transfer matrix, TM , of the burner as the coefficients of the transformation. One of well-known properties of the Mobius transformation is that it maps circles in the plane of its argument to circles in the plane of its output [19,20]. The downstream side of the burner is an acoustically passive termination, therefore $|R_{dn}| \leq 1$. Consequently, the Mobius transformation, $R_{in} = \frac{a R_{dn} + b}{c R_{dn} + d}$, transforms the unit circle in the plane of the downstream termination R_{dn} into a circle in the plane of R_{in} . This circle has a specified centre and radius,

$$M = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2}, \quad r = \frac{1}{||d|^2 - |c|^2|}, \quad (5)$$

where $a = \frac{T_{1,1}}{\sqrt{\Delta_T}}$, $b = \frac{-T_{2,1}}{\sqrt{\Delta_T}}$, $c = \frac{-T_{1,2}}{\sqrt{\Delta_T}}$, $d = \frac{T_{2,2}}{\sqrt{\Delta_T}}$, and Δ_T is the determinant of the flame transfer matrix [20]. Note that the Mobius transformation is represented here is in the normalized form.

Furthermore, the centre of R_{dn} unit circle transforms into point $O = \frac{b}{a}$ in R_{in} plane which defines to where the inside area of the unit circle of R_{dn} is mapped. It can be either the inside or outside area of the circle in R_{in} plane. Figure 2 shows four possible cases for this transformation between R_{dn} and R_{in} planes.

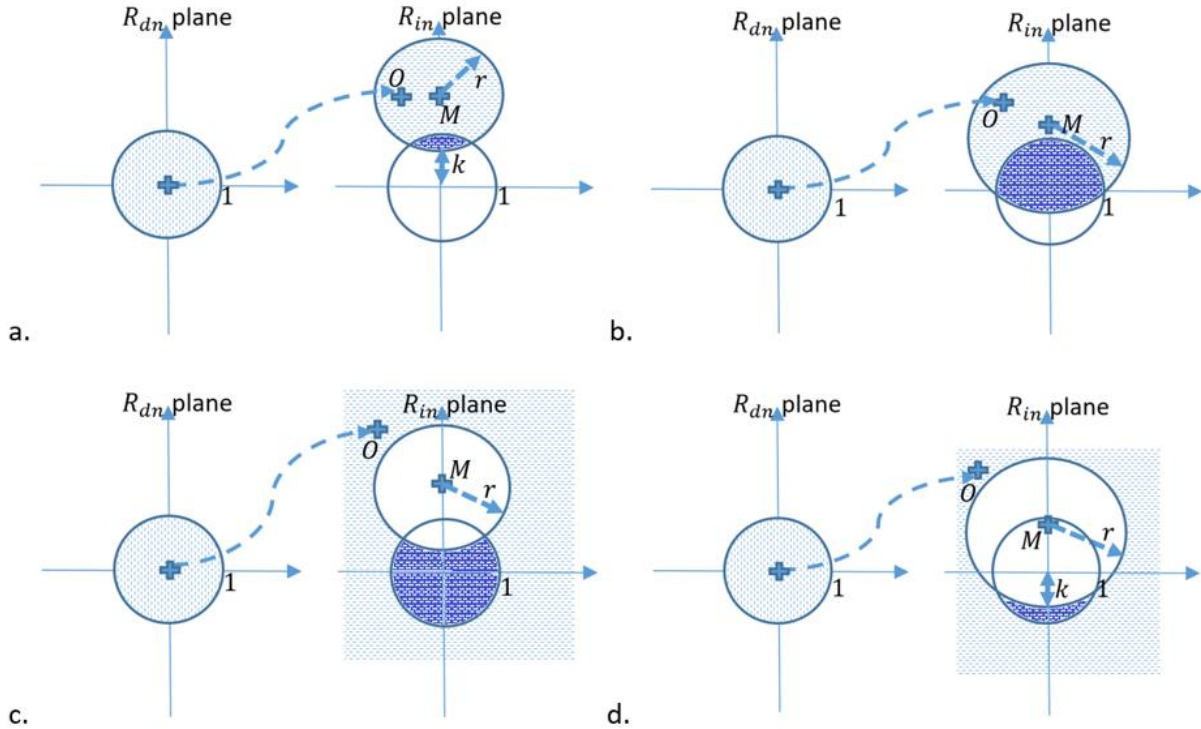


Figure 2: Four possible cases for the Möbius transformation of the unit circle of R_{dn} into R_{in} plane, double shaded areas show the joint area of the shaded area and the unit circle of R_{in} .

This kind of plot provides new insight into the design strategy one may follow to ensure system stability. For instance, considering Case *a* in Fig. 2 one may conclude that for most reflection coefficients of the downstream side of the flame, R_{dn} , the gain of R_{in} is higher than 1. Accordingly, to stabilize the system for this particular case the designing of the stabilising downstream acoustics is more demanding because there is quite limited range of R_{dn} which provides $|R_{in}| \leq 1$. Therefore, tuning of upstream side of the burner becomes more appropriate approach. Contrarily, when facing the situation similar to the Cases *b*, and *c* in Fig. 2 one may conclude that there is a wide freedom to select R_{dn} such that $|R_{in}| \leq 1$ and even it is possible to design a proper R_{dn} where $|R_{in}| = 0$. Elaboration of these design guidance ideas is the subject of next sections. It should be noted that the shaded rectangular areas in Cases *c*, and *d* in Fig. 2 are outside domains of the unit circle.

2.2 Optimal value of R_{dn}

An optimum value of R_{dn} is derived in this subsection which provides a minimum value of R_{in} at given entries of the burner TM . As shown in Fig. 2, a minimum value of R_{in} is equal to either zero for cases *b* and *c* or it is equal to k for cases *a* and *d* where $|k| = ||M| - r|$ and $\angle k = \angle M$. So, the value of R_{dn} can be restored which causes the minimum value of R_{in} by inverse mapping at point k back to R_{dn} plane. So,

$$R_{dn_{opt.}} = \frac{-dk+b}{ck-a}. \quad (6)$$

It is obvious that for cases *b* and *c* the optimum value of R_{dn} is $R_{dn_{opt.}} = -\frac{b}{a}$.

2.3 Possible range of R_{dn} to obtain a passive stable system

In this subsection, the conditions are investigated how to select a passive termination R_{dn} in such a way that for the given burner transfer matrix the corresponding magnitude of R_{in} becomes less than 1 over a certain frequency range. The aim to ensure $|R_{in}| < 1$ is motivated by the fact that in this case the burner with downstream acoustics behaves as a passive subsystem and therefore it is stable in combination with any passive upstream acoustics of the considered appliance.

The region of R_{dn} which we are looking for belongs to the joint area of the unit circle in the plane of R_{dn} and simultaneously the Mobius transformation according to Eq. 4 which maps these R_{dn} to the double shaded (stable) regions in the plane of R_{in} , see Fig. 2. To find the corresponding area in the R_{dn} plane one may invert the transformation of Eq. 4. For this case, the inverse transformation is also of Mobius kind and it is given by the expression

$$R_{dn} = \frac{-d R_{in} + b}{c R_{in} - a}. \quad (7)$$

We are interested where the unit circle of R_{in} maps into the R_{dn} plane. The mapping is similar to the one presented in Fig. 2 with the only replacement of R_{dn} and R_{in} planes. The described procedure provides the area inside the unit circle of R_{dn} which is considered as a preferable value of R_{dn} to obtain a passive stable system without considering the upstream acoustic's condition.

3. Experimental design

A test setup, shown in Fig. 3.a, was prepared to measure the flame transfer function. A constant temperature anemometer (CTA) and a photomultiplier tube (PMT) with OH filter were used to measure the acoustic velocity fluctuation before the flame and the varying heat release signal, respectively. A brass plate burner deck which has 1 mm thickness, and 5 cm diameter with a hexagonal pattern of holes was selected. The holes had a diameter of 2 mm placed at a pitch of 4.5 mm (D2- P4.5), and the total open area of the burner was 342.43 mm². Two Bronckhorst mass flow controllers metered the methane and airflow rates. A water-cooled deck holder was used to keep the outer perimeter of the burnerdeck at a fixed temperature. A set of National instruments DAQ systems connected to a PC running LabVIEW was used to transfer the data with a sampling rate of 10 kHz. Then the measured flame transfer function, TF , at the specified inlet mean velocity and equivalence ratio was used to calculate the transfer matrix's elements ($T_{i,j}$) [3] given by

$$TM = 0.5 \begin{bmatrix} \varepsilon + 1 + \theta TF & \varepsilon - 1 - \theta TF \\ \varepsilon - 1 - \theta TF & \varepsilon + 1 + \theta TF \end{bmatrix}, \quad (8)$$

where $\theta = \frac{T_2}{T_1} - 1$ is a temperature ratio, T_1, T_2 being initial and final (flame) temperature, and $\varepsilon = \frac{\rho_1 c_1}{\rho_2 c_2}$ is the jump in specific acoustics impedance across the flame. The entries of TM were used to calculate the coefficients a, b, c, d of the Mobius transformation.

4. Results and discussion

The flame transfer function measured for the burner D2P4.5 at the mean velocity of the mixture through the burner holes of 218 cm/sec and equivalence ratio of $\phi = 0.8$ is shown in Fig. 3.b. For the aforementioned flame transfer function, the Mobius transformation elements i.e. $a, b, c,$ and $d,$ are calculated in the frequency range 0-780 Hz. Then based on Eq. (7), R_{dn} circles and the possible ranges of R_{dn} at which $|R_{in}| \leq 1$ is derived. At four fixed frequencies, the R_{dn} circles and centres' locations of the circles and those preferable R_{dn} areas are shown in Fig. 4.

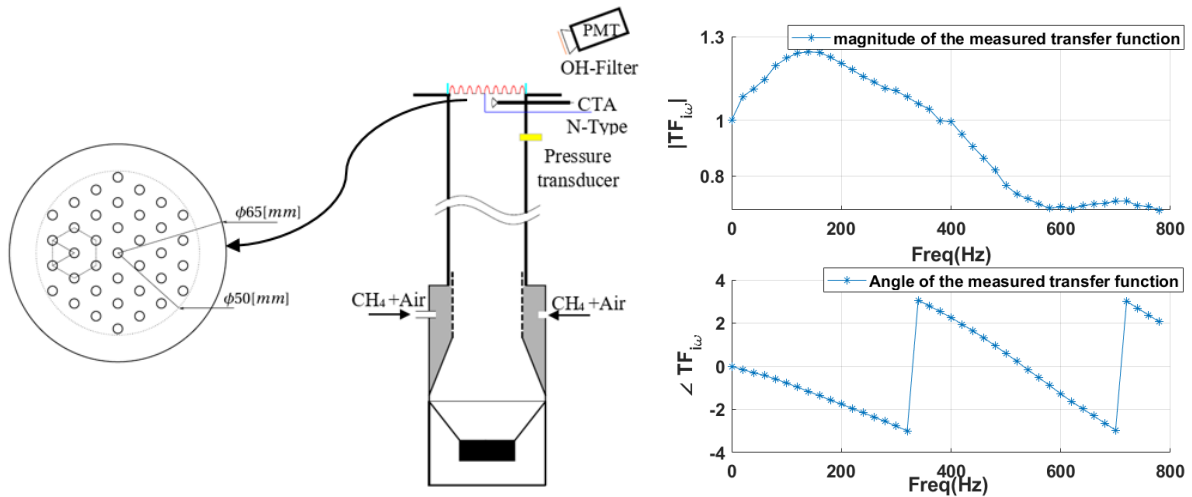


Figure 3: *a.* A test setup to measure the flame transfer function [3]; *b.* measured flame transfer function of the burner D2P4.5 at $v = 218 \frac{\text{cm}}{\text{s}}$ and $\phi = 0.8$.

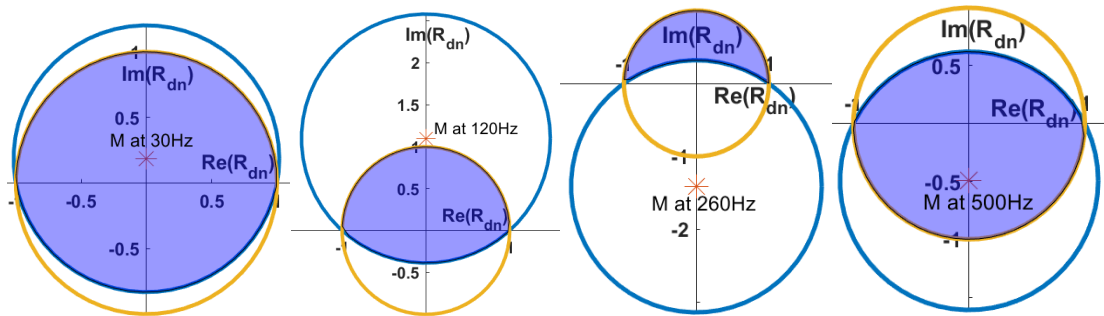


Figure 4: R_{dn} circles, centres' locations, and possible R_{dn} map to the unit circle of R_{in} at 30, 120, 260, 500 Hz.

Results show that due to a particular symmetry of the matrix TM for this kind of velocity sensitive flames, i.e. $T_{1,1} = T_{2,2}$ and $T_{1,2} = T_{2,1}$, the centre of the R_{dn} circle is lying on the imaginary axis. Fig. 5 demonstrates two 3D views of the areas in a frequency range 0-780 Hz. Four joint areas shown in Fig. 4 are also demonstrated in Fig. 5 with the red colour. If the reflection coefficient of the downstream side of the burner is inside these areas (a channel) then the system would be at a passive stability condition, otherwise the system will not be passive and the solution of dilemma of stability-instability is related to the product of upstream and inlet reflection coefficients.

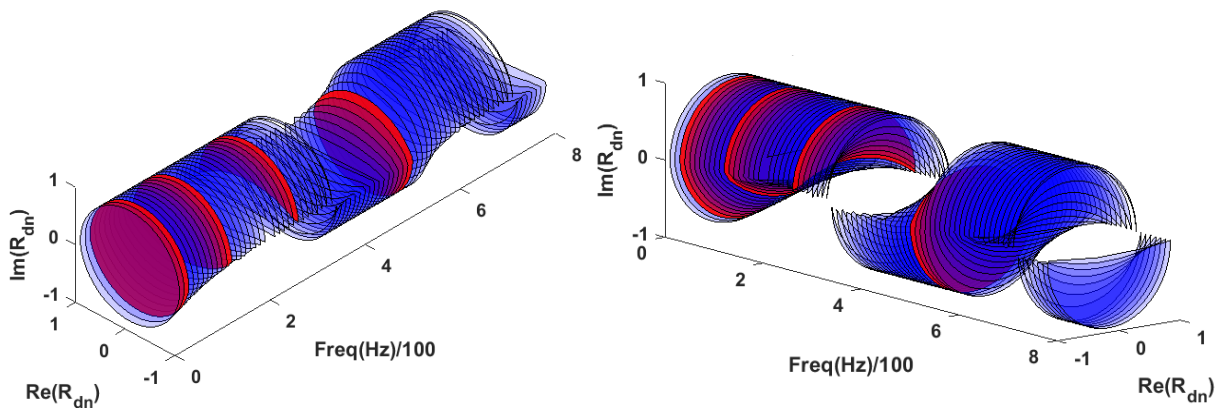


Figure 5: Two 3D views of possible R_{dn} maps to the unit circle of R_{in} in a frequency range 0-780 Hz. The four slices shown in Fig. 4 are highlighted in red.

Figure 5 shows an interesting tool for analysis. For instance, one may conclude that at low frequencies, (like 0-50 Hz), for almost any downstream termination of the mentioned burner at specified mean velocity and equivalence ratio condition, the system would be passive and, therefore, stable. The frequency ranges around 300Hz and 700Hz look the most problematic to stabilise because the areas of stabilising downstream acoustics are narrow at these frequencies.

By developing these ideas further, one may suggest that the “volume” of possible R_{dn} value (blue region shown in Fig. 5) may provide a good indication for the kind of “figure of merit” of the burner. It means that a good burner in terms of the thermo-acoustic stability would have a larger choice of possible R_{dn} values.

In addition, because of the mentioned format of matrix TM , it is possible to find a suitable R_{dn} where the system becomes passive and stable for any passive upstream termination of the system. For the sake of completeness, the optimal value of R_{dn} such that the magnitude of R_{in} becomes minimum is derived from Eq. (6) and results with corresponding minimum value of R_{in} are demonstrated in Fig. 6.

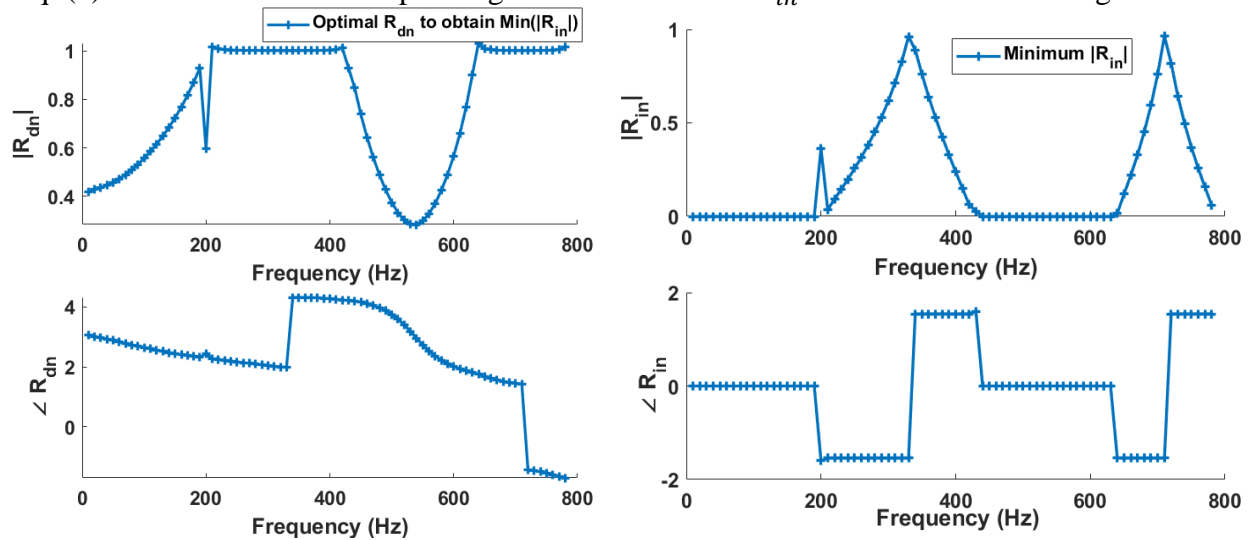


Figure 6: Optimal R_{dn} magnitude and phase (Left) to obtain a minimum amount of R_{in} (right) at 0-780 Hz.

These graphs also highlight the conclusion that for the considered burner/flame the frequencies around 320Hz and 700Hz may require special measures to ensure system stability.

5. Conclusion

A new theoretical framework to analyse the thermo-acoustic stability of combustion appliances is proposed. Within the proposed approach the system dispersion relation was expressed in terms of two reflection coefficients which can be measured at the upstream (cold) side of the burner. Using Mobius transformation properties, a requirement of passive stability of the system was translated to the requirements for the reflection coefficients of the downstream side of the burner. The results of the analysis show principal possibility to design the downstream side such that the system would be passive and stable irrespective to the upstream side of the burner. Also, the area of possible R_{dn} value to obtain a passive stable system is derived for each frequency. The area depends on the flame transfer function of the burner and the size of the area suggests a good indication for the figure of merit of the burner. Then developed theory may provide suitable criteria and guidelines for designing passive acoustic properties of the downstream side of the combustion appliance.

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