Vector algebras based on loops with reflection

VECTOR ALGEBRAS
based on
LOOPS with REFLECTION

J. de GRAAF and F.J.L. MARTENS

Preface and Summary

This note is about a special class of *(Real) Vector Algebras*. It is, in particular, about a *uniform approach* to 'number' systems like $\mathbb{C}$, $\mathbb{H}$ (Quaternions), $\mathbb{O}$ (Octonions), $\mathbb{C}_{p,q}$ (Clifford Algebras), Bicomplex Numbers, Tessarines,....etc.

A *Vector Algebra* is a vector space $\mathbb{V}$ endowed with a *Multiplication Structure*, which is compatible with addition and scalar multiplication. An undergraduate example is $\mathbb{R}^3$, equipped with an *exterior product*. The latter is fixed by a (non-commutative and non-associative) multiplication rule on the set

$$\mathfrak{M}_1 = \{\underline{0},\, +\underline{e}_1,\, -\underline{e}_1,\, +\underline{e}_2,\, -\underline{e}_2,\, +\underline{e}_3,\, -\underline{e}_3\} \subset \mathbb{R}^3,$$

followed by bi-linear extension. It represents the Lie Algebra of the orthogonal group $O(3)$.

The sort of multiplication structure we are interested in here, is somewhat different: The basic multiplication rule involves basis vectors but does NOT involve the $\underline{0}$-vector.

The standard *Group Algebra* $\mathbb{R}^{\mathfrak{G}}$, or *Loop Algebra* $\mathbb{R}^{\mathfrak{L}}$ is obtained by first imposing a multiplication structure on a basis

$$\mathfrak{M} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \ldots\} \subset \mathbb{V},$$

which is then followed by bi-linear extension to the whole of $\mathbb{V}$. In those cases the *basic* multiplication structure imposed on $\mathfrak{M}$ is that of a *GROUP* or, more generally, a *LOOP*. In the latter case only some weaker version of *associativity* holds.
Some inconvenience occurs if one wants to build $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{C}_{p,q}$, ..., in this way. For example, if we identify de standard basis in $\mathbb{R}^4$ with the complex group $\{1, \mathrm{i}, -1, -\mathrm{i}\}$ the resulting algebra is not $\mathbb{C}$, but some duplication of it.

As a remedy, roughly speaking, the set of basisvectors of $\mathbb{V}$ should first be doubled to

$$\mathfrak{L} = \{+\boldsymbol{v}_1\,, -\boldsymbol{v}_1\,, \ldots, +\boldsymbol{v}_n\,, -\boldsymbol{v}_n\,, \ldots\} \subset \mathbb{V}\,.$$

Next $\mathfrak{L}$ is turned into a R-GROUP, cq. R-LOOP, which means that there is the additional property

$$\Big[(+\boldsymbol{v}_i)\cdot(+\boldsymbol{v}_j) = (\gamma\boldsymbol{v}_k)\Big] \Rightarrow \Big[(\alpha\boldsymbol{v}_i)\cdot(\beta\boldsymbol{v}_j) = (\alpha\beta\gamma\boldsymbol{v}_k)\Big]\,,$$

$$i, j, k, \in \mathbb{N}\,, \qquad \alpha, \beta, \gamma \in \{+, -\}\,. \tag{0.1}$$

This enables a consistent bi-linear extension of the multiplication on $\mathfrak{L}$ to the whole of $\mathbb{V}$. Cf. The 'twisted product construction' as mentioned in **[B]**.

We start with a general characterization of R-LOOPS and on possible extensions of a given LOOP to an R-LOOP. Main emphasis in the underlying notes is on R-Loop extensions of (abelian) 'subset groups' and the production of Vector Algebras out of those. Our set of 'product structures' then leads to $\mathbb{C}\,, \mathbb{H}\,, \mathbb{O}\,, \mathbb{C}_{p,q}\,, \ldots$, bicomplex numbers, tessarines,....etc.

Any reader who wants to avoid the general algebra and pass to number systems and Clifford(like) algebras immediately, could start with section 3 and then continu with formula (0.1), cf. (2.19) in Thm 2.7, in the twisted product version

$$(\alpha\boldsymbol{v}_X)\cdot(\beta\boldsymbol{v}_Y) = \alpha\beta\,\sigma(X,Y)\boldsymbol{v}_{XY}\,, \quad X, Y \subset \{1, 2, \ldots, N\}\,, \quad N \in \mathbb{N}\,, \tag{0.2}$$

with

$$\begin{aligned} &\alpha, \beta \in \{+, -\} \\ &\sigma(\,\cdot\,,\,\cdot\,)\ :\ 2^{\{1,2,\ldots,N\}} \times 2^{\{1,2,\ldots,N\}} \to \{+, -\} \\ &XY = \big(X\backslash Y\big) \cup \big(Y\backslash X\big) \end{aligned}$$

Note that XY denotes the product in the subset group $2^{\{1,2,\ldots,N\}}$.

Section 3 deals precisely with the restrictions which have to be imposed on the function σ.

CONTENTS and GUIDE

The concept of R-LOOP is introduced. By means of a σ-function any LOOP can be doubled to a R-LOOP, usually in many distinct ways. All R-LOOPS arise in this way. 'Trivial extensions' and 'Non-trivial extensions' of LOOPS to R-LOOPS are distinguished and related to 'splittings' of the R-LOOP.

First, starting from a general R-LOOP $\mathfrak{L}$, the Loop Algebra $\mathbb{R}^{\mathfrak{L}}$ is constructed in a standard way. A set of natural operations (left/right actions, reversion, transposition, conjugation, etc.) is considered together with their behaviour on products.
Next the even and odd subspaces $\mathbb{R}^{\mathfrak{L}}_{+}$, $\mathbb{R}^{\mathfrak{L}}_{-}$ are introduced. They are sub-algebras. Both are ideals. Both are invariant under the 'natural operations'. The latter are studied on bases for $\mathbb{R}^{\mathfrak{L}}_{+}$, $\mathbb{R}^{\mathfrak{L}}_{-}$.

The odd subalgebras $\mathbb{R}^{\mathfrak{L}}_{-}$, combined with a member of the special class $\mathfrak{L} = 2^{\mathfrak{I}} \overset{\sigma}{\times} \{+\,,-\}$ of section 3, lead to the traditional number systems as mentioned in the preface.

The set of subsets $2^{\mathfrak{I}}$ of a fixed set $\mathfrak{I}$ is made into an Abelian group by means of the symmetric difference product $\diamond = \triangle$. For the 'low numbered cases' $\#\mathfrak{I} = 1, 2, 3, 4$, a full overview of all possible R-GROUP extensions $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+\,,-\}$ of $2^{\mathfrak{I}}$ is studied.
For $\#\mathfrak{I} \geq 5$ all standard R-GROUP extensions, underlying Clifford Algebras, are explicitly constructed: Thm (3.6).
Again: The Loop Algebra construction of section 2 performed with the special R-LOOPS of this section then leads to the traditional number systems and Clifford Algebras.

Some simple examples of multiplication tables for 5 element sets are provided.

It is shown how the octonions are an odd subalgebra $\mathbb{R}^{\mathfrak{L}}_{-}$, where the R-LOOP $\mathfrak{L}$ is a R-LOOP extension of $2^{\{A,B,C\}}$ and also how this $\mathbb{R}^{\mathfrak{L}}_{-}$ contains the group of unit quaternions 'in all possible ways'.

In a straightforward way all σ-functions leading to a R-GROUP extension of the subset group $2^{\{A,B,C\}}$ are found. Including the trivial ones as mentioned in Thm 1.8.

Fills in the details of the last part of section 3.

For the subset groups $2^{\mathfrak{I}}$, with $\mathfrak{I} = \{A, B\}$ and $\mathfrak{I} = \{A, B, C\}$, the group of their automorphisms is studied in great detail. The latter is splitted in 7-cycles, 3-cycles and 2-cycles. It could serve the study of the Clifford Algebras of type $\mathbb{R}^{\mathfrak{L}}_{-}$, with $\mathfrak{L} = 2^{\{A,B,C\}} \overset{\sigma}{\times} \{+\,,-\}$.

1 LOOPS with REFLECTION

In our first definition the concept of a "group $\mathfrak{G}$ with a (normal) subgroup $\{\mathbf{e}, \hat{\mathbf{e}}\}$" is generalized.

Definition 1.1
A *loop with reflection*, in short R-LOOP, is a set $\mathfrak{L}$ endowed with a multiplication structure, such that

$\mathbf{I_a}$ There is a left/right *unit* $\mathbf{e}$ in $\mathfrak{L}$.
$\mathbf{I_b}$ Multiplication is *dissociative*. That means: The equations $\mathbf{r} \cdot \mathbf{x} = \mathbf{s}$ and $\mathbf{y} \cdot \mathbf{r} = \mathbf{s}$ have unique solutions $\mathbf{x} \in \mathfrak{L}$ and $\mathbf{y} \in \mathfrak{L}$, respectively.
$\mathbf{I_c}$ For all $\mathbf{h} \in \mathfrak{L}$ there exists a left/right *inverse* $\mathbf{h}^{-1}$ such that: $\mathbf{h}\mathbf{h}^{-1} = \mathbf{h}^{-1}\mathbf{h} = \mathbf{e}$.

$\mathbf{II_a}$ There is an (appointed) element, named *reflector*, $\hat{\mathbf{e}}$ in $\mathfrak{L}$ with $\hat{\mathbf{e}}^2 = \mathbf{e}$.
$\mathbf{II_b}$ For all $\mathbf{s} \in \mathfrak{L}$ we have $\mathbf{s} \cdot \hat{\mathbf{e}} = \hat{\mathbf{e}} \cdot \mathbf{s}$.
$\mathbf{II_c}$ For all $\mathbf{h}, \mathbf{k} \in \mathfrak{L}$: $(\hat{\mathbf{e}} \cdot \mathbf{h}) \cdot \mathbf{k} = \hat{\mathbf{e}} \cdot (\mathbf{h} \cdot \mathbf{k}) = \mathbf{h} \cdot (\mathbf{k} \cdot \hat{\mathbf{e}})$.

The set of assumptions $\mathbf{I_{ab}}$ describes the properties of a **loop**, whereas $\mathbf{II_{abc}}$ explains what the reflection properties are. The latter mean that '*associativity* is required whenever $\hat{\mathbf{e}}$ is involved'. Note that our $\mathbf{I_c}$ is an extra. It is not required in a general loop.
For some examples see Appendix A.

Consequences of $\mathbf{I_{ab}}$:
$\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{y} \;\Rightarrow\; \mathbf{x} = \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{b} = \mathbf{y} \cdot \mathbf{b} \;\Rightarrow\; \mathbf{x} = \mathbf{y}$

Consequences of $\mathbf{II_{abc}}$:
$(\hat{\mathbf{e}} \cdot \mathbf{h}) \cdot (\hat{\mathbf{e}} \cdot \mathbf{k}) = \hat{\mathbf{e}} \cdot (\mathbf{h} \cdot (\hat{\mathbf{e}} \cdot \mathbf{k})) = \hat{\mathbf{e}} \cdot (\mathbf{h} \cdot (\mathbf{k} \cdot \hat{\mathbf{e}})) = \hat{\mathbf{e}} \cdot (\hat{\mathbf{e}} \cdot (\mathbf{h} \cdot \mathbf{k})) = (\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}) \cdot (\mathbf{h} \cdot \mathbf{k}) = \mathbf{h} \cdot \mathbf{k}$,
For left/right inverses: $(\hat{\mathbf{e}} \cdot \mathbf{g})^{-1} = \hat{\mathbf{e}} \cdot \mathbf{g}^{-1}$, and $(\hat{\mathbf{e}} \cdot \mathbf{g})^{-1} = \hat{\mathbf{e}} \cdot \mathbf{g}^{-1}$

We now list a set of examples of R-LOOP's.

Example 1.2
Any group $\mathfrak{G}$ with a normal subgroup $\{\mathbf{e}, \hat{\mathbf{e}}\}$.
The cyclic group $\{\mathbf{e}, \mathbf{a}, \ldots, \mathbf{a}^N, \ldots, \mathbf{a}^{2N-1} \,\big|\, \mathbf{a}^{2N} = \mathbf{e}\}$, for any fixed $N \in \mathbb{N}$.
Put $\mathbf{a}^N = \hat{\mathbf{e}}$.

Example 1.3
Let L be a loop with properties $\mathbf{I_{abc}}$.
Consider the sign group $\{+, -\}$ with the obvious multiplication.
Further, let $\sigma : \mathsf{L} \times \mathsf{L} \to \{+, -\}$ be any (fixed) mapping with the property:

$$\sigma(e, a) = \sigma(a, e) = +, \text{ for all } a \in \mathsf{S}. \tag{1.1}$$

The set

$$\mathsf{L} \overset{\sigma}{\times} \{+, -\}, \quad \begin{array}{ll} \text{with multiplication:} & (a\,;\alpha) \cdot (b\,;\beta) = (ab\,;\sigma(a,b)\alpha\beta), \\ \text{with unit:} & (e\,;+), \\ \text{with reflector:} & (e\,;-), \end{array} \tag{1.2}$$

is a R-LOOP, which we will name "**R-LOOP extension of a LOOP L**".

Note that

$$
\begin{aligned}
&(e\,;\,\varepsilon)\cdot(e\,;\,\varepsilon)=(e\,;\,\sigma(e,e)\,\varepsilon\varepsilon)=(e\,;\,\sigma(e,e))=(e\,;\,+),\\
&(a\,;\,\sigma(e,a)\,\varepsilon\alpha)=(e\,;\,\varepsilon)\cdot(a\,;\,\alpha)=(a\,;\,\alpha)\cdot(e\,;\,\varepsilon)=(a\,;\,\sigma(a,e)\,\alpha\varepsilon),\\
&(b\,;\beta)\cdot(a\,;\alpha)=(ba\,;\sigma(b,a)\beta\alpha),\\
&(a\,;\alpha)\cdot(a^{-1}\,;\theta)=(e\,;\sigma(a,a^{-1})\alpha\theta),\\
&(a^{-1}\,;\lambda)\cdot(a\,;\alpha)=(e\,;\sigma(a^{-1},a)\lambda\alpha),\\
&\big((a\,;\alpha)\cdot(b\,;\beta)\big)\cdot(c\,;\gamma)=\big((ab)c\,;\sigma(ab,c)\sigma(a,b)\alpha\beta\gamma\big),\\
&(a\,;\alpha)\cdot\big((b\,;\beta)\cdot(c\,;\gamma)\big)=\big(a(bc)\,;\sigma(a,bc)\sigma(b,c)\alpha\beta\gamma\big).
\end{aligned}
$$

Properties 1.4

Depending on the choice of σ, special properties of L 'persist' after extension to $\mathsf{L}\overset{\sigma}{\times}\{+\,,-\}$:

- For unique left/right inversion, $aa^{-1}=a^{-1}a=e$, to persist, require $\sigma(a,a^{-1})=\sigma(a^{-1},a)$.
- For $(ab^{-1})b=a$ to persist, require $\sigma(ab^{-1},b)\sigma(a,b^{-1})\sigma(b,b^{-1})=+$.
- For $b(b^{-1}a)=a$ to persist, require $\sigma(b,b^{-1}a)\sigma(b^{-1},a)\sigma(b,b^{-1})=+$.
- For $(a^{-1}b^{-1})(ba)=e$ to persist, require

$$\sigma(a^{-1}b^{-1},ba)\sigma(a^{-1},b^{-1})\sigma(a,a^{-1})\sigma(b,b^{-1})\sigma(b,a)=+.$$

- For $(aa)c=a(ac)$ to persist (left alternativity), require

$$\sigma(aa,c)\sigma(a,a)=\sigma(a,ac)\sigma(a,c).$$

- For $(ab)b=a(bb)$ to persist (right alternativity), require

$$\sigma(ab,b)\sigma(a,b)=\sigma(a,bb)\sigma(b,b).$$

- For commutativity to persist require $\sigma(a,b)=\sigma(b,a)$.
- For associativity to persist require $\sigma(ab,c)\sigma(a,b)=\sigma(a,bc)\sigma(b,c)$. ■

So if L happens to be a group then $\mathsf{L}\overset{\sigma}{\times}\{+\,,-\}$ is a group iff

$$\forall\; a,b,c\in\mathsf{L}\;\;:\;\;\sigma(ab,c)\sigma(a,b)=\sigma(a,bc)\sigma(b,c).$$

In such cases $\mathsf{L}\overset{\sigma}{\times}\{+\,,-\}$ will be named "**R-GROUP extension of a GROUP L**".

We will show that any R-LOOP $\mathfrak{L}$ arises from a construction with a σ-function.

Definition 1.5
By $\mathfrak{L}/\sim$ we denote the quotient space which is obtained from the equivalence relation

$$\mathbf{f} \sim \mathbf{g} \quad \Leftrightarrow \quad \mathbf{g} \in \{\mathbf{f}, \hat{\mathbf{e}}\mathbf{f}\}\,,$$

on $\mathfrak{L}$. The equivalence classes are denoted

$$\{\mathbf{g}\,,\,\hat{\mathbf{e}}\mathbf{g}\} = [\![\mathbf{g}]\!] = [\![\hat{\mathbf{e}}\mathbf{g}]\!]\,.$$

The quotient set $\mathfrak{L}/\sim$ is turned into a loop by the obvious product rule

$$[\![\mathbf{g}]\!] \cdot [\![\mathbf{h}]\!] = [\![\mathbf{gh}]\!] = [\![\hat{\mathbf{e}}(\mathbf{gh})]\!]\,.$$

It follows that $\#\{\mathfrak{L}/\sim\} = \frac{1}{2}\#\{\mathfrak{L}\}$.
If it happens that $\mathfrak{L}$ is a GROUP, then also $\mathfrak{L}/\sim$ is a GROUP.
If $\mathfrak{L}$ happens to be abelian, then so is $\mathfrak{L}/\sim$. ■

For later purposes we proceed in a way which may look somewhat primitive for genuine algebraists.
First some bookkeeping.

Definition 1.6
Consider any fixed splitting S of $\mathfrak{L}$ of type

$$\mathfrak{L} = \mathfrak{L}_S \cup \hat{\mathbf{e}}\mathfrak{L}_S \;= \{\mathbf{e}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\} \;\cup\; \{\hat{\mathbf{e}}, \hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}, \ldots\}\,, \tag{1.3}$$

where $\mathfrak{L}_S$ contains the unit element $\mathbf{e} \in \mathfrak{L}$ and $\hat{\mathbf{e}}\mathfrak{L}_S$ contains the reflector $\hat{\mathbf{e}} \in \mathfrak{L}$.
Further, $\hat{\mathbf{a}} = \hat{\mathbf{e}}\mathbf{a}$, etc.
The following mappings and conventions will be needed

- $$\mathbf{e}_S \;:\; \mathfrak{L} \;\rightarrow\; \{\mathbf{e}, \hat{\mathbf{e}}\} \;:\quad \mathbf{f} \mapsto \mathbf{e}_S(\mathbf{f}) = \begin{cases} \mathbf{e} & \text{if } \mathbf{f} \in \mathfrak{L}_S \\ \hat{\mathbf{e}} & \text{if } \mathbf{f} \in \hat{\mathbf{e}}\mathfrak{L}_S \end{cases}\,. \tag{1.4}$$

So, $\forall\, \mathbf{f} \in \mathfrak{L} : \quad \mathbf{f}\,\mathbf{e}_S(\mathbf{f}) \in \mathfrak{L}_S$. The inverse image of $\mathbf{e}_S$ yields the splitting S.

- $$\mathbf{E}_S \;: \mathfrak{L} \times \mathfrak{L} \;\rightarrow\; \{\mathbf{e}, \hat{\mathbf{e}}\} \;:\quad \mathbf{f}\,, \mathbf{g} \;\mapsto \mathbf{E}_S(\mathbf{f}, \mathbf{g}) = \mathbf{e}_S(\mathbf{fg}) \in \{\mathbf{e}, \hat{\mathbf{e}}\}\,. \tag{1.5}$$

- The isomorphisms

$$\begin{aligned} \mathbf{e}(\cdot) \;: \{+, -\} \;&\rightarrow\; \{\mathbf{e}, \hat{\mathbf{e}}\} \;:\quad \mathbf{e}(+) = \mathbf{e} \;\text{ and }\; \mathbf{e}(-) = \hat{\mathbf{e}}\,, \\ \mathbf{e}^{\leftarrow} \;: \{\mathbf{e}, \hat{\mathbf{e}}\} \;&\rightarrow\; \{+, -\} \;:\quad \mathbf{e}^{\leftarrow}(\mathbf{e}) = + \;\text{ and }\; \mathbf{e}^{\leftarrow}(\hat{\mathbf{e}}) = -\,. \end{aligned}$$

- $$\sigma_S \;:\; \mathfrak{L}_S \times \mathfrak{L}_S \;\rightarrow\; \{+, -\} \;:\; \mathbf{a}\,, \mathbf{b} \;\mapsto\; \sigma_S(\mathbf{a}, \mathbf{b}) = \mathbf{e}^{\leftarrow}(\mathbf{E}_S(\mathbf{a}, \mathbf{b})) = \begin{cases} + & \text{if } \mathbf{ab} \in \mathfrak{L}_S \\ - & \text{if } \mathbf{ab} \in \hat{\mathbf{e}}\mathfrak{L}_S \end{cases}\,.$$

• Finally, on $\mathfrak{L}_S$ a product structure $\diamond$ is introduced by

$$\mathbf{a} \diamond \mathbf{b} = \mathbf{ab}\mathbf{E}_S(\mathbf{a},\mathbf{b}) \;=\left\{ \begin{array}{ll} \mathbf{ab} & \text{if } \mathbf{ab} \in \mathfrak{L}_S\,, \\ \hat{\mathbf{e}}(\mathbf{ab}) & \text{if } \mathbf{ab} \in \hat{\mathbf{e}}\mathfrak{L}_S\,, \end{array} \right. . \tag{1.6}$$

■

Theorem 1.7
a. *For any splitting S the set $\mathfrak{L}_S$,with multiplication $\diamond$, is a LOOP which is isomorphic to $\mathfrak{L}/\!\sim$.*
b. *For any splitting S, the R-LOOP $\mathfrak{L}_S \overset{\sigma_S}{\times} \{+\,,-\}$, with multiplication structure*

$$(\mathbf{a}\,;\alpha)\cdot(\mathbf{b}\,;\beta) = (\mathbf{a}\diamond\mathbf{b}\,;\sigma_S(\mathbf{a},\mathbf{b})\alpha\beta)\,, \tag{1.7}$$

is isomorphic with the original R-LOOP $\mathfrak{L}$. The isomorphism is given by

$$\mathscr{I}_S\;:\quad \mathfrak{L}_S\times\{+,-\}\;\to\;\mathfrak{L}: \qquad (\mathbf{a};\alpha)\;\mapsto\;\mathbf{ae}(\alpha)\,,$$

with inverse

$$\mathscr{I}_S^{-1}\;:\quad \mathfrak{L}\;\to\;\mathfrak{L}_S\times\{+,-\}\;: \qquad \mathbf{g}\;\mapsto\;(\mathbf{g}\,\mathbf{e}_S(\mathbf{g});\mathbf{e}^{\leftarrow}(\mathbf{e}_S(\mathbf{g})))\,.$$

c. *The R-LOOP $\mathfrak{L}$ is isomorphic to $\mathfrak{L}/\!\sim \overset{\sigma_S}{\times} \{+\,,-\}$, where σ_S is found by means of the isomorphism in* **a.**. *In the sequel we put $\mathsf{L} \cong (\mathfrak{L}_S;\diamond) \cong \mathfrak{L}/\!\sim$.*
d. *For the inverse $\mathbf{a}_\diamond^{-1}$ of $\mathbf{a}\in\mathfrak{L}_S$, with respect to $\diamond$, we have $\mathbf{a}_\diamond^{-1} = \mathbf{a}^{-1}\mathbf{e}_S(\mathbf{a}^{-1})$.*
e.

$$\forall\,\mathbf{a},\mathbf{b},\mathbf{c}\in\mathfrak{L}_S\;: \quad \begin{array}{rcl} (\mathbf{a}\diamond\mathbf{b})\diamond\mathbf{c} &=& (\mathbf{ab})\mathbf{c}\,\mathbf{E}_S(\mathbf{a},\mathbf{b})\,\mathbf{E}_S(\mathbf{a}\diamond\mathbf{b},\mathbf{c}) \\ \mathbf{a}\diamond(\mathbf{b}\diamond\mathbf{c}) &=& \mathbf{a}(\mathbf{bc})\,\mathbf{E}_S(\mathbf{b},\mathbf{c})\,\mathbf{E}_S(\mathbf{a},\mathbf{b}\diamond\mathbf{c}) \\ (\mathbf{a}\diamond\mathbf{b})\diamond\mathbf{a}_\diamond^{-1} &=& (\mathbf{ab})\mathbf{a}_\diamond^{-1}\,\mathbf{E}_S(\mathbf{a},\mathbf{b})\,\mathbf{E}_S(\mathbf{a}\diamond\mathbf{b},\mathbf{a}_\diamond^{-1}) \\ &=& (\mathbf{ab})\mathbf{a}^{-1}\,\mathbf{e}_S(\mathbf{a}^{-1})\,\mathbf{E}_S(\mathbf{a},\mathbf{b})\,\mathbf{E}_S(\mathbf{a}\diamond\mathbf{b},\mathbf{a}_\diamond^{-1}) \end{array}$$

Proof
a. Follows from $[\![\mathbf{a}\diamond\mathbf{b}]\!] = [\![\mathbf{ab}]\!] = [\![\hat{\mathbf{e}}(\mathbf{ab})]\!]$.
b. Calculate from the definitions

$$\mathscr{I}_S((\mathbf{a};\alpha)\cdot(\mathbf{b};\beta)) = \mathscr{I}_S((\mathbf{a}\diamond\mathbf{b}\,;\sigma_S(\mathbf{a},\mathbf{b})\alpha\beta)) = (\mathbf{a}\diamond\mathbf{b})\mathbf{e}\big(\sigma_S(\mathbf{a},\mathbf{b})\alpha\beta\big) =$$

$$= (\mathbf{a}\diamond\mathbf{b})\mathbf{E}_S(\mathbf{a},\mathbf{b})\mathbf{e}(\alpha)\mathbf{e}(\beta) = \big(\mathbf{a}\,\mathbf{e}(\alpha)\big)\big(\mathbf{b}\,\mathbf{e}(\beta)\big)\,,$$

and

$$\mathscr{I}_S(\mathbf{a};\alpha)\mathscr{I}_S(\mathbf{b};\beta) = \big(\mathbf{a}\,\mathbf{e}(\alpha)\big)\big(\mathbf{b}\,\mathbf{e}(\beta)\big)\,.$$

c. Follows from a. and b.

d. calculate $\mathbf{a} \diamond \mathbf{a}_\diamond^{-1} = \mathbf{a}(\mathbf{a}^{-1}\,\mathbf{e}_S(\mathbf{a}^{-1}))\mathbf{E}_S(\mathbf{a}, \mathbf{a}_\diamond^{-1}) = \mathbf{a}(\mathbf{a}^{-1}\,\mathbf{e}_S(\mathbf{a}^{-1}))\,\mathbf{e}_S(\mathbf{a}^{-1})) = \mathbf{e}$, since $\mathbf{a}\,\mathbf{a}_\diamond^{-1} = \mathbf{e}_S(\mathbf{a}^{-1})$.
e. Straightforward. ■

We now want to investigate the σ-function if the underlying LOOP $\mathsf{L} = \mathfrak{L}/\!\sim$ happens to be a GROUP. This property persists in the R-LOOP extension, cf. Properties 1.4, *iff* the σ-function satisfies

$$\forall\, x, y, z \in \mathsf{L} \;:\; \sigma(xy, z)\sigma(x, yz) = \sigma(x, y)\sigma(y, z)\,. \tag{1.8}$$

Note that, because of (1.1), this equation is satisfied if one of the entries happens to be e. A trivial solution to (1.8) is given by

$$\forall\, x, y \in \mathsf{L} :\; \sigma(x, y) = +\,. \tag{1.9}$$

Theorem 1.8
a. *If σ_1 and σ_2 are both solutions to (1.8), then so is their (pointwise) product $\sigma_1 \cdot \sigma_2$.*

b. *For any fixed $a \in \mathsf{L}$, a non-trivial solution σ_a to (1.8) is*

$$\sigma_a(x, y) = \big(-1\big)^{[1-\delta_e(a)]\,[1-\delta_e(x)]\,[1-\delta_e(y)]\,\{\delta_a(x)+\delta_a(y)+\delta_a(xy)\}}\,, \tag{1.10}$$

where $\delta_a(x) = 1$, if $x = a$, and $\delta_a(x) = 0$, if $x \neq a$.
The solution σ_e corresponds to (1.8). If $a \neq e$, we have

$$\sigma_a(x, y) = \begin{cases} - & \textit{if} \quad x = a\,,\; y \notin \{a, e\} \\ - & \textit{if} \quad y = a\,,\; x \notin \{a, e\} \\ - & \textit{if} \quad xy = a\,,\; x \notin \{a, e\} \\ + & \qquad \text{remaining cases} \end{cases} \tag{1.11}$$

If L happens to be abelian, this solution is symmetric, i.e. $\forall\, x, y \in \mathsf{L} \;:\; \sigma_a(x, y) = \sigma_a(y, x)$.

c. *If all elements in the set $\{x, y, z, xy, xz, yz, xyz\} \subset \mathsf{L}$ are distinct from each other, we have for all $a \neq e$,*

$$\sigma_a(x, yz)\sigma_a(x, y)\sigma_a(x, z) = \begin{cases} + & \textit{if} \quad a \notin \{x, y, z, xy, xz, yz, xyz\} \\ - & \textit{if} \quad a \in \{x, y, z, xy, xz, yz, xyz\} \end{cases} \tag{1.12}$$

d. *For $a, x, z \in \mathsf{L}$, only considering the non-trivial case $a \neq e\,, x \neq e\,, z \neq e$, we have the interesting cases*

$$\sigma_a(x, xz)\sigma_a(x, x)\sigma_a(x, z) = \begin{cases} + & \textit{if} \quad x^2 = e \\ + & \textit{if} \quad x^2 = a,\; z = a \\ + & \textit{if} \quad x^2 z \neq a,\; x^2 \neq a,\; z \neq a \\ - & \textit{if} \quad x^2 = a,\; z \neq a,\; z \neq e \\ - & \textit{if} \quad x^2 z = a,\; x^2 \neq a,\; z \neq a \end{cases} \tag{1.13}$$

e. $\forall\, a \in \mathsf{L}\; \forall\, x, y \in \mathsf{L} \;:\; \sigma_{a^{-1}}(x, y) = \sigma_a(y^{-1}, x^{-1})\,.$

Proof
a. Trivial.
b. Two verifications of the solution (1.10) are mentioned:
b'. List

$$\begin{aligned}
\sigma_a(x,y) &= (-1)^{[1-\delta_e(a)]\,[1-\delta_e(x)]\,[1-\delta_e(y)]\,\{\delta_a(x)+\delta_a(y)+\delta_a(xy)\}} \\
\sigma_a(y,z) &= (-1)^{[1-\delta_e(a)]\,[1-\delta_e(y)]\,[1-\delta_e(z)]\,\{\delta_a(y)+\delta_a(z)+\delta_a(yz)\}} \\
\sigma_a(x,yz) &= (-1)^{[1-\delta_e(a)]\,[1-\delta_e(x)]\,[1-\delta_e(yz)]\,\{\delta_a(x)+\delta_a(yz)+\delta_a(xyz)\}} \\
\sigma_a(xy,z) &= (-1)^{[1-\delta_e(a)]\,[1-\delta_e(xy)]\,[1-\delta_e(z)]\,\{\delta_a(xy)+\delta_a(z)+\delta_a(xyz)\}}
\end{aligned}$$

In the sum of the terms between { } each term occurs twice, leading to an even number. Also in the cases that one or several of the factors between [ ] vanishes each of the terms between { } that remain occurs twice.
b". Extend the group L by means of the trivial σ, cf. (1.9). We find a R-GROUP $\mathfrak{L}$. For fixed $a \in \mathsf{L}\,, a \neq e$ make a new splitting of $\mathfrak{L}$ by swapping $(a;+)$ and $(a;-)$. The new σ_a has the form (1.9) and does not alter the (associative) product structure of $\mathfrak{L}$.
c. Follows from

$$\sigma_a(x,yz)\sigma_a(x,y)\sigma(x,z) = (-1)^{\{\delta_a(x)+\delta_a(y)+\delta_a(z)+\delta_a(xy)+\delta_a(xz)+\delta_a(yz)+\delta_a(xyz)\}}$$

d. The assumptions lead to

$$\sigma_a(x,xz)\sigma_a(x,x)\sigma(x,z) = (-1)^{\{\delta_a(x^2z)+\delta_a(x^2)+\delta_a(z)\}} .$$

The properties now follow straightforwardly. As a specimen, suppose in case 4 that $x^2z = a$, then $az = a$, hence $z = e$. Which has been excluded. Therefore also $x^2z \neq a$.
e. Note that $\delta_a(x) = \delta_{a^{-1}}(x^{-1})\,,\ \delta_a(xy) = \delta_{a^{-1}}(x^{-1}y^{-1})\,$, etc. ■

In these notes we are mainly interested in *associative* R-extensions of Abelian groups. For the sake of Clifford Algebras we want to obtain *non-commutative, associative* R-extensions. However, such non-commutative extensions of Abelian groups do not exist in general as the following example shows.

Example 1.9
Consider a cyclic group with one generator $\mathsf{L} = \{e, a, a^2, \ldots, a^{N-1}\}$, with $a^N = e$, in presentation notation: $\mathsf{L} = \{a \,|\, a^N = e\,\}$. We show that its possible R-GROUP extensions are necessarily of Abelian type. One of the necessary conditions for associativity of the R-LOOP extension is, cf. (1.8),

$$\forall\, x, y\, \in \mathsf{L} : \quad \sigma(xy,x)\,\sigma(x,yx) = \sigma(x,y)\,\sigma(y,x) \tag{1.14}$$

So for $k, l = 1, 2, 3, \ldots$ we necessarily have

$$\sigma(a^{k+\ell}, a^k)\,\sigma(a^k, a^{k+\ell}) = \sigma(a^\ell, a^k)\,\sigma(a^k, a^\ell)\,. \tag{1.15}$$

As starting points we take successively

$$k = 1,\ \ell = 1\,;\quad k = 2,\ 1 \le \ell \le 2\,;\quad k = 3,\ 1 \le \ell \le 3\,;\ \ \ldots\ \ ;\quad k,\ 1 \le \ell \le k\,;\ \ \ldots$$

The very first starting point is $\sigma(a,a)\,\sigma(a,a) = +$. Each new starting point is either available from earlier steps or of type $\sigma(a^k, a^k)\,\sigma(a^k, a^k) = +$.

• In case $\mathsf{L} = \{a \,|\, a^2 = e\,\}$, we find

$$\sigma_a \;=\; \sigma_e \;=\; \begin{array}{ccc} & e & a \\ e & + & + \\ a & + & + \end{array}$$

Another solution to (1.8) is

$$\sigma_1 \;= \begin{array}{ccc} & e & a \\ e & + & + \\ a & + & - \end{array}$$

• In case $\mathsf{L} = \{a \,|\, a^3 = e\,\}$, we find

$$\sigma_a \;=\; \begin{array}{cccc} & e & a & a^2 \\ e & + & + & + \\ a & + & + & - \\ a^2 & + & - & - \end{array} \qquad \sigma_{a^2} \;=\; \begin{array}{cccc} & e & a & a^2 \\ e & + & + & + \\ a & + & - & - \\ a^2 & + & - & + \end{array} \qquad \sigma_a\sigma_{a^2} \;=\; \begin{array}{cccc} & e & a & a^2 \\ e & + & + & + \\ a & + & - & + \\ a^2 & + & + & - \end{array}$$

In this case there are no other solutions to (1.8). So only the trivial R-GROUP extension exists.

• In case $\mathsf{L} = \{a \,|\, a^4 = e\,\}$, we find

$$\sigma_a \;=\; \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & + & - & - \\ a^2 & + & - & + & - \\ a^3 & + & - & - & + \end{array} \qquad \sigma_{a^2} \;=\; \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & - & - & + \\ a^2 & + & - & + & - \\ a^3 & + & + & - & - \end{array} \qquad \sigma_{a^3} \;=\; \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & + & - & - \\ a^2 & + & - & + & - \\ a^3 & + & - & - & + \end{array}$$

$$\sigma_a\sigma_{a^2} \;=\; \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & - & + & - \\ a^2 & + & + & + & + \\ a^3 & + & - & + & - \end{array} \qquad \sigma_a\sigma_{a^3} \;=\; \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & + & + & + \\ a^2 & + & + & + & + \\ a^3 & + & + & + & + \end{array} \qquad \sigma_{a^2}\sigma_{a^3} \;=\; \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & - & + & - \\ a^2 & + & + & + & + \\ a^3 & + & - & + & - \end{array}$$

In this case the class of solutions is given by products of the above and

$$\sigma_1 = \begin{array}{ccccc} & e & a & a^2 & a^3 \\ e & + & + & + & + \\ a & + & - & - & - \\ a^2 & + & - & - & + \\ a^3 & + & - & + & + \end{array}$$

So there is, up to isomorphism, only 1 non-trivial R-GROUP extension. Moreover it is Abelian. These results are obtained by Mathematica calculations. ■

2 Vector Algebras based on R-LOOPS

We start by mimicking the usual group algebra idea.

Definition 2.1
Let $\mathfrak{L}$ denote a R-LOOP as before. On the vectorspace

$$\mathbb{R}^{\mathfrak{L}} = \{\mathfrak{a} \,\big|\, \mathfrak{a} : \mathfrak{L} \to \mathbb{R}\},$$

the (usual) addition and an (obvious) multiplication is introduced according to

$$\mathfrak{a} + \mathfrak{b} \;:\; \mathfrak{L} \to \mathbb{R} \;:\; (\mathfrak{a} + \mathfrak{b})(\mathbf{g}) = \mathfrak{a}(\mathbf{g}) + \mathfrak{b}(\mathbf{g}), \tag{2.1}$$

$$\mathfrak{a} \cdot \mathfrak{b} \;:\; \mathfrak{L} \to \mathbb{R} \;:\; (\mathfrak{a} \cdot \mathfrak{b})(\mathbf{g}) = \sum_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}). \tag{2.2}$$

The standard basis of $\mathbb{R}^{\mathfrak{L}}$ is given by the set $\{\mathfrak{e}_{\mathbf{a}} \,\big|\, \mathbf{a} \in \mathfrak{L}\}$, where

$$\mathfrak{e}_{\mathbf{a}} \;:\; \mathfrak{L} \to \mathbb{R} \;:\; \mathbf{g} \mapsto \delta_{\mathbf{a}}(\mathbf{g}) = \begin{cases} 1 & \text{if} \quad \mathbf{g} = \mathbf{a} \\ 0 & \text{else} \end{cases} .$$

A relevant algebraic property of this basis is

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathfrak{L} \;:\; \mathfrak{e}_{\mathbf{a}} \cdot \mathfrak{e}_{\mathbf{b}} = \mathfrak{e}_{\mathbf{ab}}\,. \tag{2.3}$$

Note that $\mathfrak{e}_{\mathbf{e}}$ acts as a **unit**.
We gather some properties of the algebra $\mathbb{R}^{\mathfrak{L}}$.

Theorem 2.2
a. *If* $\forall\, \mathbf{g}, \mathbf{k}, \mathbf{h} \in \mathfrak{L}$ *it happens that* $(\mathbf{g}\cdot\mathbf{k}^{-1})\cdot\mathbf{k} = \mathbf{g}$, *and/or* $\mathbf{h}\cdot(\mathbf{h}^{-1}\cdot\mathbf{g}) = \mathbf{g}$, *the multiplication (2.2) can be written as a convolution, respectively,*

$$(\mathfrak{a} \cdot \mathfrak{b})(\mathbf{g}) = \sum_{\mathbf{k}\in\mathfrak{L}} \mathfrak{a}(\mathbf{gk}^{-1})\mathfrak{b}(\mathbf{k}), \qquad (\mathfrak{a} \cdot \mathfrak{b})(\mathbf{g}) = \sum_{\mathbf{h}\in\mathfrak{L}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{h}^{-1}\mathbf{g}). \tag{2.4}$$

b. *If* $\mathfrak{L}$ *happens to be, respectively, commutative, associative, the algebra* $\mathbb{R}^{\mathfrak{L}}$ *inherits these properties.*

Proof Obvious. ∎

Definition 2.3 (Some operations on $\mathbb{R}^{\mathfrak{L}}$)
The *Opposition Operator*

$$\mathscr{M} \;:\; \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}} \;:\; \mathfrak{a} \mapsto \mathscr{M}\mathfrak{a} = \{\, \mathbf{g} \mapsto \mathfrak{a}(\hat{\mathbf{e}}\mathbf{g}) = \mathfrak{a}(\mathbf{g}\hat{\mathbf{e}}) \,\}. \tag{2.5}$$

The *Left Action*

$$\mathscr{L}_{\mathbf{p}} \ : \ \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}} \ : \quad \mathfrak{a} \mapsto \mathscr{L}_{\mathbf{p}}\mathfrak{a} = \{\, \mathbf{g} \mapsto \mathfrak{a}(\mathbf{p}^{-1}\mathbf{g}) \,\} \tag{2.6}$$

The *Right Action*

$$\mathscr{R}_{\mathbf{p}} \ : \ \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}} \ : \quad \mathfrak{a} \mapsto \mathscr{R}_{\mathbf{p}}\mathfrak{a} = \{\, \mathbf{g} \mapsto \mathfrak{a}(\mathbf{g}\mathbf{p}) \,\}. \tag{2.7}$$

The *Reversion*

$$\cdot^{\dagger} \ : \ \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}} \ : \quad \mathfrak{a} \mapsto \mathfrak{a}^{\dagger} = \{\, \mathbf{g} \mapsto \mathfrak{a}(\mathbf{g}^{-1}) \,\}. \tag{2.8}$$

If $\mathfrak{L}$ happens to be a group, then, for $\mathbf{p} \in \mathfrak{L}$, there is the $\mathbf{p}$-*Conjugation*

$$\mathscr{C}_{\mathbf{p}} \ : \ \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}} \ : \quad \mathfrak{a} \mapsto \mathscr{C}_{\mathbf{p}}\mathfrak{a} = \{\, \mathbf{g} \mapsto \mathfrak{a}(\mathbf{p}^{-1}\mathbf{g}\mathbf{p}) \,\}. \tag{2.9}$$

If $\mathbf{p} \in \mathfrak{L}$ happens to be such that $\forall \mathbf{g} \in \mathfrak{L} : \mathbf{p}(\mathbf{p}^{-1}\mathbf{g}) = \mathbf{g}$ and $(\mathbf{g}\mathbf{p})\mathbf{p}^{-1} = \mathbf{g}$), application of those operations to the standard basis in $\mathbb{R}^{\mathfrak{L}}$ gives us

$$\begin{array}{rcccl} \mathscr{M}\mathfrak{e}_{\mathbf{a}} & = & \{\, \mathbf{g} \mapsto \mathfrak{e}_{\mathbf{a}}(\hat{\mathbf{e}}\mathbf{g}) = \mathfrak{e}_{\mathbf{a}}(\mathbf{g}\hat{\mathbf{e}}) = \mathfrak{e}_{\mathbf{a}\hat{\mathbf{e}}}(\mathbf{g}) \,\} & = & \mathfrak{e}_{\hat{\mathbf{e}}\mathbf{a}} \\ \mathscr{L}_{\mathbf{p}}\mathfrak{e}_{\mathbf{a}} & = & \{\, \mathbf{g} \mapsto \mathfrak{e}_{\mathbf{a}}(\mathbf{p}^{-1}\mathbf{g}) = \mathfrak{e}_{\mathbf{p}\mathbf{a}}(\mathbf{g}) \,\} & = & \mathfrak{e}_{\mathbf{p}\mathbf{a}} \\ \mathscr{R}_{\mathbf{p}}\mathfrak{e}_{\mathbf{a}} & = & \{\, \mathbf{g} \mapsto \mathfrak{e}_{\mathbf{a}}(\mathbf{g}\mathbf{p}) = \mathfrak{e}_{\mathbf{a}\mathbf{p}^{-1}}(\mathbf{g}) \,\} & = & \mathfrak{e}_{\mathbf{a}\mathbf{p}^{-1}} \\ \mathscr{C}_{\mathbf{p}}\mathfrak{e}_{\mathbf{a}} & = & \{\, \mathbf{g} \mapsto \mathfrak{e}_{\mathbf{a}}(\mathbf{p}^{-1}\mathbf{g}\mathbf{p}) = \mathfrak{e}_{\mathbf{p}\mathbf{a}\mathbf{p}^{-1}}(\mathbf{g}) \,\} & = & \mathfrak{e}_{\mathbf{p}\mathbf{a}\mathbf{p}^{-1}} \\ \mathfrak{e}_{\mathbf{a}}^{\dagger} & = & \{\, \mathbf{g} \mapsto \mathfrak{e}_{\mathbf{a}}(\mathbf{g}^{-1}) = \mathfrak{e}_{\mathbf{a}^{-1}}(\mathbf{g}) \,\} & = & \mathfrak{e}_{\mathbf{a}^{-1}} \end{array} \tag{2.10}$$

The action of the operators (2.5)-(2.9) *on products* runs as follows.

Theorem 2.4
Suppose $\mathfrak{L}$ is a R-LOOP.

a. $\mathscr{L}_{\mathbf{p}}(\mathfrak{a}+\mathfrak{b}) = \mathscr{L}_{\mathbf{p}}\mathfrak{a} + \mathscr{L}_{p}\mathfrak{b}$ *and* $\mathscr{R}_{\mathbf{p}}(\mathfrak{a}+\mathfrak{b}) = \mathscr{R}_{\mathbf{p}}\mathfrak{a} + \mathscr{R}_{\mathbf{p}}\mathfrak{b}$

b. $\mathscr{M} = \mathscr{R}_{\hat{\mathbf{e}}} = \mathscr{L}_{\hat{\mathbf{e}}}$.

c. *If* $\mathbf{p}, \mathbf{q} \in \mathfrak{L}$ *are such that* $\forall\, \mathbf{h}, \mathbf{k} \in \mathfrak{L} : \ (\mathbf{p}^{-1}\mathbf{h})\mathbf{k} = \mathbf{p}^{-1}(\mathbf{h}\mathbf{k}), \ \mathbf{h}(\mathbf{k}\mathbf{q}) = (\mathbf{h}\mathbf{k})\mathbf{q},$ *then*

$$\mathscr{L}_{\mathbf{p}}\,(\mathfrak{a}\cdot\mathfrak{b}) = (\mathscr{L}_{\mathbf{p}}\,\mathfrak{a})\cdot\mathfrak{b} \qquad \textit{and} \qquad \mathscr{R}_{\mathbf{q}}\,(\mathfrak{a}\cdot\mathfrak{b}) = \mathfrak{a}\cdot(\mathscr{R}_{\mathbf{q}}\,\mathfrak{b}). \tag{2.11}$$

d. *If it happens that* $\forall\, \mathbf{h}, \mathbf{k} \in \mathfrak{L} : \ (\mathbf{h}\mathbf{k})^{-1} = \mathbf{k}^{-1}\mathbf{h}^{-1},$ *then*

$$\big(\mathfrak{a}\cdot\mathfrak{b}\big)^{\dagger} = \mathfrak{b}^{\dagger}\cdot\mathfrak{a}^{\dagger}. \tag{2.12}$$

e. *If* $\mathfrak{L}$ *is a group, then*

$$\mathscr{L}_{\mathbf{p}}\mathscr{L}_{\mathbf{q}} = \mathscr{L}_{\mathbf{p}\mathbf{q}}, \qquad \textit{and} \qquad \mathscr{R}_{\mathbf{p}}\mathscr{R}_{\mathbf{q}} = \mathscr{R}_{\mathbf{p}\mathbf{q}}\,. \tag{2.13}$$

f. *If* $\mathfrak{L}$ *is a group, then*

$$\big((\mathscr{L}_{\mathbf{p}}\,\mathfrak{a})\cdot(\mathscr{R}_{\mathbf{q}}\,\mathfrak{b})\big)(\mathbf{g}) = \big(\mathfrak{a}\cdot\mathfrak{b}\big)(\mathbf{p}^{-1}\mathbf{g}\mathbf{q}). \tag{2.14}$$

g. *If* $\mathfrak{L}$ *is a group, then*

$$\mathscr{C}_{\mathbf{p}}\,(\mathfrak{a}\cdot\mathfrak{b}) = \mathscr{C}_{\mathbf{p}}\mathfrak{a}\cdot\mathscr{C}_{\mathbf{p}}\mathfrak{b}. \tag{2.15}$$

Proof
ab. The linearity properties are obvious.
c. Calculate

$$\mathscr{L}_{\mathbf{p}}(\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}) = \big(\mathfrak{a}\cdot\mathfrak{b}\big)(\mathbf{p}^{-1}\mathbf{g}) = \sum_{\mathbf{hk}=\mathbf{p}^{-1}\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) =$$

$$= \sum_{(\mathbf{p}^{-1}\mathbf{h})\mathbf{k}=\mathbf{p}^{-1}\mathbf{g}} \mathfrak{a}(\mathbf{p}^{-1}\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum_{\mathbf{p}^{-1}(\mathbf{hk})=\mathbf{p}^{-1}\mathbf{g}} \mathfrak{a}(\mathbf{p}^{-1}\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum_{\mathbf{hk}=\mathbf{g}} \big(\mathscr{L}_{\mathbf{p}}\mathfrak{a}\big)(\mathbf{h})\mathfrak{b}(\mathbf{k}).$$

d. Calculate

$$(\mathfrak{a}\cdot\mathfrak{b})^{\dagger}(\mathbf{g}) = (\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}^{-1}) = \sum_{\mathbf{hk}=\mathbf{g}^{-1}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) =$$

$$= \sum_{\mathbf{k}^{-1}\mathbf{h}^{-1}=\mathbf{g}^{-1}} \mathfrak{a}(\mathbf{k}^{-1})\mathfrak{b}(\mathbf{h}^{-1}) = \sum_{(\mathbf{hk})^{-1}=\mathbf{g}^{-1}} \mathfrak{a}(\mathbf{k}^{-1})\mathfrak{b}(\mathbf{h}^{-1}) = \sum_{\mathbf{hk}=\mathbf{g}} \mathfrak{b}^{\dagger}(\mathbf{h})\mathfrak{a}^{\dagger}(\mathbf{k}).$$

e. Just a specimen,

$$\big(\mathscr{L}_{\mathbf{p}}(\mathscr{L}_{\mathbf{q}}\mathfrak{a})\big)(\mathbf{g}) = (\mathscr{L}_{\mathbf{q}}\mathfrak{a})(\mathbf{p}^{-1}\mathbf{g}) = \mathfrak{a}(\mathbf{q}^{-1}(\mathbf{p}^{-1}\mathbf{g})) = \mathfrak{a}((\mathbf{pq})^{-1}\mathbf{g}) = \mathscr{L}_{\mathbf{pq}}\mathfrak{a}(\mathbf{g}).$$

f. Calculate

$$(\mathscr{L}_{\mathbf{p}}\mathfrak{a}\cdot\mathscr{R}_{\mathbf{q}}\mathfrak{b})(\mathbf{g}) = \sum_{\mathbf{hk}=\mathbf{g}} \mathscr{L}_{\mathbf{p}}\mathfrak{a}(\mathbf{h})\mathscr{R}_{\mathbf{q}}\mathfrak{b}(\mathbf{k}) = \sum_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\mathbf{p}^{-1}\mathbf{h})\mathfrak{b}(\mathbf{kq}) =$$

$$= \sum_{(\mathbf{ph})(\mathbf{kq}^{-1})=\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum_{\mathbf{hk}=\mathbf{p}^{-1}\mathbf{gq}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = (\mathfrak{a}\cdot\mathfrak{b})(\mathbf{p}^{-1}\mathbf{gq}).$$

g. Calculate

$$\mathscr{C}_{\mathbf{p}}(\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}) = \big(\mathfrak{a}\cdot\mathfrak{b}\big)(\mathbf{p}^{-1}\mathbf{gp}) = \sum_{\mathbf{hk}=\mathbf{p}^{-1}\mathbf{gp}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum_{(\mathbf{php}^{-1})(\mathbf{pkp}^{-1})=\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) =$$

$$= \sum_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\mathbf{p}^{-1}\mathbf{hp})\mathfrak{b}(\mathbf{p}^{-1}\mathbf{kp}) = \sum_{\mathbf{hk}=\mathbf{g}} (\mathscr{C}_{\mathbf{p}}\mathfrak{a})(\mathbf{h})(\mathscr{C}_{\mathbf{p}}\mathfrak{b})(\mathbf{k}) = (\mathscr{C}_{\mathbf{p}}\mathfrak{a}\cdot\mathscr{C}_{\mathbf{p}}\mathfrak{b})(\mathbf{g}).$$

■

In the next theorem some commutation properties are established.

Theorem 2.5
i) *For all* $\mathfrak{a},\mathfrak{b}\in\mathbb{R}^{\mathfrak{L}}$, *we have*

$$\begin{gathered}\mathscr{M}(\mathfrak{a}\cdot\mathfrak{b}) = (\mathscr{M}\mathfrak{a})\cdot\mathfrak{b} = \mathfrak{a}\cdot(\mathscr{M}\mathfrak{b}),\\ (\mathscr{M}\mathfrak{a})\cdot(\mathscr{M}\mathfrak{b}) = \mathfrak{a}\cdot\mathfrak{b}, \qquad \mathscr{M}(\mathscr{M}\mathfrak{a}) = \mathfrak{a}.\end{gathered} \tag{2.16}$$

ii) *For all* $\mathfrak{a} \in \mathbb{R}^{\mathfrak{L}}$ *and all* $\mathbf{p} \in \mathfrak{L}$ *there are the commutation properties*

$$\begin{aligned} \mathscr{L}_{\mathbf{p}}\mathscr{M} &= \mathscr{M}\mathscr{L}_{\mathbf{p}} \qquad & \mathscr{R}_{\mathbf{p}}\mathscr{M} &= \mathscr{M}\mathscr{R}_{\mathbf{p}} \\ (\mathscr{M}\mathfrak{a})^{\dagger} &= \mathscr{M}(\mathfrak{a}^{\dagger}) \qquad & \mathscr{C}_{\mathbf{p}}\mathscr{M} &= \mathscr{M}\mathscr{C}_{\mathbf{p}} \end{aligned} \tag{2.17}$$

Proof: In the proof the commutative and associative properties of $\hat{\mathbf{e}}$ are extensively used.

i) • $\mathscr{M}(\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}) = (\mathfrak{a}\cdot\mathfrak{b})(\hat{\mathbf{e}}\mathbf{g}) = \sum\limits_{\mathbf{hk}=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum\limits_{(\hat{\mathbf{e}}\mathbf{h})\mathbf{k}=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\hat{\mathbf{e}}\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum\limits_{(\mathbf{hk})\hat{\mathbf{e}}=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\hat{\mathbf{e}}\mathbf{h})\mathfrak{b}(\mathbf{k}) =$
$= \sum\limits_{\mathbf{hk}=\mathbf{g}} (\mathscr{M}\mathfrak{a})(\mathbf{h})\mathfrak{b}(\mathbf{k}) = (\mathscr{M}\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}).$

• $\mathscr{M}(\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}) = (\mathfrak{a}\cdot\mathfrak{b})(\hat{\mathbf{e}}\mathbf{g}) = \sum\limits_{\mathbf{hk}=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum\limits_{\mathbf{h}(\mathbf{k}\hat{\mathbf{e}})=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}\hat{\mathbf{e}}) = \sum\limits_{(\mathbf{hk})\hat{\mathbf{e}}=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}\hat{\mathbf{e}}) =$
$= \sum\limits_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\mathbf{h})(\mathscr{M}\mathfrak{b})(\mathbf{k}) = (\mathfrak{a}\cdot\mathscr{M}\mathfrak{b})(\mathbf{g}),$

• $(\mathscr{M}\mathfrak{a}\cdot\mathscr{M}\mathfrak{b})(\mathbf{g}) = \sum\limits_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\hat{\mathbf{e}}\mathbf{h})\mathfrak{b}(\hat{\mathbf{e}}\mathbf{k}) = \sum\limits_{(\hat{\mathbf{e}}\mathbf{h})(\hat{\mathbf{e}}\mathbf{k})=\mathbf{g}} \mathfrak{a}(\hat{\mathbf{e}}\mathbf{h})\mathfrak{b}(\hat{\mathbf{e}}\mathbf{k}) = \sum\limits_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = (\mathfrak{a}\cdot\mathfrak{b})(\mathbf{g}).$

ii) • $\big(\mathscr{L}_{\mathbf{k}}(\mathscr{M}\mathfrak{a})\big)(\mathbf{g}) = (\mathscr{M}\mathfrak{a})(\mathbf{k}^{-1}\mathbf{g}) = \mathfrak{a}(\hat{\mathbf{e}}(\mathbf{k}^{-1}\mathbf{g})) = \mathfrak{a}(\mathbf{k}^{-1}(\hat{\mathbf{e}}\mathbf{g})) =$
$= (\mathscr{L}_{\mathbf{k}})\mathfrak{a}(\hat{\mathbf{e}}\mathbf{g}) = \big(\mathscr{M}(\mathscr{L}_{\mathbf{k}}\mathfrak{a})\big)(\mathbf{g}),$

• $\big(\mathscr{R}_{\mathbf{k}}(\mathscr{M}\mathfrak{a})\big)(\mathbf{g}) = (\mathscr{M}\mathfrak{a})(\mathbf{gk}) = \mathfrak{a}(\hat{\mathbf{e}}(\mathbf{gk})) = \mathfrak{a}((\hat{\mathbf{e}}\mathbf{g})\mathbf{k})) = (\mathscr{R}_{\mathbf{k}}\mathfrak{a})(\hat{\mathbf{e}}\mathbf{g}) = \big(\mathscr{M}(\mathscr{R}_{\mathbf{k}}\mathfrak{a})\big)(\mathbf{g}).$

iii) • $(\mathscr{M}\mathfrak{a}^{\dagger})(\mathbf{g}) = (\mathfrak{a}^{\dagger})(\hat{\mathbf{e}}\mathbf{g}) = \mathfrak{a}((\hat{\mathbf{e}}\mathbf{g})^{-1}) = \mathfrak{a}(\hat{\mathbf{e}}\mathbf{g}^{-1}) = (\mathscr{M}\mathfrak{a})(\mathbf{g}^{-1}) = (\mathscr{M}\mathfrak{a})^{\dagger}(\mathbf{g}).$ ■

Definition 2.6
a. The *even linear subspace* $\mathbb{R}^{\mathfrak{L}}_{+}$ of $\mathbb{R}^{\mathfrak{L}}$ is defined by

$$\mathbb{R}^{\mathfrak{L}}_{+} = \{\mathfrak{a} \,\big|\, \mathfrak{a} : \mathfrak{L} \to \mathbb{R}\,,\ \mathfrak{a}(\mathbf{g}) = \mathfrak{a}(\hat{\mathbf{e}}\mathbf{g})\,\}.$$

For any $\mathbf{a} \in \mathfrak{L}$ we define the even function

$$\mathfrak{e}^{+}_{\mathbf{a}} \;:\; \mathfrak{L} \to \mathbb{R} \;:\; \mathbf{g} \mapsto \frac{1}{2}\big(\mathfrak{e}_{\mathbf{a}} + \mathfrak{e}_{\hat{\mathbf{e}}\mathbf{a}}\big)(\mathbf{g}) \;=\; \begin{cases} \frac{1}{2} & \text{if} \quad \mathbf{g} = \mathbf{a}\,,\ \mathbf{g} = \hat{\mathbf{e}}\mathbf{a}\,, \\ 0 & \text{else} \quad . \end{cases}$$

Note that $\mathfrak{e}^{+}_{\mathbf{a}} = \mathfrak{e}^{+}_{\hat{\mathbf{e}}\mathbf{a}}$.
b. The *odd linear subspace* $\mathbb{R}^{\mathfrak{L}}_{-}$ of $\mathbb{R}^{\mathfrak{L}}$ is defined by

$$\mathbb{R}^{\mathfrak{L}}_{-} = \{\mathfrak{a} \,\big|\, \mathfrak{a} : \mathfrak{L} \to \mathbb{R}\,,\ \mathfrak{a}(\mathbf{g}) = -\mathfrak{a}(\hat{\mathbf{e}}\mathbf{g})\,\}.$$

For any $\mathbf{a} \in \mathfrak{L}$ we define the odd function

$$\mathfrak{e}^{-}_{\mathbf{a}} \;:\; \mathfrak{L} \to \mathbb{R} \;:\; \mathbf{g} \mapsto \frac{1}{2}\big(\mathfrak{e}_{\mathbf{a}} - \mathfrak{e}_{\hat{\mathbf{e}}\mathbf{a}}\big)(\mathbf{g}) \;=\; \begin{cases} \frac{1}{2} & \text{if} \quad \mathbf{g} = \mathbf{a}\,, \\ -\frac{1}{2} & \text{if} \quad \mathbf{g} = \hat{\mathbf{e}}\mathbf{a}\,, \\ 0 & \text{else} \quad . \end{cases}$$

Note that $\mathfrak{e}^-_{\mathbf{a}} = -\mathfrak{e}^-_{\hat{\mathbf{e}}\mathbf{a}}$. ■

Theorem 2.7
Fix, as before, a partition

$$\mathfrak{L} = \mathfrak{L}_S \cup \hat{\mathbf{e}}\mathfrak{L}_S \;=\; \{\mathbf{e}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\} \;\cup\; \{\hat{\mathbf{e}}, \hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}, \ldots\} .$$

Denote the inverse of $\mathbf{a} \in \mathfrak{L}_S$*, with respect to the product* $\diamond$ *of* $\mathfrak{L}_S$ *, by* $\mathbf{a}_\diamond^{-1}$ *.*
a. *The set* $\{\, \mathfrak{e}^+_{\mathbf{a}} \,|\, \mathbf{a} \in \mathfrak{L}_S\}$ *is a basis for* $\mathbb{R}^{\mathfrak{L}}_+$*. This basis does NOT depend on the choice of the partition* S*.*

b. *The set* $\{\, \mathfrak{e}^-_{\mathbf{a}} \,|\, \mathbf{a} \in \mathfrak{L}_S\}$ *is a basis for* $\mathbb{R}^{\mathfrak{L}}_-$*. This basis does DEPEND on the choice of the partition.*

c. *Both linear subspaces* $\mathbb{R}^{\mathfrak{L}}_+$, $\mathbb{R}^{\mathfrak{L}}_-$ *are subalgebras. In fact they are both left/right ideals.*

d. *In* $\mathbb{R}^{\mathfrak{L}}_+$ *we have*

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathfrak{L}_S \;:\; \mathfrak{e}^+_{\mathbf{a}} \cdot \mathfrak{e}^+_{\mathbf{b}} = \mathfrak{e}^+_{\mathbf{ab}} = \mathfrak{e}^+_{\hat{\mathbf{e}}(\mathbf{ab})} = \mathfrak{e}^+_{\mathbf{a}\diamond\mathbf{b}} \tag{2.18}$$

In $\mathbb{R}^{\mathfrak{L}}_-$ *we have*

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathfrak{L}_S \;:\; \mathfrak{e}^-_{\mathbf{a}} \cdot \mathfrak{e}^-_{\mathbf{b}} = \mathfrak{e}^-_{\mathbf{ab}} = -\mathfrak{e}^-_{\hat{\mathbf{e}}(\mathbf{ab})} = \sigma_S(\mathbf{a}, \mathbf{b})\, \mathfrak{e}^-_{\mathbf{a}\diamond\mathbf{b}} \tag{2.19}$$

e. *Both linear subspaces* $\mathbb{R}^{\mathfrak{L}}_+$, $\mathbb{R}^{\mathfrak{L}}_-$ *are invariant subspaces for the operations*

$$\mathscr{M} \,,\; \mathscr{L}_{\mathbf{p}} \,,\; \mathscr{R}_{\mathbf{p}} \,,\; \mathscr{C}_{\mathbf{p}} \,,\; \dagger \,,$$

as mentioned in Def 2.3. For $\mathbf{a}\,,\,\mathbf{p} \in \mathfrak{L}_S$*,*

$$\begin{array}{lll} \mathscr{M}\mathfrak{e}^+_{\mathbf{a}} = \mathfrak{e}^+_{\mathbf{a}} & \mathscr{L}_{\mathbf{p}}\mathfrak{e}^+_{\mathbf{a}} = \mathfrak{e}^+_{\mathbf{p}\diamond\mathbf{a}} & \mathscr{R}_{\mathbf{p}}\mathfrak{e}^+_{\mathbf{a}} = \mathfrak{e}^+_{\mathbf{a}\diamond\mathbf{p}_\diamond^{-1}} \\ & \mathscr{C}_{\mathbf{p}}\mathfrak{e}^+_{\mathbf{a}} = \mathfrak{e}^+_{\mathbf{p}\diamond\mathbf{a}\diamond\mathbf{p}_\diamond^{-1}} & (\mathfrak{e}^+_{\mathbf{a}})^\dagger = \mathfrak{e}^+_{\mathbf{a}_\diamond^{-1}} \\ \mathscr{M}\mathfrak{e}^-_{\mathbf{a}} = -\mathfrak{e}^-_{\mathbf{a}} & \mathscr{L}_{\mathbf{p}}\mathfrak{e}^-_{\mathbf{a}} = \sigma_S(\mathbf{p}, \mathbf{a})\mathfrak{e}^-_{\mathbf{p}\diamond\mathbf{a}} & \mathscr{R}_{\mathbf{p}}\mathfrak{e}^-_{\mathbf{a}} = \sigma_S(\mathbf{a}, \mathbf{p}_\diamond^{-1})\mathfrak{e}^-_{\mathbf{a}\diamond\mathbf{p}_\diamond^{-1}} \\ & \mathscr{C}_{\mathbf{p}}\mathfrak{e}^-_{\mathbf{a}} = \tau_S(\mathbf{p}, \mathbf{a})\mathfrak{e}^-_{\mathbf{p}\diamond\mathbf{a}\diamond\mathbf{p}_\diamond^{-1}} & (\mathfrak{e}^-_{\mathbf{a}})^\dagger = \sigma_S(\mathbf{a}, \mathbf{a}_\diamond^{-1})\mathfrak{e}^-_{\mathbf{a}_\diamond^{-1}} \end{array} \tag{2.20}$$

Here $\tau_S(\mathbf{p}, \mathbf{a}) = \mathbf{e}^{\leftarrow}(\mathbf{e}_S(\mathbf{p}^{-1}))\, \sigma_S(\mathbf{p}, \mathbf{a})\, \sigma_S(\mathbf{p} \diamond \mathbf{a}, \mathbf{p}_\diamond^{-1})$ *. Note that* $\sigma_S(\mathbf{a}, \mathbf{a}_\diamond^{-1}) = \mathbf{e}^{\leftarrow}(\mathbf{e}_S(\mathbf{a}^{-1}))$ *.*

Proof
ab. Obvious .
c. Let $\mathfrak{a}$ be in either $\mathbb{R}^{\mathfrak{L}}_\pm$ and calculate

$$(\mathfrak{a} \cdot \mathfrak{b})(\hat{\mathbf{e}}\mathbf{g}) = \sum_{\mathbf{hk}=\hat{\mathbf{e}}\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = \sum_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\hat{\mathbf{e}}\mathbf{h})\mathfrak{b}(\mathbf{k}) = \pm \sum_{\mathbf{hk}=\mathbf{g}} \mathfrak{a}(\mathbf{h})\mathfrak{b}(\mathbf{k}) = \pm(\mathfrak{a} \cdot \mathfrak{b})(\mathbf{g}) .$$

So, both $\mathbb{R}^{\mathfrak{L}}_{\pm}$ are left/right ideals in $\mathbb{R}^{\mathfrak{L}}$.
d. Calculate, with (2.3) and $\mathbf{ab} = (\mathbf{a} \diamond \mathbf{b})\mathbf{E}_S(\mathbf{a},\mathbf{b})$,

$$\mathfrak{e}^-_{\mathbf{a}}\, \mathfrak{e}^-_{\mathbf{b}} = \frac{1}{4}(\mathfrak{e}_{\mathbf{a}} - \mathfrak{e}_{\hat{\mathbf{e}}\mathbf{a}})(\mathfrak{e}_{\mathbf{b}} - \mathfrak{e}_{\hat{\mathbf{e}}\mathbf{b}}) = \frac{1}{2}(\mathfrak{e}_{\mathbf{ab}} - \mathfrak{e}_{\hat{\mathbf{e}}(\mathbf{ab})}) = \sigma_S(\mathbf{a},\mathbf{b})\, \mathfrak{e}^-_{\mathbf{a}\diamond\mathbf{b}}\,.$$

e. Follows from the commuting properties with $\mathscr{M}$.
As a specimen calculate, with Theorem 1.7e,

$$\mathscr{C}_{\mathbf{p}}\mathfrak{e}^-_{\mathbf{a}} = \frac{1}{2}(\mathfrak{e}_{\mathbf{pap}^{-1}} - \mathfrak{e}_{\hat{\mathbf{e}}(\mathbf{pap}^{-1})}) = \tau_S(\mathbf{p},\mathbf{a})\, \mathfrak{e}^-_{\mathbf{p}\diamond\mathbf{a}\diamond\mathbf{p}_\diamond^{-1}}\,.$$

■

The final definition of this section concerns a '*grading mapping*' $\mathscr{G}$, which does not depend on a choice of the σ-function.

Definition 2.8 (Grading)
Suppose the availability of a non-trivial homomorphism

$$\mathtt{h}\,:\,\mathfrak{L} \,\to\, \{+,-\}\;:\; \mathbf{p} \mapsto \mathtt{h}(\mathbf{p}) = \mathtt{h}(\hat{\mathbf{e}}\mathbf{p}) \in \{+,-\}\,. \tag{2.21}$$

A linear mapping, *the Grading mapping*, $\mathscr{G}_{\mathtt{h}} : \mathbb{R}^{\mathfrak{L}} \to \mathbb{R}^{\mathfrak{L}}$ is then introduced by

$$\forall\, \mathbf{a} \in \mathfrak{L}\;:\; \mathfrak{e}_{\mathbf{a}} \;\mapsto\; \mathscr{G}_{\mathtt{h}}\mathfrak{e}_{\mathbf{a}} = \mathtt{h}(\mathbf{a})\mathfrak{e}_{\mathbf{a}}\,. \tag{2.22}$$

Note that also $\mathscr{G}_{\mathtt{h}}\mathfrak{e}^+_{\mathbf{a}} = \mathtt{h}(\mathbf{a})\mathfrak{e}^+_{\mathbf{a}}$ and $\mathscr{G}_{\mathtt{h}}\mathfrak{e}^-_{\mathbf{a}} = \mathtt{h}(\mathbf{a})\mathfrak{e}^-_{\mathbf{a}}$, for $\mathbf{a} \in \mathfrak{L}_S$. ■

We finish this section by shortly mentioning an **interesting modification**.
Instead of $\mathbb{R}^{\mathfrak{L}}$, cf. Def 2.1, one could start start off with the set $\mathbb{P}^{\mathfrak{L}}$, where $\mathbb{P} = (0,\infty)$, the strictly positive numbers. The set $\mathbb{P}^{\mathfrak{L}}$ is no longer a vector space, but most of the introduced concepts still make sense. The set of even functions $\mathbb{P}^{\mathfrak{L}}_{+}$ is still a left/right ideal in $\mathbb{P}^{\mathfrak{L}}$. Of course, there are no odd functions in this case.
The quotient space $\mathbb{P}^{\mathfrak{L}}/\mathbb{P}^{\mathfrak{L}}_{+}$ can be given the structure of a vector space in the following way. First, for any fixed $\alpha > 1$, $\mathbb{R}$ in imbedded in $\mathbb{P}^{\mathfrak{L}}$, by

$$\mathbb{R} \overset{\mathrm{emb}_\alpha}{\longrightarrow} \mathbb{P}^{\mathfrak{L}}\;:\; \xi \mapsto \mathrm{emb}_\alpha(\xi) = \big\{\mathbf{g} \mapsto (\xi + \alpha|\xi|)\delta_{\mathbf{e}}(\mathbf{g}) + (\alpha|\xi|)\delta_{\hat{\mathbf{e}}}(\mathbf{g})\big\}\,. \tag{2.23}$$

Then, turning to the quotient space, we arrive at the embedding emb : $\mathbb{R} \to \mathbb{P}^{\mathfrak{L}}/\mathbb{P}^{\mathfrak{L}}_{+}$, which does not depend on α. Scalar multiplication by $\xi \in \mathbb{R}$ is finally obtained from multiplication by emb(ξ).
It is straightforward to show that the vector spaces $\mathbb{P}^{\mathfrak{L}}/\mathbb{P}^{\mathfrak{L}}_{+}$ and $\mathbb{R}^{\mathfrak{L}}_{+}$ are isomorphic.
The conclusion is that the whole bunch of number systems can be constructed from the positive reals $(0,\infty)$. Such construction could replace the ugly Kapitel 4 in Edmund Landau's *Grundlagen der Analysis* by taking $\mathfrak{L} = \{\mathbf{e}, \hat{\mathbf{e}}\}$. This Kapitel 4 is preceded by a very elegant construction of the positive reals and their properties from just the positive integers $\mathbb{N}$. See [**L**].

3 A special class of R-LOOPS

We construct a particular class of R-LOOPS which plays an important role in hyper complex number theory: Quaternions, Clifford algebras, Octonions, $\cdots$.
The 'underlying loop L', that we start with, is a very special abelian Coxeter group.

Definition 3.1 The set of subsets $2^{\mathfrak{I}}$ of a fixed set $\mathfrak{I}$ is made into an abelian group with the operation of symmetric difference $\triangle$ as its product,

$$\forall\ X,Y \in 2^{\mathfrak{I}}\ :\ \ X\triangle Y = Y\triangle X = (X\backslash Y)\cup(Y\backslash X)\,, \qquad \emptyset X = X\,, \qquad X\triangle X = X^2 = \emptyset, \tag{3.1}$$

where the empty set $\emptyset$ represents the unit element.
In this section we omit $\triangle$. So, we will write XY instead of $X\triangle Y$. ■

By choosing a σ-function, as indicated in the preceding section, the group $2^{\mathfrak{I}}$ is extended to a R-LOOP

Theorem 3.2
Suppose a fixed

$$\sigma\ :\ \ 2^{\mathfrak{I}}\times 2^{\mathfrak{I}}\ \rightarrow\ \{+\,,-\}\ \ :\ \ X,Y\ \mapsto\ \sigma(X,Y)\ \in\ \{+\,,-\}\,,$$

be such that

$$\forall\ X\ \in 2^{\mathfrak{I}}\ :\ \ \sigma(\emptyset,X)=\sigma(X,\emptyset)=+\,. \tag{3.2}$$

Let the product rule on $2^{\mathfrak{I}}\overset{\sigma}{\times}\{+\,,-\}$ *be given by*

$$(X;\xi)\cdot(Y;\eta)=(XY;\sigma(X,Y)\xi\eta)\,, \tag{3.3}$$

then $2^{\mathfrak{I}}\overset{\sigma}{\times}\{+\,,-\}$ *is a R-LOOP with unit* $\mathbf{e}=(\emptyset;+)$ *and reflector* $\hat{\mathbf{e}}=(\emptyset,-)$.

Proof Looking at the special Properties (1.4), we trivially get a loop, because in this special case $X^{-1}=X$. ■

Theorem 3.3
a. $2^{\mathfrak{I}}\overset{\sigma}{\times}\{+\,,-\}$ *is a left/right-alternative R-LOOP iff, in addition to (3.2), we impose*

$$\forall\ X,Y\ \in 2^{\mathfrak{I}}\ :\ \left\{\begin{array}{rcl}\sigma(X,XY) & = & \sigma(X,X)\sigma(X,Y)\\ \sigma(XY,XY) & = & \sigma(X,Y)\sigma(Y,X)\sigma(X,X)\sigma(Y,Y)\end{array}\right. \tag{3.4}$$

b. *A full list of properties of a σ-function, leading to a left/right-alternative R-LOOP* $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+,-\}$ *is*

$$\forall\ X,Y\ \in 2^{\mathfrak{I}}\ :\ \begin{cases} \sigma(X,XY) &= \sigma(X,X)\sigma(X,Y) \\ \sigma(XY,X) &= \sigma(Y,Y)\sigma(XY,XY)\sigma(X,Y) \\ \sigma(XY,Y) &= \sigma(Y,Y)\sigma(X,Y) \\ \sigma(Y,XY) &= \sigma(X,X)\sigma(XY,XY)\sigma(X,Y) \\ \sigma(Y,X) &= \sigma(X,X)\sigma(Y,Y)\sigma(XY,XY)\sigma(X,Y) \end{cases} \tag{3.5}$$

c. *For any pair $X,Y \in 2^{\mathfrak{I}}$, the sub-loop*

$$\{\,(\phi;+)\,,\,(\phi;-)\,,\,(X;+)\,,\,(X;-)\,,\,(Y;+)\,,\,(Y;-)\,,\,(XY;+)\,,\,(XY;-)\,\}\,, \tag{3.6}$$

of a left/right-alternative R-LOOP $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+,-\}$, *is a* **R-GROUP**.
For given $\sigma(X,X), \sigma(Y,Y), \sigma(XY,XY), \sigma(X,Y)$, the σ-function for this group is fixed. (Instead of $\sigma(XY,XY)$ also $\sigma(Y,X)$ could be prescribed.)

Proof (a. $\Rightarrow$) According to the preceding section, we have left/right-alternativity iff

$$\forall\ X,Y \in 2^{\mathfrak{I}}:\ \sigma(X,Y)\sigma(X,XY) = \sigma(X,X)\,,\quad \text{and}\quad \sigma(X,Y)\sigma(XY,Y) = \sigma(Y,Y)\,, \tag{3.7}$$

or, interchanging X and Y in the 1st identity,

$$\forall\ X,Y \in 2^{\mathfrak{I}}:\ \sigma(Y,X)\sigma(Y,XY)\sigma(Y,Y) = +\,,\quad \text{and}\quad \sigma(X,Y)\sigma(XY,Y)\sigma(Y,Y) = +\,. \tag{3.8}$$

The product of the latter two reads

$$\forall\ X,Y\ \in 2^{\mathfrak{I}}\ :\ \ \sigma(Y,XY)\sigma(XY,Y) = \sigma(X,Y)\sigma(Y,X)\,. \tag{3.9}$$

Now replace in (3.7) X by XY and Y by XY, respectively,

$$\forall\, X,Y\ \in 2^{\mathfrak{I}}:\ \sigma(XY,Y)\sigma(XY,X) = \sigma(XY,XY)\,,\ \text{and}\ \sigma(X,XY)\sigma(Y,XY) = \sigma(XY,XY)\,. \tag{3.10}$$

The product of the two identities in (3.7) reads

$$\forall\ X,Y \in 2^{\mathfrak{I}}\ :\ \sigma(X,XY)\sigma(XY,Y) = \sigma(X,X)\sigma(Y,Y)\,.$$

The product of this with the 2nd identity in (3.10), together with (3.9), finally leads to

$$\sigma(X,X)\sigma(Y,Y) = \sigma(XY,Y)\sigma(X,XY) = \sigma(XY,XY)\sigma(XY,Y)\sigma(Y,XY).$$

(a. $\Leftarrow$) Left alternativity is already included in condition (3.4). Together with the other condition right alternativity follows by 'swapping the entries'.
(b.) The 1st part of the proof leads to this list.

(c.) For all X,Y,Z in the subgroup of $2^{\mathfrak{I}}$, generated by X,Y, we have to verify $\sigma(XY,Z)\sigma(X,Y) = \sigma(X,YZ)\sigma(Y,Z)$. Going through the checklist is greatly facilitated by the table (3.5). From this table also follows the final uniqueness result. ■

From Properties 1.4 (last item) it follows that the R-LOOP $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+,-\}$ is a *group* with normal sub-group $\{\mathbf{e}, \hat{\mathbf{e}}\}$, ie. a R-GROUP, iff associativity holds, that is

$$\forall\ X,Y,Z\ \in\ 2^{\mathfrak{I}}\ :\quad \sigma(XY,Z)\sigma(X,Y) = \sigma(X,YZ)\sigma(Y,Z)\,. \tag{3.11}$$

We now intend to construct a class of functions $\sigma\ :\ 2^{\mathfrak{I}} \times 2^{\mathfrak{I}}\ \to\ \{+,-\}$ which satisfies (3.11).

Definition 3.4
Let $\mathfrak{I}$ be a fixed set. Fix a finite subset $\mathfrak{S} \subset \mathfrak{I} \times \mathfrak{I}$. Define

$$\sigma_{\mathfrak{S}}\ :\ 2^{\mathfrak{I}} \times 2^{\mathfrak{I}}\ \to\ \{+\,,-\}\quad :\quad (X,Y)\ \mapsto\ \sigma(X,Y) = \big(-\big)^{\#\{(X\times Y)\cap\mathfrak{S}\}}\,, \tag{3.12}$$

where $\#\{\ \ \}$ denotes the number of points in the set $\{\ \ \}$.

Note that $\sigma_{\mathfrak{S}}(X,Y)$ indicates whether the integer $\#\{(X\times Y)\cap\mathfrak{S}\} \geq 0$ is even or odd.

Lemma 3.5
The function $\sigma_{\mathfrak{S}}$ enjoys the properties

$$\forall\, X,Y,Z\ \in 2^{\mathfrak{I}}\ :\quad \begin{cases} \sigma_{\mathfrak{S}}(X,Y\cup Z) &= \sigma_{\mathfrak{S}}(X,Y)\sigma_{\mathfrak{S}}(X,Z)\sigma_{\mathfrak{S}}(X,Y\cap Z)\,, \\ \sigma_{\mathfrak{S}}(X\cup Y,Z) &= \sigma_{\mathfrak{S}}(X,Z)\sigma_{\mathfrak{S}}(Y,Z)\sigma_{\mathfrak{S}}(X\cap Y,Z)\,, \\ \sigma_{\mathfrak{S}}(XY,Z) &= \sigma_{\mathfrak{S}}(X,Z)\sigma_{\mathfrak{S}}(Y,Z) \\ \sigma_{\mathfrak{S}}(X,YZ) &= \sigma_{\mathfrak{S}}(X,Y)\sigma_{\mathfrak{S}}(X,Z) \end{cases} \tag{3.13}$$

Proof First note the disjoint union $Y\cup Z = (Y\backslash Z)\cup(Z\backslash Y)\cup(Y\cap Z)$ and calculate

$$\#\big\{\{X\times(Y\cup Z)\}\cap\mathfrak{S}\big\} = \#\big\{\{X\times(Y\backslash Z)\}\cap\mathfrak{S}\big\}+\#\big\{\{X\times(Z\backslash Y)\}\cap\mathfrak{S}\big\}+\#\big\{\{X\times(Y\cap ZC)\}\cap\mathfrak{S}\big\},$$

which leads to the 1st formula. The 2nd formula is obtained similarly.
In order to obtain the 3rd identity we use the disjoint unions $X = (X\backslash Y)\cup(X\cap Y)$ and $Y = (Y\backslash X)\cup(X\cap Y)$. Calculate

$$\begin{aligned} &\#\big\{\{X\times Z\}\cap\mathfrak{S}\big\} + \#\big\{\{Y\times Z\}\cap\mathfrak{S}\big\} = \\ &\quad = \#\big\{\{(X\backslash Y)\times Z\}\cap\mathfrak{S}\big\} + \#\big\{\{(Y\backslash X)\times Z\}\cap\mathfrak{S}\big\} + 2\#\big\{\{(X\cap Y)\times Z\}\cap\mathfrak{S}\big\} = \\ &\qquad\qquad = \#\big\{\{(XY)\times Z\}\cap\mathfrak{S}\big\} + 2\#\big\{\{(X\cap Y)\times Z\}\cap\mathfrak{S}\big\}. \end{aligned}$$

The last term is even. Its presence in the exponent to $\big(-\big)$ does not contribute to $\sigma(XY,Z)$. The 4th identity is obtained in a similar way. ■

Theorem 3.6
The function $\sigma_{\mathfrak{S}}$, defined by (3.12), satisfies the associativity condition (3.11).

Proof Starting from the obvious identity

$$\Big(\sigma(X,Z)\sigma(Y,Z)\Big)\sigma(X,Y) = \Big(\sigma(X,Y)\sigma(X,Z)\Big)\sigma(Y,Z)\,,$$

and Lemma 3.5, we find the wanted result

$$\sigma(XY,Z)\sigma(X,Y) = \sigma(X,YZ)\sigma(Y,Z).$$

■

Examples
1. Take $\mathfrak{S} = \emptyset$, the empty set, then $\forall\, X, Y \in 2^{\mathfrak{I}}$ one has $\sigma_{\emptyset}(X,Y) = +$.
2. If $\mathfrak{S}$ is a symmetric set in $\mathfrak{I} \times \mathfrak{I}$ we have $\sigma(X,Y) = \sigma(Y,X)$ and hence commutativity. This happens e.g. if $\mathfrak{S}$ is (a part of) the diagonal in $\mathfrak{I} \times \mathfrak{I}$.
3. Let $\mathfrak{I}$ be finite. Take $\mathfrak{S} = \mathfrak{I} \times \mathfrak{I}$. Then $\sigma_{\mathfrak{I}\times\mathfrak{I}}(X,Y) = \big(-\big)^{\#\{X\}\#\{Y\}}$.
So if both X, Y happen to be singletons one has $\sigma_{\mathfrak{I}\times\mathfrak{I}}(X,Y) = -$.

4. Take $\mathfrak{I} = \{1, 2, \ldots, N\}$. Denote the standard basis vectors in $\mathbb{R}^{2^N}$ by $\{\mathbf{e}_X\}_{X \in 2^{\mathfrak{I}}}$. In **[BLS]** a 'Clifford product' is introduced by, in our notation,

$$\mathbf{e}_X \mathbf{e}_Y = \big(-\big)^{\#\big\{(X\cap Y)\setminus\{1,\ldots,s\}\big\}}\big(-\big)^{p(X,Y)}\mathbf{e}_{XY},$$

with fixed s, $0 \le s \le N$, and

$$p(X,Y) = \sum_{j\in Y} \#\{i \in X \,\big|\, i > j\}\,.$$

In our approach this comes down to a special choice for $\mathfrak{S}$: The union of the sub-diagonal set in $\mathfrak{I} \times \mathfrak{I}$ with $\{(s+1, s+1), \ldots (N,N)\}$. The latter is a part of the diagonal in $\mathfrak{I} \times \mathfrak{I}$. Our construction, Definition 3.4, of the $\sigma_{\mathfrak{S}}$-functions has been inspired by the formulae in **[BLS]**.

Remark As we shall see, if $2^{\mathfrak{I}}$ has 3 or more generators, there are solutions to (3.11) which are NOT obtained by the construction of Lemma 3.5.

3.1 The special case $\#\{\Im\} = 2$

Table (2.5) tells us that for any choice $\alpha, \beta, \gamma, \delta \in \{+, -\}$, the σ-function rendered by the table

	$\emptyset$	A	B	AB
$\emptyset$	$+$	$+$	$+$	$+$
A	$+$	α	δ	$\alpha\delta$
B	$+$	$\alpha\beta\gamma\delta$	β	$\alpha\gamma\delta$
AB	$+$	$\beta\gamma\delta$	$\beta\delta$	γ

and $\Im = \{A, B\}$ leads to a group $2^{\Im} \overset{\sigma}{\times} \{+\,,-\}$.
In this special case we have, cf. Thm 1.8b,

$\sigma_A = \sigma_B = \sigma_{AB} =$

	$\emptyset$	A	B	AB
$\emptyset$	$+$	$+$	$+$	$+$
A	$+$	$+$	$-$	$-$
B	$+$	$-$	$+$	$-$
AB	$+$	$-$	$-$	$+$

Note that, in the 2-generator case, the whole of the matrix is fixed by the data on the A-B-subsquare. Note also the full correspondence with the construction of Lemma (3.5). The special choice $\alpha = \beta = \gamma = -$, $\delta = +$, leads to the unit-quaternion group. $\{\pm 1, \pm i, \pm j, \pm k\}$.

3.2 The special case $\#\{\Im\} = 3$

In this subsection we concentrate on the group generated by $\Im = \{A, B, C\}$. If we take the singletons in the set $\{A, B, C\}$ as generators, we get

$$2^{\{A,B,C\}} = \{\emptyset, A, B, C, AB, AC, BC, ABC\}.$$

There are $\frac{\binom{7}{2}}{\binom{3}{2}} = 7$ 2-generator subgroups. They are

$$\begin{array}{ccc} \{\emptyset\,,\,A\,,\,B\,,\,AB\} & \{\emptyset\,,\,A\,,\,C\,,\,AC\} & \{\emptyset\,,\,B\,,\,C\,,\,BC\} \\ & \{\emptyset\,,\,AB\,,\,AC\,,\,BC\} & \\ \{\emptyset\,,\,A\,,\,BC\,,\,ABC\} & \{\emptyset\,,\,B\,,\,AC\,,\,ABC\} & \{\emptyset\,,\,C\,,\,AB\,,\,ABC\} \end{array} \tag{3.14}$$

Note there are $\frac{\binom{7}{2}4}{3} = 28$ triples of generators of $2^{\{A,B,C\}}$. Expressed in $\{A,B,C\}$ they are of type

$$\begin{array}{llll} 1 & & \{A\,,\,B\,,\,C\} & \\ 3 & \{A\,,\,B\,,\,ABC\} & \{A\,,\,AB\,,\,AC\} & \{AB\,,\,BC\,,\,ABC\} \\ 6 & \{A\,,\,B\,,\,AC\} & \{AB\,,\,B\,,\,AC\} & \{A\,,\,AB\,,\,ABC\} \end{array} \tag{3.15}$$

The numbers in the 1st column indicates the number of generating triples there are of each type in the corresponding row.

Theorem 3.7
Let $\mathfrak{I} = \{A,B,C\}$. Construct $\sigma : 2^{\mathfrak{I}} \times 2^{\mathfrak{I}} \to \{+\,,\,-\}$ in the following way:
Step 1 *Take for the boundary entries: $\sigma(\emptyset, X) = \sigma(X, \emptyset) = +\,,\quad X \in 2^{\mathfrak{I}}$.*

Step 2 *Pick the diagonal elements* $\left\{ \begin{array}{ccc} \sigma(A,A) & \sigma(B,B) & \sigma(C,C) \\ \sigma(AB,AB) & \sigma(AC,AC) & \sigma(BC,BC) \\ & \sigma(ABC,ABC) & \end{array} \right\} \in \{+\,,\,-\}.$

Step 3 *Pick the off-diagonal elements* $\left\{ \begin{array}{ccc} \sigma(A,B) & \sigma(A,C) & \sigma(B,C) \\ & \sigma(AB,AC) & \\ \sigma(A,BC) & \sigma(B,AC) & \sigma(C,AB) \end{array} \right\} \in \{+\,,\,-\}.$

Step 4 *Calculate the remaining 35 values of $\sigma \in \{+\,,\,-\}$ by means of table (3.5).*
Then, *with such σ, the loop $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+\,,-\}$ is a left/right alternative R-LOOP.*

Proof Left/right alternativity always plays in a 2-generator subspace. So it is enough to apply Theorem (3.3) to the subgroups (3.14). ■

We now turn to the conditions on σ for associativity of $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+\,,-\}$. Of course Theorem 3.6 provides us with lots of σ-functions leading to associativity. However, not all possibilities are exhausted this way. Starting from the preceding Theorem (3.7) we explore a few more conditions on the diagonal and off-diagonal elements in the respective sets of Step 2 and Step 3 in Theorem(3.7).

Theorem 3.8
Let $\mathfrak{I} = \{A,B,C\}$. Consider the left/right alternative R-LOOP $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+\,,-\}$ as constructed in Theorem (3.7).

A. *For $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+\,,-\}$ to have the property of associativity, i.e. to be a group, the following three extra conditions on σ are all* **necessary**

- **i.** *For all triples of generators $\{X,Y,Z\}$ we have*

  $\sigma(X,X)\sigma(Y,Y)\sigma(Z,Z)\sigma(XY,XY)\sigma(XZ,XZ)\sigma(YZ,YZ)\sigma(XYZ,XYZ) =$

$$= \sigma(A,A)\sigma(B,B)\sigma(C,C)\sigma(AB,AB)\sigma(AC,AC)\sigma(BC,BC)\sigma(ABC,ABC) = +. \tag{3.16}$$

ii. *For all triples of generators* $\{X, Y, Z\}$ *we have*

$$\sigma(XY, YZ) = \sigma(X,Y)\,\sigma(X,Z)\,\sigma(Y,Y)\,\sigma(Y,Z)\,. \tag{3.17}$$

$$\sigma(X, XYZ) = \sigma(XY,Z)\,\sigma(X,X)\,\sigma(Y,Z)\,\sigma(X,Y). \tag{3.18}$$

iii. Either *for all (ordered) triples of generators* $\{X, Y, Z\}$ *we have*

$$\sigma(XY,Z)\sigma(X,Z)\sigma(Y,Z) = + \qquad \textit{and} \qquad \sigma(X,YZ)\sigma(X,Y)\sigma(X,Z) = +\ , \tag{3.19}$$

or, *for all (ordered) triples of generators* $\{X, Y, Z\}$ *we have*

$$\sigma(XY,Z)\sigma(X,Z)\sigma(Y,Z) = - \qquad \textit{and} \qquad \sigma(X,YZ)\sigma(X,Y)\sigma(X,Z) = -\ , \tag{3.20}$$

B. *For* $2^{\mathfrak{I}} \overset{\sigma}{\times} \{+,-\}$ *to have the property of associativity, i.e. to be a group, it is* **sufficient** *that for some triple of generators* $\{A, B, C\}$, *the following three properties hold*

i. $\sigma(A,A)\sigma(B,B)\sigma(C,C)\sigma(AB,AB)\sigma(AC,AC)\sigma(BC,BC)\sigma(ABC,ABC) = +\,.$

ii. $\sigma(AB,BC) = \sigma(A,B)\sigma(A,C)\sigma(B,B)\sigma(B,C).$

iii. $\sigma(AB,C)\sigma(A,C)\sigma(B,C) = \sigma(BC,A)\sigma(B,A)\sigma(C,A) = \sigma(CA,B)\sigma(C,B)\sigma(A,B).$

C. *By choosing the set of values*

$$\left\{ \begin{array}{ccc} \sigma(A,A) & \sigma(B,B) & \sigma(C,C) \\ \sigma(AB,AB) & \sigma(AC,AC) & \sigma(BC,BC) \\ \sigma(A,B) & \sigma(A,C) & \sigma(B,C) \end{array} \right\} \in \{+\,,\,-\}\ , \tag{3.21}$$

or, equivalently,

$$\left\{ \begin{array}{ccc} \sigma(A,A) & \sigma(B,B) & \sigma(C,C) \\ \sigma(A,B) & \sigma(A,C) & \sigma(B,C) \\ \sigma(B,A) & \sigma(C,A) & \sigma(C,B) \end{array} \right\} \in \{+\,,\,-\}\ , \tag{3.22}$$

ànd *choosing the value* $\alpha = \sigma(AB,C)\sigma(A,C)\sigma(B,C)$, *leads to a uniquely defined associative multiplication structure on* $2^{\{A,B,C\}} \overset{\sigma}{\times} \{+,-\}$.

Proof
ad A. The associativity condition says

$$\forall\, X, Y, Z \in 2^{\mathfrak{I}}\ :\quad \sigma(XY,Z)\sigma(X,YZ)\sigma(X,Y)\sigma(Y,Z) = + \tag{3.23}$$

ad Ai. Applying even permutations $X \to Y \to Z \to X$, we find

$$\left\{ \begin{array}{lcl} \sigma(X,YZ)\sigma(XY,Z) & = & \sigma(X,Y)\sigma(Y,Z) \\ \sigma(Y,ZX)\sigma(YZ,X) & = & \sigma(Y,Z)\sigma(Z,X) \\ \sigma(Z,XY)\sigma(ZX,Y) & = & \sigma(Z,X)\sigma(X,Y) \end{array} \right. \tag{3.24}$$

The product of those lines gives us

$$\sigma(XY,Z)\sigma(ZX,Y)\sigma(YZ,X) = \sigma(X,YZ)\sigma(Y,ZX)\sigma(Z,XY).$$

Swopping the arguments on the left, cf. the last line in (3.5), leads to (3.16).
ad Aii. In (3.23) put $X = XY,\ Y = Y,\ Z = Z$. We find, together with swopping,

$$\sigma(XY,YZ) = \sigma(X,Z)\sigma(XY,Y)\sigma(Y,Z) = \sigma(X,Z)\sigma(X,Y)\sigma(Y,Y)\sigma(Y,Z).$$

Next, in (3.23), put $X = X,\ Y = XYZ,\ Z = Z$. Then

$$\sigma(X,XYZ) = \sigma(XY,Z)\,\sigma(X,X)\,\sigma(Y,Z)\,\sigma(X,Y).$$

ad Aiii. Pick a generating triple $\{A,B,C\}$. Put

$$\sigma(AB,C)\sigma(A,C)\sigma(B,C) = \alpha \in \{+,-\}. \tag{3.25}$$

From (3.24)

$$\sigma(A,BC)\sigma(A,B)\sigma(A,C) = \alpha\,. \tag{3.26}$$

Swopping this and applying (3.16), it turns out that the alternative (3.19)-(3.20) is OK for *all orderings* of our *chosen generating triple* A,B,C. Starting from this result, together with the obtained (3.17)-(3.18), we now prove that the alternative holds for all orderings of any generating triple. That means we have to work through the full list of generators (3.15). Two specimen are presented here:

- $\sigma((AB)B,AC)\sigma(AB,AC)\sigma(B,AC) =$

$$= \sigma(A,A)\sigma(A,C)\sigma(A,A)\sigma(A,C)\Big(\sigma(B,A)\sigma(B,C)\sigma(B,AC)\Big) = \alpha.$$

- $\sigma((AB)(BC),ABC)\sigma(AB,ABC)\sigma(BC,ABC) = \sigma(AC,AC)\sigma(AC,B)\sigma(AB,AB)\cdot$

$$\cdot\,\sigma(AB,C)\sigma(BC,BC)\sigma(BC,A) =$$

$$= \Big(\sigma(AC,B)\sigma(A,B)\sigma(C,B)\Big)\Big(\sigma(BC,A)\sigma(B,A)\sigma(C,A)\Big)\cdot\big(\sigma(AB,C)\sigma(A,C)\sigma(B,C)\Big)\cdot$$

$$\cdot\Big(\sigma(A,B)\sigma(B,A)\sigma(A,A)\sigma(B,B)\sigma(AB,AB)\Big)\Big(\sigma(C,B)\sigma(B,C)\sigma(C,C)\sigma(B,B)\sigma(BC,BC)\Big)\cdot$$

$$\cdot\Big(\sigma(A,C)\sigma(C,A)\sigma(A,A)\sigma(C,C)\sigma(AC,AC)\Big) = \alpha\,.$$

For a full list of such verifications see Appendix C.

ad B. First note that (B.ii) also holds for any permutation of $\{A,B,C\}$. For example

$$\sigma(AB,AC) = \sigma(AB,(AB)(BC)) = \Big(\sigma(AB,AB)\sigma(A,B)\sigma(B,B)\Big)\sigma(A,C)\sigma(B,C) =$$

$$= \sigma(A,A)\sigma(B,A)\sigma(A,C)\sigma(B,C).$$

For associativity we will verify, cf. (3.23),

$$\forall\ X,Y \in 2^{\{A,B,C\}}\ :\ \ \sigma(XY,Z)\sigma(X,Z)\sigma(Y,Z) = \sigma(X,YZ)\sigma(X,Y)\sigma(X,Z)\,. \qquad (3.27)$$

If the triple $\{X,Y,Z\}$ belongs to a 2-generator subgroup each side of (3.27) equals $+$. See (3.5) of Thm 3.3.
In (A.iii) we effectively showed that, starting from the assumptions of Part B, for all generating triples $\{X,Y,Z\}$, all $\sigma(XY,Z)\sigma(X,Z)\sigma(Y,Z)$ and all $\sigma(X,YZ)\sigma(X,Y)\sigma(X,Z)$ have the same sign. This proves associativity.

ad C. Given a choice for $\alpha \in \{+,-\}$, all values of σ can be uniquely expressed as products of the in the entries of either (3.21) or (3.22). For example $\sigma(ABC,AB) = \sigma(AB,AB)\sigma(C,AB) = \alpha\,\sigma(A,A)\sigma(B,B)\sigma(A,B)\sigma(B,A)\sigma(C,A)\sigma(C,B)$. ■

Example A σ-function of 'type $\alpha = -$' is, cf. Theorem 1.8,

$$\sigma_{ABC} \ = \ \begin{array}{c|cccccccc} & \emptyset & A & B & C & AB & BC & CA & ABC \\ \emptyset & & & & & & & & \\ A & & & & & & - & & - \\ B & & & & & & & - & - \\ C & & & & & - & & & - \\ AB & & & & - & & & & - \\ BC & & - & & & & & & - \\ CA & & & - & & & & & - \\ ABC & & - & - & - & - & - & - & \end{array}$$

The empty entries are $+$.

The next theorem characterizes the solutions σ of the associativity condition (3.23).

Notation
• $\mathfrak{D}$ denotes a fixed subset in the *diagonal* of $\{A,B,C\} \times \{A,B,C\}$. So outside the diagonal in $\{A,B,C\} \times \{A,B,C\}$, the function $\sigma_{\mathfrak{D}}$ only assumes the value $+$.
• $\mathfrak{U}$ denotes a fixed subset in the *strict upper triangular part* of $\{A,B,C\} \times \{A,B,C\}$. So, within $\{A,B,C\} \times \{A,B,C\}$, but below the diagonal, $\sigma_{\mathfrak{U}}$ has only $+$'s .

Theorem 3.9

A. *If σ satisfies the associativity condition (3.11) and also satisfies (3.19), ie. $\alpha = +$, it corresponds to Lemma (3.5). For suitable special subsets $\mathfrak{D}, \mathfrak{U} \subset \{A,B,C\} \times \{A,B,C\}$ such σ can be written as a product of $\sigma_{\mathfrak{D}}\sigma_{\mathfrak{U}}$, cf. Def 3.4, and some of the σ_{AB}, σ_{AC}, σ_{BC}, σ_{ABC}, where the number of factors of the latter type is even.*

B. *Suppose a* symmetric *σ satisfies the associativity condition (3.11) and also satisfies (3.20), ie. $\alpha = -$. Then such σ can be written as a product of $\sigma_{\mathfrak{D}}$ and some of the σ_{AB}, σ_{AC}, σ_{BC}, σ_{ABC}, where the number of the latter factors is odd.*

C. *All Abelian R-GROUP extensions of $2^{\{A,B,C\}}$ of type $2^{\{A,B,C\}} \overset{\sigma}{\times} \{+,-\}$ are isomorphic to such an extension with a σ-function of type $\sigma_{\mathfrak{D}}$.*

D. *All R-GROUP extensions of $2^{\{A,B,C\}}$ of type $2^{\{A,B,C\}} \overset{\sigma}{\times} \{+,-\}$ are isomorphic to such an extension with a σ-function of type $\sigma_{\mathfrak{D}}\sigma_{\mathfrak{U}}$.*

Proof

ad A. As we have seen the σ-function is of type $\sigma_{\mathfrak{S}}$ with $\mathfrak{S} \subset \{A,B,C\} \times \{A,B,C\}$. If the lower diagonal part of this $\mathfrak{S}$ contains a $-$ at A, B, say, we multiply $\sigma_{\mathfrak{S}}$ by σ_{AB}. Thus this entry becomes $+$. Continuing this way all $-$'s in the lower diagonal part are removed. In case an odd number of such multipliers is needed we regain $\alpha = +$ by multiplying finally with σ_{ABC}, which does not affect the previous operations.

ad B. First we construct $\mathfrak{D}$ from the diagonal values of the given σ. For each $-$ outside the diagonal a factor of type σ_{AB} is introduced. If an even number of such factors does the job an extra factor σ_{ABC} enters the game in order to regain $\alpha = -$.

ad CD. Factors of type $\sigma_{AB}, \ldots, \sigma_{ABC}$ don't alter the algebraic structure of the obtained R-GROUP. Cf. the proof of Thm 1.8. ■

3.3 The special case $\#\{\Im\} = 4$

In this subsection we restrict to *associative* doublings of the group $2^{\{A,B,C,D\}}$. If we take the singletons in the set $\{A, B, C, D\}$ as generators, we get

$$2^{\{A,B,C,D\}} = \{\emptyset, A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, ABCD\}.$$

In order to apply the associativity considerations of the preceding section we have to look at 3-generator subgroups. Their number is

$$\frac{\binom{15}{2}(15-3)}{\binom{7}{2}(7-2-1)} = 15\,.$$

We list them here. In each subgroup a lexicographically ordered set of generators is chosen. They are printed in boldface.

I.	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	AB	AC	BC	ABC	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{D}$	AB	AD	BD	ABD	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{C}$	$\boldsymbol{D}$	AC	AD	CD	ACD	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$	BC	BD	CD	BCD	}
II.	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{B}$	AB	$\boldsymbol{CD}$	ACD	BCD	$ABCD$	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{C}$	AC	$\boldsymbol{BD}$	ABD	BCD	$ABCD$	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{D}$	AD	$\boldsymbol{BC}$	ABC	BCD	$ABCD$	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{AD}$	BC	ABD	ACD	$ABCD$	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{D}$	$\boldsymbol{AC}$	BD	ABC	ACD	$ABCD$	}
	{	ϕ	$\boldsymbol{C}$	$\boldsymbol{D}$	$\boldsymbol{AB}$	CD	ABC	ABD	$ABCD$	}
III.	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{BC}$	$\boldsymbol{BD}$	CD	ABC	ABD	ACD	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{AC}$	$\boldsymbol{AD}$	CD	ABC	ABD	BCD	}
	{	ϕ	$\boldsymbol{C}$	$\boldsymbol{AB}$	$\boldsymbol{AD}$	BD	ABC	ACD	BCD	}
	{	ϕ	$\boldsymbol{D}$	$\boldsymbol{AB}$	$\boldsymbol{AC}$	BC	ABD	ACD	BCD	}
IV.	{	ϕ	$\boldsymbol{AB}$	$\boldsymbol{AC}$	$\boldsymbol{AD}$	BC	BD	CD	$ABCD$	}

Note that every 2-generator subgroup is a subgroup of three 3-generator subgroups.

The construction of Theorem 3.6 tells us that a class of σ-functions

$$\sigma \; : \; 2^{\{A,B,C,D\}} \times 2^{\{A,B,C,D\}} \;\rightarrow\; \{+,-\}\,,$$

leading to associativity of $2^{\{A,B,C,D\}} \overset{\sigma}{\times} \{+\,,-\}$ is found by fixing a subset $\mathfrak{S} \subset \{A,B,C,D\} \times \{A,B,C,D\}$. Then $\sigma_{\mathfrak{S}}$, which takes the value $-$ on $\mathfrak{S}$, does the job. In treatments on *Clifford Algebras*, such as DeLanghe **[L]** only this type of σ-function seems to occur.

In practical applications of *Clifford Algebras* one often prefers to fix the values of σ on the diagonal of $2^{\{A,B,C,D\}} \times 2^{\{A,B,C,D\}}$ first:

The diagonal entries in the σ-matrix

First, pick at will $\sigma(A_k\,,A_k\,) \in \{+,-\}$, $\sigma(A_kA_\ell\,,A_kA_\ell\,) \in \{+,-\}$, for $1 \leq k < \ell \leq 4$.
In each subgroup the product of the diagonal elements should equal $+$, cf. Thm 3.8-B.i .
This fixes $\sigma(A_kA_\ell A_m\,,\,A_kA_\ell A_m) \in \{+,-\}$, for $1 \leq k < \ell < m \leq 4$, by requiring associativity of the R-GROUPS obtained from subgroups of type I.
Next $\sigma(ABCD, ABCD)$ by means of IV.
By taking products one finds that all diagonal products of II and III also equal $+$.

The non-diagonal entries in the σ-matrix

Pick the 'upper diagonal' entries $\sigma(A_k\,,A_\ell\,) \in \{+,-\}$, for $1 \leq k < \ell \leq 4$.

For considerations on an 'overall consistent/coherent choice' for $\alpha \in \{+,-\}$ in the 3-generator subgroups we refer to Appendix D.

A APPENDIX on LOOPS and LATIN SQUARES

We list some examples of Latin Squares
1: Cyclic group of order 5. **2:** Abelian group of order 5. **3:** Group

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 4 & 5 & 1 & 2 \\
4 & 5 & 1 & 2 & 3 \\
5 & 1 & 2 & 3 & 4
\end{array}
\qquad
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 1 & 4 \\
3 & 5 & 4 & 2 & 1 \\
4 & 1 & 2 & 5 & 3 \\
5 & 4 & 1 & 3 & 2
\end{array}
\qquad
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 4 & 1 & 5 & 3 \\
3 & 1 & 5 & 2 & 4 \\
4 & 5 & 2 & 3 & 1 \\
5 & 3 & 4 & 1 & 2
\end{array}
\tag{A.1}
$$

4: Non-associative loop with all $\mathbf{x}^2 = \mathbf{e}$. It has $2 = (23)4 \neq 2(34) = 4$.
5: Distinct left/right inverses in general.

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 5 & 3 & 4 \\
3 & 4 & 1 & 5 & 2 \\
4 & 5 & 2 & 1 & 3 \\
5 & 3 & 4 & 2 & 1
\end{array}
\qquad
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 1 & 5 & 4 \\
3 & 5 & 4 & 1 & 2 \\
4 & 1 & 5 & 2 & 3 \\
5 & 4 & 2 & 3 & 1
\end{array}
\tag{A.2}
$$

Note that 1's on the diagonal ànd being symmetric is not possible.

6: Non-left/right alternative loops with equal left/right inverse.

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 5 & 2 & 1 & 4 \\
4 & 1 & 5 & 3 & 2 \\
5 & 4 & 1 & 2 & 3
\end{array}
\qquad
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 5 & 1 & 2 & 4 \\
4 & 1 & 5 & 3 & 2 \\
5 & 4 & 2 & 1 & 3
\end{array}
\tag{A.3}
$$

In these examples we have, respectively, $2(23) = 24 = 5\,,\ (22)3 = 33 = 2$ and $3(22) = 33 = 2\,,\ (32)2 = 52 = 4$.

B APPENDIX Octonions

Theorem 3.7 tells us for which constructions of σ the R-LOOP $2^{\{A,B,C\}} \overset{\sigma}{\times} \{+,-\}$ is left/right alternative. With the special choice of 'diagonal elements'

$$\sigma(A,A) = \ldots = \sigma(ABC,ABC) = - \,, \qquad \text{and further}$$

$$\sigma(A,B) = \sigma(A,C) = \sigma(B,C) = \sigma(A,BC) = \sigma(AC,B) = \sigma(C,AB) = \sigma(AB,BC) = + \,,$$

we find a traditional multiplication table for octonions:

	$\phi = e_0$	$A = e_1$	$B = e_2$	$C = e_3$	$AB = e_4$	$BC = e_5$	$ABC = e_6$	$AC = e_7$
$\phi = e_0$	+	+	+	+	+	+	+	+
$A = e_1$	+	−	+	+	−	+	−	−
$B = e_2$	+	−	−	+	+	−	+	−
$C = e_3$	+	−	−	−	+	+	−	+
$AB = e_4$	+	+	−	−	−	+	+	−
$BC = e_5$	+	−	+	−	−	−	+	+
$ABC = e_6$	+	+	−	+	−	−	−	+
$AC = e_7$	+	+	+	−	+	−	−	−

Note: $e_0 e_0 = 1\,,\ \ e_k e_k = -1\,,\ 1 \leq k \leq 7\,,\qquad e_k e_{k+1} = e_{k+3}\,,\ 1 \leq k \leq 7$, cycl.
$e_k e_\ell = -e_\ell e_k\,,\ 1 \leq k,\ell \leq 7.$

$e_1e_2 = e_4$	$e_2e_3 = e_5$	$e_3e_4 = e_6$	$e_4e_5 = e_7$	$e_5e_6 = e_1$	$e_6e_7 = e_2$	$e_7e_1 = e_3$
$e_4e_1 = e_2$	$e_5e_2 = e_3$	$e_6e_3 = e_4$	$e_7e_4 = e_5$	$e_1e_5 = e_6$	$e_2e_6 = e_7$	$e_3e_7 = e_1$
$e_2e_4 = e_1$	$e_3e_5 = e_2$	$e_4e_6 = e_3$	$e_5e_7 = e_4$	$e_6e_1 = e_5$	$e_7e_2 = e_6$	$e_1e_3 = e_7$

Note: With each 2-generator subgroup in $2^{\{A,B,C\}} \overset{\sigma}{\times} \{+,-\}$ a unit-quaternion group is associated. The **order** in their multiplication rules is prescribed by the choice of the $+$-signs above. Cf. section 3.1. The necessary information is contained in the sequence

$$A \to B \to C \to BC \to AC \to AB \to ABC \to A\,,$$

where each pair of ordered neighbours corresponds to a $+$-sign in the σ-table. Another possibility is

$$A \to C \to BC \to ABC \to AC \to AB \to ABC \to A\,,$$

which is less elegant because ABC occurs twice.
We mention the overview article **[B]**.

C ASSOCIATIVITY in $\mathbf{2^{\{A,B,C\}}\overset{\sigma}{\times}\{+\,,-\}}$

Let

$$\sigma \ : \ 2^{\{A,B,C\}} \to \{+,-\} \tag{C.1}$$

be such that

$$\forall\ X,Y \ \in 2^{\{A,B,C\}} \ : \ \left\{ \begin{array}{rcl} \sigma(X,Y)\sigma(Y,X) &=& \sigma(X,X)\sigma(Y,Y)\sigma(XY,XY) \\ \sigma(X,XY) &=& \sigma(X,X)\sigma(X,Y) \end{array} \right. \tag{C.2}$$

and, suppose that for some fixed $\alpha \in \{+,-\}$ and for all permutations $A \to B \to C \to A$,

$$\left\{ \begin{array}{l} \sigma(A,A)\sigma(B,B)\sigma(C,C)\sigma(AB,AB)\sigma(AC,AC)\sigma(BC,BC)\sigma(ABC,ABC) = +\,, \\ \sigma(AB,BC) = \sigma(A,B)\sigma(A,C)\sigma(B,B)\sigma(B,C), \\ \sigma(AB,C)\sigma(A,C)\sigma(B,C) = \alpha\,, \end{array} \right. \tag{C.3}$$

then for all ordered generating triples $\{U,V,W\} \subset 2^{\{A,B,C\}}$ we have

$$\sigma(UV,W)\sigma(U,W)\sigma(V,W) = \alpha \qquad \text{and} \qquad \sigma(U,VW)\sigma(U,V)\sigma(U,W) = \alpha\,. \tag{C.4}$$

Note first that (C.2) implies that also $\sigma(XY,X) = \sigma(X,X)\sigma(Y,X)$ and, secondly, that we only have to verify the first identity in (C.4) because of (C.2), 'swopping'.
We now check the list of generating triples (3.15).

$\{A\,,\ B\,,\ ABC\}$

- $\sigma(B(ABC),A)\sigma(B,A)\sigma(ABC,A) = \sigma(AC,A)\sigma(B,A)\sigma(A,A)\sigma(BC,A) = \alpha$
- $\sigma(A(ABC),B)\sigma(A,B)\sigma(ABC,B) = \sigma(B,B)\sigma(C,B)\sigma(A,B)\sigma(B,B)\sigma(AC,B) = \alpha$
- $\sigma(AB,ABC)\sigma(A,ABC)\sigma(B,ABC) = \sigma(AB,AB)\sigma(AB,C)\sigma(A,A)\sigma(A,BC)\sigma(B,B)\cdot$
  $\cdot\,\sigma(B,AC) = \sigma(AB,AB)\sigma(A,C)\sigma(B,C)\sigma(A,A)\sigma(A,B)\sigma(A,C)\sigma(B,B)\sigma(B,A)\cdot$
  $\cdot\,\sigma(B,C)\alpha^3 = \alpha$

$\{A\,,\ AB\,,\ AC\}$

- $\sigma((AB)(AC),A)\sigma(AB,A)\sigma(AC,A) = \sigma(BC,A)\sigma(A,A)\sigma(B,A)\sigma(A,A)\sigma(C,A) = \alpha$
- $\sigma(A(AC),AB)\sigma(A,AB)\sigma(AC,AB) = \sigma(C,AB)\sigma(A,A)\sigma(A,B)\sigma(A,A)\sigma(A,B)\cdot$
  $\cdot\,\sigma(C,A)\sigma(C,B) = \alpha$
- $\sigma(A(AB),AC)\sigma(A,AC)\sigma(AB,AC) = \alpha$ from above and $B \leftrightarrow C$

$\{AB\,,\ BC\,,\ ABC\}$

- $\sigma(BC)(ABC),AB)\sigma(BC,AB)\sigma(ABC,AB) = \sigma(A,A)\sigma(A,B)\sigma(B,A)\sigma(B,B)\cdot$
  $\cdot\,\sigma(C,A)\sigma(C,B)\sigma(AB,AB)\sigma(C,AB) = \alpha$
- $\sigma((AB)(ABC),BC)\sigma(AB,BC)\sigma(ABC,BC) = \sigma(C,B)\sigma(C,C)\sigma(A,B)\sigma(A,C)\cdot$
  $\cdot\,\sigma(B,B)\sigma(B,C)\sigma(BC,BC)\sigma(A,BC) = \alpha$
- $\sigma((AB)(BC),ABC)\sigma(AB,ABC)\sigma(BC,ABC) = \sigma(AC,AC)\sigma(AC,B)\sigma(AB,AB)\cdot$
  $\cdot\,\sigma(AB,C)\sigma(BC,BC)\sigma(BC,A) = \sigma(AC,AC)\sigma(A,B)\sigma(C,B)\sigma(AB,AB)\cdot$
  $\cdot\,\sigma(A,C)\sigma(B,C)\sigma(BC,BC)\sigma(B,A)\sigma(C,A)\alpha^3 = \alpha$

$\{\, A\,,\ B\,,\ AC\,\}$

- $\sigma(B(AC),A)\sigma(B,A)\sigma(AC,A) = \sigma(A,A)\sigma(BC,A)\sigma(B,A)\sigma(A,A)\sigma(C,A) = \alpha$
- $\sigma(A(AC),B)\sigma(A,B)\sigma(AC,B) = \sigma(C,B)\sigma(A,B)\sigma(AC,B) = \alpha$
- $\sigma(AB,AC)\sigma(A,AC)\sigma(B,AC) = \sigma(A,A)\sigma(A,C)\sigma(B,A)\sigma(B,C)\cdot$
  $\cdot\,\sigma(A,A)\sigma(A,C)\sigma(B,AC) = \alpha$

$\{\, AB\,,\ B\,,\ AC\,\}$

- $\sigma(B(AC),AB)\sigma(B,AB)\sigma(AC,AB) = \sigma(C,AB)\sigma(AB,AB)\sigma(B,A)\sigma(B,B)\cdot$
  $\cdot\,\sigma(A,A)\sigma(A,B)\sigma(C,A)\sigma(C,B) = \alpha$
- $\sigma((AB)(AC),B)\sigma(AB,B)\sigma(AC,B) = \sigma(B,B)\sigma(C,B)\sigma(A,B)\sigma(B,B)\sigma(AC,B) = \alpha$
- $\sigma((AB)B,AC)\sigma(AB,AC)\sigma(B,AC) = \sigma(A,A)\sigma(A,C)\sigma(A,A)\sigma(A,C)\cdot$
  $\cdot\,\sigma(B,A)\sigma(B,C)\sigma(B,AC) = \alpha$

$\{\, A\,,\ AB\,,\ ABC\,\}$

- $\sigma((AB(ABC),A)\sigma(AB,A)\sigma(ABC,A) = \sigma(C,A)\sigma(A,A)\sigma(B,A)\sigma(A,A)\sigma(BC,A) = \alpha$
- $\sigma((ABC)A,AB)\sigma(ABC,AB)\sigma(A,AB) = \sigma(BC,AB)\sigma(AB,AB)\sigma(C,AB)\sigma(A,A)\sigma(A,B)\cdot$
  $\cdot\,\sigma(B,A)\sigma(B,B)\sigma(C,A)\sigma(C,B)\sigma(AB,AB)\sigma(C,AB)\sigma(A,A)\sigma(A,B) = \alpha$
- $\sigma(A(AB),ABC)\sigma(A,ABC)\sigma(AB,ABC) = \sigma(B,B)\sigma(B,AC)\sigma(A,A)\sigma(A,BC)\cdot$
  $\cdot\,\sigma(AB,AB)\sigma(AB,C)\sigma(AB,AB)\sigma(AB,C) =$
  $= \sigma(A,B)\sigma(B,A)\sigma(B,AC)\sigma(A,BC)\sigma(AB,C)\alpha^3 = \alpha$

Remark
Instead of making the assumptions (C.3) on the singleton triple $\{A,B,C\}$ and its permutations we could have started with *any* generating triple and its permutations!

D ASSOCIATIVITY in $\mathbf{2^{\{A,B,C,D\}}} \overset{\sigma}{\times} \{+\,,-\}$

The 3-generator subgroups of $2^{\{A,B,C,D\}}$ are listed below. The generators of our choice are in boldface.

I.	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	AB	AC	BC	ABC	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{D}$	AB	AD	BD	ABD	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{C}$	$\boldsymbol{D}$	AC	AD	CD	ACD	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{D}$	BC	BD	CD	BCD	}
II.	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{B}$	AB	$\boldsymbol{CD}$	ACD	BCD	$ABCD$	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{C}$	AC	$\boldsymbol{BD}$	ABD	BCD	$ABCD$	}
	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{D}$	AD	$\boldsymbol{BC}$	ABC	BCD	$ABCD$	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{AD}$	BC	ABD	ACD	$ABCD$	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{D}$	$\boldsymbol{AC}$	BD	ABC	ACD	$ABCD$	}
	{	ϕ	$\boldsymbol{C}$	$\boldsymbol{D}$	$\boldsymbol{AB}$	CD	ABC	ABD	$ABCD$	}
III.	{	ϕ	$\boldsymbol{A}$	$\boldsymbol{BC}$	$\boldsymbol{BD}$	CD	ABC	ABD	ACD	}
	{	ϕ	$\boldsymbol{B}$	$\boldsymbol{AC}$	AD	$\boldsymbol{CD}$	ABC	ABD	BCD	}
	{	ϕ	$\boldsymbol{C}$	AB	$\boldsymbol{AD}$	$\boldsymbol{BD}$	ABC	ACD	BCD	}
	{	ϕ	$\boldsymbol{D}$	$\boldsymbol{AB}$	$\boldsymbol{AC}$	BC	ABD	ACD	BCD	}
IV.	{	ϕ	$\boldsymbol{AB}$	$\boldsymbol{AC}$	$\boldsymbol{AD}$	BC	BD	CD	$ABCD$	}

Note that every 2-generator subgroup is a subgroup of three 3-generator subgroups.

The diagonal entries in the σ-matrix
Pick $\sigma(A_k\,,A_k\,) \in \{+,-\}$, $\sigma(A_kA_\ell\,,A_kA_\ell\,) \in \{+,-\}$, for $1 \le k < \ell \le 4$.
In each subgroup the product of the diagonal elements should equal $+$.
So, first calculate $\sigma(A_kA_\ell A_m\,,\,A_kA_\ell A_m) \in \{+,-\}$, for $1 \le k < \ell < m \le 4$ by means of I.
Then calculate $\sigma(ABCD, ABCD)$ by means of IV.
By taking products one finds that all diagonal products of II and III also equal $+$.

The non-diagonal entries in the σ-matrix
Pick the 'upper diagonal' entries $\sigma(A_k\,,A_\ell\,)\in\{+,-\}$, for $1\leq k<\ell\leq 4$.
On the $+$ and $-$ -signs in the σ-matrix of $2^{\{A\,B\,C\,D\}}$.
First note that the number of 3-generator subgroups is

$$\frac{\binom{15}{2}(15-3)}{\binom{7}{2}(7-2-1)}=15\,.$$

The σ-function has to be chosen such that each of the subgroups in the table extends to an associative R-LOOP, hence a group. The choice of (2.18) has to be made for each 3-subgroup, be it in a coherent way!
1. The $\pm$-choice in each subgroup will be denoted, in an obvious way, by α_{ABC}, $\alpha_{AB(CD)}$, $\alpha_{A(BC)(BD)}$, $\ldots\in\{+,-\}$. Our first task is to investigate which coherent choices are possible, that is to fix a relation between those $\alpha_{\ldots}$'s.
2. From part II of the table we derive, applying (3.19)-(3.20),

$$\begin{aligned}
\sigma(A,B(CD))&=\alpha_{AB(CD)}\,\sigma(A,B)\sigma(A,CD)=\alpha_{AB(CD)}\,\alpha_{ACD}\,\sigma(A,B)\sigma(A,C)\sigma(A,D)\\
\sigma(A,C(BD))&=\alpha_{AC(BD)}\,\sigma(A,C)\sigma(A,BD)=\alpha_{AC(BD)}\,\alpha_{ABD}\,\sigma(A,C)\sigma(A,B)\sigma(A,D)\\
\sigma(A,D(BC))&=\alpha_{AD(BC)}\,\sigma(A,D)\sigma(A,BC)=\alpha_{AD(BC)}\,\alpha_{ABC}\,\sigma(A,D)\sigma(A,B)\sigma(A,C)
\end{aligned}$$

$$\begin{aligned}
\sigma(B,A(CD))&=\alpha_{AB(CD)}\,\sigma(B,A)\sigma(B,CD)=\alpha_{AB(CD)}\,\alpha_{BCD}\,\sigma(B,A)\sigma(B,C)\sigma(B,D)\\
\sigma(B,C(AD))&=\alpha_{BC(AD)}\,\sigma(B,C)\sigma(B,AD)=\alpha_{BC(AD)}\,\alpha_{ABD}\,\sigma(B,C)\sigma(B,A)\sigma(B,D)\\
\sigma(B,D(AC))&=\alpha_{BD(AC)}\,\sigma(B,D)\sigma(B,AC)=\alpha_{BD(AC)}\,\alpha_{ABC}\,\sigma(B,D)\sigma(B,A)\sigma(B,C)
\end{aligned}$$

$$\begin{aligned}
\sigma(C,A(BD))&=\alpha_{AC(BD)}\,\sigma(C,A)\sigma(C,BD)=\alpha_{AC(BD)}\,\alpha_{BCD}\,\sigma(C,A)\sigma(C,B)\sigma(C,D)\\
\sigma(C,B(AD))&=\alpha_{BC(AD)}\,\sigma(C,B)\sigma(C,AD)=\alpha_{BC(AD)}\,\alpha_{ACD}\,\sigma(C,B)\sigma(C,A)\sigma(C,D)\\
\sigma(C,D(AB))&=\alpha_{CD(AB)}\,\sigma(C,D)\sigma(C,AB)=\alpha_{CD(AB)}\,\alpha_{ABC}\,\sigma(C,D)\sigma(C,A)\sigma(C,B)
\end{aligned}$$

$$\begin{aligned}
\sigma(D,A(BC))&=\alpha_{AD(BC)}\,\sigma(D,A)\sigma(D,BC)=\alpha_{AD(BC)}\,\alpha_{BCD}\,\sigma(D,A)\sigma(D,B)\sigma(D,C)\\
\sigma(D,B(AC))&=\alpha_{BD(AC)}\,\sigma(D,B)\sigma(D,AC)=\alpha_{BD(AC)}\,\alpha_{ACD}\,\sigma(D,B)\sigma(D,A)\sigma(D,C)\\
\sigma(D,C(AB))&=\alpha_{CD(AB)}\,\sigma(D,C)\sigma(D,AB)=\alpha_{CD(AB)}\,\alpha_{ABD}\,\sigma(D,C)\sigma(D,A)\sigma(D,B)
\end{aligned}$$

$$\begin{aligned}
\sigma(AB,CD)&=\alpha_{CD(AB)}\,\sigma(AB,C)\sigma(AB,D)=\alpha_{CD(AB)}\,\alpha_{ABC}\,\alpha_{ABD}\sigma(A,C)\sigma(B,C)\sigma(A,D)\sigma(B,D)\\
\sigma(CD,AB)&=\alpha_{AB(CD)}\,\sigma(CD,A)\sigma(CD,B)=\alpha_{AB(CD)}\,\alpha_{ACD}\,\alpha_{BCD}\sigma(C,A)\sigma(D,A)\sigma(C,B)\sigma(D,B)
\end{aligned}$$

$$\begin{aligned}
\sigma(AC,BD)&=\alpha_{BD(AC)}\,\sigma(AC,B)\sigma(AC,D)=\alpha_{BD(AC)}\,\alpha_{ABC}\,\alpha_{ACD}\sigma(A,B)\sigma(C,B)\sigma(A,D)\sigma(C,D)\\
\sigma(BD,AC)&=\alpha_{AC(BD)}\,\sigma(BD,A)\sigma(BD,C)=\alpha_{AC(BD)}\,\alpha_{ABD}\,\alpha_{BCD}\sigma(B,A)\sigma(D,A)\sigma(B,C)\sigma(D,C)
\end{aligned}$$

$$\begin{aligned}
\sigma(AD,BC)&=\alpha_{BC(AD)}\,\sigma(AD,B)\sigma(AD,C)=\alpha_{BC(AD)}\,\alpha_{ABD}\,\alpha_{ACD}\sigma(A,B)\sigma(D,B)\sigma(A,C)\sigma(D,C)\\
\sigma(BC,AD)&=\alpha_{AD(BC)}\,\sigma(BC,A)\sigma(BC,D)=\alpha_{AD(BC)}\,\alpha_{ABC}\,\alpha_{BCD}\sigma(B,A)\sigma(C,A)\sigma(B,D)\sigma(C,D)
\end{aligned}$$

We conclude

$$
\begin{aligned}
\alpha_{AC(BD)} &= \alpha_{AB(CD)}\,\alpha_{ACD}\,\alpha_{ABD} \\
\alpha_{AD(BC)} &= \alpha_{AB(CD)}\,\alpha_{ACD}\,\alpha_{ABC} \\
\alpha_{BC(AD)} &= \alpha_{AB(CD)}\,\alpha_{BCD}\,\alpha_{ABD} \\
&= \alpha_{AC(BD)}\,\alpha_{BCD}\,\alpha_{ACD} \\
\alpha_{BD(AC)} &= \alpha_{AB(CD)}\,\alpha_{BCD}\,\alpha_{ABC} \\
&= \alpha_{AD(BC)}\,\alpha_{BCD}\,\alpha_{ACD} \\
\alpha_{CD(AB)} &= \alpha_{AC(BD)}\,\alpha_{BCD}\,\alpha_{ABC} \\
&= \alpha_{AD(BC)}\,\alpha_{BCD}\,\alpha_{ABD}
\end{aligned}
\tag{D.1}
$$

From the last three double rows we find, by interchanging the arguments,

$$
\begin{aligned}
\alpha_{CD(AB)}\,\alpha_{ABC}\,\alpha_{ABD} &= \alpha_{AB(CD)}\,\alpha_{ACD}\,\alpha_{BCD} \\
\alpha_{BD(AC)}\,\alpha_{ABC}\,\alpha_{ACD} &= \alpha_{AC(BD)}\,\alpha_{ABD}\,\alpha_{BCD} \\
\alpha_{BC(AD)}\,\alpha_{ABD}\,\alpha_{ACD} &= \alpha_{AD(BC)}\,\alpha_{ABC}\,\alpha_{BCD}
\end{aligned}
\tag{D.2}
$$

Happily, the set (D.1), (D.2) can be solved

$$
\begin{aligned}
\alpha_{AB(CD)} &= \alpha_{AB(CD)} \\
\alpha_{AC(BD)} &= \alpha_{ABD}\,\alpha_{ACD}\,\alpha_{AB(CD)} \\
\alpha_{AD(BC)} &= \alpha_{ABC}\,\alpha_{ACD}\,\alpha_{AB(CD)} \\
\alpha_{BC(AD)} &= \alpha_{ABD}\,\alpha_{BCD}\,\alpha_{AB(CD)} \\
\alpha_{BD(AC)} &= \alpha_{ABC}\,\alpha_{BCD}\,\alpha_{AB(CD)} \\
\alpha_{CD(AB)} &= \alpha_{ABC}\,\alpha_{ABD}\,\alpha_{ACD}\,\alpha_{BCD}\,\alpha_{AB(CD)}
\end{aligned}
\tag{D.3}
$$

Note that for the consistency check one needs 'the diagonal property of subgroup IV'. Up to this point we found that associativity holds in I and II if we freely choose

$$
\alpha_{ABC}\,,\alpha_{ABD}\,,\alpha_{ACD}\,,\alpha_{BCD}\,,\alpha_{AB(CD)} \;\in\; \{+,-\}
$$

and take the remaining α's in II according to (D.3).

3. We will show now that the α's in III and IV are fixed by the latter choice if associativity is required. First a sample in III is considered. On one hand we put

$$
\sigma(A,(BC)(BD)) = \alpha_{A(BC)(CD)}\,\sigma(A,BC)\,\sigma(A,BD) =
$$

$$
= \alpha_{A(BC)(CD)}\,\alpha_{ABC}\,\alpha_{ABD}\,\sigma(A,B)\,\sigma(A,C)\,\sigma(A,B)\,\sigma(A,D)\,.
$$

Confrontation of this expression with

$$
\sigma(A,CD) = \alpha_{ACD}\,\sigma(A,C)\,\sigma(A,D)\,,
$$

leads to the necessary condition

$$\alpha_{A(BC)(CD)} \;=\; \alpha_{ABC}\,\alpha_{ABD}\,\alpha_{ACD}\,. \tag{D.4}$$

We perform a consistency check by confrontation

$$\sigma(BD, A(BC)) = \alpha_{A(BC)(CD)}\,\sigma(BD, A)\sigma(BD, BC) =$$

$$= \alpha_{ABC}\,\alpha_{ABD}\,\alpha_{ACD}\,\alpha_{ABD}\,\sigma(B, A)\,\sigma(D, A)\,\sigma(B, B)\,\sigma(B, C)\,\sigma(D, B)\,\sigma(D, C)\,,$$

with, cf. subgroup II.5,

$$\sigma(BD, B(AC)) = \sigma(B, B)\,\sigma(B, AC)\,\sigma(D, B)\,\sigma(D, AC) \;=$$

$$= \alpha_{ABC}\,\alpha_{ACD}\,\sigma(B, B)\,\sigma(D, B)\,\sigma(D, A)\,\sigma(D, C)\,\sigma(B, A)\,\sigma(B, C)\,.$$

4. Finally we consider a sample in IV. First we put

$$\sigma(AD, (AB)(AC)) = \alpha_{(AD)(AB)(AC)}\,\sigma(AD, AB)\,\sigma(AD, AC) =$$

$$= \alpha_{(AD)(AB)(AC)}\,\sigma(A, A)\,\sigma(A, B)\,\sigma(D, A)\,\sigma(D, B)\,\sigma(A, A)\,\sigma(A, C)\,\sigma(D, A)\,\sigma(D, C)\,.$$

Confrontation of this expression with, cf. subgroup II.4,

$$\sigma(AD, BC) = \alpha_{BC(AD)}\,\alpha_{ABD}\,\alpha_{ACD}\,\sigma(A, B)\,\sigma(D, B)\,\sigma(A, C)\,\sigma(D, C)\,,$$

leads to

$$\alpha_{(AD)(AB)(AC)} \;=\; \alpha_{BC(AD)}\,\alpha_{ABD}\,\alpha_{ACD} \;=\; \alpha_{ACD}\,\alpha_{BCD}\,\alpha_{AB(CD)}\,. \tag{D.5}$$

We do the following sample check. Calculate

$$\sigma(AC, (AB)(AD)) = \alpha_{(AC)(AB)(AD)}\,\sigma(AC, AB)\,\sigma(AC, AD) =$$

$$= \alpha_{ACD}\,\alpha_{BCD}\,\alpha_{AB(CD)}\,\sigma(A, A)\,\sigma(A, B)\,\sigma(C, A)\,\sigma(C, B)\,\sigma(A, A)\,\sigma(A, D)\,\sigma(C, A)\,\sigma(C, D).$$

On the other hand, cf. subgroup II.2,

$$\sigma(AC, BD) = \alpha_{AC(BD)}\,\sigma(A, BD)\,\sigma(C, BD) =$$

$$= \alpha_{ABD}\,\alpha_{ACD}\,\alpha_{AB(CD)}\,\alpha_{ABD}\,\alpha_{BCD}\,\sigma(A, B)\,\sigma(A, D)\,\sigma(C, B)\,\sigma(C, D),$$

which is the same.

E APPENDIX Automorphisms of $2^{\mathfrak{I}}$

E.1 Case $\mathfrak{I} = \{A, B\}$

$$\begin{matrix} A & & B \\ B & \xmapsto{W_1} & A \\ AB & & AB \end{matrix} \qquad \begin{matrix} A & & AB \\ B & \xmapsto{W_2} & B \\ AB & & A \end{matrix} \qquad \begin{matrix} A & & A \\ B & \xmapsto{W_3} & AB \\ AB & & B \end{matrix} \tag{E.1}$$

$$\begin{matrix} A & & A \\ B & \xmapsto{I} & B \\ AB & & AB \end{matrix} \qquad \begin{matrix} A & & B \\ B & \xmapsto{C_1} & AB \\ AB & & A \end{matrix} \qquad \begin{matrix} A & & AB \\ B & \xmapsto{C_2} & A \\ AB & & B \end{matrix} \tag{E.2}$$

The group $\{I, C_1, C_2, W_1, W_2, W_3\}$ is just the permutation group. The elements have the properties

$$W_1^2 = W_2^2 = W_3^2 = I\,, \quad C_1^3 = C_2^3 = I\,, \quad W_2 \circ W_3 = C_2\,, \quad W_2 \circ W_1 = C_1\,, \quad W_1 \circ W_3 = C_1\,,$$
$$C_1^2 = C_2\,, \quad C_1^{-1} = C_1^2\,, \ldots \text{etc.}$$

$\{I, C_1, C_2\}$ is the subgroup of even permutations. The left/right coset $\{I, W_1, W_2, W_3\}$ consists of odd permutations.
All permutations of $2^{\{A,B\}}$, for which $\emptyset \mapsto \emptyset$, are automorphisms in this case.

E.2 Case $\mathfrak{I} = \{A, B, C\}$

Any automorphism of $2^{\{A,B,C\}}$ is uniquely fixed by a bijection $\{A, B, C\} \to \{X, Y, Z\}$, where $\{X, Y, Z\} \subset 2^{\{A,B,C\}}$ denotes an *ordered* generating triple, cf. (3.15). Therefore the automorphism group of $2^{\{A,B,C\}}$ consists of $6 \times 28 = 168$ elements.

There are 4 types of automorphisms $\mathcal{P} : 2^{\{A,B,C\}} \to 2^{\{A,B,C\}}$:
1. $\mathcal{P} = \mathcal{I}$, the identity map.
2. $\mathcal{P}$ has **no** fixed points. There are 48 such automorphisms. They are contained in the union of 8 abelian subgroups of 7 elements each, which are all 7-cycles.
3. $\mathcal{P}$ has precisely **1** fixed point. There are altogether 98 such automorphisms. A given point is kept fixed by 3 abelian automorphism-subgroups of 4 elements (all 4-cycles) **and also** by 5 abelian automorphism-subgroups of 3 elements (all 3-cycles).
4. $\mathcal{P}$ has precisely **3** fixed points. There are altogether 21 such automorphisms. The set of those fixed points is necessarily a subgroup of $2^{\{A,B,C\}}$. There are 21 such automorphisms. For each given 3-subgroup of $2^{\{A,B,C\}}$ there are 3 idempotent automorphisms (2-cycles).

Those 4 types are now considered in detail
ad 2. Any automorphism is uniquely fixed by $(A, B, C) \mapsto (X, Y, Z)$. It is a good start to look for ordered generating triples $(X, Y, Z) = (B, Y, Z) \subset 2^{\{A,B,C\}}$ with the properties

$$Y \neq B, \quad Z \neq C, \quad BY \neq AB, \quad BZ \neq AC, \quad YZ \neq BC, \quad BYZ \neq ABC\,.$$

They are readily found to be

$$\begin{bmatrix} B \\ ABC \\ AB \end{bmatrix}, \begin{bmatrix} B \\ ABC \\ BC \end{bmatrix}, \begin{bmatrix} B \\ AC \\ A \end{bmatrix}, \begin{bmatrix} B \\ AC \\ BC \end{bmatrix}, \begin{bmatrix} B \\ BC \\ A \end{bmatrix}, \begin{bmatrix} B \\ BC \\ AC \end{bmatrix}, \begin{bmatrix} B \\ C \\ AB \end{bmatrix}, \begin{bmatrix} B \\ C \\ AC \end{bmatrix}. \tag{E.3}$$

Each of them generates an abelian subgroup of 7 automorphisms of type $\{\mathcal{I}, \mathcal{P}, \mathcal{P}^2, \ldots, \mathcal{P}^6, \mathcal{P}^7 = \mathcal{I}\}$.
As a specimen we mention

$$\begin{array}{ccccccccccccccc} A & & B & & ABC & & BC & & C & & AB & & AC & & A \\ B & & ABC & & BC & & C & & AB & & AC & & A & & B \\ C & & AB & & AC & & A & & B & & ABC & & BC & & C \\ AB & \stackrel{\mathcal{P}}{\longmapsto} & AC & \stackrel{\mathcal{P}}{\longmapsto} & A & \stackrel{\mathcal{P}}{\longmapsto} & B & \stackrel{\mathcal{P}}{\longmapsto} & ABC & \stackrel{\mathcal{P}}{\longmapsto} & BC & \stackrel{\mathcal{P}}{\longmapsto} & C & \stackrel{\mathcal{P}}{\longmapsto} & AB \\ AC & & A & & B & & ABC & & BC & & C & & AB & & AC \\ BC & & C & & AB & & AC & & A & & B & & ABC & & BC \\ ABC & & BC & & C & & AB & & AC & & A & & B & & ABC \end{array} \tag{E.4}$$

In cycle notation the 8 cycles constructed from (E.3) are given by, respectively,:

$$\begin{array}{l} A \mapsto B \mapsto ABC \mapsto BC \mapsto C \mapsto AB \mapsto AC \mapsto A \\ A \mapsto B \mapsto ABC \mapsto AB \mapsto AC \mapsto C \mapsto BC \mapsto A \\ A \mapsto B \mapsto AC \mapsto AB \mapsto ABC \mapsto BC \mapsto C \mapsto A \\ A \mapsto B \mapsto AC \mapsto C \mapsto BC \mapsto AB \mapsto ABC \mapsto A \\ A \mapsto B \mapsto BC \mapsto AB \mapsto C \mapsto AC \mapsto ABC \mapsto A \\ A \mapsto B \mapsto C \mapsto AB \mapsto BC \mapsto ABC \mapsto AC \mapsto A \\ A \mapsto B \mapsto C \mapsto AC \mapsto ABC \mapsto AB \mapsto BC \mapsto A \end{array} \tag{E.5}$$

ad 3.
If A is supposed to be the only fixed point, we need automorphisms produced by $(A, B, C) \mapsto (A, Y, Z)$, where the ordered generating triples $(A, Y, Z) \subset 2^{\{A,B,C\}}$ have the properties

$$Y \neq B, \quad Z \neq C, \quad YZ \neq BC, \quad \text{hence:} \quad AY \neq AB, \quad AZ \neq AC, \quad AYZ \neq ABC\,.$$

They are readily found to be

$$\begin{bmatrix} A \\ C \\ AB \end{bmatrix}, \begin{bmatrix} A \\ AB \\ BC \end{bmatrix}, \begin{bmatrix} A \\ AB \\ ABC \end{bmatrix}, \begin{bmatrix} A \\ AC \\ B \end{bmatrix}, \begin{bmatrix} A \\ BC \\ AC \end{bmatrix}, \begin{bmatrix} A \\ ABC \\ AC \end{bmatrix}, \tag{E.6}$$

and

$$\begin{bmatrix} A \\ C \\ BC \end{bmatrix}, \begin{bmatrix} A \\ C \\ ABC \end{bmatrix}, \begin{bmatrix} A \\ AC \\ BC \end{bmatrix}, \begin{bmatrix} A \\ AC \\ ABC \end{bmatrix}, \begin{bmatrix} A \\ BC \\ B \end{bmatrix}, \begin{bmatrix} A \\ BC \\ AB \end{bmatrix}, \begin{bmatrix} A \\ ABC \\ B \end{bmatrix}, \begin{bmatrix} A \\ ABC \\ AB \end{bmatrix}. \tag{E.7}$$

Each of the generating triples in (E.6) leads to an abelian subgroup of 4 automorphisms of type $\{\mathcal{I}, \mathcal{Q}, \mathcal{Q}^2, \mathcal{Q}^3, \mathcal{Q}^4 = \mathcal{I}\}$. As a specimen of such a cycle we mention

$$\begin{array}{ccccccccc} A & & A & & A & & A & & A \\ B & & C & & AB & & AC & & A \\ C & & AB & & AC & & B & & C \\ AB & \stackrel{\mathcal{Q}}{\longmapsto} & AC & \stackrel{\mathcal{Q}}{\longmapsto} & B & \stackrel{\mathcal{Q}}{\longmapsto} & C & \stackrel{\mathcal{Q}}{\longmapsto} & AB \\ AC & & B & & C & & AB & & AC \\ BC & & ABC & & BC & & ABC & & BC \\ ABC & & BC & & ABC & & BC & & ABC \end{array} \tag{E.8}$$

Note that $\mathcal{Q}^2$ has $\{\emptyset, A, BC, ABC\}$ as invariant set. It does not belong to class 3. Note also that $\mathcal{Q}^3$ comes from $(A, B, C) \mapsto (A, AC, B)$, which also appears in the list (E.6). We gather that there are 3 distinct abelian subgroups of 4 automorphisms which keep A fixed. However, in each of those only 2 automorphisms have A as their *only* fixed point. In cycle notation the 3 cycles constructed from (E.6) are given by, respectively,:

$$\begin{array}{ccccc} A \mapsto A & , & B \mapsto BC \mapsto AB \mapsto ABC \mapsto B & , & AC \mapsto C \mapsto AC \\ A \mapsto A & , & B \mapsto C \mapsto AB \mapsto AC \mapsto B & , & BC \mapsto ABC \mapsto BC \\ A \mapsto A & , & C \mapsto BC \mapsto AC \mapsto ABC \mapsto C & , & B \mapsto AB \mapsto B \end{array} \tag{E.9}$$

Next, from each of the generating triples in (E.7) an abelian subgroup of 3 automorphisms of type $\{\mathcal{I}, \mathcal{R}, \mathcal{R}^2, \mathcal{R}^3 = \mathcal{I}\}$ is produced. As a specimen we mention

$$\begin{array}{ccccccc} A & & A & & A & & A \\ B & & C & & BC & & B \\ C & & BC & & B & & C \\ AB & \stackrel{\mathcal{R}}{\longmapsto} & AC & \stackrel{\mathcal{R}}{\longmapsto} & ABC & \stackrel{\mathcal{R}}{\longmapsto} & AB \\ AC & & ABC & & AB & & AC \\ BC & & B & & C & & BC \\ ABC & & AB & & AC & & ABC \end{array} \tag{E.10}$$

Note that $\mathcal{R}^2$ comes from $(A, B, C) \mapsto (A, BC, B)$, which also appears in the list (E.7). We gather that there are 4 abelian subgroups of type $\{\mathcal{I}, \mathcal{R}, \mathcal{R}^2, \mathcal{R}^3 = \mathcal{I}\}$ for which A is the only fixed point.
In cycle notation they can be written

$$\begin{array}{ccccc} A \mapsto A & , & AB \mapsto AC \mapsto BC \mapsto AB & , & B \mapsto C \mapsto ABC \mapsto B \\ A \mapsto A & , & B \mapsto AC \mapsto ABC \mapsto B & , & AB \mapsto C \mapsto BC \mapsto AB \\ A \mapsto A & , & B \mapsto C \mapsto BC \mapsto B & , & AB \mapsto AC \mapsto ABC \mapsto AB \\ A \mapsto A & , & C \mapsto AB \mapsto ABC \mapsto C & , & B \mapsto BC \mapsto AC \mapsto B \end{array} \tag{E.11}$$

Altogether, from (E.9) and (E.11) we find $6+8 = 14$ automorphisms which have A as their *only* fixed point. Thus $7 \times 14 = 98$ automorphisms are found which have exactly 1 fixed point.

ad 4.

If, precisely, the subgroup $\{\emptyset, A, B, AB\} \subset 2^{\{A,B,C\}}$ is supposed to be fixed, we need automorphisms produced by $(A, B, C) \mapsto (A, B, Z)$, with $Z \neq C$. Only $Z = AC\,,\ BC\,,\ ABC$ are possible. Together they lead to an abelian automorphism-subgroup consisting of the identity and the three 2-cycles

$$
\begin{array}{ccc}
A & & A \\
B & & B \\
C & & AC \\
AB & \overset{\mathcal{S}_1}{\longmapsto} & AB \\
AC & & C \\
BC & & ABC \\
ABC & & BC
\end{array}
\qquad
\begin{array}{ccc}
A & & A \\
B & & B \\
C & & BC \\
AB & \overset{\mathcal{S}_2}{\longmapsto} & AB \\
AC & & ABC \\
BC & & C \\
ABC & & ABC
\end{array}
\qquad
\begin{array}{ccc}
A & & A \\
B & & B \\
C & & ABC \\
AB & \overset{\mathcal{S}_3}{\longmapsto} & AB \\
AC & & BC \\
BC & & AC \\
ABC & & C
\end{array}
\quad ,
$$

with $\mathcal{S}_1^2 = \mathcal{S}_2^2 = \mathcal{S}_3^2 = \mathcal{I}$ and $\mathcal{S}_1\mathcal{S}_2 = \mathcal{S}_2\mathcal{S}_1 = \mathcal{S}_3$, cycl.
According to (3.14) there are 7 subgroups in $2^{\{A,B,C\}}$. So there are $7 \times 3 = 21$ automorphisms of type 4.

For convenient reference we collect all 3-generator sets of $2^{\{A,B,C\}}$ together with their products:

$$
\begin{array}{ccc}
 & \{A, B, C\} : (ABC) & \\
\{A, B, ABC\} : C & \{A, AB, AC\} : (ABC) & \{AB, BC, ABC\} : B \\
\{B, C, ABC\} : A & \{B, BC, AB\} : (ABC) & \{AB, AC, ABC\} : A \\
\{A, C, ABC\} : B & \{C, AC, BC\} : (ABC) & \{BC, AC, ABC\} : C \\
\{A, B, AC\} : (BC) & \{AB, B, AC\} : C & \{A, AB, ABC\} : (AC) \\
\{B, C, AB\} : (AC) & \{BC, C, AB\} : A & \{B, BC, ABC\} : (AB) \\
\{A, C, BC\} : (BC) & \{AC, A, BC\} : B & \{C, AC, ABC\} : (BC) \\
\{A, B, BC\} : (AC) & \{AB, A, BC\} : C & \{B, AB, ABC\} : (BC) \\
\{A, C, AB\} : (BC) & \{AC, C, AB\} : B & \{A, AC, ABC\} : (AB) \\
\{B, C, AC\} : (AB) & \{BC, B, AC\} : A & \{C, BC, ABC\} : (AC)
\end{array}
\tag{E.12}
$$

Rows 3,4 are obtained from even permutations $\{A, B, C\} \to \{A, B, C\}$ of row 2. Rows 6,7 are obtained from even permutations $\{A, B, C\} \to \{A, B, C\}$ of row 5. Rows 9,10 are obtained from even permutations $\{A, B, C\} \to \{A, B, C\}$ of row 8.

Acknowledgement

The 'interesting modification', as mentioned at the end of section 2 came up in discussions with Herman van de Kuit.

References

[B] J.C. Baez: The Octonions. Bull Am. Math. Soc. Vol 39, #2, 2001, pp. 145-205.

[BLS] F. Brackxs, R. de Langhe, F. Sommen: Clifford Analysis. Pitman AAP. Boston-London-Melbourne, 1982.

[L] E. Landau: Grundlagen der Analysis. 3rd ed. Chelsea. New York 1960.

[P] B.O. Pierce: Linear Associative Algebra. Van Nostrand New York 1882.

Eindhoven, May 30, 2012.