

Bohrification of operator algebras and quantum logic

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Bohrification of operator algebras and quantum logic

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Abstract Following Birkhoff and von Neumann, quantum logic has traditionally been based on the lattice of closed linear subspaces of some Hilbert space, or, more generally, on the lattice of projections in a von Neumann algebra A . Unfortunately, the logical interpretation of these lattices is impaired by their nondistributivity and by various other problems. We show that a possible resolution of these difficulties, suggested by the ideas of Bohr, emerges if instead of single projections one considers elementary propositions to be families of projections indexed by a partially ordered set $\mathcal{C}(A)$ of appropriate commutative subalgebras of A . In fact, to achieve both maximal generality and ease of use within topos theory, we assume that A is a so-called Rickart C^* -algebra and that $\mathcal{C}(A)$ consists of all unital commutative Rickart C^* -subalgebras of A . Such families of projections form a Heyting algebra in a natural way, so that the associated propositional logic is intuitionistic: distributivity is recovered at the expense of the law of the excluded middle. Subsequently, generalizing an earlier computation for $n \times n$ matrices, we prove that the Heyting algebra thus associated to A arises as a basis for the internal Gelfand spectrum (in the sense of Banaschewski–Mulvey) of the “Bohrification” \underline{A} of A , which is a commutative Rickart C^* -algebra in the topos of functors from $\mathcal{C}(A)$ to the category of sets. We explain the relationship of this

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construction to partial Boolean algebras and Bruns–Lakser completions. Finally, we establish a connection between probability measures on the lattice of projections on a Hilbert space H and probability valuations on the internal Gelfand spectrum of \underline{A} for $A = B(H)$.

Keywords Quantum logic · Intuitionistic logic · C*-algebras · Locales · Bohrification

1 Introduction

As its title is meant to suggest, this paper is an attempt to reconcile the views on the logical structure of quantum mechanics of Niels Bohr on the one hand, and John von Neumann on the other. This is not an easy task, as indicated, for example, by the following two quotations:

All departures from common language and ordinary logic are entirely avoided by reserving the word “phenomenon” solely for reference to unambiguously communicable information, in the account of which the word “measurement” is used in its plain meaning of standardized comparison. (Bohr 1987)

The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. (Birkhoff and von Neumann 1936)

Another difference lies in the highly technical and advanced mathematical nature of von Neumann’s writings on quantum theory, compared with the philosophical (if not mystical) style of Bohr, who in particular used only very basic mathematics (if any) (Bohr 1985). This discrepancy implies that any attempt at reconciliation between these authors has to rely on mathematical extrapolations of Bohr’s ideas that cannot really be justified by his own writings. So be it.

It should be mentioned that in what follows, we use the so-called *semantic approach* to the axiomatization of physical theories (Suppe 1989; van Fraassen 1980), in which theories are defined through their class of models (so that a preceding stage involving an abstract logical language is lacking). This, incidentally, is exactly the way quantum mechanics was axiomatized by von Neumann (1932), who may therefore be seen as a predecessor of the semantic approach (in contrast with Hilbert (2004–2010), who is regarded as the founder of the *syntactic approach* to axiomatization in general).

The outline of this paper is as follows. The next section reviews the logic of classical physics from a semantic perspective. We then recall in Sect. 3 how Birkhoff and von Neumann were led to (if not seduced by) their concept of quantum logic, which we criticize and to which we propose an intuitionistic alternative in Sect. 4. Von Neumann not only invented quantum logic, he also generalized Hilbert space theory to the theory of operator algebras. In Sect. 5 we explain the connection between quantum logic and operator algebras, where we take the unusual step of going beyond von Neumann algebras. In fact, we propose to study both traditional quantum logic and our own intuitionistic version of it in the setting of so-called Rickart C*-algebras. This class of C*-algebras is studied in detail in Sects. 6 and 7, particularly with a view on their

internalization to topos theory. Specifically, we develop an internal Gelfand theory for commutative Rickart C^* -algebras, which refines the work of Banaschewski and Mulvey (2006) for general commutative C^* -algebras to the Rickart case. Section 8 studies the relationship between our version of intuitionistic quantum logic and partial Boolean algebras on the one hand, and so-called Bruns–Lakser completions on the other. Finally, in Sect. 9 we explain how the well-known concept of a probability measure on the projection lattice on a Hilbert space is related to various concepts intrinsic to our approach, and explicitly compute a non-probabilistic state-proposition pairing.

This paper is a continuation of our earlier work (Caspers et al. 2009; Heunen et al. 2009), which provides some background, particularly on quantum theory in a topos. However, the present paper is largely self-contained and takes our program a significant step further.

2 The logic of classical physics

To explain the basic issue, we first recall the logical structure of classical physics.¹ Let X be the phase space of a classical physical system; we assume that X is a topological space with ensuing Borel structure. We identify elements of X with (pure) states of the system. Observables are measurable functions $f : X \rightarrow \mathbb{R}$, and elementary propositions take the form $f \in \Delta$, where Δ is a measurable subset of \mathbb{R} . Further propositions are inductively built from these through the operations \neg of negation, \vee of disjunction and \wedge of conjunction. An elementary proposition $f \in \Delta$ is dictated by physics to be true in a state $x \in X$ iff $f(x) \in \Delta$, i.e., iff $x \in f^{-1}(\Delta)$; this notion of truth is defined semantically (as opposed to formal derivability in the syntactic approach). Consequently, we may introduce the notation \models of semantic entailment, meaning (sic) that $(f \in \Delta) \models (g \in \Gamma)$ whenever the truth of $f \in \Delta$ implies the truth of $g \in \Gamma$. Hence one may form the associated Lindenbaum–Tarski algebra of equivalence classes $[f \in \Delta]$, where we say that $(f \in \Delta) \sim (g \in \Gamma)$ when $(f \in \Delta) \models (g \in \Gamma)$ and $(g \in \Gamma) \models (f \in \Delta)$ both hold (in words, $f \in \Delta$ is true iff $g \in \Gamma$ is true). This yields the identification $[f \in \Delta] \cong f^{-1}(\Delta)$ and the ensuing identification of the Lindenbaum–Tarski algebra of the given system with the Boolean algebra² $\Sigma(X)$ of (Borel) measurable subsets of X . Under this identification, the logical connectives \models , \neg , \vee and \wedge descend to set-theoretic inclusion \subseteq , complementation $(-)^c$, union \cup , and intersection \cap , respectively, and these are compatible in that \cup and \cap are precisely the lattice operations \sup and \inf induced by the partial order \subseteq . Finally, $\Sigma(X)$ has bottom and top elements \emptyset and X , respectively, which play the role of falsehood \perp

¹ It is remarkable that this structure was not written down by either Boole or Hamilton in the mid 19th Century, as it clearly emerges from the conjunction of their ideas on propositional logic and on classical physics, respectively (Boole 1854; Hamilton 1835). As far as we know, however, the logical structure of classical physics was first explicated by Birkhoff and von Neumann (1936); see also Rédei (1998) for a very clear account.

² Recall that a lattice L is called *orthocomplemented* when there exists a map $\perp : L \rightarrow L$ that satisfies $x^{\perp\perp} = x$, $y^\perp \leq x^\perp$ when $x \leq y$, $x \wedge x^\perp = 0$, and $x \vee x^\perp = 1$. For example, the lattice of closed subspaces of a Hilbert space has an orthocomplement; namely, V^\perp is the orthogonal complement of V . A lattice L is called *Boolean* when it is distributive and orthocomplemented, in which case the orthocomplement \perp is called a *complement* and written as \neg , and has the logical meaning of negation.

and truth \top , and with respect to which $(-)^c$ is an orthocomplementation. This means, in particular, that besides the law of contradiction $p \wedge (\neg p) = \perp$, which in this case descends to $p \cap p^c = \emptyset$, one has the law of excluded middle $p \vee (\neg p) = \top$, descending to $p \cup p^c = X$.

This procedure is unobjectionable, in that \neg, \vee and \wedge thus interpreted in set theory indeed have their usual meaning of negation, disjunction, and conjunction, respectively. In particular (identifying propositions with their image in $\Sigma(X)$),

1. Disjunction and conjunction distribute over each other;³
2. $p \vee q$ is true iff p is true or q is true;
3. $p \wedge q$ is true iff p is true and q is true;
4. $\neg p$ is true iff p is not true;
5. There is a material implication $\Rightarrow: \Sigma(X) \times \Sigma(X) \rightarrow \Sigma(X)$ that satisfies⁴

$$p \leq (q \Rightarrow r) \text{ iff } p \wedge q \leq r, \tag{2.1}$$

namely $(q \Rightarrow r) = (q^c \cup r)$.

3 The lure of quantum logic

The quantum logic of [Birkhoff and von Neumann \(1936\)](#) is an attempt to adapt this scheme to quantum mechanics.⁵ This time, the starting point is a Hilbert space H , whose unit vectors Ψ are interpreted as pure states. Furthermore, observables are taken to be self-adjoint operators $a : \text{Dom}(a) \rightarrow H$, with dense domain $\text{Dom}(a) \subseteq H$; in what follows, we assume for simplicity that $\text{Dom}(a) = H$, so that a is bounded. Elementary propositions assume the same form “ $a \in \Delta$ ” as in classical physics, and may formally be combined using the connectives \neg, \vee , and \wedge . This time, the truth predicate on $a \in \Delta$ is determined by the associated spectral projection, which we write as $E_a(\Delta)$ (so that the map $\Delta \mapsto E_a(\Delta)$ is the spectral measure defined by a). According to [von Neumann \(1932\)](#), the proposition $a \in \Delta$ is true in a state $\Psi \in H$ iff $\Psi \in E_a(\Delta)H$, so that the equivalence classes determined by this truth condition may be written as $[a \in \Delta] = E_a(\Delta)H$. Each such class is a closed linear subspace of H , and semantic entailment of propositions obviously descends to inclusion of closed linear subspaces. Thus it is hard to resist the temptation to conclude that the lattice $\mathcal{L}(H)$ of closed linear subspaces of the Hilbert space H (with partial ordering given by inclusion) is the correct quantum-mechanical analogue of the lattice $\Sigma(X)$ of measurable subsets of the classical phase space X .

[Birkhoff and von Neumann \(1936\)](#) were indeed seduced by this perspective, and proposed that the logic of quantum mechanics is described by the lattice structure of

³ I.e., $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

⁴ If $\Sigma(X)$ is seen as a category (with a unique arrow from p to q iff $p \leq q$), then \Rightarrow is right adjoint to \wedge .

⁵ See, for example, [Chiara and Giuntini \(2002\)](#), [Chiara et al. \(2004\)](#), and [Rédei \(1998\)](#) for recent surveys of quantum logic in the tradition of Birkhoff and von Neumann. The relationship between quantum logic and projective geometry, which was a major discovery of von Neumann’s, is beautifully surveyed in [Stubbe and van Steirteghem \(2007\)](#). A good philosophical critique of quantum logic is [Stairs \(1983\)](#).

$\mathcal{L}(H)$, which, then, plays the role of the Lindenbaum–Tarski algebra of equivalence classes of quantum-mechanical propositions (Rédei 1998). Once more using the same notation for the images of propositions and logical connectives in $\mathcal{L}(H)$ as for these things themselves, the ensuing lattice operations on $\mathcal{L}(H)$ are given by $p \vee q = p \dot{+} q$ (i.e., the closure of the linear span of p and q) and $p \wedge q = p \cap q$. As to negation, Birkhoff and von Neumann decided to define $\neg p$ as the proposition that is true whenever p is false; unlike in classical physics, this is not the same as saying that p is not true. Now in quantum mechanics a proposition $a \in \Delta$ is false in a state Ψ iff $\Psi \in (E_a(\Delta)H)^\perp$ (where $(-)^\perp$ denotes the orthogonal complement), so that $\neg p = p^\perp$. With the bottom and top elements of $\mathcal{L}(H)$ given by $\{0\}$ and H , respectively, this implies that \neg is an orthocomplementation, so that the quantum logic of Birkhoff and von Neumann (1936) formally satisfies both the law of contradiction, implemented as $p \cap p^\perp = \{0\}$, and the law of excluded middle $p \dot{+} p^\perp = H$.

Nonetheless, we feel that Birkhoff and von Neumann should have resisted this temptation.⁶ Indeed, compared with the five points in favour of the propositional logic of classical physics being the Boolean algebra of measurable subsets of phase space, we now have:

1. Disjunction and conjunction do not distribute over each other;⁷
2. There are states in which $p \vee q$ is true while neither p nor q is true;⁸
3. There are propositions p and q for which $p \wedge q$ cannot be regarded as the conjunction of p and q because this conjunction is physically undefined;⁹
4. $\neg p$ is true iff p is false, rather than iff p is not true;¹⁰
5. There exists no map $\Rightarrow: \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ that satisfies (2.1).

It is important to realize that the equality $p \vee (\neg p) = \top$ is only true in quantum logic because neither \vee nor \neg has its usual logical meaning. In fact, in quantum logic this equality only formally expresses the law of excluded middle; it is semantically empty.

As to the last point, it can be shown that one has a material implication on an orthocomplemented lattice \mathcal{L} (i.e., a map $\Rightarrow: \mathcal{L} \rightarrow \mathcal{L}$ satisfying (2.1)) iff \mathcal{L} is Boolean, in which case $p \Rightarrow q = \neg p \vee q$; see, e.g., Rédei (1998, Proposition 8.1). Consequently, quantum logicians tend to weaken the property (2.1) by requiring it only for all q and r that are compatible in the sense that $q = (q \wedge r^\perp) \vee (q \wedge r)$; in $\mathcal{L}(H)$ this is the case iff q and r commute. If \mathcal{L} is orthocomplemented, the existence of such an implication forces \mathcal{L} to be orthomodular and implies that \Rightarrow takes the form of the “Sasaki hook”

⁶ In what follows, we intend to criticize the *logical* interpretation of the connectives \vee, \wedge, \neg in standard quantum logic; we do not take issue with their *operational* interpretation assigned by the Geneva school led by Moore (1999) and Piron (1976).

⁷ The lattice $\mathcal{L}(H)$ does satisfy a weakening of distributivity called *orthomodularity*; see Sect. 8.

⁸ Take any unit vector that lies in the subspace spanned by p and q without lying in either p or q . This is famously the kind of state Schrödinger’s Cat is in.

⁹ Take, for example, q to be a spectral projection for position and p to be one for momentum, or, more generally, any pair of projections that do not commute.

¹⁰ The distinction between “false” and “not true” arises from the Born rule of quantum theory, according to which the proposition $a \in \Delta$ is true in a state $\Psi \in H$ with probability $\|E_a(\Delta)\Psi\|^2$. If this probability equals one we say the proposition is true, and if it equals zero we say it is false. Hence “not true” refers to all probabilities in the semi-open interval $[0, 1)$, rather than to zero alone.

$$p \Rightarrow_s q = p^\perp \vee (p \wedge q), \tag{3.2}$$

discussed in some detail in Sect. 8 below.

In order to pave the way for the algebraic ideas to follow, we close this section by reminding the reader of the well-known connection between closed linear subspaces of H and projections p on H , defined as bounded linear operators $p : H \rightarrow H$ satisfying $p^2 = p^* = p$. Indeed, we know from elementary Hilbert space theory that there is a bijective correspondence between projections p on H and closed linear subspaces of H : a projection p defines such a subspace as its image pH , and any closed linear subspace is the image of a unique projection. For consistency with later notation, we denote the set of all projections on H by $\mathcal{P}(B(H))$ (instead of the more natural expression $\mathcal{P}(H)$), where $B(H)$ is the algebra of all bounded operators on H . If we now define a partial order on the set $\mathcal{P}(B(H))$ of $p \leq q$ iff $pH \subseteq qH$, by construction we obtain a lattice isomorphism

$$\mathcal{P}(B(H)) \cong \mathcal{L}(H). \tag{3.3}$$

In view of this, if no confusion can arise we make no notational distinction between closed linear subspaces and projections, denoting both by p etc. The partial order on $\mathcal{P}(B(H))$ may, in fact be defined without reference to (3.3): one has

$$p \leq q \text{ iff } pq = qp = p. \tag{3.4}$$

As to the ensuing lattice operations, defining

$$p^\perp = 1 - p, \tag{3.5}$$

the inf and sup derived from \leq may be expressed by

$$p \wedge q = \text{s-lim}_{n \rightarrow \infty} (pq)^n; \tag{3.6}$$

$$p \vee q = (p^\perp \wedge q^\perp)^\perp, \tag{3.7}$$

where s-lim denotes the limit in the strong operator topology.¹¹ If p and q happen to commute, these expressions reduce to

$$p \wedge q = pq; \tag{3.8}$$

$$p \vee q = p + q - pq. \tag{3.9}$$

¹¹ The strong operator topology on $B(H)$ is induced by the seminorms $p_\Psi(a) = \|a\Psi\|$, $\Psi \in H$, so that $\text{s-lim}_n a_n = a$ iff $\lim_n \|(a_n - a)\Psi\| = 0$ for all $\Psi \in H$.

4 Intuitionistic quantum logic

We now return to Bohr for guidance towards the solution of the problems with von Neumann's quantum logic. Bohr's best-known formulation of what came to be called his "doctrine of classical concepts" (Scheibe 1973) is as follows:

However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. (...) The argument is simply that by the word *experiment* we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangements and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics. (Bohr 1949)

For simplicity, we assume in this section that our Hilbert space H is n -dimensional with $n < \infty$; the general case will be covered in the remainder of the paper. Anticipating later generalizations at least in the notation, we write $A = M_n(\mathbb{C})$ for the algebra of $n \times n$ matrices. Our mathematical translation of Bohr's doctrine, then, is to study A through its *commutative* subalgebras C , where for technical reasons we assume C to contain the unit matrix and to be closed under the involution $*$ (i.e., Hermitian conjugation, often denoted by a dagger by physicists); that is, if $a \in C$, then $a^* \in C$. Thus we define $\mathcal{C}(A)$ to be the set of all unital commutative $*$ -subalgebras of A . This set is partially ordered by inclusion, i.e., for $C, D \in \mathcal{C}(A)$ we say that $C \leq D$ iff $C \subseteq D$. The poset $\mathcal{C}(A)$ is merely a so-called meet-semilattice rather than a lattice: although infima exist in the form $C \wedge D = C \cap D$, there are no suprema, since C and D will not, in general, be contained in a commutative subalgebra of A (unless $cd = dc$ for all $c \in C$ and $d \in D$).

It is much harder to make mathematical sense of Bohr's idea of "complementarity", especially as his formulation of this notion remained vague and in fact changed over time.¹² Be it as it may, we interpret the idea of complementarity in the following way: rather than following von Neumann (1932) in defining an elementary quantum-mechanical proposition as a *single* projection on H , we follow (the spirit of) Bohr in defining such a proposition as a *family* $\{p_C\}_{C \in \mathcal{C}(A)}$ of projections, one for each "classical context" C , with p_C pertinent to that context in requiring that $p_C \in \mathcal{P}(C)$. For the moment, we simply postulate this idea, but in the main body of the paper we will actually derive it from the doctrine of classical concepts (rephrased mathematically as explained above). Adding minimal mathematical structure, our proposal means that we replace the lattice $\mathcal{P}(A)$ of all projections in A as the codification of quantum logic by

$$\mathcal{O}(\Sigma) = \{S : \mathcal{C}(A) \rightarrow \mathcal{P}(A) \mid S(C) \in \mathcal{P}(C), S(D) \leq S(E) \text{ if } D \subseteq E\}, \quad (4.10)$$

where $\mathcal{P}(C)$ is the (Boolean) lattice of projections in C . As already mentioned, we regard each $S \in \mathcal{O}(\Sigma)$ as a single proposition as far as logical structure is concerned;

¹² The literature on complementarity is abundant, but we recommend the critical studies (Held 1994; Lahti 1980).

physically, S breaks down into a family $\{S(C)\}_{C \in \mathcal{C}(A)}$. This could either mean that one invents a question for each context C separately (compatible with the monotonicity in (4.10)), or that one constructs such a family from a single proposition in the sense of von Neumann. The latter may be done in at least two ways:

1. For $p \in \mathcal{P}(A)$, one defines

$$\begin{aligned} S_p(C) &= p \quad \text{if } p \in C; \\ &= 0 \quad \text{if } p \notin C. \end{aligned} \tag{4.11}$$

2. One uses the “inner Daseinisation” map of Döring and Isham (2008), which associates the best approximation in each C to a proposition $a \in \Delta$; see also Heunen et al. (2009). In fact, (4.11) may be seen as a crude analogue of this procedure.

In order to unravel its logical structure, we turn $\mathcal{O}(\Sigma)$ into a poset under pointwise partial ordering with respect to the usual ordering of projections, i.e., for $S, T \in \mathcal{O}(\Sigma)$ we put $S \leq T$ iff $S(C) \leq T(C)$ for all $C \in \mathcal{C}(A)$, where \leq is defined by (3.4). The main observation is that $\mathcal{O}(\Sigma)$ is a *complete Heyting algebra*¹³ under this partial ordering.

The whole point now is that in being a (complete) Heyting algebra, $\mathcal{O}(\Sigma)$ defines an *intuitionistic propositional logic*, which in fact is not Boolean (Caspers et al. 2009).¹⁴ First, the inf and sup derived from \leq are given by the pointwise expressions

$$(S \wedge T)(C) = S(C) \wedge T(C); \tag{4.12}$$

$$(S \vee T)(C) = S(C) \vee T(C). \tag{4.13}$$

The top and bottom elements are $\top : C \mapsto 1$ and $\perp : C \mapsto 0$ for all C , where 1 and 0 are seen as elements of $\mathcal{P}(C)$. Material implication is defined by

$$S \Rightarrow T = \bigvee \{U \in \mathcal{O}(\Sigma) \mid U \wedge S \leq T\}, \tag{4.14}$$

and is explicitly given by the nonlocal formula

$$(S \Rightarrow T)(C) = \bigwedge_{D \supseteq C}^{\mathcal{P}(C)} S(D)^\perp \vee T(D). \tag{4.15}$$

Here the right-hand side denotes the greatest lower bound of all $S(D)^\perp \vee T(D)$, $D \supseteq C$, that lies in $\mathcal{P}(C)$. The derived operation of negation, which in any Heyting algebra is given in terms of \Rightarrow by

¹³ A *Heyting algebra* is just a lattice \mathcal{L} with a map $\Rightarrow : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ satisfying (2.1); it is automatically a distributive lattice. It is complete when \mathcal{L} is complete as a lattice. The interpretation of \Rightarrow as a right adjoint to \wedge , as in footnote 4, remains valid. In particular, every Boolean lattice is a Heyting algebra with $x \Rightarrow y = \neg x \vee y$.

¹⁴ A Heyting algebra is Boolean iff the negation \neg defined by (4.16) below is an orthocomplementation.

$$\neg x = (x \Rightarrow \perp), \tag{4.16}$$

is then equal to

$$(\neg S)(C) = \bigwedge_{D \supseteq C}^{\mathcal{P}(C)} S(D)^\perp. \tag{4.17}$$

The natural semantics for the intuitionistic propositional logic $\mathcal{O}(\Sigma)$ is of Kripke type (Kripke 1965) (see also Dummett 2000; Goldblatt 1984). First, we take the Kripke frame to be the poset $\mathcal{C}(A)$, and denote the set of upper sets in $\mathcal{C}(A)$ by $\mathcal{O}_A(\mathcal{C}(A))$.¹⁵ Each unit vector $\Psi \in \mathbb{C}^n$ defines a state on A , i.e., a linear functional $\psi : A \rightarrow \mathbb{C}$ that satisfies $\psi(1) = 1$ and $\psi(a^*a) \geq 0$ for all $a \in A$ by $\psi(a) = (\Psi, a\Psi)$; more generally, each density matrix defines a state on A by taking expectation values. This, in turn, defines a map

$$V_\psi : \mathcal{O}(\Sigma) \rightarrow \mathcal{O}_A(\mathcal{C}(A)) \tag{4.18}$$

by¹⁶

$$V_\psi(S) = \{C \in \mathcal{C}(A) \mid \psi(S(C)) = 1\}. \tag{4.19}$$

This map is to be compared with the traditional truth attribution

$$W_\psi : \mathcal{P}(A) \rightarrow \{0, 1\} \tag{4.20}$$

in quantum logic, given by $W_\psi(p) = 1$ iff $\psi(p) = 1$.¹⁷ Consequently, (4.19) lists the “possible worlds” C in which $S(C)$ is true in the usual sense.

However, unless A is Abelian, neither V_ψ nor W_ψ is a lattice homomorphism;¹⁸ even the restrictions of W_ψ to Boolean sublattices of $\mathcal{P}(A)$ fail to be lattice homomorphisms. In fact, for $n > 2$ there are no lattice homomorphisms $W : \mathcal{P}(A) \rightarrow \{0, 1\}$ or $V : \mathcal{O}(\Sigma) \rightarrow \mathcal{O}_A(\mathcal{C}(A))$ altogether; the first claim is the content of the original Kochen–Specker Theorem (Kochen and Specker 1967), and the second is its generalization by the authors (Heunen et al. 2009; Caspers et al. 2009) (see also Butterfield and Isham 1998; Döring 2005 for predecessors of this generalization).

In any case, we are now in a position to compare the quantum logic of Birkhoff and von Neumann with our own version, at least as far as the five points listed in both Sects. 2 and 3 are concerned:

¹⁵ This notation reflects the fact that the upper sets in a poset just form its Alexandrov topology.

¹⁶ Note that (4.19) indeed defines an upper set in $\mathcal{C}(A)$. If $C \subseteq D$ then $S(C) \leq S(D)$, so that $\psi(S(C)) \leq \psi(S(D))$ by positivity of states, so that $\psi(S(D)) = 1$ whenever $\psi(S(C)) = 1$ (given that $\psi(S(D)) \leq 1$, since $\psi(p) \leq 1$ for any projection p).

¹⁷ This is a slight generalization from the example $A = B(H)$, where a proposition p is called true in a pure state Ψ if $\Psi \in pH$. This is equivalent to $\psi(p) = (\Psi, p\Psi) = 1$.

¹⁸ More precisely, V_ψ is not a frame homomorphism, see below.

1. The lattice $\mathcal{O}(\Sigma)$ is distributive;
2. Defining a proposition $S \in \mathcal{O}(\Sigma)$ to be true in a state ψ if $V_\psi(S) = \mathcal{C}(A)$ (i.e., the top element of the Kripke frame $\mathcal{O}_A(\mathcal{C}(A))$), it follows that $S \vee T$ is true iff either S or T is true;¹⁹
3. The conjunction $S \wedge T$ is always defined physically, as it only involves “local” conjunctions $S(C) \wedge T(C)$ for which $S(C)$ and $T(C)$ both lie in $P(C)$ and hence commute;
4. Defining $S \in \mathcal{O}(\Sigma)$ to be false in ψ if $V_\psi(S) = \emptyset$ (i.e., the bottom element of $\mathcal{O}_A(\mathcal{C}(A))$), one has that $\neg S$ is true iff S is false.
5. There exists a map $\Rightarrow: \mathcal{O}(\Sigma) \rightarrow \mathcal{O}(\Sigma)$ that satisfies (2.1), namely (4.15).²⁰

To restore the balance a little, let us draw attention to a good side of traditional quantum logic, namely its essentially topological character. This is especially clear in its original incarnation, where propositions are identified with closed subspaces of Hilbert space. This aspect is somewhat obscured in the reformulation in terms of projections, and looks truly remote in our version (4.10). However, the lattice defined by (4.10) is topological in a more subtle sense, in that it defines the “topology” of a “pointless space”. To explain this, we note that the topology $\mathcal{O}(X)$ on a space X has the structure of a so-called *frame*,²¹ i.e., a complete distributive lattice such that $x \wedge \bigvee_\lambda y_\lambda = \bigvee_\lambda x \wedge y_\lambda$ for arbitrary families $\{y_\lambda\}$. Here the partial order on the opens in X is simply given by inclusion. For a large class of spaces (namely, the so-called sober ones), one may recover X from its frame of opens in two steps: first, the points of X correspond to the set $\text{pt}(\mathcal{O}(X))$ of lattice homomorphisms $\varphi: \mathcal{O}(X) \rightarrow \{0, 1\}$ that preserve arbitrary suprema, and second, the topology is recovered in stating that the open sets in $\text{pt}(\mathcal{O}(X))$ are those of the form $\{\varphi \in \text{pt}(\mathcal{O}(X)) \mid \varphi(U) = 1\}$, for each $U \in \mathcal{O}(X)$. Compare the discussion following Proposition 2.

Our notation $\mathcal{O}(\Sigma)$ for the lattice defined by (4.10) is meant to suggest that it is a frame, and indeed it is: the Heyting algebra structure of $\mathcal{O}(\Sigma)$ is actually derived from its frame structure by (2.1). More generally, any frame is at the same time a complete Heyting algebra with implication (2.1), and in fact frames and complete Heyting algebras are essentially the same things.²² Due to the Kochen–Specker Theorem of Heunen et al. (2009) and Caspers et al. (2009), the frame $\mathcal{O}(\Sigma)$ cannot be of the type given by the opens of some genuine topological space Σ , but even though it isn’t, one may reason about $\mathcal{O}(\Sigma)$ *as if it were* the collection of opens of a space. This underlying space, Σ , is so to speak “virtual”, or “pointfree”; it only exists through its associated frame $\mathcal{O}(\Sigma)$. The upshot is that while a classical physical system has an actual topological space associated with it, namely its phase space, a quantum system

¹⁹ This has the rather trivial origin that $V_\psi(S) = \mathcal{C}(A)$ iff $S(\mathbb{C} \cdot 1) = 1$, which forces $S(C) = 1$ for all C .

²⁰ Note that, compared with the Sasaki hook (3.2), one has $(S \Rightarrow T)(C) \neq S(C) \Rightarrow_S T(C) = S(C)^\perp \vee T(C)$, as the left-hand side is nonlocal in C .

²¹ This notion is not to be confused with that of a Kripke frame; the latter is not an instance of the former at all.

²² The infinite distributivity law in a frame is automatically satisfied in a Heyting algebra. Frames and Heyting algebras do not form isomorphic or even equivalent categories, though, for frame maps do not necessarily preserve the implication \Rightarrow defining the Heyting algebra structure.

still defines a space, albeit a pointfree one that only exists through its “topology”, namely the frame defined by (4.10).

Our proposal, then, is that quantum logic should not be described by an orthomodular lattice of the type $\mathcal{P}(A)$, but by a frame or Heyting algebra of the type (4.10). Thus the “Bohrification” of quantum logic is intuitionistic. In this light, it is interesting to note that Birkhoff and von Neumann actually considered this possibility, but rejected it:

The models for propositional calculi which have been considered in the preceding sections are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have had to guide logicians hitherto. Hence it is interesting to compare the modifications which they introduce into Boolean algebra, with those which logicians on “intuitionist” and related grounds have tried introducing.

The main difference seems to be that whereas logicians have usually assumed that properties [...] of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities* [...] as the weakest link in the algebra of logic. (Birkhoff and von Neumann 1936)

5 Generalization to operator algebras

The technical thrust of this paper lies in the generalization of the above ideas to infinite-dimensional Hilbert spaces H and to more general algebras of operators than $A = B(H)$. As we shall see, this generalization is quite interesting mathematically, but we also envisage future physical applications to infinite quantum systems and other systems with so-called superselection rules (Haag 1996), as well as to quantization and the classical limit of quantum mechanics (Landsman 2007).

The natural setting for our work is the theory of operator algebras, created by none other than von Neumann. The class of operator algebras he introduced is now aptly called von Neumann algebras (older names are rings of operators and W^* -algebras), and incorporates not only the highly noncommutative world of the $n \times n$ matrices and their infinite-dimensional generalization $B(H)$, but also covers the commutative case, with a direct link to Boolean algebras and hence classical logic. The main reference for the general theory of von Neumann algebras is Takesaki’s three-volume treatise Takesaki (2003a,b,c); the relationship between von Neumann algebras and quantum logic has been beautifully described by Rédei (1998).

Definition 1 For any Hilbert space H , a *von Neumann algebra* of operators on H is a subalgebra A of $B(H)$ that contains the unit of $B(H)$, contains the adjoint a^* whenever it contains a , and in addition satisfies one (and hence both) of the following equivalent conditions:

1. $A'' = A$;

2. A is closed in the strong operator topology.²³

In the first condition, we write $A'' \equiv (A')'$, where A' is the commutant of A , consisting of all $a \in B(H)$ that commute with any $b \in A$.

To see how von Neumann algebras lead to a generalization of quantum logic Rédei (1998), we note that a von Neumann algebra is generated by its projections: if

$$\mathcal{P}(A) = \{p \in A \mid p^2 = p^* = p\} \tag{5.21}$$

is the set of projections in A , then $\mathcal{P}(A)'' = A$; equivalently, the strong closure of the (algebraic) linear span of $\mathcal{P}(A)$ equals A .²⁴ Moreover, for any von Neumann algebra A , the set $\mathcal{P}(A)$ is an orthomodular lattice under the ordering defined by (3.4), with orthocomplementation, inf and sup given by (3.5), (3.6), and (3.7), respectively, and bottom and top elements $\perp = 0, \top = 1$. One may continue to identify $p \in \mathcal{P}(A)$ with an elementary quantum-mechanical proposition, and look at $\mathcal{P}(A)$ as a generalized quantum logic in the sense of Birkhoff and von Neumann. It is important to note that the lattice $\mathcal{P}(A)$ is always complete (in that infima and suprema of arbitrary subsets exist).

Inspired by both von Neumann’s operator algebras and the theory of commutative Banach algebras, Gelfand and Naimark introduced the concept of a *C*-algebra* in 1943. Unlike a von Neumann algebra, a C*-algebra is defined without reference to a Hilbert space, namely as an involutive Banach algebra A for which $\|a^*a\| = \|a\|^2$ for each $a \in A$. For any Hilbert space H , the algebra $B(H)$ satisfies these axioms. More generally, each von Neumann algebra is a C*-algebra, but even if a C*-algebra is concretely given as an algebra of operators on some Hilbert space, it need not be strongly closed and hence need not be a von Neumann algebra. In fact, the class of all C*-algebras is not directly relevant to quantum logic, as a generic C*-algebra may not have enough projections.

One can already see this in the commutative case, where (in the unital case) one already has the so-called Gelfand isomorphism

$$A \cong C(\Sigma_A) \equiv C(\Sigma_A, \mathbb{C}), \tag{5.22}$$

for some compact Hausdorff space Σ_A , called the (Gelfand) *spectrum* of A . Now, under this isomorphism the projections in A correspond to characteristic functions of (Borel) subsets of Σ_A , so we immediately see that if Σ_A is connected, $A \cong C(\Sigma_A)$ has no nontrivial projections (i.e., except 0 and 1).

For later use, we briefly recall how the isomorphism (5.22) comes about. One may define Σ_A as the space of *characters* of A , i.e., nonzero multiplicative linear functionals $\varphi : A \rightarrow \mathbb{C}$ that satisfy $\varphi(ab) = \varphi(a)\varphi(b)$; such functionals are automatically

²³ Here $A \subset B(H)$ is strongly closed if for any strongly convergent net (a_λ) in A with limit a in $B(H)$ (in the sense that $\|a_\lambda \Psi - a\Psi\| \rightarrow 0$ for all $\Psi \in H$), the limit a in fact lies in A .

²⁴ Another good way of looking at von Neumann algebras is to see them as symmetries: any von Neumann algebra on a Hilbert space H arises as the algebra of invariants of some group action on H , in the sense that $A = U(G)'$ for some group G acting on H through a unitary representation U . To see this, note in one direction that $U(G)''' = U(G)'$, so that $U(G)'$ is indeed a von Neumann algebra. In the opposite direction, given A , let G be the group of all unitary operators in A' and take U to be the defining representation.

continuous and hence Σ_A inherits the weak*-topology on the Banach space dual A^* .²⁵ Subsequently, one defines a map

$$\begin{aligned} A &\xrightarrow{\cong} C(\Sigma_A); \\ a &\mapsto \hat{a}; \\ \hat{a}(\varphi) &= \varphi(a). \end{aligned} \tag{5.23}$$

This map is called the *Gelfand transform* and turns out to be an isomorphism when A is a commutative C^* -algebra with unit, and $C(\Sigma_A)$ is equipped with pointwise operations and the supremum norm. The space Σ_A is homeomorphic to the set of all regular maximal ideals of A ,²⁶ topologized by letting each $a \in A$ define a basic open that consists of all regular maximal ideals of A not containing a . The pertinent homeomorphism is then given by $\varphi \leftrightarrow \varphi^{-1}(\{0\})$.

Interestingly, it is also possible to directly describe this topology $\mathcal{O}(\Sigma_A)$ as a frame (up to frame isomorphism), without taking recourse to the initial construction of Σ_A as a set. In the special case that A has sufficiently many projections, for example, when it is a commutative von Neumann algebra (or, more generally, a commutative Rickart C^* -algebra, as in Definition 3 below), this description is given by²⁷

$$\mathcal{O}(\Sigma_A) \cong \text{Idl}(\mathcal{P}(A)), \tag{5.24}$$

where $\text{Idl}(L)$ is the usual frame of ideals of a lattice L ,²⁸ and $\mathcal{P}(A)$ is the lattice of projections in A , as above (in the present case, where A is assumed to be commutative, this lattice is Boolean, see below). This result (which may be unfamiliar even to specialists in C^* -algebras) is a special case of Theorem 16 below.

The absence of sufficiently many projections in a general C^* -algebra inspires the search for extra conditions on a C^* -algebra that do have an ample supply of projections and hence provide a good home for quantum logic. As we have seen, von Neumann algebras indeed do have enough projections. Although we will work with the more general class of Rickart C^* -algebras later on, since the former are much more familiar it is instructive to first review the connection between commutative von Neumann algebras and classical propositional logic. In the latter direction, let us recall the *Stone representation theorem* (see Johnstone 1982, passim or Mac Lane and Moerdijk 1992, § IX.10):

Any Boolean lattice \mathcal{L} is isomorphic to the lattice $\mathcal{B}(\hat{\Sigma}_{\mathcal{L}})$ of clopen subsets of a *Stone space* $\hat{\Sigma}_{\mathcal{L}}$, i.e., a compact Hausdorff space that is *totally* disconnected, in

²⁵ This is the weakest topology under which each \hat{a} defined below is continuous.

²⁶ In this context, an ideal I of a commutative Banach algebra A is by definition closed, and is called regular if the quotient algebra A/I admits an identity.

²⁷ See Coquand (2005), Coquand and Spitters (2005), and Heunen et al. (2009) for the case of general commutative C^* -algebras.

²⁸ This is the collection of nonempty lower closed subsets $I \subset L$ such that $x, y \in I$ implies $x \vee y \in I$, ordered by inclusion (Johnstone 1982, p. 59).

that the only connected subsets of $\hat{\Sigma}_{\mathcal{L}}$ are single points. (Equivalently, a Stone space is compact, T_0 , and has a basis of clopen sets.)

Here $\hat{\Sigma}_{\mathcal{L}} = \text{pt}(\mathcal{L})$, called the *Stone spectrum* of \mathcal{L} , arises as the space of ‘points’ of \mathcal{L} , which by definition are homomorphisms $\varphi : \mathcal{L} \rightarrow \{0, 1\}$ of Boolean lattices (where $\{0, 1\} \equiv \{\perp, \top\}$ as a lattice, i.e., $0 \leq 1$ and $0 \neq 1$), topologized by declaring that the basic open sets in $\hat{\Sigma}_{\mathcal{L}}$ are those of the form $U_x = \{\varphi \in \hat{\Sigma}_{\mathcal{L}} \mid \varphi(x) = 1\}$, for each $x \in \mathcal{L}$. Such ‘points’ $\varphi \in \hat{\Sigma}_{\mathcal{L}}$ may be identified with maximal ideals²⁹ $I_\varphi = \varphi^{-1}(\{0\}) \subset \mathcal{L}$, topologized by saying that each $x \in \mathcal{L}$ defines a basic open consisting of all maximal ideals not containing x . As in (5.24), one has a direct description of this topology as a frame (up to frame isomorphism), which turns out to be given by

$$\mathcal{O}(\hat{\Sigma}_{\mathcal{L}}) \cong \text{Idl}(\mathcal{L}); \tag{5.25}$$

see Corollaries II.4.4 and II.3.3 and Proposition II.3.2 in [Johnstone \(1982\)](#).

The following result describes the relationship between Boolean lattice and von Neumann algebras:³⁰

Proposition 2 *Let A be a von Neumann algebra. The following conditions are equivalent:*

1. A is commutative;
2. The lattice $\mathcal{P}(A)$ of projections in A is Boolean.

In that case, the Gelfand spectrum Σ_A of A is homeomorphic to (and hence may be identified with) with the Stone spectrum $\hat{\Sigma}_{\mathcal{P}(A)}$ of $\mathcal{P}(A)$, and $\mathcal{P}(A)$ is isomorphic with the Boolean lattice $\mathcal{B}(\Sigma_A)$ of clopens in Σ_A .

Proof For the equivalence between 1 and 2 see [Rédei \(1998, Proposition 4.16\)](#). The homeomorphism

$$\Sigma_A \cong \hat{\Sigma}_{\mathcal{P}(A)} \tag{5.26}$$

is clear from (5.24) and (5.25). The isomorphism of Boolean lattices

$$\mathcal{P}(A) \xrightarrow{\cong} \mathcal{B}(\Sigma_A); \tag{5.27}$$

$$p \mapsto \mathcal{D}(\hat{p}) \tag{5.28}$$

then follows from Stone’s Theorem. □

Here and in what follows, for any $a \in C(\Sigma_A)$ we write

$$\mathcal{D}(a) = \{\sigma \in \Sigma \mid a(\sigma) \neq 0\}. \tag{5.29}$$

²⁹ In this usage, an ideal I in a lattice L denotes a subset of L such that $x, y \in I$ implies $x \vee y \in I$, and $y \leq x \in I$ implies $y \in I$. In a Boolean lattice, prime ideals and maximal ideals coincide, so that the Stone spectrum of a Boolean lattice is often described as the space of its prime ideals (which are those ideals that not contain 1 and where $x \wedge y \in I$ implies either $x \in I$ or $y \in I$).

³⁰ More generally, the proposition holds for Rickart C^* -algebras, with the same proof.

The homeomorphism (5.26) arises as follows:

- each character $\varphi : A \rightarrow \mathbb{C}$, $\varphi \in \Sigma_A$, restricts to a point $\hat{\varphi} : \mathcal{P}(A) \rightarrow \{0, 1\}$, $\hat{\varphi} \in \hat{\Sigma}_{\mathcal{P}(A)}$;
- conversely, each $\hat{\varphi} \in \hat{\Sigma}_{\mathcal{P}(A)}$ extends to a character $\varphi \in \Sigma_A$ by the spectral theorem.

Proposition 2 suggests that the projection lattices $\mathcal{P}(A)$ of general von Neumann algebras may be seen as noncommutative generalizations of classical propositional logic (in its semantic guise of Boolean algebras). Despite the conceptual drawbacks we mentioned in Sect. 3, this gives a clear mathematical status to quantum logic in the style of Birkhoff and von Neumann. However, for various technical reasons the class of von Neumann algebras is not optimal in this respect. First, Proposition 2 does not identify the class of Boolean lattices with the class of commutative von Neumann algebras; in fact, if A is a commutative von Neumann algebra, then the lattice $\mathcal{P}(A)$ is complete, so that Σ_A is not merely Stone but *Stonean*, i.e., compact, Hausdorff and *extremely* disconnected, in that the closure of every open set is open (and hence clopen).³¹ But one does not obtain an identification of complete Boolean lattices (or, equivalently, Stonean spaces) with commutative von Neumann algebras either, since the Gelfand spectrum of a commutative von Neumann algebra is not merely Stonean but has the stronger property of being *hyperstonean*, in admitting sufficiently many positive normal measures (Takesaki 2003a, Definition 1.14). This is the situation: a commutative C*-algebra is a von Neumann algebra iff its Gelfand spectrum (and hence the Stone spectrum of its projection lattice) is hyperstonean. Second, our use of constructive mathematics in the main body of this paper leads to certain difficulties with the class of von Neumann algebras, mainly because they are defined on a given Hilbert space (as opposed to an abstract C*-algebra).³²

To survey the landscape, we mention the basic classes of C*-algebras that are potentially relevant to logic in having sufficiently many projections, in order of increasing generality:³³

Definition 3 A unital C*-algebra A is said to be:

1. a *von Neumann algebra* if it is the dual of some Banach space (Sakai 1971);
2. an *AW*-algebra* if for each nonempty subset $S \subset A$ there is a projection $p \in A$ so that $R(S) = pA$ (Kaplansky 1968);
3. a *Rickart C*-algebra* if for each $x \in A$ there is a projection $p \in A$ so that $R(x) = pA$ (Rickart 1946);

³¹ The Stone spectrum of a Boolean lattice \mathcal{L} is Stonean iff \mathcal{L} is complete.

³² Sakai's abstract characterization of von Neumann algebras as C*-algebras that are the dual of some Banach space obviates this problem, but introduces others (notably the problem of internalizing the so-called ultraweak or σ -weak topology on a von Neumann algebra), which we are unable to deal with constructively at the moment. A constructive theory of von Neumann algebras actually exists (Dediú and Douglas 2001; Spitters 2005), but this theory relies on the use of the strong operator topology, which has awkward continuity properties (e.g., the map $s \mapsto E_s$, where E_s is the spectral projection associated to $(-\infty, s)$, need not be strongly continuous). Furthermore, it uses the axiom of dependent choice, which although available in our presheaf topos defined below, is not valid in arbitrary toposes in which C*-algebras can be defined.

³³ These definitions were originally motivated by the desire to find a purely algebraic analogue of the theory of von Neumann algebras, rather than by quantum logic.

4. a *spectral C*-algebra* if for each $a \in A$, $a \geq 0$, and each $\lambda, \mu \in (0, \infty)$, $\lambda < \mu$, there exists a projection $p \in A$ so that $ap \geq \lambda p$ and $a(1 - p) \leq \mu(1 - p)$ (Stratila and Zsido 2009).

Here the *right-annihilator* $R(S)$ of $S \subset A$ is defined as $R(S) = \{a \in A \mid xa = 0 \forall x \in S\}$ and $R(x) \equiv R(\{x\})$; in view of the presence of an involution, equivalent definitions may be given in terms of the left-annihilator. In all cases, the projection p is unique. It is known that if a C*-algebra A has a faithful representation on a separable Hilbert space, then it is a Rickart C*-algebra iff it is an AW*-algebra, but otherwise these classes are different.³⁴ Let us note that the equivalence between the original definition of a von Neumann algebra and the one given here is quite a deep result in the theory of operator algebras.

We now have the following results, of which the first has already been mentioned. Recall that $\mathcal{B}(\Sigma)$ is the Boolean lattice of clopens of a Stone space Σ ; as in the case of von Neumann algebras, if Σ_A is the Gelfand spectrum of a commutative C*-algebra A , then $\mathcal{B}(\Sigma_A)$ is isomorphic to the lattice $\mathcal{P}(A)$ of projections in A .

Theorem 4 *Let A be a commutative C*-algebra with Gelfand spectrum Σ_A . Then A is:*

1. a von Neumann algebra iff Σ_A is hyperstonean (Takesaki 2003a, § III.1);
2. an AW*-algebra iff Σ_A is Stonean (equivalently, Stone with the additional property that $\mathcal{B}(\Sigma_A)$ is complete) (Berberian 1972, Thm. 1.7.1);
3. a Rickart C*-algebra iff Σ_A is Stone with the additional property that $\mathcal{B}(\Sigma_A)$ is σ -complete (Berberian 1972, Thm. 1.8.1);
4. a spectral C*-algebra iff Σ_A is Stone (Stratila and Zsido 2009, §. 9.7).

Restricting Gelfand duality to each of the above cases results in a categorical duality (e.g., for case 3 above, between commutative Rickart C-algebras and Stone spaces X for which $\mathcal{B}(X)$ is σ -complete).*

The completeness of $\mathcal{B}(\Sigma)$ is equivalent to the property that the closure of the union of any family of clopens in Σ is clopen; similarly, $\mathcal{B}(\Sigma)$ is σ -complete iff the closure of the union of a countable family of clopens in Σ is clopen.

It appears that in the commutative case spectral C*-algebras form the most general class to work with from the point of view of classical logic, but unfortunately, the projections in a noncommutative spectral C*-algebra may not form a lattice. A major advantage of Rickart C*-algebras is that they do (Berberian 1972, Prop. 1.3.7 and Lemma 1.8.3):

Proposition 5 *The set of projections $\mathcal{P}(A)$ in a Rickart C*-algebra A form a σ -complete lattice under the ordering $p \leq q$ iff $pA \subseteq qA$.*

The ensuing lattice operations are given by

$$p \wedge q = q + \text{RP}[(p(1 - q))]; \tag{5.30}$$

$$p \vee q = p - \text{LP}[(p(1 - q))], \tag{5.31}$$

³⁴ It is generally believed that a C*-algebra is Rickart iff it is monotone σ -complete. In that case, one may also define a C*-algebra A to be Rickart if each maximal Abelian *-subalgebra of A is monotone σ -complete (Saitô and Wright 2003).

where for any $x \in A$ the projections $\text{RP}[x]$ and $\text{LP}[x]$ are defined by $R(x) = (1 - \text{RP}[x])A$ and $L(x) = A(1 - \text{LP}[x])$, respectively (i.e., $\text{RP}[x] = 1 - p$ where $R(x) = pA$, etc.). We also have properties that guarantee the availability of spectral theory (the strong limits in the usual constructions are just replaced by limits of monotone positive sequences):

- Proposition 6** 1. A commutative Rickart C^* -algebra is the (norm-)closed linear span of its projections (Berberian 1972, Prop. 1.8.1.(3));
2. A commutative Rickart C^* -algebra C is monotone σ -complete, in that each increasing bounded sequence of self-adjoint elements of C has a supremum in C (Stratila and Zsidó 2009, Prop. 9.2.6.1).³⁵

In our search for a suitable class of operator algebras to lie at the basis of intuitionistic quantum logic, and in particular to generalize the Heyting algebra (or frame) (4.10) to all elements A of this class, we also require certain constructions to work internally in a topos; in particular, the “Bohrification” \underline{A} of A (defined in the next section) should internally lie in the same class as A itself. This will indeed be the case for Rickart C^* -algebras; see Theorem 7 below. Summing up, we generalize the usual algebraic approach to quantum logic (Rédei 1998) in proposing that instead of von Neumann algebras, we prefer to work with Rickart C^* -algebras. All one loses in this generalization is the completeness of the projection lattice $\mathcal{P}(A)$ of A , but since one does have the slightly weaker property of σ -completeness (which, if A has a faithful representation on a separable Hilbert space, actually implies the completeness of $\mathcal{P}(A)$), this is not a source of tremendous worry.

6 Internal Rickart C^* -algebras

In this section we assume familiarity with basic category and topos theory; the Appendix to Caspers et al. (2009) is tailor-made for this purpose, and also the first few chapters of Mac Lane (1998) and Mac Lane and Moerdijk (1992) contain all necessary background. See also Bell (1988) and Goldblatt (1984) for introductions that emphasise the connection between topos theory and intuitionistic logic. In some technical arguments we will also use the so-called internal language of a topos and its Kripke–Joyal semantics, for which (Mac Lane and Moerdijk 1992, Chap. VI) is our basic reference. Briefly, a topos may be seen as a generalization of the category **Sets** (whose objects are sets and whose arrows are functions, subject to the usual ZFC axiom system) in which most set-theoretic reasoning can be carried out, with the restriction that all proofs need to be constructive in the limited sense that one cannot make use of the law of the excluded middle or the Axiom of Choice. In what follows, we will use the term ‘constructive’ in this way.³⁶

Let A be a Rickart C^* -algebra, with associated poset $\mathcal{C}(A)$ of all unital commutative Rickart C^* -subalgebras of A , partially ordered by set-theoretic inclusion. The poset

³⁵ Quoted in D’Antoni and Zsidó (2008, p. 4728). Similarly, a commutative AW^* -algebra is monotone complete. It is an open question whether any Rickart C^* -algebra C is monotone σ -complete.

³⁶ The reader be warned that topos theory makes extensive use of the power set construction, which is avoided in so-called predicative constructive mathematics.

$\mathcal{C}(A)$ defines a category, called $\mathcal{C}(A)$ as well, in which C and D are connected by a unique arrow $C \rightarrow D$ iff $C \subseteq D$, and are not connected by any arrow otherwise. In this paper, the only relevant topos besides **Sets** is the category

$$\mathcal{T}(A) = \mathbf{Sets}^{\mathcal{C}(A)} \tag{6.32}$$

of (covariant) functors from $\mathcal{C}(A)$, seen as a category, to **Sets**. We will underline objects in $\mathcal{T}(A)$. As a case in point, the tautological functor

$$A : C \mapsto C, \tag{6.33}$$

maps a point $C \in \mathcal{C}(A)$ to the corresponding commutative C^* -algebra $C \subset A$ (seen as a set); for $C \subseteq D$ the map $\underline{A}(C \subseteq D) : \underline{A}(C) \rightarrow \underline{A}(D)$ is just the inclusion $C \hookrightarrow D$. We call A the *Bohrification* of A .

Theorem 7 *Let A be a Rickart C^* -algebra. Then \underline{A} is a commutative Rickart C^* -algebra in $\mathcal{T}(A)$.*

Proof Since A is, in particular, a C^* -algebra, it follows from Heunen et al. (2009, Thm. 5) that \underline{A} is a commutative C^* -algebra in $\mathcal{T}(A)$. To prove that it is internally Rickart, we spell out Definition 3.3 in logical notation, with $x \in A$ as a free variable:

$$\exists_{p \in A} xp = 0 \wedge \forall_{y \in A} xy = 0 \Rightarrow y = py. \tag{6.34}$$

Here we have changed the condition inherent in Definition 3.3 that $xy = 0$ implies that there exists $a \in A$ such that $y = pa$, to the equivalent condition that $xy = 0$ implies $y = py$; see Berberian (1972, Prop. 1.3.3). This is not necessary, but simplifies the argument somewhat.

We regard (6.34) as a formula ϕ in the internal language of $\mathcal{T}(A)$ with a single free variable x of type \underline{A} . By Kripke–Joyal semantics, ϕ is true if $C \Vdash \phi(\tilde{x})$ for all $C \in \mathcal{C}(A)$ and all $\tilde{x} \in \underline{A}(C) = C$ (Mac Lane and Moerdijk 1992, § VI.7). By the rules for this semantics, $C \Vdash \phi(\tilde{x})$ is true iff there exists a projection $\tilde{p} \in C$ such that for all $D \supseteq C$, all $\tilde{y} \in D$, and all $E \supseteq D$ one has: if $\tilde{x}\tilde{y} = 0$, then $\tilde{y} = \tilde{p}\tilde{y}$. In the latter part, the elements $\tilde{x} \in C$, $\tilde{p} \in C$, and $\tilde{y} \in D$ are all regarded as elements of E , but clearly the if ...then statement holds at all $E \supseteq D$ iff it holds at D . The truth of $C \Vdash \phi(\tilde{x})$, and hence of Theorem 7 now follows from the following lemma.

Lemma 8 *Let C and D be commutative Rickart C^* -algebras with $C \subseteq D$, and take $x \in C$. If one regards x as an element of D , then the projection p for which $R(x) = pD$ lies in C . In other words: if $x \in C \subseteq D$, then the projection $RP[x]$ as computed in D actually lies in C .*

Proof We have $C \cong C(\Sigma_C)$ and $D \cong C(\Sigma_D)$ through the Gelfand transform. As we have seen, Σ_C is a Stone space, whose topology has a basis $\mathcal{B}(\Sigma_C)$ consisting of all clopen sets in Σ_C . This basis is isomorphic as a Boolean lattice to the projection lattice $\mathcal{P}(C)$ of C , with isomorphism (5.28) (for $A = C$), and analogously for D .

One has a canonical map $r_{DC} : \Sigma_D \rightarrow \Sigma_C$ given by restriction, i.e., $(r_{DC}\varphi)(a) = \varphi(a)$ for $a \in C$, or $r_{DC}\varphi = \varphi|_C$. Being continuous, this map induces the inverse image map

$$r_{DC}^{-1} : \mathcal{O}(\Sigma_C) \rightarrow \mathcal{O}(\Sigma_D) \tag{6.35}$$

as well as the pullback

$$r_{DC}^* : C(\Sigma_C) \rightarrow C(\Sigma_D). \tag{6.36}$$

Restricted to basic opens and projections, respectively, these maps are related to each other and to the inclusion $\iota_{CD} : C \hookrightarrow D$ by

$$r_{DC}^*(\hat{p}) = \widehat{\iota_{CD}(p)}; \tag{6.37}$$

$$r_{DC}^{-1}(\mathcal{D}(\hat{p})) = \mathcal{D}(r_{DC}^*(\hat{p})). \tag{6.38}$$

By Berberian (1972, Prop. 1.8.1.(4)), the projection $p \in D$ in the statement of the Lemma has Gelfand transform $\hat{p} = 1 - \chi_{\mathcal{D}(r_{DC}^*x)^-}$, where for any $U \subset \Sigma$, U^- is the closure of U . But by (6.38) one then has $\hat{p} = r_{DC}^*(\hat{q})$ with $\hat{q} = (1 - \chi_{\mathcal{D}(x)^-})$, and (6.37) yields $p = \iota_{CD}(q)$. Hence $p \in C$. This concludes the proof of Lemma 8 as well as of Theorem 7. □

We now initiate a constructive theory of Rickart C*-algebras. Our constructive approach is crucial for what follows, for any constructive result may be used internally, i.e., in an arbitrary topos. In addition, it leads to an alternative proof of Theorem 7, which may be rederived from the Proposition 9 below.³⁷

Proposition 9 *Let A be a commutative C*-algebra. The following are equivalent:*

1. *for each $a \in A$ there exists a (unique) projection p such that i) $ap = 0$ and ii) if $ab = 0$, then there exists c such that $b = cp$.*
2. *for each a there exists a (unique) projection p such that $ap = 0$ and if $ab = 0$, then $b = bp$.*
3. *for each self-adjoint a there exists a (unique) projection, denoted $[a > 0]$, such that $[a > 0]a = a^+$ and $[a > 0] \wedge [-a > 0] = 0$.*

Let us note that since A is commutative, the infimum \wedge in 3 is the same as the product.

Proof The equivalence of 1 and 2 is in Berberian (1972, Prop. 1.3.3). We denote the projection p in 2 by $[a = 0]$. By the decomposition of arbitrary elements of A in four positives, it suffices to require the existence of $[a = 0]$ only for positive elements a ; for general $a \in A$ we obtain the required projection by multiplication of the four projections for its positive components.

³⁷ Proposition 9 shows that Rickart C*-algebras are C*-algebras equipped with an extra (partial) operation $a \mapsto [a > 0]$. A proof of Thm. 7 may then be obtained by a simple extension of Heunen et al. (2009, Theorem 5) by observing that the definition of f-algebras with such an operation is Cartesian and hence geometric.

2→3 For a self-adjoint a we define $[a > 0] := 1 - [a^+ = 0]$. Then

$$[a > 0]a = (1 - [a^+ = 0])(a^+ - a^-) = a^+.$$

By definition, $[a > 0] = [a^+ > 0]$. By 2 and $a^-a^+ = 0$, $a^-[a > 0] = 0$. Again by 2, but applied to a^- , $[a^- > 0][a > 0] = 0$. Since $(-a)^+ = a^-$, $[a > 0] \wedge [-a > 0] = 0$.

3→2 For positive a we define $[a = 0] := 1 - [a > 0]$. Then $a[a = 0] = a(1 - [a > 0]) = 0$. We may assume that $a, b \geq 0$ and $ab = 0$. Then $b[a > 0] \leq 0$ (see part 1 of Lemma 12 below), and since $b[a > 0]$ is the product of commuting positive operators, this implies $b[a > 0] = 0$. □

Thus A is a commutative Rickart C^* -algebra if any (and hence all) of the three conditions in this proposition is satisfied. Our earlier proof of Theorem 7 can now be reformulated in a simple way by applying the above proposition to \underline{A} : since the existence of the projection $[a > 0]$ in part 3 of Proposition 9 is interpreted locally, \underline{A} satisfies the condition in 3 if each $C \in \mathcal{C}(A)$ does. Hence \underline{A} is Rickart.

Similarly, the σ -completeness of the projection lattice of a commutative Rickart C^* -algebra (cf. Proposition 5) is immediate from the following analogue of Berberian (1972, Lem. 1.8.2):

Lemma 10 *A sequence p_n of mutual orthogonal projections has a supremum.*

Proof The sum $a := \sum 2^{-n} p_n$ converges in the C^* -algebra. The supremum of the sequence is the projection $[a > 0]$. □

Finally, Saitô and Wright (2003) define a C^* -algebra to be Rickart if each maximal Abelian $*$ -subalgebra of A is Rickart (or, equivalently, monotone σ -complete). Equivalently, one may require that every Abelian $*$ -subalgebra is contained in an Abelian Rickart C^* -algebra. This definition captures essential parts of the theory of von Neumann algebras and, being formulated entirely in terms of commutative subalgebras, is very much in the spirit of our “Bohrification” program. Unfortunately, although every Rickart C^* -algebra in the sense of Definition 3 is a Rickart C^* -algebra in the sense of Saitô and Wright, the converse has not been shown to date.³⁸ In any case, upon the definition of Saitô and Wright, Rickart C^* -algebras admit a nice internal characterization, provided we use classical meta-logic and use the original definition of the poset $\mathcal{C}(A)$ from Heunen et al. (2009), according to which $\mathcal{C}(A)$ is the collection of all commutative unital C^* -subalgebras of A .

Proposition 11 *Let A be a C^* -algebra in Sets. Then A is a Rickart C^* -algebra in the sense of Saitô and Wright iff \underline{A} satisfies: for all self-adjoint a , not there exists a projection p such that $p = [a > 0]$.*

Proof By Lemma 19 below, the right hand side means that for all $D \in \mathcal{C}(A)$ and $\tilde{a} \in D_{sa}$ there exists $E \supset D$ and a projection $\tilde{p} \in E$ such that $E \Vdash (p = [a > 0])(\tilde{p}, \tilde{a})$, i.e., $\tilde{p} = [\tilde{a} > 0]$ in E . This is precisely our earlier definition of a Rickart C^* -algebra. □

³⁸ Private communications from Saitô and Wright.

The use of Proposition 11 derives from the fact that in both classical and intuitionistic logic, the propositions $A \rightarrow \neg B$ and $\neg\neg A \rightarrow \neg B$ are equivalent. Hence negative statements for Rickart algebras may be proved by assuming that the projection $[a > 0]$ actually exists.

7 Gelfand theory for commutative Rickart C*-algebras

The Gelfand theory for commutative C*-algebras A that in the classical case leads to the isomorphism $A \cong C(\Sigma_A)$ for some compact Hausdorff space Σ_A , generalizes to the constructive or topos-theoretic setting in producing a frame (see Sect. 4) $\mathcal{O}(\Sigma_A)$, rather than the space Σ itself, with the property that A is isomorphic as a commutative C*-algebra with the object of all frame maps from $\mathcal{O}(\mathbb{C})$ (i.e., the frame of Dedekind complex numbers, interpreted in the ambient topos) to $\mathcal{O}(\Sigma_A)$. In the classical case, since Σ_A is Hausdorff and hence sober, each frame map $\varphi^* : \mathcal{O}(\mathbb{C}) \rightarrow \mathcal{O}(\Sigma_A)$ arises as the inverse image $\varphi^* = \varphi^{-1}$ of some continuous map $\varphi : \Sigma_A \rightarrow \mathbb{C}$, so that one recovers the usual Gelfand isomorphism, but in general this isomorphism involves the frame $\mathcal{O}(\Sigma_A)$ in the said way; an underlying space Σ_A may not even exist (indeed, due to the Kochen–Specker Theorem this is precisely the case in our application to quantum theory).³⁹

The abstract theory of internal C*-algebras and Gelfand duality in a topos is due to Banaschewski and Mulvey (2006). In order to explicitly compute the frame $\mathcal{O}(\Sigma_A)$ for given A , we use the constructive formulation of Gelfand duality due to Coquand and Spitters (2005, 2009a), building on fundamental insights into Stone duality by Coquand (2005); see also Heunen et al. (2009). First, define a relation \preceq on the self-adjoint part $A_{sa} = \{a \in A \mid a^* = a\}$ of A by putting $a \preceq b$ iff there exists an $n \in \mathbb{N}$ such that $a \leq nb^+$. This yields an associated equivalence relation $a \approx b$, defined by $a \preceq b$ and $b \preceq a$. The lattice L_A is defined as

$$L_A = A^+ / \approx, \tag{7.39}$$

where $A^+ = \{a \in A \mid a \geq 0\}$ is the positive cone of A .

The key results are that L_A is a so-called normal distributive lattice and that $\mathcal{O}(\Sigma_A)$ arises as the frame $\text{RI}d(L_A)$ of regular ideals in L_A . We shall not define these notions here (see Coquand 2005; Coquand and Spitters 2005; Heunen et al. 2009), since in the case at hand the situation simplifies according to Theorem 16 below, but we will need the following information. We denote the equivalence class of $a \in A_{sa}$ in L_A by $D(a)$; we have $D(a) = D(a^+)$, so that we may restrict a to lie in A^+ , i.e., $a \geq 0$. Furthermore, we denote the map $L_A \rightarrow \text{RI}d(L_A)$ that assigns the regular closure of the principal down set $\downarrow D(a)$ to $D(a) \in L_A$ (see Cederquist and Coquand 2000, Thm. 27 or Heunen et al. 2009, Eq. (80)) by $D(a) \mapsto \mathcal{D}(a)$; upon the identification $\mathcal{O}(\Sigma_A) \cong \text{RI}d(L_A)$, this map simply injects $D(a)$ into $\mathcal{O}(\Sigma_A)$ as a basis open, and in the classical case this notation is consistent with (5.29). On then has the following relations:

³⁹ The notation $\mathcal{O}(\Sigma_A)$ for a frame whose underlying point set Σ_A may not exist may appear odd, but is generally used in order to stress that one may reason with $\mathcal{O}(\Sigma_A)$ as if it were the topology of some space.

$$\mathcal{D}(1) = \top; \tag{7.40}$$

$$\mathcal{D}(a) \wedge \mathcal{D}(-a) = \perp; \tag{7.41}$$

$$\mathcal{D}(-b^2) = \perp; \tag{7.42}$$

$$\mathcal{D}(a + b) \leq \mathcal{D}(a) \vee \mathcal{D}(b), \tag{7.43}$$

$$\mathcal{D}(ab) = (\mathcal{D}(a) \wedge \mathcal{D}(b)) \vee (\mathcal{D}(-a) \wedge \mathcal{D}(-b)), \tag{7.44}$$

$$\mathcal{D}(a) = \bigvee_{s>0} \mathcal{D}(a - s). \tag{7.45}$$

In fact, the first five relations already hold for the $\mathcal{D}(\cdot)$ and may be used to define L_A , whereas the complete set may be used as a definition of $\mathcal{O}(\Sigma_A)$.

We now work towards the explicit formula for the external description of the Gelfand spectrum of the Bohrification of a Rickart C^* -algebra in Theorem 16 below.

Lemma 12 *Let A be a commutative Rickart C^* -algebra, and $a, b \in A$ self-adjoint. If $a \leq ab$, then $a \preceq b$, i.e. $\mathcal{D}(a) \leq \mathcal{D}(b)$.*

Proof If $a \leq ab$ then certainly $a \preceq ab$. Hence $\mathcal{D}(a) \leq \mathcal{D}(ab) = \mathcal{D}(a) \wedge \mathcal{D}(b)$. In other words, $\mathcal{D}(a) \leq \mathcal{D}(b)$, whence $a \preceq b$. □

Definition 13 (Goldblatt 1984; Johnstone 1982) *A pseudocomplement on a distributive lattice L is an antitone (i.e., anti-monotone) function $\neg: L \rightarrow L$ satisfying $x \wedge y = 0$ iff $x \leq \neg y$.⁴⁰*

Proposition 14 *For a commutative Rickart C^* -algebra A , the lattice L_A has a pseudocomplement, determined by $\neg \mathcal{D}(a) = \mathcal{D}([a = 0])$ for $a \in A^+$.*

Proof Without loss of generality, let $b \leq 1$. Then

$$\begin{aligned} \mathcal{D}(a) \wedge \mathcal{D}(b) = 0 &\iff \mathcal{D}(ab) = \mathcal{D}(0) \\ &\iff ab = 0 \\ &\iff b[a = 0] = b && (\implies \text{by Proposition 9}) \\ &\iff b \preceq [a = 0] && (\Leftarrow \text{ since } b \leq 1, \implies \text{by Lemma 12}) \\ &\iff \mathcal{D}(b) \leq \mathcal{D}([a = 0]) = \neg \mathcal{D}(a). \end{aligned}$$

To see that \neg is antitone, suppose that $\mathcal{D}(a) \leq \mathcal{D}(b)$. Then $a \preceq b$, so $a \leq nb$ for some $n \in \mathbb{N}$. Hence $[b = 0]a \leq [b = 0]bn = 0$, so that $\neg \mathcal{D}(b) \wedge \mathcal{D}(a) = \mathcal{D}([b = 0]a) = 0$, and therefore $\neg \mathcal{D}(b) \leq \neg \mathcal{D}(a)$. □

Lemma 15 *If A is a commutative Rickart C^* -algebra, then the lattice L_A satisfies $\mathcal{D}(a) = \bigvee_{r \in \mathbb{Q}^+} \mathcal{D}([a - r > 0])$ for all $a \in A^+$.*

⁴⁰ The construction of the Boolean algebra of projections as the pseudocomplements in the lattice L is reminiscent of the construction of the Boolean algebra of pseudocomplements which can be carried out in a Heyting algebra; e.g. Johnstone (1982, I.1.13). However, as L need not be a Heyting algebra, our construction is not an instance of this general method.

Proof Since $[a > 0]a = a^+ \geq a$, Lemma 12 gives $a \preccurlyeq [a > 0]$ and therefore $D(a) \leq D([a > 0])$. Also, for $r \in \mathbb{Q}^+$ and $a \in A^+$, one has $1 \leq \frac{2}{r}((r - a) \vee a)$, whence

$$[a - r > 0] \leq \frac{2}{r}((r - a) \vee a)[a - r > 0] = \frac{2}{r}(a[a - r > 0]).$$

Lemma 12 then yields $D([a - r > 0]) \leq D(\frac{2}{r}a) = D(a)$. In total, we have $D([a - r > 0]) \leq D(a) \leq D([a > 0])$ for all $r \in \mathbb{Q}^+$, from which the statement follows. \square

Theorem 16 *The Gelfand spectrum $\mathcal{O}(\Sigma_A)$ of a commutative Rickart C^* -algebra A is isomorphic to the frame $\text{Idl}(\mathcal{P}(A))$ of ideals of $\mathcal{P}(A)$.*

Proof Form the sublattice $\mathcal{P}_A = \{D(a) \in L_A \mid a \in A^+, \neg\neg D(a) = D(a)\}$ of ‘closed elements’ of L_A , which is Boolean by construction. Since $\neg D(p) = D(1 - p)$ for $p \in \mathcal{P}(A)$, we have $\neg\neg D(p) = D(p)$. Conversely, $\neg\neg D(a) = D([a > 0])$, so that each element of \mathcal{P}_A is of the form $D(a) = D(p)$ for some $p \in \mathcal{P}(A)$. So $\mathcal{P}_A = \{D(p) \mid p \in \mathcal{P}(A)\} \cong \mathcal{P}(A)$, since each projection $p \in \mathcal{P}(A)$ may be selected as the unique representative of its equivalence class $D(p)$ in L_A . By Lemma 15, we may use $\mathcal{P}(A)$ instead of L_A as the generating lattice for $\mathcal{O}(\Sigma_A)$. So $\mathcal{O}(\Sigma_A)$ is the collection of regular ideals of $\mathcal{P}(A)$ by Heunen et al. (2009, Theorem 26). But since $\mathcal{P}(A) \cong \mathcal{P}_A$ is Boolean, all its ideals are regular, as $D(p) \ll D(p)$ for each $p \in \mathcal{P}(A)$ (Johnstone 1982). This establishes the statement. \square

Internalized to the topos $\mathcal{T}(A)$, Theorem 16 enables us to compute the spectrum $\mathcal{O}(\underline{\Sigma}_A)$ of the Bohrification \underline{A} of A . As a functor $\mathcal{O}(\underline{\Sigma}_A) : \mathcal{C}(A) \rightarrow \mathbf{Sets}$, this spectrum is completely determined by its component at $\mathbb{C} \cdot \bar{1}$, which is the frame in \mathbf{Sets} that provides the so-called external description of $\mathcal{O}(\underline{\Sigma}_A)$ (Joyal and Tierney 1983) (see also Heunen et al. 2009, Thm. 29). We write

$$\mathcal{O}(\Sigma_A) \equiv \mathcal{O}(\underline{\Sigma}_A)(\mathbb{C} \cdot \bar{1}), \tag{7.46}$$

and draw attention to the notation (6.35).

Theorem 17 *The frame $\mathcal{O}(\Sigma_A)$ is given by*

$$\mathcal{O}(\Sigma_A) = \{S : \mathcal{C}(A) \rightarrow \mathbf{Sets} \mid S(C) \in \mathcal{O}(\Sigma_C), r_{DC}^{-1}(S(C)) \subseteq S(D) \text{ if } C \subseteq D\}, \tag{7.47}$$

and has a basis given by

$$\mathcal{B}(\Sigma_A) = \{\tilde{S} : \mathcal{C}(A) \rightarrow \mathcal{P}(A) \mid \tilde{S}(C) \in \mathcal{P}(C), \tilde{S}(C) \leq \tilde{S}(D) \text{ if } C \subseteq D\}, \tag{7.48}$$

in the sense that under the (injective) map $f : \mathcal{B}(\Sigma_A) \rightarrow \mathcal{O}(\Sigma_A)$ given by

$$f(\tilde{S})(C) = \widehat{\mathcal{D}(\tilde{S}(C))}, \tag{7.49}$$

each $S \in \mathcal{O}(\Sigma_A)$ may be expressed as $S = \bigvee \{f(\tilde{S}) \mid \tilde{S} \in \mathcal{B}(\Sigma_A), f(\tilde{S}) \leq S\}$.

Proof We interpret Theorem 16 in the topos $\mathcal{T}(A)$, where \underline{A} plays the role of the general commutative C^* -algebra A in the above analysis (not to be confused with the noncommutative C^* -algebra A in **Sets** whose Bohrification is \underline{A}). The internal version of the lattice L_A is the functor \underline{L}_A , which according to Heunen et al. (2009, Theorem 20) is simply given by $\underline{L}_A(C) = C$. Consequently, the subobject $\underline{\mathcal{P}}_A$ is given by $\underline{\mathcal{P}}_A(C) = \mathcal{P}(C)$ (as the algebraic conditions $p^2 = p^* = p$ defining a projection are interpreted locally).

Combining Theorem 16 with Theorem 29 in Heunen et al. (2009), we find that

$$\mathcal{O}(\Sigma_A) \cong \text{Idl}(\underline{\mathcal{P}}_A), \tag{7.50}$$

where the right-hand side by definition is the subset of $\text{Sub}(\underline{\mathcal{P}}_A)$ that consists of subfunctors \underline{U} of $\underline{\mathcal{P}}_A$ for which $\underline{U}(C) \in \text{Idl}(\mathcal{P}(C))$ for all $C \in \mathcal{C}(A)$. Now, internalizing Theorem 16 to **Sets** and applying it to $A = C$, we obtain the frame isomorphism $\text{Idl}(\mathcal{P}(C)) \cong \mathcal{O}(\Sigma_C)$; the identification is given by mapping $I \in \text{Idl}(\mathcal{P}(C))$ to $\bigcup \{\mathcal{D}(\hat{p}) \mid p \in I\} \in \mathcal{O}(\Sigma_C)$. The requirement that \underline{U} be a subfunctor of $\underline{\mathcal{P}}_A$ then immediately yields (7.47). Part 2 is obvious from the fact that the order in $\mathcal{O}(\Sigma_A)$ and in $\mathcal{B}(\Sigma_A)$ is defined pointwise. \square

Now let $A = M_n(\mathbb{C})$. By the Kochen–Specker theorem in the version given in Heunen et al. (2009) and Caspers et al. (2009), the frame (more precisely, the locale) $\mathcal{O}(\underline{\Sigma}_A)$ does not have any point. In particular, it cannot have n points. Classically, of course, one has $\Sigma_{\mathbb{C}^n} = \mathbf{n} \equiv \{1, 2, \dots, n\}$ and hence

$$\mathcal{O}(\Sigma_{\mathbb{C}^n}) \cong \mathcal{P}(\mathbb{C}^n) \cong \text{Pow}(\mathbf{n}) \tag{7.51}$$

(i.e., the power set of \mathbf{n}).⁴¹ The points of $\Sigma_{\mathbb{C}^n}$ are in bijective correspondence with the completely prime filters of $\text{Pow}(\mathbf{n})$, and hence, once again, with the elements of \mathbf{n} . Remarkably, we can prove that it is not the case that internally $\mathcal{O}(\underline{\Sigma}_A)$ has precisely the same structure.

Proposition 18 *Let $A = M_n(\mathbb{C})$. Then it is impossible that the Gelfand spectrum $\underline{\Sigma}_A$ does not have n points. More precisely, noting that in $\mathcal{T}(A)$ the set \mathbf{n} is internalized as the constant functor $\underline{\mathbf{n}} : C \mapsto \mathbf{n}$, we internally have*

$$\neg\neg(\underline{\mathcal{P}}_A \cong \underline{\Omega}^{\underline{\mathbf{n}}}); \tag{7.52}$$

$$\neg\neg(\mathcal{O}(\underline{\Sigma}_A) \cong \underline{\Omega}^{\underline{\mathbf{n}}}). \tag{7.53}$$

Proof The proof relies on the following lemma.

Lemma 19 *Let ϕ be a formula in the internal language of $\mathcal{T}(A)$ (for simplicity without free variables). Then $C \Vdash \neg\neg\phi$ iff ϕ holds eventually, in that for all $D \supseteq C$ there exists $E \supseteq D$ such that $E \Vdash \phi$. In particular, ϕ is true if $E \Vdash \phi$ for any maximal commutative C^* -subalgebra E of A .*

⁴¹ We use the notation $\text{Pow}(X)$ for the power set of X to distinguish it – in a constructive setting – from 2^X , which is used to denote the set of decidable subsets of X , i.e., subsets Y such that for all x in X , $x \in Y$ or $x \notin Y$. In the presence of classical logic all subsets are decidable, so that $2^X \equiv \text{Pow}(X)$.

Proof By Kripke–Joyal semantics, we have $C \Vdash \neg\neg\phi$ iff for all $D \supseteq C$, not $D \Vdash \neg\phi$, which is the case iff for all $D \supseteq C$, not for all $E \supseteq D$ not $D \Vdash \phi$. If we now use classical meta-logic, we have $\neg\forall_x \neg\phi(x)$ iff $\exists_x \phi(x)$. Then the last condition holds iff for all $D \supseteq C$ there exists $E \supseteq D$ such that $E \Vdash \phi$. \square

The power set $\text{Pow}(\mathbf{n})$ internalizes as the functor

$$\underline{\Omega}^{\mathbf{n}} : C \mapsto \text{Sub}(\mathbf{n}_{\uparrow C}),$$

where the right-hand side is the set of all subfunctors of the functor \mathbf{n} truncated to $\uparrow C \subset \mathcal{C}(A)$. In particular, if E is a maximal commutative C^* -subalgebra of $M_n(\mathbb{C})$, using (7.51) and $\dim(E) = n$ we have

$$\underline{\Omega}^{\mathbf{n}}(E) = \text{Sub}(\mathbf{n}_{\uparrow E}) \cong \mathcal{P}(E) \cong \text{Pow}(\mathbf{n}) \tag{7.54}$$

as (Boolean) lattices in **Sets**. We now show that we may rewrite (7.54) as $E \Vdash \underline{\mathcal{P}}_A \cong \underline{\Omega}^{\mathbf{n}}$. Namely, $\mathcal{P}(E) \cong \text{Pow}(\mathbf{n})$ iff there are $f : \text{Pow}(\mathbf{n}) \rightarrow \mathcal{P}(E)$ and $g : \mathcal{P}(E) \rightarrow \text{Pow}(\mathbf{n})$ such that $f(g(p)) = p$ for all $p \in \mathcal{P}(E)$ and $g(f(Y)) = Y$ for all $Y \in \text{Pow}(\mathbf{n})$. Now $E \Vdash \forall p \in \mathcal{P}. f(g(p)) = p$ iff for all $F \supseteq E$ and p in $\mathcal{P}(E)$, $F \Vdash f(g(p)) = p$. Since E is maximal this is just: for all p in $\mathcal{P}(E)$, $E \Vdash f(g(p)) = p$, which is true. Similarly, $E \Vdash g \circ f = \text{id}$. Lemma 19 then gives $C \Vdash \neg\neg(\underline{\mathcal{P}}_A \cong \underline{\Omega}^{\mathbf{n}})$ for each $C \in \mathcal{C}(A)$, and hence (7.52).

We now show that this implies (7.53). Indeed, to prove $\neg\neg A \rightarrow \neg\neg B$ it suffices to show that $A \rightarrow B$, so that for the purpose of proving (7.53) we may assume $\underline{\mathcal{P}}_A \cong \underline{\Omega}^{\mathbf{n}}$. By Theorem 16, one then has $\mathcal{O}(\underline{\Sigma}_A) \cong \text{Idl}(\underline{\mathcal{P}}_A) \cong \text{Idl}(\underline{\Omega}^{\mathbf{n}}) \cong \underline{\Omega}^{\mathbf{n}}$ (where the last isomorphism is most easily proved internally). \square

8 Partial boolean algebras and Bruns–Lakser completions

This section compares the construction of our (complete) Heyting algebra $\mathcal{O}(\underline{\Sigma}_A)$ of Theorem 17 to some more traditional descriptions of the logical structure of quantum-mechanical systems, notably as far as distributivity and implication are involved. Furthermore, we compare our approach to that of Coecke (2002), which also gives an intuitionistic logic for quantum mechanics.

The projections $\mathcal{P}(A)$ of any von Neumann algebra A form a complete orthomodular lattice (Rédei 1998), and those in a Rickart C^* -algebra form a σ -complete orthomodular lattice (Berberian 1972).⁴² Recall that a lattice \mathcal{L} is called *orthomodular* when it is equipped with a function $\perp : \mathcal{L} \rightarrow \mathcal{L}$ that satisfies:

1. $x^{\perp\perp} = x$;
2. $y^{\perp} \leq x^{\perp}$ when $x \leq y$;
3. $x \wedge x^{\perp} = 0$ and $x \vee x^{\perp} = 1$;
4. $x \vee (x^{\perp} \wedge y) = y$ when $x \leq y$.

⁴² Orthomodularity is not mentioned in Berberian (1972), but follows from the existence of a faithful representation.

The first three requirements are sometimes called (1) “double negation”, (2) “contraposition”, (3) “noncontradiction” and “excluded middle”, but, as argued in Sect. 3, one should refrain from names suggesting a logical interpretation. If these are satisfied, the lattice is called *orthocomplemented*. The requirement (4), called the orthomodular law, is a weakening of distributivity.

Any Boolean algebra is an orthomodular lattice, and any orthomodular lattice is a combination of its Boolean sublattices, as follows (Kochen and Specker 1967; Finch 1969; Kalmbach 1983). A *partial Boolean algebra* is a family $(B_i)_{i \in I}$ of Boolean algebras whose operations coincide on overlaps:

- each B_i has the same least element 0;
- $x \Rightarrow_i y$ if and only if $x \Rightarrow_j y$, when $x, y \in B_i \cap B_j$;
- if $x \Rightarrow_i y$ and $y \Rightarrow_j z$ then there is a $k \in I$ with $x \Rightarrow_k z$;
- $\neg_i x = \neg_j x$ when $x \in B_i \cap B_j$;
- $x \vee_i y = x \vee_j y$ when $x, y \in B_i \cap B_j$;
- if $y \Rightarrow_i \neg_i x$ for some $x, y \in B_i$, and $x \Rightarrow_j z$ and $y \Rightarrow_k z$, then $x, y, z \in B_l$ for some $l \in I$.

These requirements imply that

$$X = \bigcup_{i \in I} B_i \tag{8.55}$$

carries a well-defined amalgamated structure $\vee, \wedge, 0, 1, \perp$, under which it becomes an orthomodular lattice. For example, $x^\perp = \neg_i x$ for $x \in B_i \subseteq X$. Conversely, any orthomodular lattice X is a partial Boolean algebra, in which I is the collection of all bases of X , and B_i is the sublattice of X generated by I . Here, $B \subseteq X$ is called a basis of X when pairs (x, y) of different elements of B are orthogonal, in the sense that $x \leq y^\perp$. The generated sublattices B_i are therefore automatically Boolean. If we order I by inclusion, then $B_i \subseteq B_j$ when $i \leq j$. Thus there is an isomorphism between the categories of orthomodular lattices and partial Boolean algebras.

A similar phenomenon occurs in the Heyting algebra defined by (7.48) when this is complete, which is the case for AW*-algebras and in particular for von Neumann algebras (provided, of course, that we require $\mathcal{C}(A)$ to consist of commutative subalgebras in the same class). Indeed, we can think of $\mathcal{B}(\Sigma_A)$ as an amalgamation of Boolean algebras: just as every B_i in (8.55) is a Boolean algebra, every $\mathcal{P}(C)$ in (7.48) is a Boolean algebra. Hence the fact that the set I in (8.55) is replaced by the partially ordered set $\mathcal{C}(A)$ in (7.48) and the requirement in (7.48) that S be monotone are responsible for making the partial Boolean algebra $\mathcal{O}(\Sigma)$ into a Heyting algebra (which by definition is distributive). Indeed, this construction works more generally, as the following proposition shows. Compare also Graves and Selesnick (1973).

Proposition 20 *Let (I, \leq) be a partially ordered set, and B_i an I -indexed family of complete Boolean algebras such that $B_i \subseteq B_j$ if $i \leq j$. Then*

$$Y = \{f : I \rightarrow \bigcup_{i \in I} B_i \mid \forall i \in I. f(i) \in B_i \text{ and } f \text{ monotone}\} \tag{8.56}$$

is a complete Heyting algebra, with Heyting implication

$$(g \Rightarrow h)(i) = \bigvee \{x \in B_i \mid \forall_{j \geq i}. x \leq g(j)^\perp \vee h(j)\}. \tag{8.57}$$

It is remarkable that the lattice operations on (8.56) are defined pointwise, whereas the Heyting implication (8.57) is not. But this “nonlocality” is necessary, since a pointwise attempt $(g \Rightarrow h)(i) = g(i) \Rightarrow h(i)$ would not provide a monotone function. We will also write (8.57) as

$$(g \Rightarrow h)(i) = \bigwedge_{j \geq i}^{B_i} g(j)^\perp \vee h(j),$$

as in (4.15).

Proof Defining operations pointwise makes Y into a frame. For example, $(f \wedge g)(i) = f(i) \wedge_i g(i)$ is again a well-defined monotone function whose value at i lies in B_i . Hence by a standard construction, Y is a complete Heyting algebra by $g \Rightarrow h = \bigvee \{f \in Y \mid f \wedge g \leq h\}$. We now rewrite this Heyting implication to the form (8.57):

$$\begin{aligned} (g \Rightarrow h)(i) &= \left(\bigvee \{f \in Y \mid f \wedge g \leq h\} \right)(i) \\ &= \bigvee \{f(i) \mid f \in Y, f \wedge g \leq h\} \\ &= \bigvee \{f(i) \mid f \in Y, \forall_{j \in I}. f(j) \wedge g(j) \leq h(j)\} \\ &= \bigvee \{f(i) \mid f \in Y, \forall_{j \in I}. f(j) \leq g(j)^\perp \vee h(j)\} \\ &\stackrel{*}{=} \bigvee \{x \in B_i \mid \forall_{j \geq i}. x \leq g(j)^\perp \vee h(j)\}. \end{aligned}$$

To finish the proof, we establish the marked equation. First, suppose that $f \in Y$ satisfies $f(j) \leq g(j)^\perp \vee h(j)$ for all $j \in I$. Take $x = f(i) \in B_i$. Then for all $j \geq i$ we have $x = f(i) \leq f(j) \leq g(j)^\perp \vee h(j)$. Hence the left-hand side of the marked equation is less than or equal to the right-hand side. Conversely, suppose that $x \in B_i$ satisfies $x \leq g(j)^\perp \vee h(j)$ for all $j \geq i$. Define $f: I \rightarrow \bigcup_{i \in I} B_i$ by $f(j) = x$ if $j \geq i$ and $f(j) = 0$ otherwise. Then f is monotone and $f(i) \in B_i$ for all $i \in I$, whence $f \in Y$. Moreover, $f(j) \leq g(j)^\perp \vee h(j)$ for all $j \in I$. Since $f(i) \leq x$, the right-hand side is less than or equal to the left-hand side. \square

Hence every complete orthomodular lattice gives rise to a complete Heyting algebra. The following proposition shows that the former sits inside the latter.

Proposition 21 *Let (I, \leq) be a partially ordered set. Let $(B_i)_{i \in I}$ be a partial Boolean algebra, and suppose that every B_i is complete with $B_i \subseteq B_j$ for $i \leq j$. Then there is an injection $D: X \rightarrow Y$, where X is the complete orthomodular lattice as defined by (8.55), and Y is the corresponding complete Heyting algebra as defined by (8.56). This injection reflects the order: if $D(x) \leq D(y)$ in Y , then $x \leq y$ in X .*

Proof Define $D(x)(i) = x$ if $x \in B_i$ and $D(x)(i) = 0$ if $x \notin B_i$. Suppose that $D(x) = D(y)$. Then for all $i \in I$ we have $x \in B_i$ iff $y \in B_i$. Since $x \in X = \bigcup_{i \in I} B_i$, there is some $i \in I$ with $x \in B_i$. For that i , we have $x = D(x)(i) = D(y)(i) = y$. Hence D is injective.

If $D(x) \leq D(y)$ for $x, y \in X$, pick $i \in I$ such that $x \in B_i$. We have $x = D(x)(i) \leq D(y)(i) \leq y$. □

The injection $D: X \rightarrow Y$ of the previous proposition is canonical; for example, in the case of Theorem 17 the lattice $Y = \mathcal{B}(\Sigma_A)$ is generated by the elements $D(x)$. We can use this to compare the logical structures of X and Y . Let us start with negation. The Heyting algebra Y of (8.56) has a negation $(\neg f) = (f \Rightarrow 0)$. Explicitly:

$$(\neg f)(i) = \bigwedge_{j \geq i}^{B_i} f(j)^\perp. \tag{8.58}$$

One then readily calculates:

$$D(x^\perp)(i) = \begin{bmatrix} 0 & \text{if } x \notin B_i \\ x^\perp & \text{if } x \in B_i \end{bmatrix}, \quad (\neg(D(x)))(i) = \bigwedge_{j \geq i}^{B_i} \begin{bmatrix} 1 & \text{if } x \notin B_j \\ x^\perp & \text{if } x \in B_j \end{bmatrix}.$$

For $x \notin B_j$ and any $j \geq i$, we have $D(x^\perp)(i) = 0 \neq 1 = (\neg(D(x)))(i)$. This situation already occurs for $A = M_n(\mathbb{C})$ with $I = \mathcal{C}(A)$ and $X = \mathcal{P}(A)$. Hence D does not preserve negation.

We now turn to implication. The Heyting algebra Y of course has a Heyting implication \Rightarrow satisfying $f \wedge g \leq h$ iff $f \leq g \Rightarrow h$. The orthomodular lattice X cannot have an implication, in general. The best possible approximation of the Heyting implication \Rightarrow is the *Sasaki hook* \Rightarrow_S (Chiara and Giuntini 2002, already given in (3.2)). This operation satisfies the adjunction $x \leq y \Rightarrow_S z$ iff $x \wedge y \leq z$ only for y and z that are compatible, in the sense that $y = (y \wedge z^\perp) \vee (y \wedge z)$. In fact, y and z are compatible if and only if they generate a Boolean subalgebra, if and only if $y, z \in B_i$ for some $i \in I$. In that case, the Sasaki hook \Rightarrow_S coincides with the implication \Rightarrow_i of B_i . Hence we find that

$$\begin{aligned} (D(x) \Rightarrow D(y))(i) &= \bigvee \{z \in B_i \mid \forall_{j \geq i}. z \leq D(x)(j) \Rightarrow_j D(y)(j)\} \\ &= \bigvee \{z \in B_i \mid z \leq x \Rightarrow_i y\} \\ &= (x \Rightarrow_S y). \end{aligned}$$

Thus the Sasaki hook $x \Rightarrow_S y$ coincides with the Heyting implication $D(x) \Rightarrow D(y)$ defined by (8.57) at i if x and y are compatible. In particular, we find that \Rightarrow and \Rightarrow_S coincide on $B_i \times B_i$ for $i \in I$; furthermore, this is precisely the case in which the Sasaki hook satisfies the defining adjunction for implications. However, the canonical injection D need not turn Sasaki hooks into implications in general. One finds:

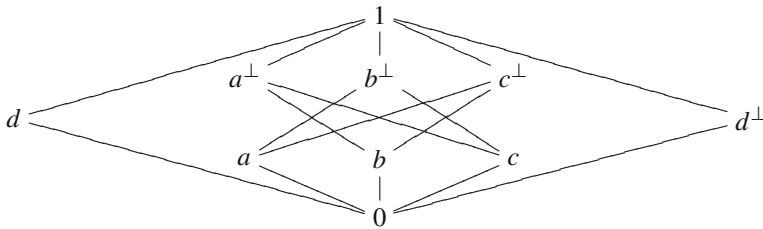
$$D(x \Rightarrow_S y)(i) = \begin{bmatrix} 0 & \text{if } x \notin B_i \\ x^\perp & \text{if } x \in B_i, y \notin B_i \\ x^\perp \vee (x \wedge y) & \text{if } x, y \in B_i \end{bmatrix},$$

$$(D(x) \Rightarrow D(y))(i) = \bigwedge_{j \geq i}^{B_j} \begin{bmatrix} 1 & \text{if } x \notin B_j \\ x^\perp & \text{if } x \in B_j, y \notin B_j \\ x^\perp \vee y & \text{if } x, y \in B_j \end{bmatrix}.$$

So for $x \notin B_j$ and each $j \geq i$, we have $D(x \Rightarrow_S y)(i) = 0 \neq 1 = (D(x) \Rightarrow D(y))(i)$.

Thus the canonical injection D does not preserve negation in general, nor does it turn Sasaki hooks into implications in general. This shows that our intuitionistic quantum logic (8.56) is of a very different nature than the traditional quantum logic (8.55), and argues in favour of the Heyting implication (8.57).

Another approach to intuitionistic quantum logic is to start with a complete lattice and perform the Bruns–Lakser completion (Bruns and Lakser 1970; Coecke 2002; Stubbe 2005). The result is a complete Heyting algebra which contains the original lattice join-densely, in such a way that distributive joins that already exist are preserved. Explicitly, the Bruns–Lakser completion of a lattice L is the collection $DI(L)$ of its distributive ideals, ordered by inclusion. Here, an ideal (lower set) M is called distributive when $(\bigvee M \text{ exists and}) (\bigvee M) \wedge l = \bigvee_{m \in M} (m \wedge l)$ for all $l \in L$. We will now compare this Heyting algebra with the one resulting from Proposition 20, on the example given by the orthomodular lattice X that has the following Hasse diagram.



This orthomodular lattice X contains precisely five Boolean algebras, namely $B_0 = \{0, 1\}$ and $B_i = \{0, 1, i, i^\perp\}$ for $i \in \{a, b, c, d\}$. Hence we take $I = \{0, a, b, c, d\}$ in (8.55), ordered by $i \leq j$ iff $B_i \subseteq B_j$. Hence $i \leq j$ and $i \neq j$ imply $i = 0$, and the monotony requirement $\forall_{i \leq j}. f(i) \leq f(j)$ in (8.56) becomes $\forall_{i \in \{a, b, c, d\}}. f(0) \leq f(i)$. If $f(0) = 0 \in B_0$, this requirement is vacuous. But if $f(0) = 1 \in B_0$, the other values of f are already fixed. Thus one finds

$$Y \cong (B_1 \times B_2 \times B_3 \times B_4) + \{1\},$$

which has 257 elements.

On the other hand, the distributive ideals of X are given by

$$DI(X) = \left\{ \left(\bigcup_{x \in A} \downarrow x \right) \cup \left(\bigcup_{y \in B} \downarrow y \right) \mid A \subseteq \{a, b, c, d, d^\perp\}, B \subseteq \{a^\perp, b^\perp, c^\perp\} \right\} - \{\emptyset\} + \{X\}.$$

In the terminology of Stubbe (2005),

$$\mathcal{J}_{\text{dis}}(x) = \{S \subseteq \downarrow x \mid x \in S\},$$

i.e., the covering relation is the trivial one, and $DI(X)$ is the Alexandrov topology (as a frame/locale). We are unaware of instances of the Bruns–Lakser completion of orthomodular lattices that occur naturally in quantum physics but lead to Heyting algebras different from ideal completions. The set $DI(X)$ has 72 elements.

The canonical injection D of Proposition 21 need not preserve the order, and hence does not satisfy the universal requirement of which the Bruns–Lakser completion is the solution. Therefore, it is unproblematic to conclude that the construction in Proposition 20 differs from the Bruns–Lakser completion.

9 Measures on projections and pairing formula

Theorem 14 in Heunen et al. (2009) gives a bijective correspondence between quasi-states on a C^* -algebra A and internal probability valuations on the Gelfand spectrum $\mathcal{O}(\Sigma_A)$. In case that A is a Rickart C^* -algebra, we can say a bit more. We start by recalling a few definitions, in which $[0, 1]_l$ is the collection of lower reals between 0 and 1, and $[0, 1]$ denotes the Dedekind reals.

Definition 22 1. A *probability measure* on a σ -complete orthomodular lattice \mathcal{L} is a function $\mu : \mathcal{L} \rightarrow [0, 1]$ that on any σ -complete Boolean sublattice of \mathcal{L} restricts to a probability measure (in the traditional sense).

2. A *probability valuation* on a Boolean lattice L is a function $\mu : L \rightarrow [0, 1]_l$ such that
 - (a) $\mu(0) = 0, \mu(1) = 1$;
 - (b) if $x \leq y$, then $\mu(x) \leq \mu(y)$;
 - (c) $\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y)$.
3. A *continuous probability valuation* on a compact regular frame $\mathcal{O}(X)$ is a monotone function $\nu : \mathcal{O}(X) \rightarrow [0, 1]_l$ that satisfies $\nu(1) = 1$ as well as $\nu(U) + \nu(V) = \nu(U \wedge V) + \nu(U \vee V)$ and $\nu(\bigvee_\lambda U_\lambda) = \bigvee_\lambda \nu(U_\lambda)$ for every directed family.

We will apply part 1 of this definition to $\mathcal{L} = \mathcal{P}(A)$ in **Sets**; see Proposition 5 for its σ -completeness. Part 2 will be applied internally to $L = \underline{\mathcal{P}}_A$ in $\mathcal{T}(A)$ (i.e., the functor $C \mapsto \mathcal{P}(C)$). As to part 3, if X is a compact Hausdorff space in **Sets**, a continuous probability valuation on $\mathcal{O}(X)$ is essentially the same thing as a regular probability measure on X . We will actually apply the definition internally to the frame $\mathcal{O}(\Sigma_A)$ in $\mathcal{T}(A)$.

Theorem 23 *Let A be a Rickart C^* -algebra. There is a bijective correspondence between:*

1. *quasi-states on A ;*
2. *probability measures on $\mathcal{P}(A)$;*
3. *probability valuations on the Boolean lattice $\underline{\mathcal{P}}_A$ in $\mathcal{T}(A)$;*
4. *continuous probability valuations on the Gelfand spectrum $\mathcal{O}(\underline{\Sigma}_A)$ in $\mathcal{T}(A)$.*

Proof We include the first item only for completeness; the equivalence between 1 and 4 is contained in Theorem 14 in Heunen et al. (2009). The equivalence between 3 and 4 follows from Theorem 16 and the observation in Coquand and Spitters (2009b, §3.3) that valuations on a compact regular frame are determined by their behaviour on a generating lattice; indeed, if a frame $\mathcal{O}(X)$ is generated by L , then a probability measure μ on L yields a continuous probability valuation ν on $\mathcal{O}(X)$ by $\nu(U) = \sup\{\mu(u) \mid u \in U\}$, where $U \subset L$ is regarded as an element of $\mathcal{O}(X)$.

To prove the equivalence between 2 and 3 we use the following lemma, which holds in the internal logic of any topos.⁴³

Lemma 24 *Let L be a Boolean algebra and μ a valuation on L . Then $\mu(x)$ is a Dedekind real for every $x \in L$.*

Proof Let $s + \epsilon < t$ in \mathbb{Q} . We need to prove that $s < \mu(x)$ or $\mu(x) \leq t$. The last statement is defined as $\mu(y) > 1 - t$ for some y such that $x \wedge y = 0$. We choose $y = x^\perp$, the complement of x . Now,

$$1 - \epsilon < \mu(\top) \quad \text{and} \quad s + \epsilon - t < 0 = \mu(\perp),$$

equivalently,

$$1 - \epsilon \leq \mu(x \vee x^\perp) \quad \text{and} \quad s + \epsilon - t < \mu(x \wedge x^\perp).$$

By the modular law for valuations and Lemma 2.2 in Coquand and Spitters (2009b), if $p + q < \mu(z \wedge w) + \mu(z \vee w)$, then $p < \mu(w)$ or $q < \mu(z)$. Choosing $z = x$, $w = x^\perp$, $p = s$, $q = 1 - t$ we have

$$s < \mu(x) \quad \text{or} \quad 1 - t < \mu(x^\perp).$$

That is,

$$s < \mu(x) \quad \text{or} \quad \mu(x) \leq t.$$

It follows that $\mu(x)$ is a Dedekind real. □

Since the Dedekind reals in $\mathcal{T}(A)$ are internalized as the constant functor $\mathbb{R} : C \mapsto \mathbb{R}$ (as opposed to the lower reals), according to this lemma an internal probability

⁴³ Classically, this lemma is trivial as the lower reals and the Dedekind reals coincide.

valuation $v : \underline{\mathcal{P}}_{B(H)} \rightarrow [0, 1]$ is defined by its components (as a natural transformation) $v_C : \mathcal{P}(C) \rightarrow [0, 1]$. By naturality, for $p \in \mathcal{P}(C)$, the number $v_C(p) \equiv \mu(p)$ is independent of C , from which the equivalence between 2 and 3 in Theorem 23 is immediate. \square

Finally, we justify the formula (4.19) in case $A = B(H)$ for some Hilbert space H , by identifying $V_\psi(S)$ with the nonprobabilistic state-proposition pairing $\langle S, \psi \rangle$ defined in Heunen et al. (2009); see Sect. 6 of that paper for the background of the following computation. By definition, $C \in \langle S, \psi \rangle$ iff $C \Vdash v^\psi(S) = 1$, where v^ψ is the probability valuation on $\mathcal{O}(\underline{\Sigma}_{B(H)})$ defined by a normal state ψ on $B(H)$, seen as a probability measure on $\mathcal{P}(B(H))$. Using (7.50), we describe $S \in \mathcal{O}(\underline{\Sigma}_{B(H)})$ as a subfunctor \underline{U} of $\underline{\mathcal{P}}_A$, which (lying in the set of ideals) is locally closed under \vee . Then the following are equivalent:

$$\begin{aligned}
 & C \Vdash v^\psi(\underline{U}) = 1 \\
 & C \Vdash \forall q < 1 \cdot v^\psi(\underline{U}) > q \\
 & \text{for all } D \supseteq C \text{ and } q < 1, D \Vdash v^\psi(\underline{U}) > q \\
 & \text{for all } D \supseteq C \text{ and } q < 1, D \Vdash \exists u \in \underline{U} \cdot v^\psi(u) > q \\
 & \text{for all } D \supseteq C \text{ and } q < 1, \text{ there exists } u \in \underline{U}(D) \text{ s.t. } D \Vdash v^\psi(u) > q \\
 & \text{for all } q < 1, \text{ there exists } u \in \underline{U}(C) \text{ s.t. } v^\psi(u) > q \\
 & \sup_{u \in \underline{U}(C)} v^\psi(u) = 1 \\
 & v^\psi(\underline{U}(C)) = 1.
 \end{aligned}$$

Now $\underline{U}(C)$ is a collection of projections. By classical meta-logic we can take its supremum $p := \bigvee \underline{U}(C)$. Then $\psi(p) = v^\psi(p) = 1$, which proves (4.19).

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