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Finite-time blow-up and variational approximation scheme for a  
Wigner-Fokker-Planck equation with a nonlocal perturbation

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# Finite-time blow-up and variational approximation scheme for a Wigner-Fokker-Planck equation with a nonlocal perturbation

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## Abstract

The paper is concerned with analysis of a Wigner-Fokker-Planck equation with a nonlocal perturbation that consists both conservative and dissipative as well as nonlinear non-local effects. We first show that the system has a finite-time blow-up phenomenon. We then introduce a variational steepest descent approximation scheme for the system. At each step, the scheme minimizes an energy functional with respect to the Kantorovich functional associated with a certain cost function which is inspired by the rate functional in the Freidlin-Wentzell theory of large deviations for the underlying stochastic system.

**Key words and phrases.** Wigner-Fokker-Planck equation, Variational principle, conservative-dissipative systems.

## 1 Introduction

### 1.1 Motivation

This paper is concerned with analysis of the following equation

$$\begin{cases} \partial_t \rho(t, x) = -\alpha \operatorname{div}(\rho(t, x) \mathbf{J} \nabla H(x)) + \operatorname{div}(\rho(t, x) (\nabla H(x) - \chi \nabla c(t, x))) + \Delta \rho(t, x), & (1a) \\ c(t, x) = -\frac{1}{2\pi d} \int_{\mathbf{R}^{2d}} \log |y - x| \rho(t, y) dy = -\frac{1}{2\pi d} \log |\cdot - x| * \rho(t, x), & (1b) \\ \rho(0, x) = \rho_0(x). & (1c) \end{cases}$$

In the equation above, the unknown is  $\rho: (t, x) \in \mathbf{R}^+ \times \mathbf{R}^{2d} \mapsto \mathbf{R}^+$ .  $\nabla$ ,  $\operatorname{div}$  and  $\Delta$  are respectively the gradient, divergence and Laplacian operators with respect to spatial variable. Finally,

$$H(x) = \frac{1}{2}|x|^2; \quad \mathbf{J} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}; \quad \alpha, \chi \geq 0 \text{ are constants.}$$

The system (1) covers and is motivated by the following relevant well-known systems.

*The Wigner-Fokker-Planck (WFP) equation and the Wigner-Poisson-Fokker-Planck equation (WPFP):* When  $\chi = 0$ , (1) is the Wigner-Fokker-Planck (WFP) equation, which is obtained as the phase-representation from the master equation for an open quantum system [SS87]. When  $\chi > 0$ , (1) can be seen as a simplified model for the Wigner-Poisson Fokker Planck equation where

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the non-local operator is replaced by convolution term  $\nabla c(t, x)$  [ADM07, AGG<sup>+</sup>12]. The WFP and the WFPF equations are widely used in semiconductor device modelling, quantum optics and quantum Brownian motion [AJ06].

*The Patlak-Keller-Segel (PKS) equation:* When  $\alpha = 0$ ,  $d = 1$ , (1) is the two dimensional Patlak-Keller-Segel equation. The PKS equation plays an important role in mathematical biology and is used to model the collective motion of cells which are attracted by a self-emitted chemical substance. The literature for the PKS is huge, e.g., [Fat13, BDP06, BCC08, DS09] and references therein. In this model,  $\rho(t, x)$  is the cells density,  $c(t, x)$  is the concentration of the chemo-attractant, and  $\chi$  is the *sensitivity* of the bacteria to the chemo-attractant. Only for  $d = 1$ ,  $c(t, x)$  solves the Poisson equation  $\partial_t c = -\Delta \rho$  and the logarithmic kernel is the Poisson kernel. However, the logarithmic kernel also has been used for any dimension in some papers [CPSt07, BCC08]. Like these papers, here we do not restrict to any specific dimension.

*A mixture of conservative-dissipative effects:* (1) is also a model of a system that consists of both conservative and dissipative effects. The conservative, Hamiltonian part is represented by the term  $-\alpha \operatorname{div}(\rho(t, x)\mathbf{J}\nabla H(x))$ , which is a classical Hamiltonian system with the Hamiltonian  $\alpha H(x)$ . While the dissipative part is depicted in the last three terms in the right hand side of (1a). Systems of this type have recently received a lot of attention [Ött05, FGY11, FSS13, DPZ12, DPZ13].

## 1.2 Aim and main results

As mentioned in the previous section, the system (1) consists of different effects such as the Hamiltonian, the dissipative and the nonlinear non-local effects. The aim of the paper is to provide an analysis of the system (1) in order to understand the behavior of these effects.

We first show that the system has a blow-up phenomenon: smooth solutions may only exist for finite time. This phenomenon depends strongly on the sensitivity  $\chi$ : there exists a critical value  $\chi_0 = \chi_0(d)$  such that if  $\chi < \chi_0$ , then solutions blow-up and if  $\chi > \chi_0$ , then solutions exist globally. This phenomenon for the PKS equation, i.e. when  $\alpha = 0$ , is well-known and has been studied intensively by many authors in the last two decades, see e.g., [Fat13] and references therein. We demonstrate that for  $\alpha > 0$ , the Hamiltonian part does not effect the blow-up phenomenon. Using the so-called moment method, we obtain the same result as the case  $\alpha = 0$  since all calculations involving  $\alpha$  will be cancelled due to the symplectic structure of  $\mathbf{J}$ , see Section 2.

We then introduce a variational scheme to approximate solutions to the system (1) in the case  $\chi < \chi_0$ . The globally existence problem for the PKS system has been addressed by a variety of methods. In particular, in [BCC08], the authors constructed a variational approximation scheme for the PKS system using the so-called Wasserstein gradient flow structure. More precisely, they showed that weak solutions to the PKS system can be approximated by the time-discrete sequence  $\rho_k$  defined recursively by

$$\rho_k \in \operatorname{argmin}_{\rho} K_h(\rho, \rho_{k-1}), \quad K_h(\rho, \rho_{k-1}) := \frac{1}{2h} d(\rho, \rho_{k-1})^2 + \mathcal{F}(\rho). \quad (2)$$

Here  $\mathcal{F}$  is an energy functional given by

$$\mathcal{F}(\rho) := \int_{\mathbf{R}^{2d}} \rho(x) \log \rho(x) dx + \frac{1}{2} \int_{\mathbf{R}^{2d}} |x|^2 \rho(x) dx + \frac{\chi}{4\pi d} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \log |y - x| \rho(x) \rho(y) dx dy, \quad (3)$$

and  $d$  is the Wasserstein distance between two probability measures  $\rho_0(x)dx$  and  $\rho(y)dy$  with finite second moment,

$$d(\rho_0, \rho)^2 := \inf_{P \in \Gamma(\rho_0, \rho)} \int_{\mathbf{R}^{4d}} |x - y|^2 P(dx dy), \quad (4)$$

where  $\Gamma(\rho_0, \rho)$  is the set of all probability measures on  $\mathbf{R}^{2d} \times \mathbf{R}^{2d}$  with marginals  $\rho_0$  and  $\rho$ ,

$$\Gamma(\rho_0, \rho) = \{P \in \mathcal{P}(\mathbf{R}^{2d} \times \mathbf{R}^{2d}) : P(A \times \mathbf{R}^{2d}) = \rho_0(A), P(\mathbf{R}^{2d} \times A) = \rho(A) \text{ for all Borel subsets } A \subset \mathbf{R}^{2d}\}. \quad (5)$$

The Wasserstein distance is an instance of a Kantorovich functional associated with the cost function  $c(x, y) = |x - y|^2$ , see [Vil03]. The scheme (2), which is basically an implicit Euler scheme in the space of probability measure, has been introduced by Jordan-Kinderlehrer-Otto in [JKO98]. It has become so famous that now it is often called JKO-scheme. We review this shortly in Section 3.

However, the JKO-scheme can not be used for the system (1) because it is not a gradient flow but a mixture of both conservative and dissipative effects as discussed in the previous subsection. In this paper, we extend the JKO-scheme to the system (1) by introducing a different optimal transport cost functional  $W_h(\rho_0, \rho)$  instead of the Wasserstein metric.  $W_h(\rho_0, \rho)$  is also a Kantorovich functional associated with a certain cost function  $C_h(x, y)$  which is inspired by the rate functional in the Freidlin-Wentzell theory of large deviations for the underlying stochastic system of the Hamiltonian part, see (16). The energy functional is again  $\mathcal{F}(\rho)$ . Thus we introduce the following variational scheme

$$\rho_k \in \underset{\rho}{\operatorname{argmin}} K_h(\rho, \rho_{k-1}), \quad K_h(\rho, \rho_{k-1}) := \frac{1}{2h} W_h(\rho, \rho_{k-1})^2 + \mathcal{F}(\rho). \quad (6)$$

The main result of the paper, Theorem 3.6, shows that solutions of this variational scheme converge to a weak solution to the system (1). The proof of the main theorem follows the procedure for the JKO-scheme, see e.g. [JKO98, Hua00, BCC08]. However, several significant technical improvements need to be developed in dealing with convergence of our scheme.

### 1.3 Organization of the paper

The rest of the paper is organized as follows. In Section 2, we study the blow-up phenomenon. We then introduce our variational scheme and main result in Section 3. The next Sections are devoted to the proof of the main theorem. We first present some properties of the cost functional in Section 4. The Euler-Lagrange equation for minimizers of the Scheme is established in Section 5. In Section 6, we provide a prior estimate. Finally, the convergence of the scheme is carried out in Section 7.

## 2 Blow up in finite time

It is well-known that due to the nonlinear non-local term, the PKS system, i.e. when  $\alpha = 0$ , has a blow-up phenomenon: solutions may only exist for finite time. In this section, we show that (1) also has a similar blow-up phenomenon. It is interesting that the Hamiltonian part does not play a role here and all the calculations involving it will be cancelled after the first line. Thus the critical value  $\chi_0$  is the same as the PKS system. We also use the classical method in this topic, namely the moment method [Nag00, BDP06].

We first notice that solutions to (1) satisfy the conservation of mass at least for smooth and sufficiently decay solutions

$$m := \int_{\mathbf{R}^{2d}} \rho_0(x) dx = \int_{\mathbf{R}^{2d}} \rho(t, x) dx \quad \text{for any } t > 0. \quad (7)$$

**Lemma 2.1.** *Let  $\rho$  be a smooth solution to the system (1) with initial value  $\rho_0(x)$  such that  $\int |x|^2 \rho_0 dx < \infty$ . Suppose that mass is conserved and let  $[0, T^*)$  be the maximal interval of existence. If  $\chi > \chi_0 := 8\pi d^2$ , then*

$$T^* \leq \frac{2\pi d}{\chi - 8\pi d^2} \int |x|^2 \rho_0(x) dx.$$

*Proof.* Without loss of generality we assume that  $m = 1$ . We formally compute the following identity

$$\frac{d}{dt} \int |x|^2 \rho(t, x) dx + 2 \int |x|^2 \rho(t, x) dx = \left(4d - \frac{\chi}{2\pi d}\right). \quad (8)$$

Indeed,

$$\begin{aligned} \frac{d}{dt} \int |x|^2 \rho(t, x) dx &= \int |x|^2 \partial_t \rho(t, x) dx \\ &= \int |x|^2 [-\alpha \operatorname{div}(\rho(t, x) \mathbf{J} \nabla H(x)) + \operatorname{div}(\rho(t, x) (\nabla H(x) - \chi \nabla c(t, x))) + \Delta \rho(t, x)] dx \\ &= 2 \int [\alpha \mathbf{J} \nabla H(x) \cdot x - \nabla H(x) \cdot x + \chi \nabla c(t, x) \cdot x + \frac{1}{2} \Delta(|x|^2)] \rho(t, x) dx \\ &= -2 \int |x|^2 \rho(t, x) dx + \left(4d - \frac{\chi}{2\pi d}\right). \end{aligned}$$

Note that to obtain the last equality we use the fact that  $\mathbf{J} \nabla H(x) \cdot x = 0$  due to the symplectic property of  $\mathbf{J}$  and use (11) with  $\phi = |x|^2$ . The above computation can be made rigorous using the approximation argument, see [Nag00, BDP06]. If  $\chi > 8\pi d^2$ , then the right hand side of (8) will be negative when  $t > T^*$ , where

$$T^* = \frac{2\pi d}{\chi - 8\pi d^2} \int |x|^2 \rho_0(x) dx.$$

As a consequence, all the solutions with finite second moment cannot be global in time.  $\square$

**Remark 2.2.** It is worth pointing out that the blow-up phenomenon does not depend on the Hamiltonian effect.  $\square$

In the next section, we construct a variational approximation scheme to the system (1) in the case  $\chi < 8\pi d^2$ .

### 3 Variational scheme for $\chi < 8\pi d^2$

In this section, we introduce a variational scheme to approximate solutions of (1) in a weak formulation which we define in the next subsection. As a consequence, the scheme shows that weak solutions to the system (1) exist globally. From now on, without loss of generality, we assume  $\alpha = 1$ .

#### 3.1 Weak formulation

A function  $\rho \in L^1(\mathbf{R}^+ \times \mathbf{R}^{2d})$  is called a weak solution of the system (1) with initial datum  $\rho_0$  if it satisfies the following weak formulation of (1) for all test functions  $\phi \in C_0^\infty((0, T) \times \mathbf{R}^{2d})$

$$\begin{aligned} & \int_0^\infty \int_{\mathbf{R}^{2d}} \partial_t \phi(t, x) \rho(t, x) dx dt - \int_0^\infty \int_{\mathbf{R}^{2d}} \mathbf{J} \nabla H(x) \cdot \nabla \phi(s, x) \rho(s, x) dx ds \\ & - \int_0^\infty \int_{\mathbf{R}^{2d}} \Delta \phi(s, x) \rho(s, x) dx + \int_0^\infty \int_{\mathbf{R}^{2d}} \nabla H(x) \nabla \phi(s, x) \rho(s, x) dx ds \\ & + \frac{\chi}{4\pi d} \int_0^\infty \int_{\mathbf{R}^{4d}} [\nabla \phi(x) - \nabla \phi(y)] \cdot \frac{x - y}{|x - y|^2} \rho(t, x) \rho(t, y) dx dy \\ & = \int_{\mathbf{R}^{2d}} \phi(0, x) \rho_0(x) dx. \end{aligned} \quad (9)$$

**Remark 3.1.** To obtain the weak formulation above, we multiply the equation (1) with the test function  $\phi$ , integrate it and use integration by parts. The linear part is straightforward. For the nonlinear term, we use a technique introduced in [SS02] (see also [BDP06, BCC08]) that use the symmetry of the interaction kernel as follows.

$$\begin{aligned} \int_{\mathbf{R}^{2d}} \chi \operatorname{div}(\rho \nabla c) \phi \, dx &= -\chi \int_{\mathbf{R}^{2d}} \rho \nabla c \cdot \nabla \phi \, dx \\ &= \frac{\chi}{2\pi d} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \frac{(x-y) \cdot \nabla \phi(x)}{|x-y|^2} \rho(x) \rho(y) \, dx \, dy \end{aligned} \quad (10)$$

$$\begin{aligned} &= -\frac{\chi}{2\pi d} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \frac{(x-y) \cdot \nabla \phi(y)}{|x-y|^2} \rho(x) \rho(y) \, dx \, dy \\ &= \frac{\chi}{4\pi d} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \frac{(x-y)}{|x-y|^2} \cdot (\nabla \phi(x) - \nabla \phi(y)) \rho(x) \rho(y) \, dx \, dy. \end{aligned} \quad (11)$$

In the weak formulation we used (11) instead of (10) as in the traditional derivation. It has an advantage since the term  $\frac{(x-y)}{|x-y|^2} \cdot (\nabla \phi(x) - \nabla \phi(y))$  is continuous and hence it is possible to handle the nonlinear term.  $\square$

## 3.2 Approximation scheme

Let us first recall the implicit Euler scheme to approximate solutions of a gradient flow in an Euclidean space  $\mathbf{R}^n$ :

$$\dot{x}(t) = -\nabla \mathcal{E}(x(t)), \quad x(0) = x_0, \quad (12)$$

where  $\mathcal{E} \in \mathcal{C}^1(\mathbf{R}^n)$  is a given function. The implicit Euler scheme for (12) is as follows. Let  $h > 0$  be a time step. Suppose we know  $x_k$ , then  $x_{k+1}$  solves

$$\frac{x_{k+1} - x_k}{h} = -\nabla \mathcal{E}(x_{k+1}).$$

$x_{k+1}$  can be found by solving the following minimization problem

$$\min_{x \in \mathbf{R}^n} \frac{1}{2h} \|x - x_k\|^2 + \mathcal{E}(x), \quad (13)$$

where  $\|\cdot\|$  is the Euclidean metric on  $\mathbf{R}^n$ .

In the seminal paper [JKO98], the authors generalised (13) to gradient flows on the space of probability measures of finite second moment using the so-called Wasserstein metric, see (2) and (5). The JKO-scheme has become a cornerstone in the field since it has connected many branches of mathematics together and has opened new research directions. In particular, the theory of Wasserstein gradient flows has been developing tremendously in the last decade, see e.g. [AGS08] for an exposition. Recently, many authors have extended the JKO-scheme to systems that are not gradient flows but are combination of both conservative and dissipative effects. The major difficulty in doing so is the introduction of an appropriate cost functional. Only a few results has been obtained in this direction, see [Hua00, FGY11, DPZ12] for different systems. A common feature in these papers is that it is not necessary to have a metric but only a cost of optimal transport type. Moreover, recent results on the connection between Large deviation principle and generalized gradient flows [DG89, ADPZ11, DLR13, DPZ12, DPZ13] reveal a method to derive a cost functional. Specifically, in [DPZ12], to construct an approximation scheme for the Kramers equation, the authors introduced a cost functional which is inspired by the rate functional in the Freidlin-Wentzell theory of large deviations for the underlying stochastic system. Their idea is to split the conservative and the dissipative part: the former is coupled to the cost functional, while the later is displayed by a free energy functional.

We apply their method to derive a cost functional to the system (1). More precisely, we consider a small random perturbation of the Hamiltonian system

$$dX_\epsilon(t) = \mathbf{J} \nabla H(X_\epsilon(t)) \, dt + \epsilon \, dW(t), \quad (14)$$



where  $W(t)$  is the standard Wiener process.

By Freidlin-Wentzell theorem (e.g. [DZ87, Th. 5.6.3]),  $\{X_\epsilon\}$  satisfies a large-deviation principle as  $\epsilon \rightarrow 0$  with a rate functional  $I: C([0, h], \mathbf{R}^{2d}) \rightarrow \mathbf{R} \cup \{+\infty\}$  given by

$$I(\xi(\cdot)) = \frac{1}{4} \int_0^h |\dot{\xi}(t) - \mathbf{J}\nabla H(\xi(t))|^2 dt.$$

Roughly speaking, the rate functional  $I$  measures the deviation of a curve from the Hamiltonian flow:  $I(\xi(\cdot))$  is always non-negative and equals to 0 if and only if  $\xi(\cdot)$  is the solution to the Hamiltonian system.

This rate functional motivates us to define the following cost functional

$$C_h(x, x') := h \inf \left\{ \int_0^h |\dot{\xi}(t) - \mathbf{J}\nabla H(\xi(t))|^2 dt : \xi \in C^1([0, h], \mathbf{R}^{2d}) \right. \\ \left. \text{such that } \xi(0) = x, \quad \xi(h) = x' \right\}. \quad (15)$$

From this formula,  $C_h(x, x') = 0$  if and only if there exists a curve  $\xi \in C^1([0, h], \mathbf{R}^{2d})$  that is a solution to the classical Hamiltonian system  $\dot{\xi}(t) - \mathbf{J}\nabla H(\xi(t)) = 0$  and with endpoints conditions  $\xi(0) = x, \xi(h) = x'$ .

In other words, given two point  $x, x' \in \mathbf{R}^{2d}$ , physically,  $C_h(x, x')$  measures the deviation from the Hamiltonian path that connects them.

Given  $\rho(x), \nu(x') \in \mathcal{P}_2(\mathbf{R}^{2d})$ , we define an optimal transport between them similarly as in the Wasserstein distance

$$W_h(\rho, \nu) = \inf_{\gamma \in \Gamma(\rho, \nu)} \int_{\mathbf{R}^{4d}} C_h(x, x') \gamma(dxdx'), \quad (16)$$

where  $\Gamma(\rho, \nu)$  as in (5) is the set of all probability measures on  $\mathbf{R}^{4d}$  having the first marginal  $\rho$  and the second marginal  $\nu$ . The optimal measure  $\gamma$  in (16), if exists, is called the optimal map. See [Vil03] for a thorough exposition on the theory of optimal transport.

Next, we recall the energy functional in (3)

$$\mathcal{F}(\rho) = \int_{\mathbf{R}^{2d}} \rho(x) \log \rho(x) dx + \frac{1}{2} \int_{\mathbf{R}^{2d}} |x|^2 \rho(x) dx + \frac{\chi}{4\pi d} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \log |y - x| \rho(x) \rho(y) dx dy, \quad (17)$$

and define a set, which will be the admissible set

$$\mathcal{A} := \left\{ \rho \in L^1_+(\mathbf{R}^{2d}) : \int_{\mathbf{R}^{2d}} \rho(x) dx = 1, \int_{\mathbf{R}^{2d}} |x|^2 \rho(x) dx < \infty, \int_{\mathbf{R}^{2d}} \rho(x) |\log \rho(x)| dx < \infty \right\}. \quad (18)$$

We now state a couple of Lemmas whose proofs are omitted and can be found in the references. The first one is about the bounded-ness of  $\mathcal{F}$  on  $\mathcal{A}$ .

**Lemma 3.2.** [BCC08, Lemma 2.1]  $\mathcal{F}$  is bounded from below on  $\mathcal{A}$  if and only if  $\chi \leq 8\pi d^2$ . In addition, if  $\chi < 8\pi d^2$ , on every sub-level set  $\{\mathcal{F} \leq C\}$  the following estimate hold

$$(a) \int_{\mathbf{R}^{2d}} \rho |\log \rho| \leq C_1 \mathcal{F}(\rho) + C_2 \leq C; \quad (b) \int_{\mathbf{R}^{2d}} |x|^2 \rho \leq C_1 \mathcal{F}(\rho) + C_2 \leq C,$$

where  $C_1, C_2$  are constants. Thus every sub-level set  $\{\mathcal{F} \leq C\}$  is weakly relatively compact in  $L^1(\mathbf{R}^{2d})$ .

We notice that conditions (a) and (b) above are to prevent the concentration and vanishing phenomena that are required to obtain a relative compact sequence in the weak topology of  $L^1$  [Bre11]. The second Lemma states that the inf in (16) is achieved. It is a basic result in the theory of optimal transportation, see e.g., [Vil03, Theorem 1.3].

**Lemma 3.3.** *Let  $\rho_0, \rho \in \mathcal{A}$  be given. There exists a unique optimal plan  $P_{\text{opt}}(dxdx') \in \Gamma(\rho_0, \rho)$  such that*

$$W_h(\rho_0, \rho) = \int_{\mathbf{R}^{4d}} C_h(x, x') P_{\text{opt}}(dxdx'). \quad (19)$$

Having the above two Lemmas, we can study a minimization of the form (13) where the Euclidean metric is replaced by the optimal transport cost  $W_h$ . The next Lemma now becomes classical, see e.g., [JKO98, Proposition 4.1], [Hua00, Lemma 4.2]), and [BCC08, Lemma 3.1].

**Lemma 3.4.** *Let  $\rho_0 \in \mathcal{A}$  be given. If  $h$  is small enough, then the minimization problem*

$$\min_{\rho \in \mathcal{A}} \frac{1}{2h} W_h(\rho_0, \rho) + \mathcal{F}(\rho), \quad (20)$$

*has a solution.*

We are now ready to introduce the approximation scheme of this paper.

**Approximation scheme.** We start with  $\rho_h^0(x) = \rho_0(x)$ . Given  $\rho_{k-1}^h$ , define  $\rho_k^h$  as a solution of the minimization problem

$$\min_{\rho \in \mathcal{A}} \frac{1}{2h} W_h(\rho_{k-1}^h, \rho) + \mathcal{F}(\rho). \quad (21)$$

**Remark 3.5.** Since the functional  $\mathcal{F}(\rho)$  is not convex, we do not know whether (20) has a unique solution or not. Thus, in the discrete scheme (21),  $\rho_k^h$  is chosen as any element that minimizes  $\frac{1}{2h} W_h(\rho_{k-1}^h, \rho) + \mathcal{F}(\rho)$ . Uniqueness of solutions in the subcritical case is still an open problem [Pou02, BCC08].  $\square$

### 3.3 Main result

The main result of the paper is the following.

**Theorem 3.6.** *Let  $\rho_0 \in \mathcal{A}$  satisfy  $\mathcal{F}(\rho_0) < \infty$ . For any  $h > 0$  sufficiently small, let  $\{\rho_k^h\}$  be a sequence of minimizers of the Scheme (21). For any  $t \geq 0$ , we define the piecewise-constant time interpolation*

$$\rho^h(t, x) = \rho_k^h(x) \quad \text{for } (k-1)h < t \leq kh. \quad (22)$$

*Then for any  $T > 0$ ,*

$$\rho^h \rightharpoonup \rho \quad \text{weakly in } L^1((0, T) \times \mathbf{R}^{2d}) \quad \text{as } h \rightarrow 0, \quad (23)$$

*where  $\rho$  is a weak solution of (1) with initial value  $\rho_0$ .*

*Outline of the proof.* The proof follows the procedure of [JKO98] (see also [Hua00, BCC08, DPZ12]) and is divided into three main steps, which are carried out in Sections 5, 6, and 7: establish the Euler-Lagrange equation for the minimizers, then estimate the second moments and entropy functionals, and finally pass to the limit  $h \rightarrow 0$ . Some technical improvements need to be developed in order to deal with our Scheme. We start in Section 4 with some properties of the cost functional.  $\square$

## 4 Properties of the cost function

In this section, we present some properties of the cost functional that are necessary for the next Sections.

**Lemma 4.1.** 1. It holds that

$$|x' - x|^2 \leq C(C_h + h^2|x|^2 + h^2|x'|^2). \quad (24)$$

2. The derivative of the cost function is

$$\nabla_{x'} C_h(x, x') = 2[(x' - x) - h\mathbf{J}\nabla H(x)] + 2h\alpha_h(x, x'), \quad (25)$$

where there exists  $C > 0$  such that

$$\alpha_h(x, x') \leq Ch(|x|^2 + |x'|^2 + 1). \quad (26)$$

*Proof.* For the length of this proof, we fix  $x, x'$  and  $h$ , and we abbreviate

$$N := |x|^2 + |x'|^2.$$

Set

$$\bar{C}_h(x, x') = h \inf \left\{ \int_0^h |\dot{\xi}(t)|^2 dt : \xi \in C^1([0, h], \mathbf{R}^{2d}) \text{ such that } \xi(0) = x, \xi(h) = x' \right\},$$

Let  $\bar{\xi}$  and  $\xi$ , respectively, be the optimal curves in the definition of  $\bar{C}_h$  and of  $C_h$ . The curve  $\bar{\xi}$  satisfies  $\ddot{\bar{\xi}} = 0$ , and hence it is a straight line

$$\bar{\xi}(t) = x + \frac{1}{h}(x' - x)t, \quad (27)$$

and we obtain

$$\bar{C}_h(x, x') = |x' - x|^2.$$

$\bar{C}_h(x, x')$  is nothing but the Wasserstein cost functional. Explicit calculations give

$$\|\dot{\bar{\xi}}\|_1 = |x' - x|, \quad \|\bar{\xi}\|_1 \leq Ch(|x| + |x'|), \quad \|\dot{\bar{\xi}}\|_\infty^2 \leq CN, \quad \|\dot{\bar{\xi}}\|_2^2 \leq Ch^{-1}N, \quad \|\bar{\xi}\|_2^2 \leq ChN. \quad (28)$$

We begin with the proof of (24).

Since  $\bar{\xi}$  is optimal in  $\bar{C}_h$ , we have

$$\begin{aligned} |x' - x|^2 = \bar{C}_h &= h\|\dot{\bar{\xi}}\|_2^2 \leq h\|\dot{\xi}\|_2^2 \\ &\leq Ch(\|\dot{\xi} - f(\xi)\|_2^2 + \|f(\xi)\|_2^2) \\ &\leq C(C_h + h\|\xi\|_2^2) \\ &\leq C(C_h + h^2N). \end{aligned}$$

Hence (24) is established. We continue with the proof of (25) and (26).

The curve  $\xi(t)$  satisfies the equation

$$\mathcal{N}(\xi)(t) := \ddot{\xi}(t) - \nabla f(\xi(t)) \cdot f(\xi(t)) = 0, \quad (29)$$

where  $f(\xi(t)) = \mathbf{J}\nabla H(\xi(t))$ . Let  $\eta \in C([0, h]; \mathbf{R}^{2d})$  satisfy  $\eta(0) = 0$ . Then

$$\begin{aligned} \frac{d}{d\epsilon} I(\xi + \epsilon\eta) \Big|_{\epsilon=0} &= 2h \int_0^h (\dot{\xi}(t) - f(\xi(t))) \cdot (\dot{\eta}(t) - \nabla f(\xi(t)) \cdot \eta(t)) dt \\ &= 2h\eta(h)(\dot{\xi}(h) - f(\xi(h))) - 2h \int_0^h \mathcal{N}(\xi)(t) \cdot \eta(t) dt \\ &= 2h\eta(h)(\dot{\xi}(h) - f(\xi(h))). \end{aligned}$$

This expression is equal to

$$\nabla_{x'} C_h(x, x') \cdot \eta(h),$$

which allows us to identify the derivative of  $C_h(x, x')$  in terms of  $\xi$ :

$$\nabla_{x'} C_h(x, x') = 2h(\dot{\xi}(h) - f(\xi(h))) = 2h(\dot{\xi}(h) - \mathbf{J}\nabla H(\xi(h))).$$

Setting  $u = \xi - \bar{\xi}$ , we rewrite these in terms of  $u$  :

$$\begin{aligned} \nabla_{x'} C_h(x, x') &= 2[h\dot{\bar{\xi}}(h) - h\mathbf{J}\nabla H(\xi(h))] + 2h\dot{u}(h) \\ &= 2[(x' - x) - h\mathbf{J}\nabla H(\xi(h))] + 2h\dot{u}(h). \end{aligned}$$

Hence (25) holds with  $\alpha_h(x, x') = \dot{u}(h)$ .

To prove (26), we need the following estimates.

$$\|\xi\|_2^2 \leq ChN, \tag{30}$$

$$\|\ddot{u}\|_1 \leq Ch(N+1) \tag{31}$$

We first prove (30). Since  $\xi$  is optimal in  $C_h$ ,

$$\begin{aligned} \|\dot{\xi}\|_2 &\leq \|\dot{\xi} - f(\xi)\|_2 + \|f(\xi)\|_2 \\ &\leq \|\dot{\bar{\xi}} - f(\bar{\xi})\|_2 + \|f(\xi)\|_2 \\ &\leq \|\dot{\bar{\xi}}\|_2 + \|f(\bar{\xi})\|_2 + \|f(\xi)\|_2 \\ &\leq \|\dot{\bar{\xi}}\|_2 + C(\|\bar{\xi}\|_2 + \|\xi\|_2) \\ &\leq \|\dot{\bar{\xi}}\|_2 + C(\|\bar{\xi}\|_2 + h^{1/2}\|\xi\|_\infty) \end{aligned} \tag{32}$$

Therefore

$$\begin{aligned} \|\xi\|_\infty &\leq |\xi(0)| + h^{1/2}\|\dot{\xi}\|_2 \\ &\leq |x| + Ch^{1/2}[\|\dot{\bar{\xi}}\|_2 + \|\bar{\xi}\|_2 + h^{1/2}\|\xi\|_\infty] \end{aligned}$$

If  $h_0$  is small enough, then  $Ch < 1/2$ , so that

$$\|\xi\|_\infty \stackrel{(28)}{\leq} 2|x| + C(\sqrt{N} + \sqrt{Nh}).$$

Therefore

$$\|\xi\|_2^2 \leq h\|\xi\|_\infty^2 \leq ChN,$$

which is (30). We now prove (31). Since  $u = \xi - \bar{\xi}$ , it satisfies the following equation

$$\ddot{u} = \nabla f(\xi) \cdot f(\xi),$$

together with the boundary conditions  $u(0) = u(h) = 0$ . It follows that

$$\|\ddot{u}\|_1 \leq C\|\xi\|_1 \leq C(\|\bar{\xi}\|_1 + \|u\|_1) \leq C(\|\bar{\xi}\|_1 + h^2\|\ddot{u}\|_1).$$

Taking  $h_0$  small enough, we have  $Ch^2 < 1/2$ , and therefore

$$\|\ddot{u}\|_1 \leq C\|\bar{\xi}\|_1 \leq Ch(|x| + |x'|) \leq Ch(N+1).$$

The estimate (26) then follows from (31) and the inequality

$$|\dot{u}(h)| \leq \|\dot{u}\|_\infty \leq \|\ddot{u}\|_1 \leq Ch(N+1).$$

□

## 5 The Euler-Lagrange equation for the minimization problem

In this section, we establish the Euler-Lagrange equations for minimizers.

Let  $\bar{\rho} \in \mathcal{A}$  be given and let  $\rho$  be the unique solution of the minimization problem

$$\min_{\mu \in \mathcal{A}} \frac{1}{2h} W_h(\bar{\rho}, \mu) + \mathcal{F}(\mu).$$

We now establish the Euler-Lagrange equation for  $\rho$ . Following the now well-established route (see e.g. [JKO98, Hua00, BCC08]), we first define a perturbation of  $\rho$  by a push-forward under an appropriate flow. Let  $\phi \in C_0^\infty(\mathbf{R}^{2d}, \mathbf{R}^{2d})$ . We define the flows  $\Phi: [0, \infty) \times \mathbf{R}^{2d} \rightarrow \mathbf{R}^{2d}$  such that

$$\begin{aligned} \frac{\partial \Phi_s}{\partial s} &= \phi(\Phi_s), \\ \Phi_0(x) &= x. \end{aligned}$$

For any  $s \in \mathbf{R}$ , let  $\rho_s(x)dx$  be the push forward of  $\rho(x)dx$  under the flow  $(\Phi_s)$ , i.e., for any  $\varphi \in C_0^\infty(\mathbf{R}^{2d}, \mathbf{R})$  we have

$$\int_{\mathbf{R}^{2d}} \varphi(x) \rho_s(x) dx = \int_{\mathbf{R}^{2d}} \varphi(\Phi_s(x)) \rho(x) dx. \quad (33)$$

Obviously  $\rho_0(x) = \rho(x)$ , and an explicit calculation gives

$$\partial_s \rho_s \Big|_{s=0} = -\operatorname{div} \rho \phi \quad \text{in the sense of distributions.} \quad (34)$$

By following the calculations in e.g. [BCC08, Hua00, JKO98] we then compute the stationarity condition on  $\rho$ ,

$$\begin{aligned} \frac{1}{2h} \int_{\mathbf{R}^{4d}} \nabla_{x'} C_h(x, x') \cdot \phi(x') P_{\text{opt}}(dx dx') + \int_{\mathbf{R}^{2d}} [\nabla H \cdot \phi - \Delta \phi] \rho(x) dx \\ + \frac{\chi}{4\pi d} \int_{\mathbf{R}^{4d}} [\nabla \phi(x) - \nabla \phi(y)] \cdot \frac{x-y}{|x-y|^2} \rho(x) \rho(y) dx dy = 0, \end{aligned} \quad (35)$$

where  $P_{\text{opt}}$  is optimal in  $W_h(\bar{\rho}, \rho)$ . For any  $\varphi \in C_0^\infty(\mathbf{R}^{2d}, \mathbf{R})$ , we choose

$$\phi(x') = \nabla \varphi(x'). \quad (36)$$

We have the following Lemma.

**Lemma 5.1.** *Let  $h > 0$  and let  $\{\rho_k^h\}$  be the sequence of the minimizers. Let  $P_k^h$  be optimal in  $W_h(\rho_{k-1}^h, \rho_k^h)$ . Then, for all  $\varphi \in C_c^\infty(\mathbf{R}^{2d})$ , there holds*

$$\begin{aligned} \frac{1}{h} \int_{\mathbf{R}^{4d}} (x' - x) \cdot \nabla \varphi(x') P_k^h(dx dx') - \int_{\mathbf{R}^{2d}} \mathbf{J} \nabla H \cdot \nabla \varphi \rho_k^h dx + \int_{\mathbf{R}^{2d}} \nabla H \cdot \nabla \varphi \rho_k^h dx - \int_{\mathbf{R}^{2d}} \Delta \varphi \rho_k^h dx \\ + \frac{\chi}{4\pi d} \int_{\mathbf{R}^{4d}} [\nabla \phi(x) - \nabla \phi(y)] \cdot \frac{x-y}{|x-y|^2} \rho_k^h(x) \rho_k^h(y) dx dy + \omega_k^h = 0, \end{aligned} \quad (37)$$

where

$$|\omega_k^h| \leq Ch \left[ M_2(\rho_{k-1}^h) + M_2(\rho_k^h) + 1 \right].$$

*Proof.* We combine (25) with (36) to yield

$$\nabla_{x'} C_h(x, x') \cdot \phi(x') = 2[(x' - x) \cdot \nabla \varphi(x') - h \mathbf{J} \nabla H \cdot \nabla \varphi(x')] + 2h \alpha_h(x, x') \cdot \nabla \varphi(x'). \quad (38)$$

Substituting (36) and (38) into the Euler-Lagrange equation (35), we obtain

$$0 = \frac{1}{h} \int_{\mathbf{R}^{4d}} (x' - x) \cdot \nabla \varphi(x') P_k^h(dx dx') - \int_{\mathbf{R}^{2d}} \mathbf{J} \nabla H \cdot \nabla \varphi \rho_k^h dx + \int_{\mathbf{R}^{2d}} [\nabla H \cdot \nabla \varphi - \Delta \varphi] \rho_k^h dx \\ + \frac{\chi}{4\pi d} \int_{\mathbf{R}^{4d}} [\nabla \phi(x) - \nabla \phi(y)] \cdot \frac{x - y}{|x - y|^2} \rho_k^h(x) \rho_k^h(y) dx dy + \int_{\mathbf{R}^{4d}} \alpha_h(x, x') P_k^h(dx dx').. \quad (39)$$

Therefore (5.1) holds with

$$|\omega_k^h| = \left| \int_{\mathbf{R}^{4d}} \alpha_h(x, x') P_k^h(dx dx') \right| \\ \stackrel{(26)}{\leq} \int_{\mathbf{R}^{4d}} Ch(|x|^2 + |x'|^2 + 1) P_k^h(dx dx') \\ = Ch[M_2(\rho_{k-1}^h) + M_2(\rho_k^h) + 1].$$

□

**Remark 5.2.** The perturbation flow of the form as in (36) is also seen in the Wasserstein gradient flow. It should not be a surprise since the dissipative part of (1) is a gradient flow. □

## 6 A priori estimate

This section includes some technical lemmas that are needed in order to prove the convergence result of Section 7.

**Lemma 6.1.** *Let  $\{\rho_k^h\}_{k \geq 1}$  be a sequence of minimizers of Scheme (21) for fixed  $h > 0$ . Then for any positive integer  $k$  and sufficiently small  $h$ , we have*

$$\frac{1}{2h} W_h(\rho_{k-1}^h, \rho_k^h) + \mathcal{F}(\rho_k^h) \leq \mathcal{F}(\rho_{k-1}^h). \quad (40)$$

As a consequence, we obtain the following estimates

$$\sup_{k \in \mathbb{N}} \mathcal{F}(\rho_k^h) \leq \mathcal{F}(\rho_0), \quad \text{and} \quad \frac{1}{2h} \sum_{k \in \mathbb{N}} W_h(\rho_{k-1}^h, \rho_k^h) \leq \mathcal{F}(\rho_0) - \inf_{k \in \mathbb{N}} \mathcal{F}(\rho_k^h). \quad (41)$$

*Proof.* We first define the operator  $\mathbf{s}_h : \mathbf{R}^{2d} \rightarrow \mathbf{R}^{2d}$  as the solution operator over time  $h$  for the Hamiltonian system

$$\xi'(t) = \mathbf{J} \nabla H(\xi(t)), \quad (42)$$

that is,  $\mathbf{s}_h(x)$  is the solution at time  $h$  given the initial datum  $x$  at time zero. It is well-known that the operator  $\mathbf{s}_h$  is bijective and volume-preserving. Moreover, since  $H(x) = \frac{1}{2}|x|^2$ , we can compute  $\mathbf{s}_h$  and its inverse explicitly

$$\mathbf{s}_h(x) = \begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix} \cdot x, \quad \mathbf{s}_h^{-1}(x) = \begin{pmatrix} \cos(h) & \sin(h) \\ \sin(h) & \cos(h) \end{pmatrix} \cdot x. \quad (43)$$

Besides the properties that  $\mathbf{s}_h$  is bijective and volume-preserving, using the above formulation, it is straightforward to see that

$$|\mathbf{s}_h^{-1}(x)| = |x|, \quad |\mathbf{s}_h^{-1}(y) - \mathbf{s}_h^{-1}(x)| = |y - x| \quad \text{for all } x, y \in \mathbf{R}^{2d}. \quad (44)$$

For any fixed  $k \geq 1$ ,  $\rho_k^h$  minimizes the functional  $\frac{1}{2h} W_h(\rho_{k-1}^h, \rho) + \mathcal{F}(\rho)$  over  $\rho \in \mathcal{A}$ , i.e.,

$$\frac{1}{2h} W_h(\rho_{k-1}^h, \rho_k^h) + \mathcal{F}(\rho_k^h) \leq \frac{1}{2h} W_h(\rho_{k-1}^h, \rho) + \mathcal{F}(\rho),$$

for every  $\rho \in \mathcal{A}$ . In particular, by taking  $\rho = (\mathbf{s}_h^{-1})_{\#} \rho_{k-1}^h =: \rho_*^h$ , we get

$$\begin{aligned} \frac{1}{2h} W_h(\rho_{k-1}^h, \rho_k^h) + \mathcal{F}(\rho_k^h) &\leq \frac{1}{2h} W_h(\rho_{k-1}^h, \rho_*^h) + \mathcal{F}(\rho_*^h) \\ &= \frac{1}{2h} W_h(\rho_{k-1}^h, \rho_*^h) + \int_{\mathbf{R}^{2d}} \rho_*^h(x) \log \rho_*^h(x) dx + \frac{1}{2} \int_{\mathbf{R}^{2d}} |x|^2 \rho_*^h(x) dx \\ &\quad + \frac{\chi}{4\pi d} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \log |y-x| \rho_*^h(x) \rho_*^h(y) dx dy. \end{aligned} \quad (45)$$

We now calculate the right hand side of (45) in terms of  $\rho_{k-1}^h$  using the fact that  $\rho_*^h = (\mathbf{s}_h^{-1})_{\#} \rho_{k-1}^h$  and (44).

By definition of  $C_h(x, x')$ , we have  $C_h(x, s_h(x)) = 0$ . This implies that

$$W_h(\rho_{k-1}^h, \rho_*^h) = 0. \quad (46)$$

For the entropy term, we have, since  $\mathbf{s}_h$  is volume-preserving and bijective,

$$\begin{aligned} \int_{\mathbf{R}^{2d}} \rho_*^h(x) \log \rho_*^h(x) dx &= \int_{\mathbf{R}^{2d}} \rho_{k-1}^h(\mathbf{s}_h(x)) \log \rho_{k-1}^h(\mathbf{s}_h(x)) dx \\ &= \int_{\mathbf{R}^{2d}} \rho_{k-1}^h(x) \log \rho_{k-1}^h(x) dx. \end{aligned} \quad (47)$$

For the second moment term, using the first equality in (44), we get

$$\int_{\mathbf{R}^{2d}} |x|^2 \rho_*^h(x) dx = \int_{\mathbf{R}^{2d}} |s_h^{-1}(x)|^2 \rho_{k-1}^h(x) dx = \int_{\mathbf{R}^{2d}} |x|^2 \rho_{k-1}^h(x) dx. \quad (48)$$

For the interaction term, using the second equality in (44), we obtain

$$\begin{aligned} \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \log |y-x| \rho_*^h(x) \rho_*^h(y) dx dy &= \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \log |s_h^{-1}(y) - s_h^{-1}(x)| \rho_{k-1}^h(x) \rho_{k-1}^h(y) dx dy \\ &= \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} \log |y-x| \rho_{k-1}^h(x) \rho_{k-1}^h(y) dx dy. \end{aligned} \quad (49)$$

From (46) to (49), it follows that

$$\frac{1}{2h} W_h(\rho_{k-1}^h, \rho_k^h) + \mathcal{F}(\rho_k^h) \leq \mathcal{F}(\rho_{k-1}^h). \quad (50)$$

Summing the above inequality over  $k \in \mathbb{N}$  and using the fact that  $W_h(\rho_{k-1}^h, \rho_k^h) \geq 0$  for all  $k \in \mathbb{N}$ , and that  $\inf_{\rho \in \mathcal{A}} \mathcal{F}(\rho) > -\infty$  we obtain the estimates in (41).  $\square$

We need the following technical Lemma when passing to the limit in the nonlinear non-local term. The proof can be found in [BCC08, Pou02, NPS01].

**Lemma 6.2.** *Assume that  $\{f_i\}$  satisfies the conditions (a) and (b) in Lemma 3.2 uniformly and that  $f_i \rightharpoonup f$  in  $L^1(\mathbf{R}^n)$ , then  $f_i \otimes f_i \rightharpoonup f \otimes f$  in  $L^1(\mathbf{R}^n) \times L^1(\mathbf{R}^n)$ .*

## 7 Convergence of the Scheme

In this section, we bring all the parts together to prove Theorem 3.6.

Throughout we fix  $T > 0$  and for each  $h > 0$  we set

$$K_h := \lceil T/h \rceil.$$

Let  $(\rho_k^h)_k$  be a sequence of minimizers constructed by Scheme (21), and let  $t \mapsto \rho^h(t)$  be the piecewise-constant interpolation (22). By Lemma 6.1 we have

$$M_2(\rho^h(t)) + \int_{\mathbf{R}^{2d}} \rho^h(t) |\log \rho^h(t)| dx \leq C, \quad \text{for all } 0 \leq t \leq T. \quad (51)$$

By Lemma 3.2, (51) guarantees that there exists a subsequence, denoted again by  $\rho^h$ , and a function  $\rho \in L^1((0, T) \times \mathbf{R}^{2d})$  such that

$$\rho^h \rightarrow \rho \text{ weakly in } L^1((0, T) \times \mathbf{R}^{2d}). \quad (52)$$

This proves (23). We now prove that the limit  $\rho$  satisfies the weak formulation (9).

Fix  $T > 0$  and  $\varphi \in C_c^\infty((-\infty, T) \times \mathbf{R}^{2d})$ ; all constants  $C$  below depend on the parameters of the problem, on the initial datum  $\rho_0$ , and on  $\varphi$ , but are independent of  $k$  and of  $h$ .

Let  $P_k^h \in \Gamma(\rho_{k-1}^h, \rho_k^h)$  be the optimal plan for  $W_h(\rho_{k-1}^h, \rho_k^h)$ . For any  $0 < t < T$ , we have

$$\begin{aligned} & \int_{\mathbf{R}^{2d}} [\rho_k^h(x) - \rho_{k-1}^h(x)] \varphi(t, x) dx \\ &= \int_{\mathbf{R}^{2d}} \rho_k^h(x') \varphi(t, x') dx' - \int_{\mathbf{R}^{2d}} \rho_{k-1}^h(x) \varphi(t, x) dx \\ &= \int_{\mathbf{R}^{4d}} [\varphi(t, x') - \varphi(t, x)] P_k^h(dx dx') \\ &= \int_{\mathbf{R}^{4d}} (x' - x) \cdot \nabla \varphi(t, x') P_k^h(dx dx') + \varepsilon_k, \end{aligned} \quad (53)$$

where

$$\begin{aligned} |\varepsilon_k| &\leq C \int_{\mathbf{R}^{4d}} |x' - x|^2 P_k^h(dx dx') \\ &\stackrel{(24)}{\leq} C W_h(\rho_{k-1}^h, \rho_k^h) + C h^2 [M_2(\rho_{k-1}^h) + M_2(\rho_k^h)] \\ &\stackrel{(51)}{\leq} C W_h(\rho_{k-1}^h, \rho_k^h) + C h^2. \end{aligned} \quad (54)$$

By combining (53) with (37) we find

$$\begin{aligned} & \int_{\mathbf{R}^{2d}} \left( \frac{\rho_k^h(t, x) - \rho_{k-1}^h(t, x)}{h} \right) \varphi(t, x) dx \\ &= \int_{\mathbf{R}^{2d}} [\mathbf{J} \nabla H(x) \cdot \nabla \varphi(t, x) - \nabla H(x) \cdot \nabla \varphi(t, x) + \Delta \varphi(t, x)] \rho_k^h(x) dx \\ &\quad - \frac{\chi}{4\pi d} \int_{\mathbf{R}^{4d}} [\nabla \varphi(t, x) - \nabla \varphi(t, y)] \cdot \frac{x - y}{|x - y|^2} \rho_k^h(x) \rho_k^h(y) dx dy + \theta_k(t), \end{aligned} \quad (55)$$

where

$$\begin{aligned} |\theta_k(t)| &\leq \frac{|\varepsilon_k|}{h} + C h [W_h(\rho_{k-1}^h, \rho_k^h) + M_2(\rho_{k-1}^h) + M_2(\rho_k^h) + 1] \\ &\stackrel{(51), (54)}{\leq} \frac{C}{h} W_h(\rho_{k-1}^h, \rho_k^h) + C h. \end{aligned} \quad (56)$$



Note that  $\theta_k$  depends on  $t$  through the  $t$ -dependence of  $\varphi$ . Next, from (55), for  $k \geq 1$  we have

$$\begin{aligned}
& \int_{(k-1)h}^{kh} \int_{\mathbf{R}^{2d}} \left( \frac{\rho_k^h(x) - \rho_{k-1}^h(x)}{h} \right) \varphi(t, x) dx dt \\
&= \int_{(k-1)h}^{kh} \int_{\mathbf{R}^{2d}} [\mathbf{J}\nabla H(x) \cdot \nabla \varphi(t, x) - \nabla H(x) \cdot \nabla \varphi(t, x) + \Delta \varphi(t, x)] \rho_k^h(x) dx dt \\
&\quad - \frac{\chi}{4\pi d} \int_{(k-1)h}^{kh} \int_{\mathbf{R}^{4d}} [\nabla \varphi(t, x) - \nabla \varphi(t, y)] \cdot \frac{x-y}{|x-y|^2} \rho_k^h(x) \rho_k^h(y) dx dy + \int_{(k-1)h}^{kh} \theta_k(t) dt \\
&= \int_{(k-1)h}^{kh} \int_{\mathbf{R}^{2d}} [\mathbf{J}\nabla H(x) \cdot \nabla \varphi(t, x) - \nabla H(x) \cdot \nabla \varphi(t, x) + \Delta \varphi(t, x)] \rho^h(t, x) dx dt \\
&\quad - \frac{\chi}{4\pi d} \int_{(k-1)h}^{kh} \int_{\mathbf{R}^{4d}} [\nabla \varphi(t, x) - \nabla \varphi(t, y)] \cdot \frac{x-y}{|x-y|^2} \rho_k^h(x) \rho_k^h(y) dx dy dt + \int_{(k-1)h}^{kh} \theta_k(t) dt.
\end{aligned}$$

Summing from  $k = 1$  to  $K_h$  we obtain

$$\begin{aligned}
& \sum_{k=1}^{K_h} \int_{(k-1)h}^{kh} \int_{\mathbf{R}^{2d}} \left( \frac{\rho_k^h(x) - \rho_{k-1}^h(x)}{h} \right) \varphi(t, x) dx dt \\
&= \int_0^T \int_{\mathbf{R}^{2d}} [\mathbf{J}\nabla H(x) \cdot \nabla \varphi(t, x) - \nabla H(x) \cdot \nabla \varphi(t, x) + \Delta \varphi(t, x)] \rho^h(t, x) dx dt \\
&\quad - \frac{\chi}{4\pi d} \int_0^T \int_{\mathbf{R}^{4d}} [\nabla \varphi(t, x) - \nabla \varphi(t, y)] \cdot \frac{x-y}{|x-y|^2} \rho^h(t, x) \rho^h(t, y) dx dy + R_h, \tag{57}
\end{aligned}$$

where

$$R_h = \sum_{k=1}^{K_h} \int_{(k-1)h}^{kh} \theta_k(t) dt. \tag{58}$$

By a discrete integration by parts, we can rewrite the left hand side of (57) as

$$- \int_0^h \int_{\mathbf{R}^{2d}} \rho_0(x) \frac{\varphi(t, x)}{h} dx dt + \int_0^T \int_{\mathbf{R}^{2d}} \rho^h(t, x) \left( \frac{\varphi(t, x) - \varphi(t+h, x)}{h} \right) dx dt. \tag{59}$$

From (57) and (59) we obtain

$$\begin{aligned}
& \int_0^T \int_{\mathbf{R}^{2d}} \rho^h(t, x) \left( \frac{\varphi(t, x) - \varphi(t+h, x)}{h} \right) dx dt \\
&= \int_0^T \int_{\mathbf{R}^{2d}} [\mathbf{J}\nabla H(x) \cdot \nabla \varphi(t, x) - \nabla H(x) \cdot \nabla \varphi(t, x) + \Delta \varphi(t, x)] \rho^h(t, x) dx dt \\
&\quad - \frac{\chi}{4\pi d} \int_0^T \int_{\mathbf{R}^{4d}} [\nabla \varphi(t, x) - \nabla \varphi(t, y)] \cdot \frac{x-y}{|x-y|^2} \rho^h(t, x) \rho^h(t, y) dx dy \tag{60}
\end{aligned}$$

$$+ \int_0^h \int_{\mathbf{R}^{2d}} \rho_0(x) \frac{\varphi(t, x)}{h} dx dt + R_h. \tag{61}$$

Now  $R_h \rightarrow 0$  as  $h \rightarrow 0$ , since

$$\begin{aligned}
|R_h| &\stackrel{(58)}{\leq} \sum_{k=1}^{K_h} \int_{(k-1)h}^{kh} |\theta_k(t)| dt \stackrel{(56)}{\leq} C \sum_{k=1}^{K_h} \int_{(k-1)h}^{kh} \left( \frac{1}{h} W_h(\rho_{k-1}^h, \rho_k^h) + h \right) dt \\
&= C \sum_{k=1}^{K_h} [W_h(\rho_{k-1}^h, \rho_k^h) + Ch^2] \stackrel{(41)}{\leq} Ch.
\end{aligned}$$

Taking the limit  $h \rightarrow 0$  in (61) yields equation (9). This finishes the proof the the main theorem.

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